Axiomatic characterizations of the weighted solidarity values

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HIGHLIGHTS
- We define and characterize the family of all weighted solidarity values.
- We present two axiomatizations, one with additivity, and the other, without it.
- We study the behavior of these values in the class of monotonic games.

ABSTRACT
We define and characterize the class of all weighted solidarity values. Our first characterization employs the classical axioms determining the solidarity value (except symmetry), that is, efficiency, additivity and the A-null player axiom, and two new axioms called proportionality and strong individual rationality. In our second axiomatization, the additivity and the A-null player axioms are replaced by a new axiom called average marginality.

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1. Introduction

When a cooperative solution is considered from an axiomatic point of view, asymmetric versions of the value appear when the property of symmetry is dropped from the set of axioms that characterizes the value. The grounds justifying each asymmetric value depend on the context at hand. It could be differences in the negotiation ability of players or because they are representatives of groups of different size, etc. Thus, it seems to be more realistic to introduce some “weights” associated to the players in order to measure these differences.

The first nonsymmetric generalization of a value in coalitional form games with transferable utility is due to Shapley (1953). He defines the family of weighted Shapley values associated to positive weights for the players. Kalai and Samet (1987) extend the notion of “weights” to “weight systems”, allowing a weight of zero for some players. They also characterize the family of all weighted Shapley values axiomatically using efficiency, additivity, null player axiom, and two new axioms called positivity and partnership consistency. Hart and Mas-Colell (1989) provide a different axiomatization with monotonicity and consistency, among other axioms, but without additivity. Nowak and Radzik (1995), assuming that the weights of the players are given exogenously, provide two axiomatic characterizations of the corresponding weighted Shapley value: the first one, using the classical axioms determining the Shapley value, but replacing symmetry by a new axiom called ω—mutual dependence; the second one, adding a property called marginality, introduced by Young (1985), but removing additivity and the null player axiom. They also provide a characterization of the family of all weighted Shapley values.

The basic principle behind the weighted Shapley values is to pay players according to their productivity. A direct consequence is that null players always receive zero payoff, and this is the content of the null player axiom. Nevertheless, it is very easy to find real-life examples where a greater degree of solidarity among players seems to be natural.

There are several values that do not satisfy the null player axiom. We focus here on the solidarity value, introduced by Sprumont (1990). The Shapley value is based on the individual marginal contributions of a player to the coalitions she belongs to. In the solidarity value the individual marginal contribution is replaced by the average of the marginal contributions of all players which are in the coalition. This means that the individual contribution of each player is also shared among her partners in the game, being this a feasible way to express a certain degree of solidarity between the players in the cooperative game. Nowak and Radzik (1994) characterize this value axiomatically by means of the same axioms as
the Shapley value but replacing the null player axiom by the A-null player axiom (A-null stands average null). In this case, a player receives zero if the average of the marginal contributions is zero for all the coalitions he belongs to.

This paper defines and characterizes the family of all weighted solidarity values associated to positive weights for the players. Our first characterization (Theorem 6) employs the classical axioms determining the solidarity value (except symmetry), that is, efficiency, additivity and the A-null player axiom, and two new axioms called proportionality and strong individual rationality. This result is analogous to that of Kalai and Samet (1987), since the proportionality axiom can be seen as a variant of their axiom of partnership consistency and strong individual rationality is a weaker version of positivity. In our second axiomatization (Theorem 7), the additivity and the A-null player axioms are dropped and replaced by a new axiom called average marginality. This last property is similar to that of Young (1985) but with the average of the marginal contributions instead of the individual marginal contribution. In these two results, axioms impy the existence of a weight system such that the value is precisely the corresponding weighted solidarity value. Thus, weights are obtained endogenously. Finally, we study the behavior of the weighted solidarity values in the class of monotonic games. Contrary to the weighted Shapley values, there is a positive relationship between players’ weights and their bargaining power (Theorem 8).

The paper is organized as follows. Section 2 is devoted to some preliminaries. Section 3 defines the family of all weighted solidarity values. Finally, we provide the axiomatic characterizations in Section 4.

2. Preliminaries

A cooperative game with transferable utility (TU-game) is a pair (N, v) where N ⊆ N is a nonempty and finite set and v : 2N → R is a characteristic function, satisfying v(∅) = 0. An element i of N is called a player and every nonempty subset S of N a coalition. The real number v(S) is called the worth of coalition S, and it is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members. Let ̃gN denote the set of all cooperative TU-games with player set N and let ̃g denote the set of all games, that is, ̃g = ∪̃gN∈N ̃gN.

For all S ⊆ N, we denote the restriction of (N, v) to S as (S, v). For simplicity, we write S ∪ i instead of S ∪ {i}, N \ i instead of N \ {i}, and v(i) instead of v({i}). For each vector x ∈ Rn, let x(S) := ∑i∈S xi for each S ⊆ N.

A value is a function γ which assigns to every TU-game (N, v) and every player i ∈ N, a real number γi(N, v), which represents an assessment made by i of his gains from participating in the game. A payoff configuration is an element of ∏S⊆N Rn.

Let (N, v) be a game. For all S ⊆ N and all i ∈ S, define

\[ δ^i(v, S) := v(S) − v(S \setminus i). \]

We call δ^i(v, S) the marginal contribution of player i to coalition S in the TU-game (N, v). The Shapley value (Shapley, 1953b) of the game (N, v) is the payoff vector Sh(N, v) ∈ RN defined by

\[ Sh(N, v) = \sum_{S⊆N \setminus i} \frac{(n-s)!(s-1)!}{n!} δ^i(v, S), \quad \text{for all } i ∈ N, \]

where s = |S| and n = |N|.

Two players i, j ∈ N are symmetric in (N, v) if v(S ∪ i) = v(S ∪ j) for all S ⊆ N \ {i, j}. Player i ∈ N is a null player in (N, v) if v(S ∪ i) = v(S) for all S ⊆ N \ {i}. For any two games (N, v) and (N, v'), the game (N, v + v') is defined by (v + v')(S) = v(S) + v'(S) for all S ⊆ N.

Consider the following properties of a value γ in gN:

**Efficiency:** for all (N, v), \( \sum_{i∈N} γ_i(N, v) = v(N) \).

**Additivity:** for all (N, v) and (N, v'), γ(N, v + v') = γ(N, v) + γ(N, v')

**Symmetry:** for all (N, v) and all [i, j] ⊆ N, if i and j are symmetric players in (N, v), then γi(N, v) = γj(N, v).

Null player axiom: for all (N, v) and all i ∈ N, if i is a null player in (N, v), then γi(N, v) = 0.

The following theorem is due to Shapley (1953b).

**Theorem 1 (Shapley, 1953b).** A value γ on gN satisfies efficiency, additivity, symmetry and null player axiom if, and only if, γ is the Shapley value.

For all \( N \neq T \subseteq N \), the unanimity game of the coalition T, (N, ut), is defined by

\[ ut(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise}. \end{cases} \]

It is well known that the family of games \( \{(N, ut)\}_{\emptyset \neq T \subseteq N} \) is a basis for gN. This allows an alternative definition of the Shapley value as the linear mapping Sh : gN → Rn, which is defined for all unanimity game (N, ut) as follows

\[ Sh(N, ut) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T, \\ 0 & \text{otherwise}. \end{cases} \]

The solidarity value, Sl, was introduced by Sprumont (1990). Section 5, in a recursive way. Let (N, v) be a game. For all S ⊆ N, define

\[ \Delta^w(v, S) := \frac{1}{S} \sum_{i∈S} δ^i(v, S), \]

Thus, \( \Delta^w(v, S) \) is the average of the marginal contributions of players within coalition S in the game (N, v). Then, the solidarity value is defined by

\[ Sl(S, v) = \frac{1}{S} \Delta^w(v, S) + \frac{1}{S} \sum_{i∈S} Sl(S \setminus j, v), \quad \text{for all } i ∈ S ⊆ N, \]

starting with

\[ Sl(\emptyset, v) = v(\emptyset), \quad \text{for all } i ∈ N. \]

Later on, Nowak and Radzik (1994) yield a different definition of Sl, similar to that of the Shapley value, but with the average of the marginal contributions instead of the individual marginal contribution:

\[ Sl(N, v) = \sum_{S⊆N \setminus i} \frac{(n-s)!(s-1)!}{n!} \Delta^w(v, S), \quad \text{for all } i ∈ N. \]

Calvo (2008) shows that both definitions, (1) and (2), are equivalent.

The solidarity value satisfies some solidarity principle, since null players can obtain positive payoffs (see Example 1.1 in Nowak and Radzik (1994)). They introduce a variation of the null player axiom in order to characterize Sl on gN. Player i ∈ N is an A-null player in (N, v) if \( \Delta^w(v, S) = 0 \) for all coalition S ⊆ N containing i. There is clearly no relation between the null player and the A-null player concepts. The solidarity value satisfies the following axiom.

A-Null player axiom: for all (N, v) and all i ∈ N, if i is an A-null player, then γi(N, v) = 0.
The following theorem is due to Nowak and Radzik (1994).

**Theorem 2 (Nowak and Radzik, 1994).** A value \( \gamma \) on \( \mathcal{N} \) satisfies efficiency, additivity, symmetry and the A-null player axiom if, and only if, \( \gamma \) is the solidarity value.

Nowak and Radzik (1994) define a new basis for \( \mathcal{N} \), denoted by \( (N, b_T) \) for all \( T \subseteq N \). For all \( \emptyset \neq T \subseteq N \), \( (N, b_T) \) is defined by

\[
b_T(s) = \begin{cases} \sum_{i \in T} b_i(s) & \text{if } T \subseteq S, \\ 0 & \text{otherwise}. \end{cases}
\]

They prove that all players in \( N \setminus T \) are A-null players in the game \( (N, b_T) \), so they receive a zero payoff, and all players in \( T \) are symmetric so they receive the same payoff. Thus, it holds that

\[
\Gamma_i(N, b_T) = \begin{cases} \frac{1}{|T|} \sum_{j \in T} b_j(T) & \text{if } i \in T, \\ 0 & \text{otherwise}. \end{cases}
\]

**Remark 1.** It is shown in the recent paper of Radzik (2013) that the solidarity value \( \Gamma \) has a very close relationship with the equal split value defined as the value of the form \( \phi_\mathcal{N}(N, v) = v(N)/|N| \) for all \( i \in N \). Namely, it turns out that for large \( |N| \) the approximation \( \Gamma_i(N, v) \approx \phi_\mathcal{N}(N, v) \) can be justified for some wide subsets of games in \( \mathcal{N} \). In that paper, the general problem of asymptotic equivalence between both values is also studied.

### 3. Weighted solidity values and its basic properties

In this section, we define the weighted solidarity values in two different ways and prove that both definitions are equivalent.

A system of positive weights is a function \( \omega : N \to \mathbb{R} \) with \( \omega(i) > 0 \) for all \( i \in N \). We denote \( \omega_i = \omega(i) \). For each \( i \in N \), \( \omega_i \) is the weight of player \( i \). A weighted value \( v^\omega \) is a function that assigns to every game \( (N, v) \) and every weight \( \omega \in \mathbb{R}^N_+ \), a vector \( v^\omega(N, v) \) in \( \mathbb{R}^N \). We say that a weighted value \( v^\omega \) extends a value \( v \) if \( v^\omega(N, v) = v(N, v) \) for all \( i \in N \) and all weight vector \( \omega \) with \( \omega_i = \omega_j \) for all \( i, j \in N \). The most important weighted generalization of the Shapley value is the weighted Shapley value \( \Gamma^\omega \) (Shapley, 1953a; Kalai and Samet, 1987). Let \( \omega \) be a system of positive weights; then the weighted Shapley value \( \Gamma^\omega \) is the linear mapping defined for each unanimity game \( (N, u_T) \), \( \emptyset \neq T \subseteq N \subseteq N \), as follows

\[
\Gamma^\omega_i(N, u_T) = \begin{cases} \frac{\omega_i}{\omega(T)} u_T(T) & \text{if } i \in T, \\ 0 & \text{otherwise}. \end{cases}
\]

The weighted Shapley value \( \Gamma^\omega \) satisfies efficiency, additivity and the null player axiom, but not symmetry.

The above definition of the weighted Shapley value is based on the unanimity games \( (N, u_T) \) which play an essential role in the axiomatization of the classical Shapley value. A similar role for the solidity value plays the games \( (N, b_T) \) of the form (3). Below in Definition 1, we will use this analogy to propose the definition of the weighted solidarity value.

**Definition 1.** Let \( \omega \) be a system of positive weights. The weighted solidity value \( \Phi^\omega \) is the linear mapping defined for each game \( (N, b_T) \), \( \emptyset \neq T \subseteq N \subseteq N \), as follows

\[
\Phi^\omega_i(N, b_T) = \begin{cases} \frac{\omega_i}{\omega(T)} b_T(T) & \text{if } i \in T, \\ 0 & \text{otherwise}. \end{cases}
\]

Therefore, the weighted solidarity value \( \Phi^\omega \) satisfies efficiency and additivity, but not symmetry.

Next we show that the weighted solidarity value can also be defined recursively.

**Definition 2.** Let \( \omega \) be a system of positive weights. For each game \( (N, v) \), we define recursively the following payoff configuration:

\[
a^\omega_i(S, v) = \frac{\omega_i}{\omega(S)} \Delta^\omega_i(S, v) + \frac{1}{|S|} \sum_{j \in S} a^\omega_j(S \setminus j, v),
\]

for all \( S \subseteq N \) and all \( i \in S \), starting with

\[
a^\omega_i(\{i\}, v) = v(i), \quad \text{for all } i \in N.
\]

**Theorem 3.** For all games \( (N, v) \) and all \( \omega \in \mathbb{R}^N_+ \), we have that \( \Phi^\omega(N, v) = a^\omega(N, v) \).

**Proof.** Since both \( \Phi^\omega \) and \( a^\omega \) are linear mappings, we only have to prove that \( \Phi^\omega(N, b_T) = a^\omega(N, b_T) \) for all \( \emptyset \neq T \subseteq N \).

Let \( \emptyset \neq T \subseteq N \). For all \( i \in N \setminus T \) we have that \( a^\omega_i(\{i\}, b_T) = b_T(i) = 0 \) and \( \Delta^\omega_i(b_T, S) = 0 \) for all \( S \subseteq N \) containing \( i \), as \( i \in N \setminus T \) is an A-null player in \( (N, b_T) \). Therefore, applying Definitions 1 and 2, we deduce that \( a^\omega_i(N, b_T) = 0 = \Phi^\omega_i(N, b_T) \) for all \( i \in N \setminus T \).

Let \( i \in T \). It holds that \( \Delta^\omega_i(b_T, S) = 0 \) for all \( S \not\supseteq T \) and then \( a^\omega_i(S, b_T) = 0 \) for all \( S \not\supseteq T \) containing \( i \). Moreover, \( \Delta^\omega_i(b_T, T) = 1 \) so then

\[
a^\omega_i(T, b_T) = \frac{\omega_i}{\omega(T)} \Phi^\omega_i(T, b_T).
\]

Suppose now that \( T \subseteq N \). Then, \( \Delta^\omega_i(b_T, S) = 0 \) for all \( S \nsubseteq T \), as \( S \) contains A-null players, that is, players that belong to \( N \setminus T \). Thus, for any \( j \in N \setminus T \),

\[
a^\omega_j(T \cup j, b_T) = \frac{\omega_j}{\omega(T \cup j)} \Delta^\omega_j(b_T, T \cup j) + \frac{1}{|T \cup j| - 1} a^\omega_j(T \cup j, b_T) = \frac{1}{\omega(T \cup j)} \Phi^\omega_j(T \cup j, b_T).
\]

Suppose by induction that \( a^\omega_i(S, b_T) = \Phi^\omega_i(S, b_T) \) for all \( S \subseteq N \) containing \( i \). Then,

\[
a^\omega_i(N, b_T) = \frac{1}{n} \sum_{k=1}^{n} a^\omega_i(N \setminus k, b_T) = \frac{1}{n} \sum_{k=1}^{n} \Phi^\omega_i(N \setminus k, b_T).
\]

By induction hypothesis, \( a^\omega_i(N \setminus k, b_T) = \frac{\omega_i}{\omega(T)} b_T(N \setminus k) \) for all \( k \in N \setminus T \). Thus, following (5):

\[
a^\omega_i(N, b_T) = \frac{1}{n} \sum_{k \in N \setminus T} \frac{\omega_i}{\omega(T)} \left( \frac{n-1}{t} \right)^i \left( \frac{n-t}{t} \right)^{n-1} \Phi^\omega_i(N \setminus k, b_T) = \frac{\omega_i}{\omega(T)} \left( \frac{n-1}{t} \right)^i \left( \frac{n-t}{t} \right)^{n-1} A^\omega_i(N, b_T).
\]

Hence, the weighted solidarity value \( \Phi^\omega \) also satisfies the A-null player axiom, which trivially follows from (4). It turns out that this value satisfies the three basic properties (excluding symmetry) of the solidarity value.

### 4. Axiomatic characterizations

Following the analogy between the Shapley value and the solidity value, we here propose two axiomatizations of the family of all weighted solidarity values.
Kalai and Samet (1987) first characterize the family of all weighted Shapley values using efficiency, additivity, the null player axiom, and two axioms called positivity and partnership consistency. Hart and Mas-Colell (1989) provide a different axiomatization with monotonicity and consistency, among other axioms, but without additivity. Furthermore, Nowak and Radzik (1995) also provide another axiomatization without additivity and the null player axiom, by adding positivity, mutual dependence and marginality.

Kalai and Samet (1987) introduce the following concept and axioms. Let \( (N, v) \) be a game. A coalition \( S \subseteq N \) is a partnership in \( (N, v) \) if for each \( T \subseteq S \) and each \( R \subseteq N \setminus S \), \( v(R \cup T) = v(R) \).

**Partnership Consistency:** For all game \( (N, v) \), if \( S \subseteq N \) is a partnership in \( (N, v) \), then \( \gamma_i(N, v) = \gamma_i(N, y(v)(S) \cup R) \), for all \( i \in S \), where \( y(v)(S) \) denotes \( \sum_{i \in S} \gamma_i(N,v) \).

**Partnership consistency** expresses the following idea: suppose we want to reallocate \( y(v)(S) \) among the members of a partnership \( S \). Since each proper subcoalition of \( S \) is powerless, it is natural to reallocate \( y(v)(S) \) by applying \( y \) to the unanimity game \( y(v)(S) \). This axiom says that each player in \( S \) receives after the reallocation exactly what he received in the original game \( (N, v) \).

**Additivity:** if \( (N, v) \) is monotonic (i.e., \( v(T) \leq v(S) \) for all \( T, S \subseteq N \) such that \( T \subseteq S \)) and has no null players, then \( \gamma_i(N, v) > 0 \) for all \( i \in N \).

**Theorem 4 (Kalai and Samet, 1987).** A value \( \gamma \) on \( N \) satisfies efficiency, additivity, the null player axiom, partnership consistency and positivity if, and only if, there exists a weight system \( \omega \in \mathbb{R}^N_+ \) such that \( \gamma \) is the weighted Shapley value \( Sh^\omega \).

**Nowak and Radzik (1995) formulate some new axioms in order to provide another axiomatization without additivity. They introduce the following concept.** Let \( (N, v) \) be a game and let \( i, j \in N \) (\( i \neq j \)). If \( v(S \cup i) = v(S) = v(S \cup j) \) for all \( S \subseteq N \setminus \{i, j\} \), then the players \( i \) and \( j \) are mutually dependent in \( (N, v) \).

**Mutual Dependence:** for all \( (N, v) \) and \( (N, v') \) and all \( i, j \in N \), if \( i \neq j \), if \( i \) and \( j \) are mutually dependent players in both \( (N, v) \) and \( (N, v') \), then \( \gamma_i(N, v) \gamma_j(N, v') = \gamma_i(N, v') \gamma_j(N, v) \).

**Lemma 1.** Let \( (N, v) \) be a game and let \( S \subseteq N \) with \( |S| \geq 2 \) be a team in \( (N, v) \). Then, either all players in \( S \) are A-null players or no player in \( S \) is an A-null player.

**Proof.** Suppose that there is a player \( j \in S \) such that \( j \) is not an A-null player. Then, there exists a coalition \( T \subseteq N \) with \( T \neq S \) and \( v(T) > 0 \). Since \( S \) is a coalition in \( (N, v) \), it holds necessarily that \( S \subseteq T \). Thus, all players \( i \in S \) are not A-null players either. 

The next axiom is weaker than positivity.

**Strong Individual Rationality:** for all game \( (N, v), v(S) = 0 \) for all \( S \neq N \) and \( v(i) > 0 \) for all \( i \in N \).

We are now ready to offer our first axiomatic characterization of the family of weighted solidarity values. This is analogous to that of Kalai and Samet (1987), since the concept of a team can be seen as a variant of the notion of partnership and the proportionality axiom, as a variant of partnership consistency.

First, we prove that, for each \( \omega \in \mathbb{R}^N_+ \), the weighted solidarity value \( S^{\omega} \) satisfies the following property.

**\( \omega \)-Proportionality:** for all \( (N, v) \) and all \( S \subseteq N \) with \( |S| \geq 2 \), if \( S \) is a team in \( (N, v) \), then

\[
\frac{\gamma_i(N, v)}{\omega_i} = \frac{\gamma_j(N, v)}{\omega_j} \quad \text{for all } i, j \in S.
\]

**Proposition 1.** For each \( \omega \in \mathbb{R}^N_+ \), the weighted solidarity value \( S^{\omega} \) satisfies \( \omega \)-proportionality.

**Proof.** Let \( (N, v) \) be a game and suppose that \( S \subseteq N \) with \( |S| \geq 2 \) is a team. First, we shall prove that \( S^{\omega}(T, \omega) = 0 \) for all \( T \subseteq N \) such that \( T \neq S \) and \( i \in T \). Indeed, taking into account Definition 2,
Let $\emptyset \neq T \subseteq N$ and $\alpha \in \mathbb{R}$. By the A-null player axiom, $\gamma(N, \alpha b_T) = 0 = S_{\omega}^\alpha(N, \alpha b_T)$ for all $i \in N \setminus T$. Thus, it only remains to show that $\gamma(N, \alpha b_T)$ is uniquely determined for players $i \in N$. If $|T| = 1$, by efficiency $\gamma(N, \alpha b_T) = \alpha b_T(N) = S_{\omega}^\alpha(N, \alpha b_T)$ for $\{i\} = T$. Suppose that $|T| \geq 2$, and then $T$ is a team in $(N, \alpha b_T)$ and in $(N, b_T)$; therefore by proportionality, $\gamma(N, \alpha b_T) \gamma_T(N, b_T) = \gamma(N, b_T) \gamma_T(N, \alpha b_T)$, for all $i, j \in T$.

Therefore, by efficiency, $\alpha b_T(N) = \sum_{i \in T} \gamma(N, \alpha b_T) = C(\omega(T)$ and then,

$$\gamma(N, \alpha b_T) = C\omega_T = \frac{\alpha \omega_T}{\omega(T)}$$

for all $i \in T$.

Our second characterization is similar to that of Nowak and Radzik (1995). The *additivity* and the *A-null player* axioms are dropped and replaced by a new axiom called average *marginality*. Average marginality: for all $(N, v)$ and $(N, v')$, if for some player $i \in N$, we have $\Delta^a(v, S \cup i) = \Delta^a(v', S \cup i)$, for all $S \subseteq N \setminus i$, then $\gamma(N, v) = \gamma(N, v')$.

If for a player $i$, the average of the marginal contributions is equal in two different games for all the coalitions he belongs to, he must receive the same payoff in both games. This last property is similar to that of Young (1985) but with the average of the marginal contributions instead of the individual marginal contribution.

We now prove the second characterization.

**Theorem 7.** A value $\gamma$ on $g^N$ satisfies efficiency, proportionality, strong individual rationality and average marginality if, and only if, there exists a weight vector $\omega \in \mathbb{R}^N_{++}$ such that $\gamma = S_{\omega}^\alpha$.

**Proof.** Existence. It only remains to prove that, for each $\omega \in \mathbb{R}^N_{++}$, $S_{\omega}^\alpha$ satisfies average marginality, but this is straightforward taking into account Definition 2.

Uniqueness. Let $\gamma$ be a value satisfying the above axioms. Let $\omega = \gamma(N, b_N)$. By strong individual rationality, $\omega_i > 0$ for all $i \in N$. We shall prove that $\gamma = S_{\omega}^\alpha$.

Let $(N, v_0)$ be the game defined as $v_0(S) = 0$ for all $S \subseteq N$. First, we prove that $\gamma(N, v_0) = 0$ for all $i \in N$. If $|N| = 1$, by efficiency, $\gamma(N, v_0) = 0$. If $|N| \geq 2$, $N$ is a team in $(N, v_0)$ and in $(N, b_N)$, then by proportionality we have that

$$\gamma(N, v_0) \gamma_T(N, b_N) = \gamma(N, b_N) \gamma_T(N, v_0),$$

for all $i, j \in N$, that is,

$$\gamma(N, v_0) \frac{\omega_i}{\omega} = C,$$

and by efficiency, it implies $\gamma(N, v_0) = 0$ for all $i \in N$.

Let $(N, v)$ be a game. If $i \in N$ is an A-null player in $(N, v)$, then $\Delta^a(v, S \cup i) = \Delta^a(v_0, S \cup i)$, for all $S \subseteq N \setminus i$; thus by average marginality, $\gamma(N, v) = \gamma(N, v_0) = 0$. Hence, it only remains to show that $\gamma(N, v)$ is uniquely determined when $i \in N$ is not an A-null player.

Now consider the game $(N, \alpha b_T)$ with $\alpha \neq 0$ and $\emptyset \neq T \subseteq N$. If $|T| = 1$, by efficiency, $\gamma(N, \alpha b_T) = \alpha b_T(N)$ for $\{i\} = T$. Suppose that $|T| \geq 2$, and then $T$ is a team in $(N, \alpha b_T)$ and in $(N, b_T)$; then by proportionality we have that

$$\gamma(N, \alpha b_T) \frac{\omega_i}{\omega} = C,$$

for all $i \in T$.
and, by efficiency,
\[ \gamma_t(N, ab_T) = \frac{\alpha_t}{\omega(T)} ab_T(N), \]
for all \( t \in T \).

We now use the fact that the games \( (N, b_T) \) \( t \neq T \in N \) form a basis for \( \mathcal{g}^N \). Thus,
\[ (N, v) = \sum_{t \neq T \subseteq N} (N, \alpha_T b_T), \]
where the constants \( \alpha_T \) are uniquely determined by the game \( (N, v) \). Let \( I \subseteq N : \alpha_T \neq 0 \). We proceed by induction over \( |I(N, v)| \). We already know that \( \gamma(N, v) \) is uniquely determined when \( |I(N, v)| \leq 1 \). Suppose that it is true for every game \( (N, v) \) with \( |I(N, v)| \leq k \). Let \( (N, v) \) be a game with \( |I(N, v)| = k + 1 \). Then, we have \( k + 1 \) nonempty coalitions \( T_1, \ldots, T_{k+1} \) such that
\[ (N, v) = \sum_{j=1}^{k+1} (N, \alpha_T b_T). \]

Let \( T = T_1 \cap \cdots \cap T_{k+1} \) and suppose that \( i \notin T \). Define a new game \( (N, v') \) as
\[ (N, v') = \sum_{j=1}^{k+1} (N, \alpha_T b_T). \]
Then, \( |I(N, v')| = k \) and \( \Delta^\omega(v, S \cup i) = \Delta^\omega(v', S \cup i) \), for all \( S \subseteq N \setminus i \); thus by average marginality, \( \gamma_t(N, v) = \gamma_t(N, v') \), but \( \gamma_t(N, v') \) is uniquely determined by induction hypothesis. Suppose now that \( i \in T \). If \( |T| = 1 \), by efficiency, \( \gamma(N, v) \) is uniquely determined. If \( |T| \geq 2 \), \( T \) is a team in \( (N, v) \) as
\[ \Delta^\omega(v, S) = \sum_{j=1}^{k+1} \alpha_T \Delta^\omega(b_T, S) = 0, \]
for all \( S \ni T \), and in \( (N, b_T) \); thus by proportionality,
\[ \frac{\gamma_t(N, v)}{\alpha_t} = C, \]
for all \( t \in T \), and by efficiency,
\[ v(N) = C \omega(T) + \sum_{k \in N \setminus T} \gamma_k(N, v). \]

Since \( \gamma_k(N, v) \) is uniquely determined for all \( k \in N \setminus T \), we conclude that \( C \) and \( \gamma_t(N, v) \) are also uniquely determined for all \( t \in T \).

**Remark 2.** In these two characterizations, axioms imply the existence of a weight system such that the value is precisely the corresponding weighted solidarity value. Thus, weights are obtained endogenously.

**Remark 3.** Radzik (2012) shows that there is a problem with the interpretation of the weight system in the context of the weighted Shapley value (see Remark 4.7 and Examples 4.2 and 4.3 there). It is usual that weights are interpreted as a measure of the “importance” or “bargaining strength” that players have in the game. However, it turns out that there are monotonic games for which the behavior of the weighted Shapley value goes in opposite direction for some players’ weights (Examples 4.2 and 4.3 there), that is, bigger weights correspond to lower payoffs. Contrary to the weighted Shapley value, in the \( \text{SF}^\omega \) value this positive relationship between weights and bargaining power is completely general for any monotonic game. This is the content of the next Theorem whose proof is a direct consequence of Definition 2 and is left to the reader.

**Theorem 8.** Let \( (N, v) \) be a monotonic game and let \( \omega, \omega' \in \mathbb{R}^N_+ \) such that \( \alpha_T \geq \alpha_0 \) and \( \omega'_j = \omega_j \) for each \( j \in N \setminus S \). Then \( \text{SF}^\omega_\omega(N, v) = \text{SF}^\omega_\omega(N, v) \).

**Acknowledgments**

Emilio Calvo thanks the Ministry of Science and Technology and the European Feder Funds under project ECO2010-20584, and the Generalitat Valenciana under the Excellence Programs Prometeo 2009/068 and ISIC2012/021 for their financial support. Esther Gutiérrez-López wishes to thank financial support from the Spanish Ministry of Science and Technology and the European Regional Development Funds under project ECO2012-33618, and from UPV/EHU (UIF 11/51). The authors would like to thank an Associate Editor and two anonymous referees for their helpful comments and criticisms.

**References**


