The Value for Action-set Games

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Abstract

Action-set games are games where the set of players is finite, every player has a finite set of actions, and the worth of the game is a function of the actions taken by the players. In this setting a rule determines individual payoffs for each combination of actions. Following an axiomatic approach, we define the set of Consistent Bargaining Equilibria.

Keywords: Action-set games, Shapley value, Prekernel, Consistent Bargaining Equilibria.

1 Action-set games

One of the features common to most economic situations is that the interaction among agents in activities like production, exchange, etc. generates benefits that are shared among the participating agents. Moreover, the productivity gains from specialization in different tasks and roles (labor division) are important within any type of production process, ranging from pin manufacture to legal practice and medical care. When the production process is performed by the combination of different tasks, cooperative agents must agree on how to assign tasks to agents and how to share the outcome between them. These two decisions are not independent because the rewards assigned to each task will condition that which would like to be chosen by each agent.

We illustrate this setting with the help of the following example.

Example 1. Suppose that two friends, say $i$ and $j$, go to deer hunting. Each of them can either line driving (action $l$), i.e., flushing deers toward the hunter, or shooting deers (action $s$). The best hunting strategy is that one of them line drives and the other shoots; if both line drive, then no deer will be shoted, and, finally, if both shoot deers, then less deers will be shoted, because nobody will flush them to the hunters. Suppose in addition that they only own one rifle; that, for simplicity, the number of deers shoted is proportional to the shooting time, and that if both want to shoot they must agree on the redistribution of time. Payoffs are represented by the following matrix.

\[
\begin{array}{c|cc}
  & s_j & l_j \\
\hline
  s_i & 4 & 8 \\
  l_i & 6 & 0 \\
\end{array}
\]
The matrix is asymmetric, stressing that the hunting team combination \((s_i, l_j)\) is better than \((l_i, s_j)\).

An example of a rule that, a priori, could be appealing is the equal payoff division \(\epsilon\), which in principle is a good candidate for a fair rule. Such a rule yields the payoff matrix:

\[
\begin{array}{ccc}
  (\epsilon, \epsilon) & s_j & l_j \\
  s_i & (2, 2) & (4, 4) \\
  l_i & (3, 3) & (0, 0)
\end{array}
\]

Suppose now that the agents choose the pair of actions \((s_i, l_j)\), where the equal payoff division rule yields \(\epsilon(s_i, l_j) = 4, \epsilon(s_i, l_j) = 4\). Here, agent \(i\) could disagree with this payoff by arguing that if she changed her action to \(l_i\), then she could inflict a loss on agent \(j\) of:

\[\epsilon_j(s_i, l_j) - \epsilon_j(l_i, l_j) = 4 - 0 = 4\]

This objection cannot be balanced by a counterobjection of \(j\), consisting of a deviation from \(l_j\) to \(s_j\), because the loss on agent \(i\)’s payoffs that agent \(j\) could inflict is:

\[\epsilon_i(s_i, l_j) - \epsilon_i(s_i, s_j) = 4 - 2 = 2\]

If we wish a rule \(\psi\) satisfying that the above kind of objections and counterobjections be balanced at \((s_i, l_j)\), the payoffs must be \(\psi_i(s_i, l_j) = 5, \psi_j(s_i, l_j) = 3\). Moreover, at \((l_i, s_j)\) the agents are in reverse position and therefore the payoffs \(\epsilon_i(l_i, s_j) = 3, \epsilon_j(l_i, s_j) = 3\) must change to \(\psi_i(l_i, s_j) = 2, \psi_j(l_i, s_j) = 4\) in order to be balanced. Finally, at \((s_i, s_j)\) and \((l_i, l_j)\) no player can threat to inflict a payoff loss on another player by a unilateral deviation of her action, so there is no reason to readjust the corresponding payoffs. To sum up, the payoff matrix \([\psi(x)]\) should be

\[
\begin{array}{ccc}
  (\psi_i, \psi_j) & s_j & l_j \\
  s_i & (2, 2) & (5, 3) \\
  l_i & (2, 4) & (0, 0)
\end{array}
\]

We would like to stress that the determination of such a rule \(\psi\) is an independent and previous issue to what profile of actions will be finally taken by the players. In our example, perhaps the players’ priority is to maximize the number of deers shooted and then they will choose \((s_i, l_j)\). Then, our claim is that the redistribution of deers will be \(\psi(s_i, l_j) = (5, 3)\). Alternatively, we can think that they are risk-neutral utility maximizers and that they agree to play a fair lottery \([\lambda(s_i, l_j); (1 - \lambda)(l_i, s_j)]\). In that case if \((l_i, s_j)\) were the lottery played, the payoffs would be \(\psi(l_i, s_j) = (2, 4)\).

Once \([\psi(x)]\) has been obtained, we may try then to determine what action profile will finally be taken, based either on evolutionary dynamic arguments, or on utilitarian/egalitarian considerations, or on strategic bargaining models.

Our goal is to show the existence of a rule \(\psi\) such that given a payoff matrix \([v]\), the payoffs at each action profile \(x\) are balanced against individual deviations as described above. For that purpose we follow an axiomatic approach, that is, we impose some properties which rationalize the behavior of the agents involved in the bargaining process.

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1This example can be seen as an illustration of a two-agent cooperation problem under a simple joint production technology with two heterogeneous working types.
2 Axiomatics

Formally, an action-set game is any $\Gamma = \left( N, \left( A_i \right)_{i \in N}, v \right)$, where $N$ is a finite set of players, with $|N| = n$, and for any player $i$ in $N$, $A_i$ is a finite non-empty set of actions available for player $i$. An action profile $a = (a_i)_{i \in N}$ is a combination of actions that the players in $N$ might choose, and denote the set of all possible action profiles $\times_{i \in N} A_i$ by $A$.

For any action profile $x = (x_i)_{i \in N}$ in $A$, the real number $v(x)$ represents the total worth that players would get if $x$ where the combination of the actions taken by the players. Therefore, $v$ is a mapping $v : A \to \mathbb{R}_+^n$.

Given an action-set game $\Gamma = (N, A, v)$, a value solution $\psi$ is a mapping that specifies a payoff vector for every action profile, that is

$$\psi : A \to \mathbb{R}^N$$

The number $\psi_i(x, v)$ represents the payoff that player $i \in N$ receives when action profile $x \in A$ is taken in $\Gamma$. When no confusion arises, we denote such a number by $\psi_i(x)$. As utility is totally transferable, a payoff vector $\psi(x)$ is feasible if $\sum_{i \in N} \psi_i(x) \leq v(x)$.

The three main features characterizing our setting are:

1.- Unanimity. The final agreements must be unanimous; hence partial cooperation is not allowed and strict subcoalitions of agents play no role in this setting.

2.- No disagreement point. The agents cannot take any particular action to guarantee themselves a minimum payoff. Whatever the actions taken by the players, they must agree on the redistribution of the outcome.

3.- Transferable utility. The outcome is a totally divisible good which can be redistributed among agents.

The first requirement that a rule $\psi$ must satisfy is the budget constraint:

**Definition 1** Efficiency: $\sum_{i \in N} \psi_i(x) = v(x)$, for all $x \in A$

To illustrate the next property suppose that players are involved in bargaining at some $x$, and some agent $i$ disagrees with the proposal at hand. She can reject that proposal by threatening to change her action to a different $y_i \in A_i$, where $v(x_{-i}, y_i) < v(x)$. In this way, she can impose a loss on the other players as far as they do not take her claim into account. Note however that, if players were in action profile $\bar{x}$ where it is impossible for a player to lower any other player’s payoff by an action change, then no player would make this kind of threat. Assuming that players have equal bargaining skills, payoffs must be the same for all of them at the worst situation.

**Definition 2** Equal minimum rights: Let $\bar{x} \in A$ such that $v(\bar{x}) = \text{Min}_{x \in A} \{v(x)\}$, then $\psi_i(\bar{x}) = \psi_j(\bar{x})$, for all $i, j \in N$.

Therefore, individual payoffs at the worst possible outcome of the game act as a reference point for the remaining possible agreements.
Moreover, since unanimity of any agreement is a desirable property, then it is required that every player obtains at least as much as she obtains at the reference point.

**Definition 3** Individual rationality: Let \( \bar{x} \in A \) such that \( v(\bar{x}) = \min_{x \in A} \{v(x)\} \), then for all \( x \in A \), \( \psi_i(x) \geq \psi_i(\bar{x}) \), for all \( i \in N \).

Finally, we are looking for a rule with the property that at any action profile, agent \( i \)'s threat to agent \( j \) -by unilaterally deviating from her action- is balanced by agents \( j \)'s counterthreat to agent \( i \). This property is called *Equal Punishments*. We define the concept of "punishment" as follows:

**Definition 4** Given a value \( \psi \), the punishment that player \( i \) can inflict on player \( j \) at \( x \), for each \( x \in A \), is given by

\[
P_i^\psi(x) := \psi_j(x) - \min_{y_{ij} \in A^j} \psi_j(x_{-i}, y_{ij}).
\]

The amount \( P_i^\psi(x) \) measures the maximum payoff losses that player \( i \) can inflict on player \( j \) at \( x \). Note that since \( \min_{y_{ij} \in A^j} \psi_j(x_{-i}, y_{ij}) \leq \psi_j(x) \), then \( P_i^\psi(x) \geq 0 \), for all \( x \in A \), \( i, j \in N \). The difference \( P_i^\psi(x) - P_{ji}^\psi(x) \) can also be interpreted as the adjustment that player \( i \) could claim at \( \psi(x) \) against player \( j \) based on their punishment comparison.

**Definition 5** Total equal punishments:

\[
\sum_{j \in N \setminus i} P_{ij}^\psi(x) = \sum_{j \in N \setminus i} P_{ji}^\psi(x), \text{ for all } i, j \in N \text{ and } x \in A.
\]

We offer a partial justification of why the total equal punishment property should hold in a bargaining equilibrium\(^2\). This justification is based on strategic considerations. Suppose an action-set game \((N, A, v)\) where the distribution payoff matrix \([\psi]\) is already given. Now at certain \( x \in A \) the players wish to reconsider how to share \( v(x) \). For that purpose consider an offer bargaining process as follows:

There is an infinite sequence of rounds. At the beginning of each round a player out of \( N \) is chosen as the proposer being equally likely to be selected. If player \( i \) is the proposer, then she has to simultaneously announce both a feasible offer \( g_i^i(x) \in \mathbb{R}^N \), that is \( \sum_{j \in N} g_{ij}^i(x) \leq v(x) \), and an action threat \( y_{ij} \in A^j \) (possibly \( y_{ij} = x_i \)) for every responder \( j \in N \setminus i \). Now players in \( N \setminus i \) are asked if they accept or reject the offer following a prespecified order. If the offer is accepted for all players, the game ends with payoffs \( g_i^i(x) \). If at least a player \( j \) rejects the offer, then with a given probability \( \rho \), \( 0 < \rho \leq 1 \), a new round starts; and with probability \((1 - \rho)\) the game ends with payoffs \( \psi(x_{-i}, y_{ij}) \), where player \( j \) has been the first player in the order which rejects the offer.

As usual we will focus on the stationary subgame perfect equilibria of the game. Denote by \( g_i(x) = \frac{1}{n} g_i^i(x) + \frac{1}{n} \sum_{j \in N \setminus i} g_j^i(x) \) for all \( i \in N \). The equilibrium offer that player \( i \) would make at each round if she were the proposer is computed by letting to every \( j \) indifferent between accepting or rejecting the offer\(^3\).

\(^2\)This is only an informal argument and it is not intended to be a complete description of a noncooperative bargaining model, yielding the matrix \([\psi]\) as the equilibrium payoffs. This complementary strategic approach exceeds the purpose of this note.

\(^3\)We wish to remark that this is a partial analysis, and thus we assume that offers are only made to obtain payoffs at \( x \) as big as possible. Hence, those \( i \)'s offers designed to be rejected for sure by the other players, and with the purpose to have a chance to obtain a better payoff at \( (x_{-i}, y_{ij}) \), are excluded.
\[ g_i(x) = \frac{1}{n} g_i^l(x) + \frac{1}{n} \sum_{j \in N \setminus i} g_i^l(x) = \frac{1}{2} \left( v(x) - \sum_{j \in N \setminus i} g_j^l(x) \right) + \frac{1}{n} \sum_{j \in N \setminus i} g_j^l(x) \]

\[ n g_i(x) = v(x) - \sum_{j \in N \setminus i} \left[ p g_j(x) + (1 - \rho) \psi_j(x_{-i}, y_{ij}) \right] + \sum_{j \in N \setminus i} \left[ p g_j(x) + (1 - \rho) \psi_j(x_{-j}, y_{ji}) \right] \]

\[ n(1 - \rho) g_i(x) = (1 - \rho) v(x) - (1 - \rho) \sum_{j \in N \setminus i} \psi_j(x_{-i}, y_{ij}) + (1 - \rho) \sum_{j \in N \setminus i} \psi_j(x_{-j}, y_{ji}) \]

which finally leads to

\[ g_i(x) = \frac{1}{n} v(x) - \frac{1}{n} \sum_{j \in N \setminus i} \psi_j(x_{-i}, y_{ij}) + \frac{1}{n} \sum_{j \in N \setminus i} \psi_j(x_{-j}, y_{ji}) \] (1)

By equation (1) the only way that \( i \) has to rise her expected payoff \( g_i(x) \) is by choosing an action threat vector \( (y_{ij})_{j \in N \setminus i} \) which minimizes \( \psi_j(x_{-i}, \cdot) \) for every \( j \in N \setminus i \). This translates to saying that player \( i \)'s payoffs are higher the higher her ability to punish the other players and the lower players \( j \)'s ability to punish player \( i \). Note also that from (1) we can further obtain

\[ n g_i(x) = \sum_{j \in N} g_j(x) - \sum_{j \in N \setminus i} \psi_j(x_{-i}, y_{ij}) + \sum_{j \in N \setminus i} \psi_j(x_{-j}, y_{ji}) \]

\[ (n - 1) g_i(x) - \sum_{j \in N \setminus i} \psi_j(x_{-j}, y_{ji}) = \sum_{j \in N \setminus i} g_j(x) - \sum_{j \in N \setminus i} \psi_j(x_{-i}, y_{ij}) \]

\[ \sum_{j \in N \setminus i} (g_j(x) - \psi_j(x_{-j}, y_{ji})) = \sum_{j \in N \setminus i} (g_j(x) - \psi_j(x_{-i}, y_{ij})) \]

The difference \( \sum_{j \in N \setminus i} (g_j(x) - \psi_j(x_{-j}, y_{ji})) \) is the sum of the punishments that the remaining players can inflict on \( i \) at \( x \), and similarly, \( \sum_{j \in N \setminus i} (g_j(x) - \psi_j(x_{-i}, y_{ij})) \) is the sum of the punishments that \( i \) can inflict of the remaining players. Hence in equilibrium the total punishments have to be balanced.

Obviously, we are looking for a rule \( \psi \) yielding payoffs \( \psi(x) \) which cannot be beaten by the above retracting process, i.e. \( \psi(x) = g(x) \), for every \( x \in A \).

### 3 Equilibrium

The equilibrium notion is defined axiomatically:

**Definition 6** A value \( \psi \) is a Consistent Bargaining Equilibrium if it satisfies efficiency, equal minimum rights, individual rationality and total equal punishments.

Given \( \Gamma = (N, A, v) \), denote by \( \mathcal{CE}(N, A, v) \) the set of all consistent bargaining equilibrium values.

The next property is useful for proving the existence of a consistent bargaining equilibrium. Let \( \Gamma = (N, A, v) \) and \( \Gamma' = (N, A, w) \), where for all \( x \in A, w(x) = v(x) + a, a \in R \).
Definition 7 A rule $\psi$ verifies strategic equivalence iff $(\psi + \left(\frac{a}{n}, \ldots, \frac{a}{n}\right)) \in CE(N, A, w)$ when $\psi \in CE(N, A, v)$.

This property says that when the worth of the game is raised in the same amount at every action $x \in A$, i.e., $w(x) = v(x) + a$, the increase in the “productivity” cannot be attributed to any player in particular, so that it should be equally redistributed among all the players. It is not difficult to show that any $\psi \in CE(N, A, v)$ satisfies strategic equivalence.

Theorem 1 $CE(N, A, v) \neq \emptyset$

Proof. Consider a game $\Gamma = (N, A, v)$. By strategic equivalence, we can assume without loss of generality that $\min_{x \in A} \{v(x)\} = 0.$

Let $\Delta[v]$ be the set:

$$\Delta[v] := \left\{ g = [g(x)]_{x \in A} : \sum_{i \in N} g_i(x) = v(x) \text{ and } g_i(x) \geq 0, \ i \in N \right\}$$

By construction $\Delta[v]$ is a compact and convex set.

Define the following (adjusting) mapping $f : \Delta[v] \rightarrow \Delta[v]$, by

$$f_i(g(x)) = g(x) + \frac{1}{n-1} \left[ \sum_{j \in N \setminus i} P^g_{ij}(x) - \sum_{j \in N \setminus i} P^g_{ji}(x) \right]$$

for all $x \in A$ and for all $i \in N$.

By the definition of punishments $P^g_{ij}$ it follows that $f$ is a continuous mapping.

Now we show that $f(g) \in \Delta[v]$. Note that by construction:

$$\sum_{j \in N \setminus i} P^g_{ij}(x) \geq 0, \text{ for all } g \in \Delta[v], \text{ and } x \in A$$

Therefore, $f_i(g(x))$ can be at least:

$$f_i(g(x)) \geq g_i(x) + \frac{1}{n-1} \left[ 0 - \sum_{j \in N \setminus i} P^g_{ij}(x) \right] =$$

$$= g_i(x) + \frac{1}{n-1} \left[ - \sum_{j \in N \setminus i} g_j(x) + \sum_{j \in N \setminus i} g_i(x-j, y_{ji}) \right] \geq$$

$$\geq g_i(x) + \frac{1}{n-1} \left[ - (n-1) g_i(x) + 0 \right] = 0$$

Moreover, since $\sum_{i \in N} \left( \sum_{j \in N \setminus i} P^g_{ij}(x) - \sum_{j \in N \setminus i} P^g_{ji}(x) \right) = 0$ it follows that $\sum_{i \in N} f_i(g(x)) = v(x)$ for all $x \in A$. Hence, the mapping $f$ goes from $\Delta[v]$ into $\Delta[v]$.

Then, by Brouwer’s fixed point Theorem, there exists $g^*$, such that $f(g^*) = g^*$.

We show now that $g^* \in CE(N, A, v)$.

Firstly, efficiency follows by construction of $f(g)$. Secondly, for all $x \in A$ such that $v(x) = 0$, it holds that $g^*_i(x) = 0$, for all $i$ and then Equal Minimum Rights is verified.
Moreover, by construction \( g^*_i(x) \geq 0 \), for all \( x \in A \) and then Individual Rationality holds. Finally, \( f(g^*) = g^* \) if and only if:

\[
\sum_{j \in N \setminus i} P^g_{ij}(x) - \sum_{j \in N \setminus i} P^g_{ji}(x) = 0
\]

and then the total equal punishments property is satisfied. \( \blacksquare \)

For instance, in Example 1, the following payoff matrix:

<table>
<thead>
<tr>
<th>( \psi )</th>
<th>( a_j )</th>
<th>( b_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_i )</td>
<td>(2, 2)</td>
<td>(5, 3)</td>
</tr>
<tr>
<td>( b_i )</td>
<td>(2, 4)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

corresponds to an equilibrium. In particular:

\[
\begin{align*}
P^g_{ij}(a_i, b_j) &= P^g_{ji}(a_i, b_j) = 3 \\
P^g_{ij}(b_i, a_j) &= P^g_{ji}(b_i, a_j) = 2 \\
P^g_{ij}(a_i, a_j) &= P^g_{ji}(a_i, a_j) = P^g_{ij}(b_i, b_j) = P^g_{ji}(b_i, b_j) = 0
\end{align*}
\]

In general, this equilibrium concept gives rise to a set of equilibria, for example the next payoff division is also an equilibrium:

<table>
<thead>
<tr>
<th>( \psi )</th>
<th>( a_j )</th>
<th>( b_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_i )</td>
<td>(1, 3)</td>
<td>(4, 5, 3)</td>
</tr>
<tr>
<td>( b_i )</td>
<td>(1.5, 4.5)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

**Example 2.** Let us consider a three player game, where \( N = \{i, j, k\} \). The set of actions are: Player \( i \) chooses between two matrices \( t_i \) and \( b_i \), \( A^i = \{t_i, b_i\} \). Player \( j \) chooses between rows \( u_j \) and \( d_j \), \( A^j = \{u_j, d_j\} \), and Player \( k \) chooses between columns \( l_k \) and \( r_k \), \( A^k = \{l_k, r_k\} \). The matrix \( [v] \) is given by

<table>
<thead>
<tr>
<th>( v )</th>
<th>( l_k )</th>
<th>( r_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_j )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( d_j )</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( v )</th>
<th>( l_k )</th>
<th>( r_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_j )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( d_j )</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>
The consistent bargaining equilibrium payoffs are

<table>
<thead>
<tr>
<th>(\psi_1, \psi_2, \psi_3)</th>
<th>(l_k)</th>
<th>(r_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_j)</td>
<td>(0,0,0)</td>
<td>(0,0,0)</td>
</tr>
<tr>
<td>(d_j)</td>
<td>(0,0,0)</td>
<td>(2/6,24/6,22/6)</td>
</tr>
<tr>
<td>(l_k)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(r_k)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(u_j)</td>
<td>(0,0,0)</td>
<td>(0,0,0)</td>
</tr>
<tr>
<td>(d_j)</td>
<td>(7/6,9/6,2/6)</td>
<td>(3/6,27/6,18/6)</td>
</tr>
</tbody>
</table>

It can be checked that at \(x = (b_i, d_j, l_k)\) it holds that

\[
\begin{align*}
P^\psi_{ij}(x) &= 9/6 + P^\psi_{ik}(x) = 2/6 = P^\psi_{jk}(x) = 7/6 + P^\psi_{ki}(x) = 4/6, \\
&= 4/6 + P^\psi_{ki}(x) = 0 = P^\psi_{ik}(x) = 2/6 + P^\psi_{ij}(x) = 2/6.
\end{align*}
\]

Every difference \(P^\psi_{ij}(x) - P^\psi_{ji}(x)\) can be interpreted as the adjustment that player \(i\) could claim at \(\psi(x)\) against player \(j\) based on their punishment comparison. Notice also that although \(i\) could make an objection against \(j\) because \(P^\psi_{ij}(x) = 9/6 > P^\psi_{ji}(x) = 7/6\) and claim to adjust her payoffs by raising \(\psi_i(x)\) with respect to \(\psi_j(x)\), there is a chain of objections among the players at \(x\), such that, at the end, the initial objection goes back against player \(i\). In particular, as \(P^\psi_{jk}(x) = 2/6 > P^\psi_{ki}(x) = 0\), player \(j\) wishes to raise \(\psi_j(x)\) with respect to \(\psi_k(x)\), and because \(P^\psi_{ki}(x) = 4/6 > P^\psi_{ik}(x) = 2/6\), then player \(k\) wishes to raise \(\psi_k(x)\) with respect to \(\psi_i(x)\). This a general property that this equilibrium concept satisfies.

Denote by \(i \triangleright_{\psi(x)} j\) iff \(P^\psi_{ij}(x) > P^\psi_{ji}(x)\), and say that \(m\) players \(\{i_1, i_2, \ldots, i_m\}\) form an objection chain from \(i_1\) to \(i_m\) at \(\psi(x)\) if \(i_1 \triangleright_{\psi(x)} i_2 \triangleright_{\psi(x)} \cdots \triangleright_{\psi(x)} i_m\).

**Theorem 2** Let \(\psi \in CE(N,A,v)\) and suppose that \(k \triangleright_{\psi(x)} l\) for some \(k,l \in N, \ |N| \geq 3\) and \(x \in A\). Then there exists an objection chain from \(l\) to \(k\).

**Proof.** Let \(\psi(x)\) be an equilibrium payoff and suppose that \(k \triangleright_{\psi(x)} l\). Assume on the contrary that there does not exist an objection chain from \(l\) to \(k\) and let \(S = \{l\} \cup \{i \in N \mid k : \exists i \text{ a chain from } l \text{ to } i\}\). By the total equal punishment condition

\[
\sum_{i \in S} \sum_{j \in N \setminus \{l\}} (P^\psi_{ij}(x) - P^\psi_{ji}(x)) = \sum_{i \in S} \sum_{j \not\in S} (P^\psi_{ij}(x) - P^\psi_{ji}(x)) = 0
\]

\(^4\)This chain of binary punishments defines a binary relation that is similar to the relation for NTU games of "justified objections" between any two players, and which leads to the ordinal bargaining set of Asscher (1976)
For every $i \in S$ and $j \notin S$, $P_{ij}^\psi(x) \leq P_{ji}^\psi(x)$, otherwise $j$ will be in $S$. Moreover by assumption, $P_{kk}^\psi(x) < P_{kl}^\psi(x)$. Therefore:

$$\sum_{i \in S} \sum_{j \in S} \left( P_{ij}^\psi(x) - P_{ji}^\psi(x) \right) < 0$$

which is a contradiction. ■

4 Concluding remarks

4.1 Independence of the axiomatic system

The axiomatic system given by efficiency, equal minimal rights, individual rationality and total equal punishments, is independent. Indeed:

(i) For all action-set game $(N, A, v)$, define $\bar{v} := \min_{x \in A} v(x)$; and let $\bar{\xi}$ be the rule defined by $\bar{\xi}_i(x, v) = \frac{v}{n}$, for all $i \in N$. Then $F^1(N, A, v) := \{\bar{\xi}\}$ satisfies all axioms, but not efficiency.

(ii) For all action-set game $(N, A, v)$, let $\epsilon$ be the equal payoff division rule defined by $\epsilon_i(x, v) = \frac{v(x)}{n}$, for all $i \in N$. Then $F^2(N, A, v) := \{\epsilon\}$ satisfies all axioms, but not total equal punishments.

(iii) Let $\Gamma^1$ be the action-set game of Example 1, and let the payoff matrix $[\gamma]$ be

<table>
<thead>
<tr>
<th></th>
<th>$s_j$</th>
<th>$l_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_i$</td>
<td>(2, 2)</td>
<td>(4.5, 3.5)</td>
</tr>
<tr>
<td>$l_i$</td>
<td>(1.5, 4.5)</td>
<td>(-1, 1)</td>
</tr>
</tbody>
</table>

Let $F^3$ be defined by

$$F^3(\Gamma) := \begin{cases} \mathcal{E}(\Gamma^1) \cup \{\gamma\}, & \text{when } \Gamma = \Gamma^1, \\ \mathcal{E}(\Gamma), & \text{Otherwise.} \end{cases}$$

Then $F^3$ satisfies all axioms, but not equal minimal rights.

(iv) Let $\Gamma^1$ be the action-set game of Example 1, and let the payoff matrix $[\delta]$ be

<table>
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<tr>
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<th>$s_j$</th>
<th>$l_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_i$</td>
<td>(-1, 5)</td>
<td>(3.5, 4.5)</td>
</tr>
<tr>
<td>$l_i$</td>
<td>(0.5, 5.5)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

Let $F^4$ be defined by

$$F^4(\Gamma) := \begin{cases} \mathcal{E}(\Gamma^1) \cup \{\delta\}, & \text{when } \Gamma = \Gamma^1, \\ \mathcal{E}(\Gamma), & \text{Otherwise.} \end{cases}$$

Then $F^4$ satisfies all axioms, but not individual rationality.

4.2 Symmetry and efficiency

It is important to emphasize that ex-ante symmetric environments, where players have the same skills to undertake different jobs, may give rise to asymmetric ex-post individual payoffs with players being asymmetrically rewarded as a function of the different performance roles in labor division.

9
To illustrate this aspect, let us consider a slight variation of the hunters’ game in Example 1 with a symmetric matrix $[v]$ (at action-pairs $(s_i, l_j)$ and $(l_i, s_j)$) and equilibrium payoff matrix $[\psi]$:

<table>
<thead>
<tr>
<th></th>
<th>$s_j$</th>
<th>$l_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_i$</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>$l_i$</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th></th>
<th>$s_j$</th>
<th>$l_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_i$</td>
<td>(2, 2)</td>
<td>(5, 3)</td>
</tr>
<tr>
<td>$l_i$</td>
<td>(3, 5)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

In this ex-ante symmetric setting, the ex-post asymmetric payoffs at $(s_i, l_j)$ and $(l_i, s_j)$ are generated by the players’ location inside the matrix. In the action-pair $(l_i, s_j)$ the bargaining position of $i$ is worse than that of $j$, and viceversa. This is so because the hunter with the shooter’s role can always threaten the line driver with doing the same activity and then getting no deer, whereas the line driver hunter’s thread of becoming a shooter is less dramatic, since it only means hunting less deers. Thus, the different players’ bargaining power at $(l_i, s_j)$ and $(s_i, l_j)$ play a key role in the asymmetric ex-post equilibrium payoffs.

Additionally, if the target were to maximize the total number of deers shoted, then this would imply choosing between $(l_i, s_j)$ and $(s_i, l_j)$, with their corresponding unequal rewards. However, if the players were not able to agree on unequal rewards, then they would be forced to play $(s_i, s_j)$. Therefore, the conclusion is that equality and efficiency are incompatible in this example. Of course, it is always possible to make efficiency and ex-ante equality compatible by agreeing to play the fair lottery $L = [1/2(l_i, s_j); 1/2(s_i, l_j)]$, which yields expected payoffs $\psi_i(L) = \psi_j(L) = 4$.

### 4.3 Related Literature

The Prekernel.

The prekernel was introduced for the class of transferable utility (TU) games in Davis and Maschler (1965), and extended to the class of non-transferable utility (NTU) games in Moldovanu (1990) and Serrano (1997). The prekernel consists of those efficient payoffs $x$ in which each player is in a "bilateral equilibrium" with any other player. This balanced condition is expressed in terms of the individual excess$^5$ of player $i$ against player $j$, $e_{ij}(x)$, that is $e_{ij}(x) = e_{ji}(x)$ for all $i, j \in N$. When considering NTU-games, these excesses must be weighted by the normal vector components at $x$, $\lambda(x)$, so that the balanced condition translates to $\lambda_i(x)e_{ij}(x) = \lambda_j(x)e_{ji}(x)$, for all $i, j \in N$. For two-person problems, this solution coincides with the Nash bargaining solution. For three or more players, as pointed out in Moldovanu (1990) and Serrano (1997), the prekernel is often an empty set. To overcome this impossibility result, Orshan and Zarzuelo (2000) define the bilateral consistent prekernel, by imposing the equilibrium condition that the average (aggregate) excesses of a player against all the others must be equal to that of all of the others against her:

$$\sum_{j \in N \setminus i} \lambda_i(x)e_{ij}(x) = \sum_{j \in N \setminus i} \lambda_j(x)e_{ji}(x), \text{ for all } i \in N.$$  

This property is similar to the total equal punishment property used in our work.

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$^5$The reader is referred to the cited bibliography for the formal definitions.
The Shapley value.

The Shapley value was introduced for the class of TU-games in Shapley (1953). Given a TU-game \( v \) with player set \( N \), the value \( \phi_i(N, v) \) is the expectation of what player \( i \) will obtain in \( v \) if, for any possible order in which players arrive to the game, all equally likely, she is paid according to her marginal contribution to her predecessors. Myerson (1980) gives a characterization of the Shapley value by imposing the property of balanced contributions, which is very close in spirit to our equal punishment property. Suppose that player \( i \) leaves the game and then compute the value in the subgame \((N \setminus i, v)\) for the remaining players. The difference \( \phi_j(N, v) - \phi_j(N \setminus i, v) \) is just the variation in \( j \)'s payoffs due to player \( i \)'s decision of leaving the game. The balanced contribution axiom says that a value \( \psi \) satisfies this property if these differences are balanced for every pair of players, i.e.

\[
\psi_i(N, v) - \psi_i(N \setminus j, v) = \psi_j(N, v) - \psi_j(N \setminus i, v), \text{ for all } i, j \in N.
\]

Myerson proves that the Shapley value is the unique value satisfying efficiency and balanced contributions. Trying to impose the balanced contributions property to NTU-games, means running into the same existence problem than that of the prekernel. For two-person NTU-games \(((i, j), V)\), it suffices to consider both a payoff vector \( \psi(i, j, V) = a \) and a vector of weights \((\lambda_i, \lambda_j)\) satisfying

\[
\text{Efficiency: } \lambda_i a_i + \lambda_j a_j \geq \lambda_i b_i + \lambda_j b_j, \text{ for all feasible } (b_i, b_j),
\]

and

\[
\text{Balanced contributions: } \lambda_i (a_i - d_i) = \lambda_j (a_j - d_j),
\]

where \( d_i = \psi_i(i, V) \equiv \psi_i(i, j \setminus j, V) \). As is well known, the Nash bargaining solution (Nash, 1950) is the only one satisfying these two properties (Harsanyi, 1963). Unfortunately again, in coalitional games with three or more players the condition

\[
\lambda_i (\psi_i(N, V) - \psi_i(N \setminus j, V)) = \lambda_j (\psi_j(N, V) - \psi_j(N \setminus i, V)), \text{ for all } i, j \in N,
\]

cannot be imposed, because the existence of such payoff vectors is not guaranteed. Remarkably, replacing the above property by the average (aggregate) balanced contributions:

\[
\sum_{j \in N \setminus i} \lambda_j (\psi_j(N, V) - \psi_j(N \setminus i, V)) = \sum_{j \in N \setminus i} \lambda_i (\psi_i(N, V) - \psi_i(N \setminus j, V)), \text{ for all } i \in N,
\]

as in Hart and Mas-Colell (1996) makes it possible to show the existence of payoff allocations satisfying this property jointly with efficiency; this is the set of Consistent allocations. This concept was previously (and independently) introduced by Maschler and Owen (1989) for hyperplane games, and later defined in Maschler and Owen (1992) for the more general setting of NTU-games.

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