ON MATHEMATICS AND PHYSICS: A CONVERSATION WITH

SIR MICHAEL ATIYAH

J. Adolfo de Azcárraga

Sir Michael Francis Atiyah is one of the most distinguished mathematicians of our time. With I. Singer, he proved the famous Atiyah-Singer index theorem (1963) connecting analysis and topology, later very important in modern theoretical physics. Among other awards, he is the holder of the Fields Medal (1966) and the Abel Prize (with Singer, 2009); he has been President of the Royal Society, Master of Trinity College of Cambridge Univ. and President of the Royal Society of Edinburgh. While Master of Trinity Coll., he and the theoretical physicist Peter Goddard, of St John's Coll., promoted the creation of the Isaac Newton Institute for Mathematical Sciences, of which he became its first Director in 1990 (and Goddard the first Deputy Director). He is one of the 24 people who may hold the Commonwealth Order of Merit at any given time. The first half of Professor Atiyah's brilliant scientific career was that of a pure mathematician; the second half, that of a mathematician with a leaning towards important physics problems. At present, he is at the Edinburgh School of Mathematical Sciences where he intensely keeps doing research. If he were asked about retirement I guess that, as the 1967 pioneer of electroweak theory and Nobel laureate S. Weinberg answered recently, he would reply: "I plan to retire shortly after I die".

Professor Atiyah visited the Spanish Royal Physics Society (RSEF) at the beginning of May, a stay sponsored through the RSEF-Fundación Ramón Areces cooperation agreement. He did not wish to cancel his trip in spite of the sad passing of his wife only a few weeks earlier, and the 3rd of May delivered a Public Lecture at the Facultad de Ciencias of the Universidad Complutense (Madrid), *Have Fun with Numbers and become Rich or Famous*. A great conversationalist, I suggested to interview him on the interplay of mathematics and physics and on his very recent research, which he touched in his talk; our exchanges follow. Some of the questions are intentionally long to provide a suitable background.

[J. Adolfo de Azcárraga] Let me begin with the relation between mathematics and physics. In an essay in the *Bull. Am. Math. Soc.* in 1993, A. Jaffe and F. Quinn coined the expression 'theoretical mathematics' to emphasize the dangers of 'speculative mathematics' since, in their view, "modern mathematics is nearly characterized by the use of rigorous proofs". You were one of the mathematicians who, without disregarding rigour, reacted to their article. Why?

Published in Spanish in the *Revista Española de Física,* vol. **32**, #3 p. 32-38 (2018). This English text is available in http://www.revistadefisica.es/index.php/ref/article/view/2457/1948.

[Sir Michael F. Atiyah] I thought they were warning mathematicians away from physicists because the new dialogue was on doubtful foundations. In broad terms it was topologically based, not analytically based. It was soft, not hard analysis as practiced by Glimm and Jaffe. I resisted being told not to speak to my good friends in the physics community like Edward Witten [physicist and 1990 Fields medallist]. They may not have been rigorous in the Jaffe style, but I had a gut instinct that their work would become rigorous in due course, as happened in the past when Riemann used analytic continuation to justify Euler's brilliant theorems. That is indeed what has happened and the hard analysts have been left behind.

[AA] Yes, it is proper to speak of the very successful *physical mathematics* to recognize the influence of physics in the development of mathematics. Twentieth century examples include distribution theory, which may be said to begin in 1927 with Dirac's δ and its derivatives; Élie Cartan's then (1913) esoteric idea of spinors, that grew enormously with physics; Dirac's 1931 monopole paper, which may be argued to contain the seeds of fibre bundle theory and the role of topology; the flourishing of non-commutative geometry and of supermanifolds including Grassmann integration, which Berezin introduced in the sixties having fermion physics in mind; many developments of group theory motivated by physics (no longer a *Gruppenpest* there), etc. Which are in your view the most recent and important examples in this line?

[MA] I agree with you, but you have simplified history and the interaction between new mathematics and physics has been an iterated two-way process.

Before Dirac's delta there was Heaviside (an engineer). Spinors are really due to Hamilton (both mathematician and physicist); Élie Cartan built on Sophus Lie; fibre bundles and U(1) gauge theory are due to Clifford. Grassmann used Maxwell's understanding of Hamilton's quaternions. Maxwell derived all the notation of *div*, *grad* and *curl* from Hamilton, who first wrote down the equation $(i\, d/dx + j\, d/dy + k\, d/dz)^2 = -$ Laplacian and said that this must have deep physical meaning. This was many decades before Dirac. The dialogues continue and we now have a better understanding of fermions, supersymmetry and Morse theory. Also of topological quantum field theory in three-dimensions and Jones' knot invariants; Donaldson invariants in four dimensions, and the unique role of dimension four, both in physics and geometry. Also moduli spaces in Yang-Mills theory; monopoles and instantons; anomalies, cohomology and index theory; holography and geometry, etc. It would take me many lectures to explain the latest ideas. My lecture in Madrid was one example.

[AA] I was mentioning XXth century pioneers, but indeed O. Heaviside introduced the step 'function'. Also, by removing the vector potential from the original formulation (as Hertz also did), he reduced in the 1880s Maxwell's equations to the familiar set of four. Maxwell is a truly towering figure. Of his equations, Feynman rightly said in 1964: "the American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade". Maxwell's equations also implied an important change of perspective by dismissing Ampère's 'action at a distance' in favour of field theory, and inspired special relativity too; they are a perfect example of the successful interplay of mathematics and physics. Nevertheless, C.N. Yang, who (with T.T. Wu) translated in 1975 the structure of gauge theories to the language of principal fibre bundles with a connection, cautioned in 1979: "deep as the relationship is between mathematics and physics, it would be wrong, however, to think that the two disciplines overlap that much. They do not. And they have their separate aims and tastes. They have distinctly different value judgements and different traditions". Indeed, it seems difficult to imagine a mathematical hint in 1900 that could have led to quantum mechanics (QM). But with the advent of Einstein's relativities (1905, 1915), Weyl's gauge principle (1929), the widespread importance of all kinds of symmetries, and string theory in the more recent past, geometry and physics seem closer than ever. However, in 1972 F. Dyson mistakenly wrote that "the [enormously fruitful] marriage between mathematics and physics has recently ended in divorce". Could a post-string theory divorce between physics and mathematics be possible?

[MA] Dyson is mistaken because mathematics and physics are not like husband and wife, they are more like Siamese twins. A divorce in their case usually leads to death of at least one twin, so it should not be contemplated.

Yang also oversimplifies by regarding mathematics and physics as separate organisms, and by implication that mathematicians and physicists are different people. How does he decide which of the great figures of history was a mathematician or a physicist: Archimedes, Newton, Gauss, Hamilton, Maxwell, Riemann, Poincaré, Weyl?

[AA] True: for instance, Pauli mocked for a while the merits of mathematician H. Weyl as a physicist, but had to change his view after Weyl introduced the (final) gauge principle in 1929. Let me now move to your old friend and Oxford colleague R. Penrose, who has recently published *Fashion, Faith, and Fantasy in the New Physics of the Universe*, three words that supposedly apply to some theoretical physics research areas, as the 'fashionable' string theory. Some people have also pointed out that E. Witten, the proponent (1995) and leader of M-theory (in which all five superstring theories and eleven-dimensional supergravity appear as limiting cases), has diverted his attention to other interests as solid state theory. Nevertheless, S. Weinberg stated last year that string theory still is "the best hope we have" for a high energy theory of elementary particles and that he "hopes that people are not giving up". Which is your view on M-theory?

[MA] Roger Penrose is one of the deepest thinkers on gravitation. He is a true disciple of Einstein, but he has limited understanding of particle physics and of string theory. Weinberg by contrast dismissed geometry as the wrong way to study gravitation but is a believer in string theory. Witten in my view is the deepest thinker of all, and the only person who has a unified if incomplete picture of M-theory. His recent interest in solid state physics is not inconsistent with a belief in M-theory, but at present M-theory seems to have got stuck so it makes sense to focus on a different phase of physics. I entirely sympathize with Witten. My only difference with him is that I think we need more mathematics for the next break-through. Since I started life as a mathematician it is natural that I should search there for inspiration.

I have acquired a new perspective which looks at M-theory differently and extracts soluble problems one by one. It has roots in the de Broglie-Bohm pilot wave theory and a non-local approach to physics, away from the unphysical point particle with all its infinities, to more realistic geometric models.

The deepest question of all is whether the universe is fundamentally 4-dimensional or multi-dimensional. Penrose believes the former, M-theorists believe the latter. I believe both are true and recall the dictum of Pauli that "a deep truth is one whose opposite is also true" and the logical mind-bender of asking whether Pauli's dictum is itself a "deep truth".

It is best to remain an agnostic. All 'faiths' see an aspect of the truth, there is no single truth.

[AA] Could you broadly explain your view of the de Broglie-Bohm approach?

[MA] The Bohmian approach is the best way of understanding the particle-wave dichotomy, with its local and non-local aspects. It ran into difficulties with quantum-field theory, but with new ideas I think the difficulties can be circumvented. I predict that Bohm will be seen as far ahead of his time.

[AA] The landscape problem in string theory, the practically infinite number of consistent string vacua —like points in an earthly landscape—

makes it difficult to explain why there should be one set of laws governing physics rather than many. As often pointed out —*e.g.* by Penrose above— the landscape casts doubts on the very possibility of determining *the* theory that describes all interactions in Nature, something originally considered as a motivation for string theory. Is this lack of unicity a real difficulty for deriving the laws of Nature from first principles or, as some argue, it may be surmountable by resorting *e.g.* to multiverse cosmologies?

[MA] The landscape is not a problem. It does not describe different universes, just different states of our universe, in which temperature, pressure and other physical properties have been specified. The question of whether one universe in all its diverse states is one 'object' or a 'multiverse' is mere semantics misunderstood. Plato would never have bought the idea of parallel universes. We have been brow-beaten by the notion of 'quantum states'.

[AA] Yes, but we do live in a specific universe with a definite set of constants. Are you implying that, by the same token, ideas like H. Everett's many worlds approach to QM are merely a semantic way out?

[MA] There is only one universe but if a ficticious 'observer' travels round he will encounter many different local conditions. He will see the universe in many different states. I do not see any philosophical problem. It is the quantum mechanical straightjacket that creates the problem. Discard QM as a philosophical basis, regard it as a useful approximation, and the problems disappear.

[AA] Let us change the subject. Nobody would dream of teaching relativity to a chimpanzee. For those who have an evolutionary understanding of human nature (and setting aside artificial intelligence), this is a sobering thought on our possibilities. Nevertheless some people still speak of the 'theory of everything' as an attainable goal. Could we reach it?

[MA] Monkeys can only be understood through biological evolution and the best place to start is with the famous book [1944] written by Schrödinger entitled *What is Life*?

All the famous biologists I know (and I have known most of them from Max Perutz to Francis Crick) said that they read the book, they were inspired by it but that it was wrong! This has always intrigued me and now I understand why the biologists said this, why they were wrong and why Schrödinger was essentially right.

The explanation is very simple. Schrödinger, being a physicist, thought in terms of *Energy* and the First Law of Thermodynamics. Biologists think in terms of *Entropy* and the Second Law. In analytic models, Energy and Entropy are the modulus-squared and the phase of an analytic function and hence, as real and imaginary parts, determine each other. But this is an idealized mathematical extreme which does not represent Nature. In the real world functions may be very smooth, even infinitely differentiable but they are not analytic.

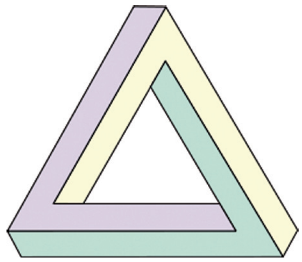
Maxwell, the great natural philosopher at the dawn of Darwinian evolution, but long before quantum theory, saw the distinction clearly. He insisted that the First Law was mathematical and certain, while the Second Law was probabilistic and uncertain. He failed to persuade Kelvin but he did persuade me as you can read in my paper on the fine structure constant.

The subsequent interpretation of entropy as information was Shannon's response to Maxwell's demon. This fits very well into modern biology with DNA encoding the information of heredity. But we now know that genes can mutate through radio-activity which is itself probabilistic, so Maxwell's demon was not really captured by Shannon. We should all re-read Maxwell.

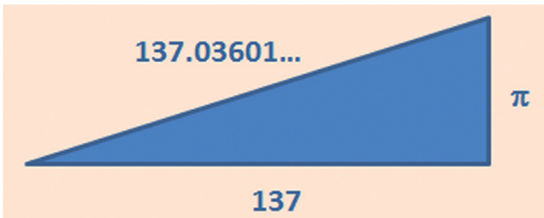
Some physicists talk about the "theory of everything". I dislike the term and think it is hubris, we can never know everything. We can just build approximate structures for particular purposes. Humility is the hallmark of great scientists like Maxwell.

[AA] Or Faraday... Consider now the Holy Grail of theoretical physics, the quantum theory of gravity, one of the hopes of string theory. It is likely that Einstein did not succeed in unifying gravity and electromagnetism -two classical theories- because he left QM out of the picture, something not surprising since he was not quite happy with it. However, your view concerning the unification of all interactions seems to be that this is so difficult not because gravity resists being treated quantum mechanically and added to the other three, but rather because gravity was not taken into account from the very beginning. If I reflect your view properly, you sustain that gravity cannot be ignored even at the microscopic level if, besides computing, one wants understanding. Could you elaborate on your view?

[MA] The difficulty of gravity is just that of feedback. Matter curves space-time and this new curved space-time affects matter. The mathematical model that best captures this problem is provided by the octonions, which lead to the famous Penrose triangle and the engravings of Escher. Quantum theory is hopeless in this situation except if gravity is very weak. The fact that weak gravity is always attractive can be understood from the octonions and the famous Bernoulli function, which we may discuss later.



Penrose impossible triangle



A 'Pythagorical' and not quite correct 1/α

[AA] We shall go back to gravity, but let us consider first one aspect of your Madrid talk, the mathematical determination of Sommerfeld's (1916) fine structure constant $\alpha = e^2/\hbar c = 1/137.035\,999\,074(44) \sim 1/137$. There have been many numerological —by which I mean here just playing with numbers— attempts to compute it, even Cabbalistic (137 appears often in the Cabbala). As you know, many great physicists such as Pauli —a student of Sommerfeld born in a Christian family of Jewish heritage— were deeply frustrated by α. In your lecture you quoted Feynman in 1985: "[The value of α] has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists... worry about it. Immediately you would like to know where this number for a coupling comes from: is it related to π or perhaps to the base of natural logarithms? Nobody knows. It's one of the greatest damn mysteries of physics". Your approach to α is purely mathematical, but not 'numerological' in the above sense (and, as it turns out, not alien to π and *e* !). But, before we come into it, why did you believe that such a calculation could be possible to begin with?

[MA] This was clearly a mathematical challenge which I could not resist. Something so important for physicists and so mathematical had to have a very good reason to exist. For me that meant beauty and simplicity if seen the right way.

[AA] Your conclusion is that α appears in the expression $e^{2Ж\omega} = 1$ where the Cyrillic letter Ж (*zhe*) has to be computed mathematically and is identified with 1/α. Your 'Euler-Hamilton' formula $e^{2Ж\omega} = 1$ follows the pattern of Euler's $e^{2\pi i} = 1$, in which π has been 'renormalized' to the positive real number Ж (and *i* to *ω*, the meaning of which we may perhaps ignore here). As you well know, renormalization for a quantum physicist is a process that allows to extract finite answers in a higher order quantum field theory computation, as Schwinger's famous quantum electrodynamics calculation (1948) of the anomalous magnetic moment of the electron. Could you explain what 'renormalization' means in your case and why the real number Ж should be *a priori* identified with the physical 1/α, setting aside your claim that it agrees with it to the calculated decimal places?

[MA] Renormalization to a mathematician just means a change of norm, often given by a simple change of coordinate such as x goes to 1/x. It is natural to interpolate skilfully between the two coordinates. This is done most beautifully by the Bernoulli function $x/(1-e^{-x})$, with x replaced by mx, where m is a positive parameter (which models the effect of weak gravity). Near the origin it looks like x shifted by m, near infinity it looks like the shifted inverse coordinate y =1/x. This is simple to understand when x is a real or complex variable because the algebra of polynomials is commutative. My task was to imitate this story for the non-commutative algebra of quaternions invented by Hamilton. Fortunately for me all the hard work had already been done by von Neumann, Hirzebruch and Alain Connes. All I had to do was to put the pieces of the jigsaw together.

Combining the challenge problem set by physicists with the non-commutativity of Hamilton and the role of u(2) in electro-weak interactions it was just one small step (of faith/guess-work/inspiration) to predict that Ж should be the inverse of the fine-structure constant α. All the pieces of the jigsaw fitted together to make the perfect picture designed by God. My former teacher J.A. Todd used to say, when he had come across a beautiful formula "If there is any justice in the world it must be true". Maxwell, Einstein and Dirac all felt the same way about their insights into the nature of light.

$$e^{2\pi i} = 1$$

"Hamlet's 'to be or not to be' is in its brevity and depth the equivalent of Euler's formula" [MA]

[AA] Your computation of α would exemplify what the Nobel laureate E.P. Wigner called in 1960 "the unreasonable effectiveness of mathematics in the natural sciences"; earlier, H. Poincaré had worried in *La Science et l'Hypothèse* (1902) on why differential equations were so prominent in the laws of physics. Your Hamilton-Euler formula $e^{2Ж\omega} =1$ looks, like Euler's $e^{2\pi i} = 1$, both beautiful and magical. Could you comment on the importance of beautiful formulae in physics, a point you have mentioned already, and why, in your view, mathematics is indeed so successful in describing Nature?

[MA] The purpose of science is for humanity to understand nature. We understand by formulating laws which organize knowledge. A law must be clear and intelligible. Beauty is our criterion of clarity. Had we been large electronic computers we would have had no need of beauty, or beauty would have meant something quite different.

The reason why mathematics is so successful is closely related to its beauty, but also to evolutionary biology. Humans are a product of long evolution, in which powerful brains were an advantage. Such brains evolved in the physical world, so evolutionary success was measured by physical success. Hence human brains evolved to solve physical problems and this required the brain to develop the right kind of mathematics. Why was Wigner surprised? Because he focused on mathematics and physics, but ignored biology.

[AA] Your mathematical approach to 1/α is bound to produce a number with no physical dimensions. Also, an experimental measure of a physical quantity necessarily produces a rational number, but this is no problem; we also use rational approximations to π. Nevertheless, should we expect the dimensionless Ж to be irrational? Is Ж perhaps even a transcendental number, like π and *e* are?

[MA] Measured quantities are not rational, they are a sequence of rationals which converge to a real number as measurement improves. π is the classic example and it has been proved that it is transcendental. Ж should be the same but it might be logically hard, in the Godel sense, to prove it. I also believe that about Euler's constant γ.

[AA] Your identification of Ж with 1/α puts the fine structure constant α in a purely mathematical —even Platonic— realm, very much as π, being the length of a circle divided by its diameter, has nothing to do with physics (although to our bewil-

derment it appears everywhere). Does this have a bearing on Hoyle's nucleosynthesis path to the C^{12} element necessary for life or on the *anthropic principle* (B. Carter, 1967, 1974) in general? Weinberg said recently that this principle was a "somewhat desperate suggestion" to explain the very small value of the cosmological constant.

[MA] I have disposed of the anthropic principle. I do not know about Hoyle and C^{12} but I can guess. All these questions are non-sense. A mathematical constant like π or Ж is not for sale. It makes no sense to ask what the universe would be like if these numbers were different from what they are —unless you are radical enough to believe mathematics might be different in other universes, but then all the anthropic arguments just show that any such universe would be a totally barren place and certainly have no life or mathematicians!

[AA] Yes, Penrose discusses the anthropic principle in the 'fantasy' chapter of his book.

You claim that your approach produces a definite value for the fine structure constant which satisfies Good's criterion; you also said that it restores Eddington's reputation. I assume that you refer to his numerological calculation (1930) of 1/α when it was thought to be 136, soon adjusted to the better 137 value. This motivated a sarcastic poem by V. Fock in the weekly *Punch* (1930), 'Sir Arthur Eddington Adding One'. Could you comment on this?

[MA] Good's criterion is that a numerological explanation is acceptable only if it comes from a more fundamental theory, providing a Platonic explanation of the numerical value. Yes, Eddington got 136 from powers of 2: $136 = 2^3 + 2^7 = 8 + + 128$ which occur in my paper on α and can be explained in terms of Clifford algebras (close to Eddington's reasoning). The extra 1 to make 137 just comes from starting the sequence one step back: $137 = 2^0 + 2^3 + 2^7$.

[AA] Curiously, there are as many division algebras (reals $\mathbb{R}$, complex $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$) as types of interactions. In your view, the electroweak force would be related to $\mathbb{R}$ and $\mathbb{C}$, the strong force to $\mathbb{H}$ and gravity to $\mathbb{O}$. $\mathbb{H}$ is not commutative and $\mathbb{O}$ not even associative. Would not this create insurmountable difficulties with gravity? And, if not, why should this non-commutativity and non-associativity be needed physically?

[MA] Exactly so. The non-commutativity of the quaternions is at the heart of the problem I deal with in my calculation of α. The non-associativity of the octonions is much harder and will be in my next paper. Gravity is much harder than gauge theories of compact Lie groups. The division algebras and the physical forces are a perfect fit.

Let me explain. The compact groups that act on $\mathbb{R}^2$, $\mathbb{C}^2$, $\mathbb{H}^2$ are SO(2), U(2), U(3). The first gives electromagnetism, the second gives the electroweak theory and the subgroup SU(3) is the gauge group of strong interactions. But $\mathbb{O}^2$ is acted on by octonions which do not give a group because they are non-associative. That is why gravity is harder than gauge theories.

[AA] The electroweak gauge algebra of the standard model is not simple; it is rather the direct sum of two algebras since $u(2) \approx su(2) \oplus u(1)$. As a result, the construction of the electroweak model begins with two constants rather than one. You state that your approach leads to α, which contains the electric charge *e* of the electromagnetic interaction. Could one find also a trace of the weak coupling in it?

[MA] Certainly, the weak coupling has a simple geometric value which is not renormalized in pure electro-magnetic theory. But it can be renormalized to get improved values for the electroweak theory.

[AA] Do you mean that Fermi's coupling constant G_F could also be included in the same scheme? Making it dimensionless would require the introduction of a $(\text{mass})^2$ parameter.

[MA] In general a coupling constant can be made dimensionless by introducing an appropriate compensating parameter. If the parameter has the right geometric features fitting with Einstein's GR then we can call it a mass parameter. If we allow ourselves to use multi-dimensional spaces which incorporate GR and gauge theories then every parameter is a 'mass' parameter in a generalized sense.

[AA] Newton's gravitational constant G also has physical dimensions. What would be the physical dimensionless constant playing the role of α in your purely mathematical scheme?

[MA] The dimensionless number analogous to α is the ratio of G to $c^3/\hbar e$ where *e* is now not Euler's e but the charge of the electron. Mass is related to (gravitational) energy by Einstein's famous formula $E = mc^2$. Charge is related to (electro-magnetic) energy by Maxwell's equations. Since energy is universal, whatever its source, mass and charge must be related. This implies a relation between G and α.

[AA] Physical coupling constants are not really constant. The electromagnetic, weak and strong coupling constants depend on the energy and converge at high energies; for instance, at the energy of the W-boson, α ∽ 1/128. Is this *physical* variability compatible with your unique, *mathematical* value for 1/α?

[MA] Absolutely. Any coupling constant is not really constant, but the physics can be squeezed out of it by idealization which lands us in the world of mathematics. The only pure number that remains (for the standard model) is α. When gravity is incorporated we then get Newton's constant G. But I have postponed that to my next paper.

[AA] Perhaps this is a good point to stop. Would you like to add something?

[MA] Plenty, but life is short.

[AA] *Ars longa, vita brevis*... I wish you that, unlike É. Galois who had one night to put his thoughts on paper before his tragic duel, you have many years ahead of you to think and write about these fundamental questions. It was a pleasure having you at the RSEF; thank you very much.