Acotación de operadores de Cesáro generalizados en espacios de diferencias fraccionarias

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1. Introduction

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Let ℓ^p, $1 \leq p < \infty$, the usual Lebesgue space of sequences

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and ℓ^∞, the set of bounded sequences with the norm

$$\ell^\infty := \{f = (f(n))_{n\geq 0} \subset \mathbb{C} \ : \ \|f\|_\infty := \sup_{n\geq 0} |f(n)| < \infty\}.$$

The continuous embedding $\ell^1 \hookrightarrow \ell^p \hookrightarrow \ell^\infty$ holds.

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A Banach algebra $\mathcal{A}$ is a Banach space with an associative and distributive product such that $\lambda(xy) = (\lambda x)y = x(\lambda y)$ and $\|xy\| \leq \|x\|\|y\|$ for all $\lambda \in \mathbb{C}$ and $x, y \in \mathcal{A}$.

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Note that ℓ^1 is a commutative Banach algebra endowed with their natural convolution product

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Moreover $\ell^p * \ell^1 \hookrightarrow \ell^p$ $(1 \leq p \leq \infty)$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p, \qquad f \in \ell^1,\ g \in \ell^p.$$

The space ℓ^p is a module over the algebra ℓ^1.

The Cesàro operator $\mathcal{C} : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$, $f \mapsto \mathcal{C}f$, is defined by

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Note that $\mathcal{C} : \ell^1 \not\to \ell^1$, $\mathcal{C} : \ell^p \to \ell^p$, with $1 < p \leq \infty$ due to

$$\sum_{n=0}^{\infty} \left| \frac{1}{n+1} \sum_{j=0}^{n} f(n) \right|^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=0}^{\infty} |f(n)|^p$$

(Hardy inequality, 1930)

For $\beta > 0$, the β-Cesàro operator $\mathcal{C}^\beta : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$, is defined by

$$\mathcal{C}^\beta f(n) = \frac{1}{k^{\beta+1}(n)} \sum_{j=0}^{n} k^\beta(n-j) f(j) = \frac{1}{k^{\beta+1}(n)} \left(k^\beta * f\right)(n), \quad n \in \mathbb{N}_0,$$

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where $k^\beta(n) = \dfrac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)}$. (Stempak (1994), Zygmund (1959))

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(Stempak(1994), Andersen (1996), Xiao (1997))

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(iv) k^α is increasing for $\alpha>1$, decreasing for $0<\alpha<1$, $k^1(n)=1$ and $k^0 = \lim_{\alpha\to 0^+} k^\alpha = e_0 = (1,0,0,\ldots)$, where $e_j(n) = \delta_{j,n}$ is the Kronecker delta.

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(v) $k^{\alpha+1}(n)\binom{n}{j}=k^{\alpha+1}(j)\binom{n+\alpha}{j+\alpha}$.

Motivation: Césaro sums and Cesáro means [ALMV]

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Let X be a complex Banach space, $T \in \mathcal{B}(X)$ and denote by $\mathcal{T}$ the discrete semigroup given by $\mathcal{T}(n) := T^n$ for $n \in \mathbb{N}_0$.

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$$S_\alpha \mathcal{T}(n)x = (k^\alpha * \mathcal{T})(n)x = \sum_{j=0}^{n} k^\alpha(n-j)T^j x, \qquad x \in X, \quad n \in \mathbb{N}_0.$$

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For example $S_0\mathcal{T}(n) = T^n$ and $S_1\mathcal{T}(n) = \sum_{j=0}^{n} T^j$.

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The Cesàro means of order $\alpha > 0$ of T, $\{M_\alpha \mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$, is defined by

$$M_\alpha \mathcal{T}(n)x = \frac{1}{k^{\alpha+1}(n)}\left(k^\alpha * \mathcal{T}\right)(n)x, \qquad x \in X, \quad n \in \mathbb{N}_0.$$

In the case that $\|S_\alpha \mathcal{T}(n)\| \leq Ck^{\alpha+1}(n)$, (i.e., Cesaro means are uniformly bounded), the operator T is called (C, α)-bounded.

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However the inverse result is not true. For example, the matrix

$$T = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

defines a $(C, 1)$-bounded operator, that is,

$$\|S_1 \mathcal{T}(n)\| = \|\sum_{j=0}^{n} T^j\| \leq C(n+1), \quad n \in \mathbb{N}_0$$

but T does not satisfy the power-boundedness condition.

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The main aim of this talk is to study the boundedness of Cesáro operator $\mathcal{C}^\beta$ (and its adjoint $(\mathcal{C}^\beta)^*$) in some fractional finite difference spaces, τ_p^α. We estimate their norms and describe their spectrum sets.

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The main aim of this talk is to study the boundedness of Cesáro operator $\mathcal{C}^\beta$ (and its adjoint $(\mathcal{C}^\beta)^*$) in some fractional finite difference spaces, τ_p^α. We estimate their norms and describe their spectrum sets.

(i) We introduce some fractional finite difference in the sense of Weyl and a scale of Banach modules, τ_p^α, contained in ℓ^p.

(ii) We define some C_0-semigroups of contractions in τ_p^α.

(iii) We express the operators $\mathcal{C}^\beta$ and its adjoint, $(\mathcal{C}^\beta)^*$, in terms of the C_0-semigroups.

(iv) These representations allow us to estimate $\|\mathcal{C}^\beta\|$ and $\|(\mathcal{C}^\beta)^*\|$ and to describe their spectrum sets via a spectral mapping theorem for C_0-semigroups and we draw them.

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Let $f : \mathbb{N}_0 \to \mathbb{C}$, we denote the usual differences by

$$\nabla(n) = f(n) - f(n-1),$$

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and for $m \in \mathbb{N}$,

$$W_+^m f(n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(n+j).$$

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$$W_+^{-m} f(n) = \sum_{j=m}^{\infty} \frac{\Gamma(j-n+m)}{\Gamma(j-n+1)\Gamma(m)} f(j) = \sum_{j=n}^{\infty} k^m(j-n) f(j), \quad n \in \mathbb{N}_0.$$

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Definition.
Let $f : \mathbb{N}_0 \to \mathbb{C}$ and $\alpha > 0$ be given. The Weyl sum of order α of f, $W_+^{-\alpha}f$, is defined by

$$W_+^{-\alpha}f(n) := \sum_{j=n}^{\infty} k^{\alpha}(j-n)f(j), \quad n \in \mathbb{N}_0.$$

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The Weyl difference of order α of f, $W_+^\alpha f$, is defined by

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for $m = [\alpha] + 1$, whenever the right hand sides make sense.

In particular

(i) $W_+^\alpha : c_{0,0} \to c_{0,0}$ for $\alpha \in \mathbb{R}$.

(ii) $W_+^\alpha W_+^\beta f = W_+^{\alpha+\beta} f = W_+^\beta W_+^\alpha f$ for $\alpha, \beta \in \mathbb{R}$ and $f \in c_{00}$.

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(i) Let $\lambda \in \mathbb{C}\backslash\{0\}$, and $p_\lambda(n) := \lambda^{-(n+1)}$ for $n \in \mathbb{N}_0$.

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Example.

(i) Let $\lambda \in \mathbb{C}\backslash\{0\}$, and $p_\lambda(n) := \lambda^{-(n+1)}$ for $n \in \mathbb{N}_0$. The sequences p_λ are eigenfunctions for the operator W_+^α for $\alpha \in \mathbb{R}$ if $|\lambda| > 1$:

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$$W_+^\alpha p_\lambda = \frac{(\lambda-1)^\alpha}{\lambda^\alpha} p_\lambda, \qquad |\lambda| > 1.$$

(ii) Let $\alpha \geq 0$ be given. We define

$$h_n^\alpha(j) := \begin{cases} k^\alpha(n-j), & 0 \leq j \leq n \\ 0, & j > n, \end{cases}$$

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for $n \in \mathbb{N}_0$. Then

$$W_+^\beta h_n^\alpha = h_n^{\alpha-\beta},$$

for $\beta \leq \alpha$ and $n \in \mathbb{N}_0$.

3. Convolution Banach modules τ_p^{α}

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For $\alpha > 0$, we define $q_{\alpha,p} : c_{0,0} \to [0,\infty)$ by

$$q_{\alpha,p}(f) := \left(\sum_{n=0}^{\infty} \left(k^{\alpha+1}(n) | W_+^\alpha f(n) | \right)^p \right)^{\frac{1}{p}}, \quad f \in c_{0,0}.$$

Note that for $\alpha = 0$, $q_{0,p} = \| \quad \|_p$.

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Note that for $\alpha = 0$, $q_{0,p} = \| \quad \|_p$.

Theorem.

Let $\alpha > 0$. Then $q_{\alpha,p}$ defines a norm in $c_{0,0}$ and

$$q_{\alpha,p}(f * g) \le C_\alpha \, q_{\alpha,p}(f) \, q_{\alpha,1}(g), \qquad f, g \in c_{0,0}(\mathbb{N}_0).$$

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$$q_{\alpha,p}(f * g) \leq C_\alpha \, q_{\alpha,p}(f) \, q_{\alpha,1}(g), \qquad f, g \in c_{0,0}(\mathbb{N}_0).$$

Denote by τ_p^α the completion of $c_{0,0}$ in the norm $q_{\alpha,p}$. Then

$$\tau_p^\beta \hookrightarrow \tau_p^\alpha \hookrightarrow \ell^p, \qquad \tau_1^\alpha \hookrightarrow \tau_p^\alpha \hookrightarrow \tau_\infty^\alpha, \qquad (\tau_p^\alpha)' = \tau_{p'}^\alpha, 1 < p < \infty,$$

for $0 < \alpha < \beta$ and $\lim_{\alpha \to 0^+} q_{\alpha,p}(f) = \|f\|_p$.

Example.

Let $p_\lambda(n) = \lambda^{-(n+1)}$. For $1 \le p \le \infty$ and $|\lambda| > 1$, the function $p_\lambda \in \tau_p^\alpha$ and

$$q_{\alpha,p}(p_\lambda) \le C_{\alpha,p} \left(\frac{|\lambda^p - \lambda^{p-1}|}{|\lambda|^p - 1} \right)^\alpha \frac{1}{(|\lambda|^p - 1)^{\frac{1}{p}}},$$

for $1 \le p < \infty$ and $|\lambda| > 1$.

4. Semigroups of composition on τ_p^α

Theorem.

Take $1 \le p \le \infty$ and $\alpha \ge 0$. The one-parameter operator families $(T_p(t))_{t\ge 0}$ and $(S_p(t))_{t\ge 0}$ defined by

$$T_p(t)f(n) := e^{-\frac{t}{p}} \sum_{j=0}^{n} \binom{n}{j} e^{-tj}(1-e^{-t})^{n-j} f(j),$$

$$S_p(t)f(n) := e^{-t(n+1-\frac{1}{p})} \sum_{j=n}^{\infty} \binom{j}{n} (1-e^{-t})^{j-n} f(j)$$

are contraction adjoint C_0-semigroups on τ_p^α whose generators A and B are given by

$$Af(0) := -\frac{1}{p} f(0), \; Af(n) := -n\nabla f(n) - \frac{1}{p} f(n), \quad n \in \mathbb{N},$$

$$Bf(n) := (n+1)\Delta f(n) + \frac{1}{p} f(n), \quad n \in \mathbb{N}_0.$$

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Theorem

Let A and B the generators of $(T_p(t))_{t\geq 0}$ and $(S_p(t))_{t\geq 0}$ on τ_p^α $(1 \leq p < \infty)$.

(i) The point spectra are $\sigma_p(A) = \emptyset$ and $\sigma_p(B) = \mathbb{C}_-$.

(ii) The spectrum of B is $\sigma(B) = \mathbb{C}_- \cup i\mathbb{R}$.

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5. Generalized Cesáro operators $\mathcal{C}_\beta$ and $\mathcal{C}_\beta^*$ on τ_p^α

Let $\beta > 0$, we consider the Cesàro operator of order β given by

$$\mathcal{C}_\beta f(n) := \frac{1}{k^{\beta+1}(n)} \sum_{j=0}^{n} k^\beta(n-j) f(j) \quad n \in \mathbb{N}_0,$$

and the adjoint Cesàro operator of order β given by

$$\mathcal{C}_\beta^* f(n) := \sum_{j=n}^{\infty} \frac{1}{k^{\beta+1}(j)} k^\beta(j-n) f(j) \quad n \in \mathbb{N}_0.$$

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$$\mathcal{C}_\beta f(n) = \beta \int_0^\infty (1-e^{-t})^{\beta-1} e^{-t(1-\frac{1}{p})} T_p(t) f(n)\, dt, \qquad f \in \tau_p^\alpha.$$

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(ii) The operator $\mathcal{C}_\beta^*$ is a bounded operator on τ_p^α, for $1 \leq p < \infty$,
$$\|\mathcal{C}_\beta^*\| \leq \frac{\Gamma(\beta+1)\Gamma(\frac{1}{p})}{\Gamma(\beta+\frac{1}{p})}$$ and

$$\mathcal{C}_\beta^* f(n) = \beta \int_0^\infty (1-e^{-t})^{\beta-1} e^{-\frac{t}{p}} S_p(t) f(n)\, dt, \qquad f \in \tau_p^\alpha.$$

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6. Spetrum sets of $\mathcal{C}_\beta$ and $\mathcal{C}_\beta^*$

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For $p=1$ and $\beta=n\in\mathbb{N}$, we draw the sets

$$\left\{\frac{n!}{(n+it)(n-1+it)\cdots(1+it)} \,:\, t\in\mathbb{R}\right\}.$$

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$\sigma(\mathcal{C}_1^*)$

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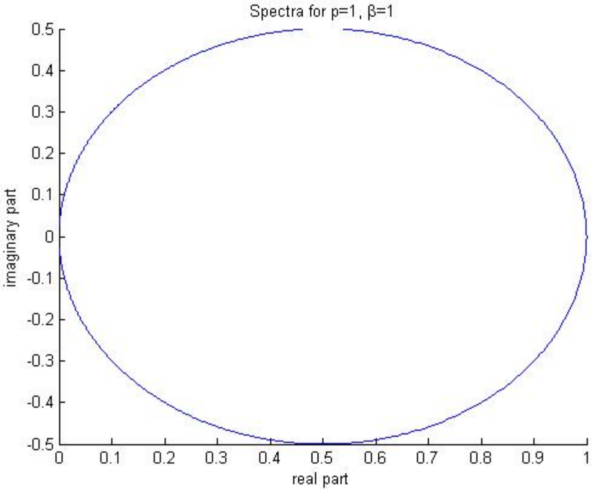

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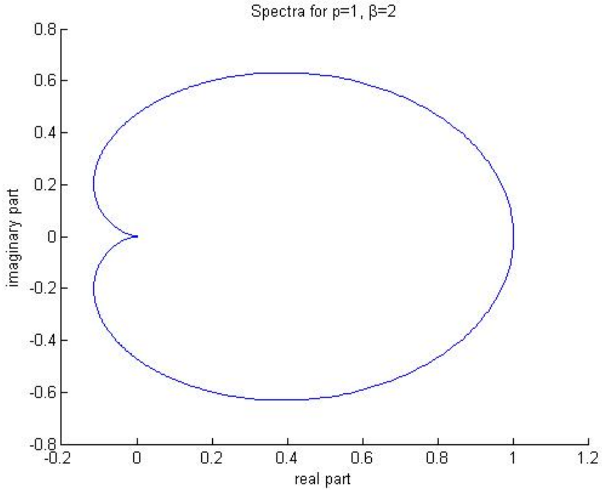
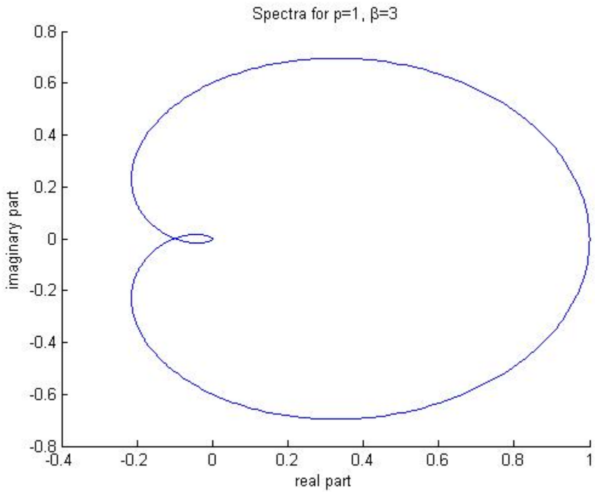
$\sigma(\mathcal{C}_2^*)$

$\sigma(\mathcal{C}_2^*)$



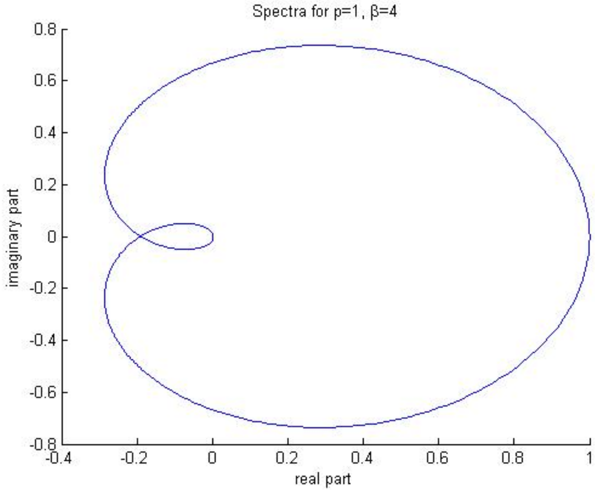
$\sigma(\mathcal{C}_3^*)$

$\sigma(\mathcal{C}_3^*)$



$\sigma(\mathcal{C}_4^*)$

$\sigma(\mathcal{C}_4^*)$



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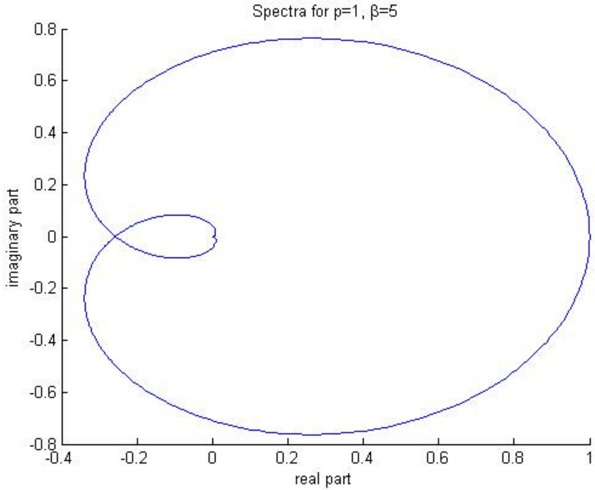
$\sigma(\mathcal{C}_5^*)$

$\sigma(\mathcal{C}_5^*)$



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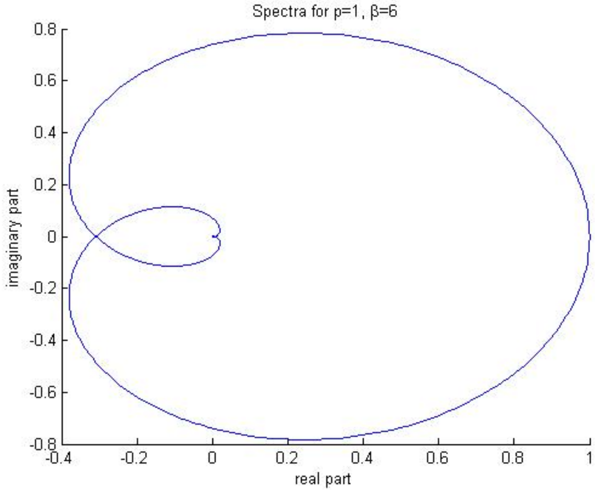
$\sigma(\mathcal{C}_6^*)$

$\sigma(\mathcal{C}_6^*)$



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$\sigma(\mathcal{C}_\beta^*)$, $1 \leq \beta \leq 6$

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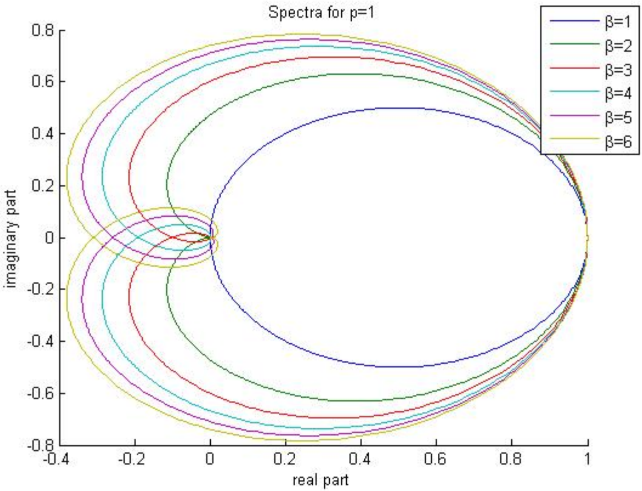
$\sigma(\mathcal{C}_\beta^*)$, $1 \leq \beta \leq 6$



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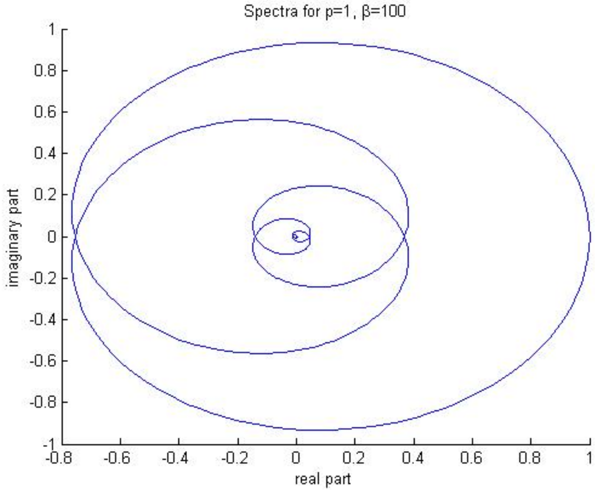
$\sigma(\mathcal{C}^*_{100})$

$\sigma(\mathcal{C}^*_{100})$



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