Absolutely summing Carleson embeddings on Hardy spaces.

Luis Rodríguez Piazza

Universidad de Sevilla
Spain

Valencia, 4th June 2013
WFAV2013
On the occasion of the 60th birthday of A. Defant.
I will present some results obtained in collaboration with Pascal Lefèvre (Université d’Artois).

This is a work still in progress.
Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ the open unit disk and $\phi : \mathbb{D} \to \mathbb{D}$ an holomorphic function.

The composition operator with symbol $\phi$ is $C_\phi$, defined on $\mathcal{H}(\mathbb{D})$ by

$$C_\phi : f \mapsto f \circ \phi$$
Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ the open unit disk and $\phi : \mathbb{D} \to \mathbb{D}$ an holomorphic function.

The composition operator with symbol $\phi$ is $C_\phi$, defined on $\mathcal{H}(\mathbb{D})$ by

$$C_\phi : f \mapsto f \circ \phi$$

If $\mathcal{E}$ is a Banach space of analytic functions over the disk one tries to characterize the properties of the operator $C_\phi : \mathcal{E} \to \mathcal{E}$ in terms of the properties of the symbol $\phi$.

In that way one can study when the operator is well defined (boundedness), when it is compact, weakly compact, $q$-summing, nuclear,...
Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) the open unit disk and \( \phi : \mathbb{D} \to \mathbb{D} \) an holomorphic function.

The composition operator with symbol \( \phi \) is \( C_\phi \), defined on \( \mathcal{H}(\mathbb{D}) \) by

\[
C_\phi : f \mapsto f \circ \phi
\]

If \( \mathcal{E} \) is a Banach space of analytic functions over the disk one tries to characterize the properties of the operator \( C_\phi : \mathcal{E} \to \mathcal{E} \) in terms of the properties of the symbol \( \phi \).

In that way one can study when the operator is well defined (boundedness), when it is compact, weakly compact, \( q \)-summing, nuclear,\ldots

In this talk we will be dealing with the study of the \( q \)-summingness when \( \mathcal{E} \) is a Hardy space \( H^p \), \( 1 \leq p < +\infty \).
Suppose $1 \leq q < +\infty$ and let $T : X \to Y$ be bounded linear operator between two Banach spaces. We say $T$ is a $q$-summing operator if there exists $C > 0$ such that

$$\sum_{j=1}^{n} \|Tx_j\|^q \leq C \sup_{x^* \in B_{X^*}} \sum_{j=1}^{n} |\langle x^*, x_j \rangle|^q,$$

for every finite sequence $x_1, x_2, \ldots, x_n$ in $X$. The $q$-summing norm of $T$ is $\pi_q(T) = \inf \{ C^{1/q} : C > 0, C$ occurs in above $\}$.
Suppose $1 \leq q < +\infty$ and let $T : X \to Y$ be a bounded linear operator between two Banach spaces. We say $T$ is a \textit{$q$-summing operator} if there exists $C > 0$ such that

$$
\sum_{j=1}^{n} \| Tx_j \|^q \leq C \sup_{x^* \in B_{X^*}} \sum_{j=1}^{n} |\langle x^*, x_j \rangle|^q,
$$

(♣)

for every finite sequence $x_1, x_2, \ldots, x_n$ in $X$. The $q$-summing norm of $T$ is

$$
\pi_q(T) = \inf \{ C^{1/q} : C > 0, C \text{ occurs in (♣)} \}.
$$

1-summing operators are also called absolutely summing operators.
If $1 \leq p < +\infty$, the Hardy space $H^p = H^p(\mathbb{D})$ is formed by the holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$
\|f\|_{H^p} = \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} < +\infty.
$$

$H^\infty(\mathbb{D})$ is the space of bounded analytic functions on $\mathbb{D}$. 
If $1 \leq p < +\infty$, the Hardy space $H^p = H^p(\mathbb{D})$ is formed by the holomorphic functions $f: \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} < +\infty.$$

$H^\infty(\mathbb{D})$ is the space of bounded analytic functions on $\mathbb{D}$.

Let $T = \partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \}$. On the torus $T$ we consider the normalized arc–length measure $m$. Every $f \in H^p(\mathbb{D})$ has almost everywhere radial limit $f^*$

$$f^*(e^{it}) = \lim_{r \to 1^-} f(re^{it}).$$

It is known that $f^* \in L^p(T) = L^p(m)$ and $\|f\|_{H^p} = \|f^*\|_{L^p}$. 

Luis Rodríguez Piazza

Absolutely summing Carleson embeddings.
For every $\phi: \mathbb{D} \to \mathbb{D}$ and all $p$, the operator $C_\phi: H^p \to H^p$ is bounded (and well defined).

This is obvious for $p = \infty$. For $1 \leq p < +\infty$ it is a consequence of Littlewood’s Subordination Principle.
For every $\phi : \mathbb{D} \to \mathbb{D}$ and all $p$, the operator $C_\phi : H^p \to H^p$ is bounded (and well defined).

This is obvious for $p = \infty$. For $1 \leq p < +\infty$ it is a consequence of Littlewood’s Subordination Principle.

Other properties of $C_\phi$ depends on the symbol $\phi$. 
For every $\phi: \mathbb{D} \rightarrow \mathbb{D}$ and all $p$, the operator $C_\phi: H^p \rightarrow H^p$ is bounded (and well defined).

This is obvious for $p = \infty$. For $1 \leq p < +\infty$ it is a consequence of Littlewood’s Subordination Principle.

Other properties of $C_\phi$ depends on the symbol $\phi$.

This happens for instance to compactness, which was characterized in two different ways in the middle of the eighties:

1) Using the Nevanlinna counting function (Shapiro).
2) Using vanishing Carleson measures (MacCluer).
For every $\phi: \mathbb{D} \to \mathbb{D}$ and all $p$, the operator $C_\phi: H^p \to H^p$ is bounded (and well defined).

This is obvious for $p = \infty$. For $1 \leq p < +\infty$ it is a consequence of Littlewood’s Subordination Principle.

Other properties of $C_\phi$ depend on the symbol $\phi$.

This happens for instance to compactness, which was characterized in two different ways in the middle of the eighties:

1) Using the Nevanlinna counting function (Shapiro).
For every $\phi: \mathbb{D} \to \mathbb{D}$ and all $p$, the operator $C_\phi: H^p \to H^p$ is bounded (and well defined).

This is obvious for $p = \infty$. For $1 \leq p < +\infty$ it is a consequence of Littlewood’s Subordination Principle.

Other properties of $C_\phi$ depends on the symbol $\phi$.

This happens for instance to compactness, which was characterized in two different ways in the middle of the eighties:

1) Using the Nevanlinna counting function (Shapiro).
2) Using vanishing Carleson measures (MacCluer).
For $f \in H^p$ we have

$$\|C_\phi f\|_{H^p}^p = \|(f \circ \phi)^*\|_{L^p(T)}^p =$$
For $f \in H^p$ we have

$$\| C_\phi f \|_{H^p}^p = \| (f \circ \phi)^* \|_{L^p(\mathbb{T})}^p = \int_{\mathbb{T}} |f|^p \circ \phi^* \, dm.$$
For $f \in H^p$ we have

$$\| C_\phi f \|_{H^p}^p = \| (f \circ \phi)^* \|^p_{L^p(T)} = \int_T |f|^p \circ \phi^* \, dm.$$ 

Let us denote $\mu_\phi$ to the image measure of $m$ by the map $\phi^*$; that is, $\mu_\phi(B) = m(\{\phi^* \in B\})$, for all Borel set $B \subset \overline{D}$. We have

$$\| C_\phi f \|_{H^p} = \| f \|_{L^p(\mu_\phi)}.$$
For $f \in H^p$ we have
\[
\| C_\phi f \|_{H^p}^p = \| (f \circ \phi)^* \|_{L^p(\mathbb{T})}^p = \int_{\mathbb{T}} |f|^p \circ \phi^* \, dm.
\]

Let us denote $\mu_\phi$ to the image measure of $m$ by the map $\phi^*$; that is, $\mu_\phi(B) = m(\{\phi^* \in B\})$, for all Borel set $B \subset \overline{D}$. We have
\[
\| C_\phi f \|_{H^p} = \| f \|_{L^p(\mu_\phi)}.
\]

This allows to see that the properties of the operator $C_\phi$ are the same that the properties of the inclusion operator
\[
j_{\mu_\phi} : H^p \hookrightarrow L^p(\mu_\phi).
\]
Let $0 < h < 1$. We define the window of center $\xi \in \mathbb{T}$ and radius $h$ as

$$W(\xi, h) = \{ z \in \mathbb{D} : 1 - h < |z|, \arg(\xi z) < h \}.$$
Let $0 < h < 1$. We define the window of center $\xi \in \mathbb{T}$ and radius $h$ as

$$W(\xi, h) = \{z \in \overline{D} : 1 - h < |z|, |\text{arg}(\overline{\xi}z)| < h\}.$$

Carleson windows

Absolutely summing Carleson embeddings.
Carleson’s Theorem

\begin{theorem}[Carleson, 1962] Let $\mu$ be a finite measure on the Borel sets of $D$. For $1 \leq p < \infty$, we have the inclusion $H^p(D) \subset L^p(\mu)$ if and only if there exists $C > 0$ such that

$$\mu(W(\xi, h)) \leq Ch, \quad \forall \xi \in \mathbb{T}, \; \forall h \in (0, 1).$$

\end{theorem}

A measure satisfying (♣) is called a Carleson measure.
Carleson’s Theorem

Theorem (Carleson, 1962)

Let $\mu$ be a finite measure on the Borel sets of $\overline{D}$. For $1 \leq p < \infty$, we have the inclusion $H^p(D) \subset L^p(\mu)$ if and only if there exists $C > 0$ such that

$$\mu\left(W(\xi, h)\right) \leq Ch, \quad \forall \xi \in \mathbb{T}, \forall h \in (0, 1).$$

A measure satisfying (♣) is called a Carleson measure.

Putting $\rho_{\mu}(h) = \sup_{\xi \in \mathbb{T}} \mu\left(W(\xi, h)\right)$, we have that $\mu$ is a Carleson measure if and only if

$$\frac{\rho_{\mu}(h)}{h}$$

is bounded for $0 < h < 1$. 
Carleson’s Theorem

**Theorem (Carleson, 1962)**

Let $\mu$ be a finite measure on the Borel sets of $\overline{D}$. For $1 \leq p < \infty$, we have the inclusion $H^p(D) \subset L^p(\mu)$ if and only if there exists $C > 0$ such that

$$\mu\left(W(\xi, h)\right) \leq Ch, \quad \forall \xi \in \mathbb{T}, \forall h \in (0, 1).$$

(♣)

A measure satisfying (♣) is called a **Carleson measure**.

Putting $\rho_\mu(h) = \sup_{\xi \in \mathbb{T}} \mu\left(W(\xi, h)\right)$, we have that $\mu$ is a Carleson measure if and only if

$$\frac{\rho_\mu(h)}{h}$$

is bounded for $0 < h < 1$.

Moreover we have

$$\| j_\mu : H^p \hookrightarrow L^p(\mu) \| \approx \left( \sup_{0 < h < 1} \frac{\rho_\mu(h)}{h} \right)^{1/p}.$$
MacCluer’s Theorem

The measure $\mu$ is called to be a vanishing Carleson measure if

$$\lim_{h \to 0^+} \frac{\rho_\mu(h)}{h} = 0.$$
The measure $\mu$ is called to be a vanishing Carleson measure if

$$\lim_{h \to 0^+} \frac{\rho_{\mu}(h)}{h} = 0.$$ 

MacCluer (1985)

The composition operator $C_{\phi}: H^p \to H^p$ is compact if and only if $\mu_{\phi}$ is a vanishing Carleson measure.

Actually we have that, for any finite measure $\mu$, the inclusion of $H^p(\mathbb{D})$ in $L^p(\mu)$ defines a compact operator if and only if $\mu$ is a vanishing Carleson measure.
Carleson embeddings

Assume from now on that $\mu$ is concentrated in the open disk $\mathbb{D}$. For $\mu$ a Carleson measure, our aim is to characterize when the Carleson embedding

$$j_{\mu} : H^p(\mathbb{D}) \hookrightarrow L^p(\mu)$$

is a $q$-summing operator.
Assume from now on that $\mu$ is concentrated in the open disk $\mathbb{D}$.

For $\mu$ a Carleson measure, our aim is to characterize when the Carleson embedding

$$j_\mu : H^p(\mathbb{D}) \hookrightarrow L^p(\mu)$$

is a $q$-summing operator.

Observe that the conditions in Carleson’s and MacCluer’s theorems do not depend on $p$. So compactness and boundedness of Carleson embeddings do not depend on $p$.

We will see that this is not the case for $q$-summingness.
If $q_1 \leq q_2$, every $q_1$-summing operator is $q_2$-summing.
Known facts

If $q_1 \leq q_2$, every $q_1$-summing operator is $q_2$-summing.

For $1 \leq p \leq 2$, $H^p$ and $L^p$ have cotype 2. For $p > 2$ they only have cotype $p$. So it is known that

$$j_\mu \text{ is } q_1\text{-summing} \iff j_\mu \text{ is } q_2\text{-summing}$$

in the following cases:

$\mu$
If $q_1 \leq q_2$, every $q_1$-summing operator is $q_2$-summing.

For $1 \leq p \leq 2$, $H^p$ and $L^p$ have cotype 2. For $p > 2$ they only have cotype $p$. So it is known that

$$j_\mu \text{ is } q_1\text{-summing} \iff j_\mu \text{ is } q_2\text{-summing}$$

in the following cases:

- For $1 \leq p \leq 2$ and $q_1, q_2 \geq 1$. 

Known facts

If \( q_1 \leq q_2 \), every \( q_1 \)-summing operator is \( q_2 \)-summing.

For \( 1 \leq p \leq 2 \), \( H^p \) and \( L^p \) have cotype 2. For \( p > 2 \) they only have cotype \( p \). So it is known that

\[
j_\mu \text{ is } q_1\text{-summing } \iff j_\mu \text{ is } q_2\text{-summing}
\]

in the following cases:

- For \( 1 \leq p \leq 2 \) and \( q_1, q_2 \geq 1 \).
- For \( p > 2 \), and \( 1 \leq q_1, q_2 < p' \), where \( p' \) is the conjugate exponent of \( p \).
Theorem (Shapiro-Taylor, 1973)

Let $p \geq 2$. The composition operator $C_\phi : H^p \to H^p$ is $p$-summing if and only if

$$\int_{\mathbb{T}} \frac{1}{1 - |\phi^*|} \, dm < +\infty.$$
Theorem (Shapiro-Taylor, 1973)

Let $p \geq 2$. The composition operator $C_\phi : H^p \to H^p$ is $p$-summing if and only if

$$\int_T \frac{1}{1 - |\phi^*|} \, dm < +\infty.$$ 

In the Carleson embedding setting the condition is

$$\int_D \frac{1}{1 - |z|} \, d\mu(z) < +\infty$$

($\spadesuit$)
Theorem (Shapiro-Taylor, 1973)

Let $p \geq 2$. The composition operator $C_\phi : H^p \to H^p$ is $p$-summing if and only if

$$
\int_T \frac{1}{1 - |\phi^*|} \, dm < +\infty.
$$

In the Carleson embedding setting the condition is

$$
\int_\mathbb{D} \frac{1}{1 - |z|} \, d\mu(z) < +\infty \quad (\spadesuit)
$$

It is known that (\spadesuit) also implies $j_\mu : H^p \to L^p(\mu)$ is $p$-summing for $1 \leq p < 2$. But the converse is not true.
Decompose the disk $\mathbb{D}$ into the family of annulus $\{\Gamma_n\}_{n \geq 0}$ where

$$\Gamma_n = \{ z \in \mathbb{D} : 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \} \quad n = 0, 1, 2, \ldots$$

Then decompose each annulus into $2^n$ equal pieces with the shape of "round" rectangles. We will call them Luecking rectangles.

$$R_{n,j} = \{ z = re^{i\theta} : 1 - 2^{-n} \leq r < 1 - 2^{-n-1}, 2\pi(j-1)/2^n \leq \theta < 2\pi j/2^n \}$$

with $n = 0, 1, 2, 3, \ldots$ and $1 \leq j \leq 2^n$. 
Luecking rectangles

These sets $R_{n,j}$ were used by D. Luecking to characterize the membership of composition operators on $H^2$ to the Schatten classes.
Let us fix a finite measure $\mu$ on $\mathbb{D}$. We denote by $\mu_n$ the restriction of $\mu$ to the annulus $\Gamma_n$, and by $j_n$ the inclusion of $H^p(\mathbb{D})$ into $L^p(\mu_n)$. 

Finally let $\alpha_n$ be the restriction of $j_n$ to $X_n$. 

Absolutely summing Carleson embeddings.
First results

Let us fix a finite measure \( \mu \) on \( \mathbb{D} \). We denote by \( \mu_n \) the restriction of \( \mu \) to the annulus \( \Gamma_n \), and by \( j_n \) the inclusion of \( H^p(\mathbb{D}) \) into \( L^p(\mu_n) \).

Now consider, for \( n \geq 0 \), the \( 2^n \)-dimensional subspace \( X_n \) of \( H^p(\mathbb{D}) \) generated by the monomials \( z^k \), with \( 2^n \leq k < 2^{n+1} \). We have, the decomposition

\[
H^p_0(\mathbb{D}) = \{ f \in H^p(\mathbb{D}) : f(0) = 0 \} = \bigoplus_{n \geq 0} X_n
\]

which is an orthogonal decomposition in the case of \( H^2 \).
First results

Let us fix a finite measure $\mu$ on $\mathbb{D}$. We denote by $\mu_n$ the restriction of $\mu$ to the annulus $\Gamma_n$, and by $j_n$ the inclusion of $H^p(\mathbb{D})$ into $L^p(\mu_n)$.

Now consider, for $n \geq 0$, the $2^n$-dimensional subspace $X_n$ of $H^p(\mathbb{D})$ generated by the monomials $z^k$, with $2^n \leq k < 2^{n+1}$. We have, the decomposition

$$H^p_0(\mathbb{D}) = \{ f \in H^p(\mathbb{D}) : f(0) = 0 \} = \bigoplus_{n \geq 0} X_n$$

which is an orthogonal decomposition in the case of $H^2$.

Finally let $\alpha_n$ be the restriction of $j_n$ to $X_n$. 

Luis Rodríguez Piazza

Absolutely summing Carleson embeddings.
Proposition

For $1 < p < +\infty$, the following quantities are equivalent:

1. $\pi_q(j_n: H^p \to L^p(\mu_n))$,
Proposition

For $1 < p < +\infty$, the following quantities are equivalent:

1. $\pi_q(j_n: H^p \to L^p(\mu_n))$,
2. $\pi_q(\alpha_n: X_n \to L^p(\mu_n))$, and
First result

Proposition

For $1 < p < +\infty$, the following quantities are equivalent:

1. $\pi_q(j_n: H^p \to L^p(\mu_n))$,
2. $\pi_q(\alpha_n: X_n \to L^p(\mu_n))$, and
3. $\pi_q(D_a)$, where $D_a: \ell^2_p \to \ell^2_p$ is the diagonal operator

$$x = (x_j)_j \mapsto D_a(x) = (a_jx_j)_j,$$

with $a_j = (2^n\mu(R_{n,j}))^{1/p}, \quad j = 1, 2, \ldots, 2^n$. 

Luis Rodríguez Piazza
Absolutely summing Carleson embeddings.
First results

In consequence we have:

$$1 < p \leq 2: \quad \pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} 2^n \mu(R_{n,j})^{2/p} \right)^{1/2}.$$
First results

In consequence we have:

\[1 < p \leq 2:\quad \pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} \left[ 2^n \mu(R_{n,j}) \right]^{2/p} \right)^{1/2}.
\]

\[p > 2:\quad \pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} \left[ 2^n \mu(R_{n,j}) \right]^{p'/p} \right)^{1/p'}, \quad \text{if } 1' \leq q \leq p'.\]
First results

In consequence we have:

1 \leq p \leq 2: \quad \pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} [2^n \mu(R_{n,j})]^{2/p} \right)^{1/2}.

p > 2: \quad \pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} [2^n \mu(R_{n,j})]^{p'/p} \right)^{1/p'}, \quad \text{if } 1' \leq q \leq p'.

\pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} [2^n \mu(R_{n,j})]^{q/p} \right)^{1/q}, \quad \text{if } p' \leq q \leq p.
In consequence we have:

\[
1 < p \leq 2: \quad \pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} \left[ 2^n \mu(R_{n,j}) \right]^{2/p} \right)^{1/2}.
\]

\[
p > 2: \quad \pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} \left[ 2^n \mu(R_{n,j}) \right]^{p'/p} \right)^{1/p'}, \quad \text{if } 1' \leq q \leq p'.
\]

\[
\pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} \left[ 2^n \mu(R_{n,j}) \right]^{q/p} \right)^{1/q}, \quad \text{if } p' \leq q \leq p.
\]

\[
\pi_q(j_n) \approx \left( \sum_{j=1}^{2^n} \left[ 2^n \mu(R_{n,j}) \right] \right)^{1/p}, \quad \text{if } p \leq q.
\]
First results

**Theorem**

In the case $p \geq 2$ and $q \geq p$ we have:

\[
\pi_q(j_\mu) \approx \left( \sum_n [\pi_q(j_n)]^p \right)^{1/p} \approx \left( \sum_{n,j} [2^n \mu(R_{n,j})] \right)^{1/p}
\]

\[
\approx \left( \int_{\mathbb{D}} \frac{1}{1 - |z|} \, d\mu(z) \right)^{1/p}.
\]
First results

Theorem

In the case $p \geq 2$ and $q \geq p$ we have:

$$
\pi_q(j_\mu) \approx \left( \sum_n [\pi_q(j_n)]^p \right)^{1/p} \approx \left( \sum_{n,j} [2^n \mu(R_{n,j})]^p \right)^{1/p}
$$

$$
\approx \left( \int_\mathbb{D} \frac{1}{1 - |z|} \, d\mu(z) \right)^{1/p}.
$$

In the case $p \geq 2$ and $2 \leq q \leq p$ we have:

$$
\pi_q(j_\mu) \approx \left( \sum_n [\pi_q(j_n)]^q \right)^{1/q} \approx \left( \sum_{n,j} [2^n \mu(R_{n,j})]^{q/p} \right)^{1/p}.
$$
First results

Theorem

In the case \( p \geq 2 \) and \( q \geq p \) we have:

\[
\pi_q(j_\mu) \approx \left( \sum_n [\pi_q(j_n)]^p \right)^{1/p} \approx \left( \sum_{n,j} [2^n \mu(R_{n,j})] \right)^{1/p} \\
\approx \left( \int_{\mathbb{D}} \frac{1}{1 - |z|} \, d\mu(z) \right)^{1/p}.
\]

In the case \( p \geq 2 \) and \( 2 \leq q \leq p \) we have:

\[
\pi_q(j_\mu) \approx \left( \sum_n [\pi_q(j_n)]^q \right)^{1/q} \approx \left( \sum_{n,j} [2^n \mu(R_{n,j})]^{q/p} \right)^{1/p}.
\]

For \( p > 2 \), the case \( 1 \leq q < 2 \) is still open.
The case $p \leq 2$.

Littlewood-Paley theorem says that, if $f_n \in X_n$, $n = 0, 1, \ldots$ we have

$$\left\| \sum_n f_n \right\|_{H^p} \approx \left\| \left( \sum_n |f_n^*|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})}$$
The case $p \leq 2$.

Littlewood-Paley theorem says that, if $f_n \in X_n$, $n = 0, 1, \ldots$ we have

$$\left\| \sum_n f_n \right\|_{H^p} \approx \left\| \left( \sum_n |f_n^*|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})}$$

and then

$$\left( \sum_n \left\| f_n \right\|_{H^p}^2 \right)^{1/2} \lesssim \left\| \sum_n f_n \right\|_{H^p} \lesssim \left( \sum_n \left\| f_n \right\|_{H^p}^p \right)^{1/p}$$

This can be used to prove

$$\left( \sum_n \pi_2^2 \left( j_n \mu \right) \right)^{1/2} \lesssim \pi_2 \left( j \mu \right) \lesssim \left( \sum_n \pi_2 \left( j_n \right) \right)^{1/p}$$

But none of these two estimates is the correct one.
The case $p \leq 2$.

Littlewood-Paley theorem says that, if $f_n \in X_n$, $n = 0, 1, \ldots$ we have

$$\left\| \sum_n f_n \right\|_{H^p} \approx \left\| \left( \sum_n |f_n^*|^2 \right)^{1/2} \right\|_{L^p(\mathbb{T})}$$

and then

$$\left( \sum_n \| f_n \|_{H^p}^2 \right)^{1/2} \lesssim \left\| \sum_n f_n \right\|_{H^p} \lesssim \left( \sum_n \| f_n \|_{H^p}^p \right)^{1/p}$$

This can be used to prove

$$\left( \sum_n \pi_2(j_n)^2 \right)^{1/2} \lesssim \pi_2(j_\mu) \lesssim \left( \sum_n \pi_2(j_n)^p \right)^{1/p}$$

But none of these two estimates is the correct one.
The case $p \leq 2$.

Theorem A

For $1 < p \leq 2$, the Carleson embedding $j_{\mu} : H^p(\mathbb{D}) \to L^p(\mu)$ is absolutely summing if and only if the space $H^1(\mathbb{D})$ is included in $L^r(\nu)$, where

$$r = 1 - \frac{p}{2}$$

and

$$d_{\nu}(z) = \frac{d_{\mu}(z)}{(1 - |z|)^{p/2}}$$
The case $p \leq 2$.

**Theorem A**

For $1 < p \leq 2$, the Carleson embedding $j_{\mu} : H^p(D) \to L^p(\mu)$ is absolutely summing if and only if the space $H^1(D)$ is included in $L^r(\nu)$, where

$$r = 1 - \frac{p}{2}$$

and

$$d_{\nu}(z) = \frac{d_{\mu}(z)}{(1 - |z|)^{p/2}}$$

Applying a result of Blasco and Jarchow, we obtain:
The case $p \leq 2$.

**Theorem A’**

For $1 < p \leq 2$, the Carleson embedding $j_\mu : H^p(\mathbb{D}) \to L^p(\mu)$ is absolutely summing if and only if

$$
\int_{\mathbb{T}} \left( \int_{\Gamma(\xi)} \frac{d\mu(z)}{(1 - |z|)^{1+p/2}} \right)^{2/p} d\mu(\xi) < +\infty
$$
Proof of Theorem A.

Proposition 1

Suppose $1 < p \leq 2$. The necessary and sufficient condition for the natural injection $j: H^p(\mathbb{D}) \to L^2(\mu)$ to be a 2-summing operator is that

$$
\int_T \left( \int_{\mathbb{D}} \frac{1}{|z - w|^2} \, d\mu(z) \right)^{p'/2} \, dm(w) < +\infty,
$$

In fact we have

$$
\pi_2(j: H^p(\mathbb{D}) \to L^2(\mu)) \approx \left( \int_T \left( \int_{\mathbb{D}} \frac{d\mu(z)}{|z - w|^2} \right)^{q/2} \, dm(w) \right)^{1/q}.
$$
Proposition 2

Suppose $1 < p < 2$ and let $r > 1$ be such that $1/r + 1/2 = 1/p$. Let $X$ be a Banach space, and $T : X \to L^p(\mu)$ a bounded operator. The necessary and sufficient condition for $T$ to be a 2-summing operator is that there exists $F \in L^r(\mu)$, with $F > 0 \mu$-a.e., such that $T : X \to L^2(\nu)$ is well defined and 2-summing, where $\nu$ is the measure defined by

$$d_\nu(z) = \frac{1}{F(z)^2} \, d_\mu(z).$$

Moreover, we have

$$\pi_2(T : X \to L^p(\mu)) \approx \inf \left\{ \pi_2(T : X \to L^2(\nu)) : \right.$$ 

$$d_\nu = d_\mu/F^2, F \geq 0, \int F^r \, d_\mu \leq 1 \right\}.$$
Proof of Theorem A.

\( j_\mu : H^p(\mathbb{D}) \to L^p(\mu) \) is 2-summing \iff the following is finite:

\[
\inf \left\{ \int_T \left( \int_D \frac{d\mu(z)}{|z - w|^2 \cdot F(z)^2} \right)^{p'/2} \, dm(w) : F \geq 0, \int F^r \, d\mu \leq 1 \right\}
\]
Proof of Theorem A.

\( j_\mu : H^p(\mathcal{D}) \to L^p(\mu) \) is 2-summing \iff the following is finite:

\[
\inf \left\{ \int_T \left( \int_D \frac{d\mu(z)}{|z-w|^2 \cdot F(z)^2} \right)^{p'/2} \, dm(w) : F \geq 0, \int F^r \, d\mu \leq 1 \right\}
\]

\iff the following is finite:

\[
\inf_{F \in B^+_L(T)} \sup_{g \in B^+_L(r/2)(\mu)} \int_T \int_D \frac{g(w)}{|z-w|^2 \cdot F(z)} \, d\mu(z) \, dm(w), \quad (♣)
\]

where \( t \) is the conjugate exponent of \( p'/2 \), and \( 1/r + 1/2 = 1/p \).
Proof of Theorem A.

By Ky Fan’s lemma the order of taking the sup and the inf can be interchanged.
By Ky Fan’s lemma the order of taking the sup and the inf can be interchanged.

Using Fubini and the fact that $\frac{1-|z|^2}{|z-w|^2}$ is the Poisson kernel, we obtain that $\clubsuit$ is finite if and only if
Proof of Theorem A.

By Ky Fan’s lemma the order of taking the sup and the inf can be interchanged.

Using Fubini and the fact that $\frac{1-|z|^2}{|z-w|^2}$ is the Poisson kernel, we obtain that (♣) is finite if and only if

Poisson integral sends $L^t(\mathbb{T})$ into $L^{p/2}(\nu)$, for $d\nu(z) = \frac{d\mu(z)}{(1-|z|)^{p/2}}$.
Proof of Theorem A.

By Ky Fan’s lemma the order of taking the sup and the inf can be interchanged.

Using Fubini and the fact that \( \frac{1-|z|^2}{|z-w|^2} \) is the Poisson kernel, we obtain that (♣) is finite if and only if

Poisson integral sends \( L^t(\mathbb{T}) \) into \( L^{p/2}(\nu) \), for \( d\nu(z) = \frac{d\mu(z)}{(1-|z|)^{p/2}} \)

if and only if \( H^t(\mathbb{D}) \subset L^{p/2}(\nu) \).