Convolution operators on spaces of real analytic functions

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Workshop on Functional Analysis, Valencia June 2013
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Convolution operators

\[ E(\Omega) = \text{a lcs. of (generalized) functions on } \Omega \subset \mathbb{R}^d. \]

For \( \mu \in E(\Omega)' \)

\[ T_\mu : E(\Omega - \text{supp}(\mu)) \rightarrow E(\Omega) \]

is defined by

\[ T_\mu(g)(x) := \mu \ast g(x) := \langle y \mu, g(x - y) \rangle, x \in \Omega. \]
Convolution operators

\( E(\Omega) = \) a lcs. of (generalized) functions on \( \Omega \subset \mathbb{R}^d \).

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\]

Examples. a) pdo (of finite or infinite order)
b) shift operators
c) mollifiers
Problems:

a) When is $T_\mu$ surjective?

b) When does $T_\mu$ admit a (continuous linear) right inverse i.e. when exists a continuous linear operator

$$R : E(\Omega) \to E(\Omega - \text{supp}(\mu))$$

such that $T_\mu \circ R = id$ ?
Known results

- **holomorphic functions**: \( E(\Omega) := \mathcal{H}(\Omega), \Omega \subset \mathbb{C}^d \text{ open} \)

  '55 Malgrange: \( T_\mu : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}) \) is always surjective.

  Further results (also for holomorphic germs): Korobeinik, Krivosheev, Gramtsev, Napalkov, Okada, Momm, Melikhov, . . .
Known results

• **holomorphic functions:** $E(\Omega) := \mathcal{H}(\Omega), \Omega \subset \mathbb{C}^d$ open

  ’55 Malgrange: $T_\mu : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ is always surjective.

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• **real analytic functions:** $E(\Omega) := \mathcal{A}(\Omega), \Omega \subset \mathbb{R}^d$ open

  ’60 Ehrenpreis: $T_\mu(f) = g \in \mathcal{A}(\mathbb{R}^d)$ solvable with $f \in C^\infty(\mathbb{R}^d)$

  ’68 Hörmander: $\Omega$ convex
 pdo in $A(\Omega)$, $\Omega$ convex:

'73 Piccinini: first example in the negative

'73 Hörmander: Characterization via PL - condition

wide range of applications to several problems in analysis:

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• **pdo in** $A(\Omega)$, $\Omega$ convex:

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  wide range of applications to several problems in analysis:

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• **pdo in** $A(\Omega)$, general $\Omega$

  '70 Kawai: sufficient condition for certain non convex $\Omega$

  '04 L.: characterization for general $\Omega$
Open problems.

a) Surjectivity of pdo $P(D)$ on $\mathcal{A}(\Omega)$ versus surjectivity of principal part $P_m(D)$ on $\mathcal{A}(\Omega)$

b) Surjectivity of convolution operators on $\mathcal{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open
• **convolution operators on** $\mathcal{A}(I)$

notation: $I \subset \mathbb{R}$ always is an open interval and $\mu \in \mathcal{A}(\mathbb{R})'$. 
• convolution operators on $\mathcal{A}(I)$

notation: $I \subset \mathbb{R}$ always is an open interval and $\mu \in \mathcal{A}(\mathbb{R})'$.

**Theorem.** (Napalkov/Rudakov '91, Meyer '92)

Let $\text{supp}(\mu) = \{0\}$ (i.e. $\mu$ is a differential operator). Then $T_\mu : \mathcal{A}(I) \to \mathcal{A}(I)$ is surjective iff there are $\delta > 0$, $r(t) = o(|t|)$:

$$\mu(z) \neq 0 \text{ if } r(|\text{Re}(z)|) \leq |\text{Im}(z)| \leq \delta|\text{Re}(z)|.$$  \hspace{1cm} (1)
convolution operators on $A(I)$

notation: $I \subset \mathbb{R}$ always is an open interval and $\mu \in A(\mathbb{R})'$.

**Theorem.** (Napalkov/Rudakov ’91, Meyer ’92)
Let $\text{supp}(\mu) = \{0\}$ (i.e. $\mu$ is a differential operator). Then $T_\mu : \mathcal{A}(I) \rightarrow \mathcal{A}(I)$ is surjective iff there are $\delta > 0$, $r(t) = o(|t|)$:

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\mu(z) \neq 0 \text{ if } r(|\text{Re}(z)|) \leq |\text{Im}(z)| \leq \delta |\text{Re}(z)|.
$$

(1)

**Theorem.** (L. ’95)
$T_\mu$ is surjective on $A(\mathbb{R})$ iff $\hat{\mu}$ satisfies (1) and

$$
\forall x \in \mathbb{R}, |x| \geq C \exists t \in \mathbb{C} : |t - x| \leq r(x) \text{ and } |\hat{\mu}(t)| \geq e^{-r(t)}.
$$

(E)
The characterization

For $G := \text{conv}(\text{supp}(\mu))$ let $H_G(z) := \sup_{x \in G} \langle x, \text{Im } z \rangle$.

**Theorem.** The following are equivalent:

a) $T_\mu : A(I - G) \to A(I)$ is surjective for some bounded $I$.

b) $T_\mu : A(I - G) \to A(I)$ is surjective for any $I$.

c) For any $\eta > 0$ there are $\eta_0 > 0$ and $\rho_\eta(t) = o(t)$ such that

$$|\hat{\mu}(z)| \geq e^{H_G(z) - \eta |\text{Im}(z)|}$$

if $\rho_\eta(|\text{Re}(z)|) \leq |\text{Im}(z)| \leq \eta_0 |\text{Re } z|$.
Hyperfunctions and elementary solutions

For $c \geq 0$ and $U \subset \mathbb{R}$ open let

$$\mathcal{B}_c(U) := \mathcal{H}(U \times (\mathbb{R} \setminus [-c, c])) / \mathcal{H}(U \times \mathbb{R})$$

= generalized hyperfunctions on $U$

$$\mathcal{B}_0(U) =: \mathcal{B}(U) = \text{hyperfunctions on } U$$
$\mu \in \mathcal{A}(\mathbb{R})'$ canonically defines a convolution operator

$$S_\mu : \mathcal{B}_c(U - G) \rightarrow \mathcal{B}_c(U) \text{ via}$$

$$S_\mu(u)(z) := \langle y\mu, u(z - y) \rangle \text{ for } u \in \mathcal{B}_c(U - G).$$
\( \mu \in \mathcal{A}(\mathbb{R})' \) canonically defines a convolution operator

\[
S_{\mu} : \mathcal{B}_c(U - G) \rightarrow \mathcal{B}_c(U) \text{ via }
S_{\mu}(u)(z) := \langle y\mu, u(z - y) \rangle \text{ for } u \in \mathcal{B}_c(U - G).
\]

\( E \in \mathcal{B}_c(I - G) \) is a \( \{t\} \)-elementary solution for \( S_{\mu} \) if

\[
S_{\mu}(E) = \delta_t = \text{ point evaluation at } t \in I.
\]

Notice that elementary solutions are \( \{0\} \)-elementary solutions.
Theorem. The following are equivalent:

a) $T_\mu : A(I - G) \rightarrow A(I)$ is surjective for some bounded $I$.

b) $\forall K \subset I \exists J \subset I, \gamma > 0 \forall t \in I \setminus J, 0 < c < \gamma$:
   there is a $\{t\}$-elementary solution $E \in \mathcal{B}_c(I - G)$ such that
   
   $E|_{K - G} \in \mathcal{H}((K - G) \times ]) - \gamma, \gamma[).$

\[ E|_{K - G} \in \mathcal{H}((K - G) \times ]) - \gamma, \gamma[). \]

c) $\hat{\mu}$ satisfies (2).

d) $S_\mu$ admits hyperfunction elementary solutions $E_\pm$ such that
   $E_+|_{-a_+, \infty[} \text{ is holomorphic and bounded on the shifted cone}$
   
   $-a_+ + \varepsilon + \{z \in \mathbb{C}_+ \mid \vert \text{Im}(z)\vert \leq \varepsilon \vert \text{Re}(z)\vert\}, \varepsilon > 0$

(similarly for $E_-|_{-\infty, a_-[}$).
Remark. ”d) ⇒ e) ⇒ b)” where

b) $\forall K \subset I \exists J \subset I, \gamma > 0 \forall t \in I \setminus J, 0 < c < \gamma$ : there is a $\{t\}$-elementary solution $E \in B_c(I - G)$ such that

$$E|_{K - G} \in \mathcal{H}((K - G) \times ] - \gamma, \gamma[).$$

d) $S_\mu$ admits hyperfunction elementary solutions $E_\pm$ such that $E_+ |_{-a_+, \infty}$ is holomorphic and bounded on the shifted cone

$$-a_+ + \varepsilon + \{z \in \mathbb{C}_+ \mid \| \text{Im}(z) \| \leq \varepsilon \| \text{Re}(z) \| \}, \varepsilon > 0$$

(similarly for $E_- |_{-\infty, a_-}$).

e) $S_\mu$ admits hyperfunction elementary solutions $E_\pm$ such that $E_+ |_{-a_+, \infty}$ and $E_- |_{-\infty, a_-}$ are real analytic.
Theorem. The following are equivalent:

a) $T_\mu : A(I - G) \to A(I)$ is surjective for some bounded $I$.

b) $T_\mu : A(I - G) \to A(I)$ is surjective for any $I$.

c) For any $\eta > 0$ there are $\eta_0 > 0$ and $\rho_\eta(t) = o(t)$ such that

$$|\hat{\mu}(z)| \geq e^{H_G(z) - \eta |\text{Im}(z)|}$$

(2)

if $\rho_\eta(|\text{Re}(z)|) \leq |\text{Im}(z)| \leq \eta_0 |\text{Re} z|$.

d) $S_\mu$ admits hyperfunction elementary solutions $E_\pm$ such that $E_+|_{-\infty, a-}$ and $E_-|_{-a+, \infty}$ are real analytic.
Theorem. The following are equivalent.

a) $T_\mu : A(I - G) \rightarrow A(I)$ admits a continuous linear right inverse for some bounded $I$.

b) $T_\mu : A(I - G) \rightarrow A(I)$ admits a continuous linear right inverse for any $I$.

c) There is $\rho(t) = o(t)$ such that

$$|\hat{\mu}(z)| \geq e^{HG(z) - \rho(|z|)} \text{ if } |\text{Im}(z)| \geq \rho(|\text{Re}(z)|).$$

(3)

d) $S_\mu$ has hyperfunction fundamental solutions $E_\pm$ such that $\text{supp}(E_+) \subset \left]-\infty, -a_+\right]$ and $\text{supp}(E_-) \subset \left]-a_-, \infty\right].$
References
