Nikodym boundedness property in set-algebras

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Outline

1. Introduction
2. Deep unboundedness and property N
3. Web Nikodym property in $\mathcal{J}(K)$

Outline

Notations

Let Ω be a set and $\mathcal{B}$ a subset of a set-algebra $\mathcal{A}$ of subsets of Ω.

$L(\mathcal{B})$ is linear hull of $\{e_C : C \in \mathcal{B}\}$ with the supremum norm $\|\cdot\|$
$ba(\mathcal{A})$ (the Banach space of bounded variation finitely additive measures on $\mathcal{A}$ with the variation norm $(|\cdot| := |\cdot|(\Omega))$) is isometric to $L(\mathcal{A})'$ with the dual norm. Whence $L(\mathcal{A})'$ is identified with $ba(\mathcal{A})$.

The $\mathcal{A}$-supremum norm in $ba(\mathcal{A})$, i.e.,
$\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$, $\mu \in ba(\mathcal{A})$, is equivalent to the variation norm.

Nikodym property $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition (Schachermayer-Valdivia)

$\mathcal{B}$ has *Nikodym property*, property N in brief, *if each $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is bounded in $ba(A)$, i.e.,*

$$\sup_{\mu \in M} |\mu(A)| < \infty, \forall A \in \mathcal{B} \Longrightarrow \sup_{A \in \mathcal{A}, \mu \in M} |\mu(A)| < \infty$$

Then $L(\mathcal{B})$ is dense subspace of $L(\mathcal{A})$ ($\{e_C : C \in \mathcal{B}\}^\circ$ is bounded$\Longrightarrow$ $\{e_C : C \in \mathcal{B}\}^{\circ\circ}$ neighborhood of 0).

Definition (Valdivia)

$\mathcal{B}$ has strong Nikodym property if for each increasing covering $\cup_m \mathcal{B}_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has property N.

Web Nikodym property of $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition

Increasing web in a set A is a family $\mathcal{W} := \{A_{m_1 m_2 \ldots m_p} : (m_1, m_2, \ldots, m_p) \in \cup_{s\in\mathbb{N}} \mathbb{N}^s\}$ of subsets of A such that

1. $A = \cup_{m_1} A_{m_1}$
2. $A_{m_1 m_2 \ldots m_p} = \cup_{m_{p+1}} A_{m_1, m_2, \ldots, m_p m_{p+1}}$, $\forall p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p+1$.

Each sequence $(A_{m_1 m_2 \ldots m_p})_p$ is call a *strand* in $\mathcal{W}$.

Definition (Kakol-LP)

$\mathcal{B}$ has web Nikodym property, property *wN* in brief, if *each increasing web $\{\mathcal{B}_t : t \in \cup_s \mathbb{N}^s\}$ in $\mathcal{B}$ has a strand composed of sets which have property N*

Properties N, sN and wN in σ-algebras

Theorem

For a σ-algebra $\mathcal{S}$ of subsets of a set Ω it was shown:

1. *$\mathcal{S}$ has property N (Nikodym-Dieudonné-Grothendieck).*
2. *$\mathcal{S}$ has property sN (Valdivia).*
3. *$\mathcal{S}$ has property wN (Kakol-LP).*

Properties N, sN and wN in algebras

Example

The algebra of finite and co-finite subsets of $\mathbb{N}$ fails to have property N.

Example

The algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has property N (Schachermayer)

Example

The algebra $\mathcal{J}(K)$ of Jordan measurable subsets of $K := \Pi_{1 \leqslant i \leqslant k}[a_i, b_i]$ has property sN (Valdivia 2013)

The aim of this talk is to prove that $\mathcal{J}(K)$ has property wN.

Outline

2 Deep unboundedness and property N

More notations

Let $t = (t_1, t_2, \ldots, t_p) \in T \subset \mathbb{N}^p$ and
$u = (u_1, u_2, \ldots, u_q) \in U \subset \mathbb{N}^q$.
p is the *length of* t. If $t = (t_1)$ then $t = t_1$.
$t(i) := (t_1, t_2, \ldots, t_i)$ is the *section of length* i of t, $1 \leqslant i \leqslant p$.
$t(i) := \emptyset$ if $i > p$.
$t \times u := (t_1, t_2, \ldots, t_p, t_{p+1}, t_{p+2}, \ldots, t_{p+q})$, with $t_{p+j} := u_j$, for $1 \leqslant j \leqslant q$.

Definition

$(t^n = (t_1^n, t_2^n, \ldots, t_n^n, \ldots) \in T)_n$ is an *infinite chain* if $t^{n+1}(n) = t^n(n) \neq \emptyset$, $\forall n \in \mathbb{N}$.

Last notations

$t \times s$ ($t \times s \in U$) is *an extension of* t (of t *in* U).
U is *increasing* at $t = (t_1, t_2, \ldots, t_p) \in \cup_s \mathbb{N}^s$ if there exists

$$\{t^1 = (t_1^1, t_2^1, \ldots), t^i = t(i-1) \times (t_i^i, t_{i+1}^i, \ldots), 1 < i \leqslant p\} \subset U$$

such that $t_i < t_i^i$, for each $1 \leqslant i \leqslant p$.

Definition

U is increasing (increasing respect to a subset V of $\cup_s \mathbb{N}^s$) if U is increasing at each $t \in U$ (at each $t \in V$).

U is increasing if $|U(1)| = \infty$ and
$|\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty$, $\forall t = (t_1, t_2, \ldots, t_p) \in U$ and $1 \leqslant i < p$.

NV-tree

Definition

An NV-tree T is an increasing subset of $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ without infinite chains such that each $t = (t_1, t_2, \ldots, t_p) \in T$ verifies that the length of each extension of $t(p-1)$ in T is p and $\{t(i) : 1 \leqslant i \leqslant p\} \cap T = \{t\}$.

Example

The infinite subsets of $\mathbb{N}$ are NV-trees, named trivial NV-trees.

Example

$\mathbb{N}^i$, $i \in \mathbb{N}\backslash\{1\}$, and $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are non trivial NV-trees.

Example

The product of a finite family of NV-trees is an NV-tree

Elementary properties of a NV-tree T

Proposition

Each increasing subset S of T is an NV-tree.
Whence if $S_n \subset T$ and S_{n+1} is increasing respect to S_n then $\cup_n S_n$ is an NV-tree.

Proposition

If no NV-tree is contained in U $(\subset T)$ then $T \backslash U$ contains an NV-tree.

Elementary properties of a NV-tree T

Proposition

$\{B_u : u \in \cup_s \mathbb{N}^s\} \uparrow$ *web in* $B \Longrightarrow B = \cup\{B_t : t \in T\}$.

Proof.

This equality follows from the following trivial facts:

1. $B = \cup_{u \in T} B_{u(1)} \uparrow$, $B_{u(i)} = \cup_{u(i) \times n \in T(i+1)} B_{u(i) \times n}$
2. $b \in B \Longrightarrow \exists t \in T : b \in B_t$, ($T$ does not contain infinite chains).

□

Deep B-unbounded sets

Definition (Deep B-unbounded set)

Let $B \in \mathcal{A}$. $M \subset ba(\mathcal{A})$ is deep B-unbounded if for each finite subset $\mathcal{Q}$ of $\{e_A : A \in \mathcal{A}\}$

$$\sup\{|\mu(C)| : \mu \in M \cap \mathcal{Q}^\circ,\ C \in \mathcal{A},\ C \subset B\} = \infty.$$

Proposition

Let $M(\subset ba(\mathcal{A}))$ be deep B-unbd and $\{B_i \in \mathcal{A} : 1 \leqslant i \leqslant q\}$ a partition of B. There exists j, $1 \leqslant j \leqslant q$, such that M is deep B_j-unbounded.

Proof.

If $\exists \mathcal{Q}^i \subset \{e_A : A \in \mathcal{A}\}$, finite, $\sup_{\mu \in M \cap (\mathcal{Q}^i)^\circ} |\mu| (C_i) < H_i$, $i \in \{1, 2, \ldots, q\}$, then $\sup_{\mu \in M \cap \mathcal{Q}^\circ} |\mu| (B) < \Sigma_{1 \leqslant i \leqslant q} H_i$, with $\mathcal{Q} = \cup_{1 \leqslant i \leqslant q} \mathcal{Q}^i$. □

Webs without N-strands

Proposition

Let $\mathcal{B} := \{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ *be an increasing web in* $\mathcal{A}$ *without N-strands.*
Then there exists an NV-tree T such that for each $t = (t_1, t_2, \ldots, t_q) \in T$ *the set* $\mathcal{B}_t$ *does not have property N and if* $p > 1$ *then* $\mathcal{B}_{t(i)}$ *has property N if* $i < p$.

Webs without N-strands

Proof.

Either $\mathcal{B}_{m_1}$, $m_1 \in \mathbb{N}$ not prop N (then $T := \mathbb{N}$) or $\mathcal{B}_{m'_1}$ prop N. $Q_1 := \emptyset$ and $Q'_1 := \{t_1 \in \mathbb{N} : t_1 \geqslant m'_1\}$.
Suppose $j \leqslant i$, $t \in Q_j \cup Q'_j \implies t(j-1) \in Q'_{j-1}$,

- $t \in Q_j \implies \mathcal{B}_t$ not prop N; $t(j-1) \times \mathbb{N} \subset Q_j$; $S_{t(j-1)} := \mathbb{N}$ and $S'_{t(j-1)} := \emptyset$,
- $t \in Q'_j \implies \mathcal{B}_t$ prop N; $\exists \left|S'_{t(j-1)}\right| = \infty$ with $t(j-1) \times S'_{t(j-1)} \subset Q'_j$ and $(t(j-1) \times \mathbb{N}) \cap Q_j = \emptyset$. Then $S_{t(j-1)} := \emptyset$.

□

Webs without N-strands

Proof.

If $t \in Q_i'$ then $\mathcal{B}_t$ prop N and $\mathcal{B}_t = \cup_n \mathcal{B}_{t\times n} \uparrow \Longrightarrow$

- Either each $\mathcal{B}_{t\times n}$ not prop N. Then $S_{t_1 t_2 \ldots t_i} := \mathbb{N}$ and $S'_{t_1 t_2 \ldots t_i} := \emptyset$,
- or $\exists m'_{i+1} \in \mathbb{N} : \mathcal{B}_{t\times n}$ prop N if $n \geqslant m'_{i+1}$. Then $S_{t_1 t_2 \ldots t_i} := \emptyset$ and $S'_{t_1 t_2 \ldots t_i} := \{n \in \mathbb{N} : m'_{i+1} \leqslant n\}$.

$Q_{i+1} := \cup\{t \times S_t : t \in Q_i'\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q_i'\}$.
$\mathcal{B}$ without N-strands$\Longrightarrow$ $T := \cup\{Q_i : i \in \mathbb{N}\}$ does not contain infinite chains

$$(t \in Q_p \Longrightarrow t(p-1) \in Q'_{t(p-1)} \Longrightarrow \mathcal{B}_{t(p-1)} \text{ prop } N)$$

□

Webs without N-strands

Proof.

Each $t \in Q'_k$ has an extension $\tilde{t} \in Q_{k+q}$, hence $T(k) = Q_k \cup Q'_k$, $\forall k \in \mathbb{N}$. Hence T increasing prop, because

- $|T(1)| = |Q'_1| = \infty$
- $t = (t_1, t_2, \ldots, t_p) \in T$ then $\left|S'_{t(i-1)}\right| = \infty$, $1 < i < p$, and $\left|S_{t(p-1)}\right| = \infty$,

Further, $t(i) \in Q'_i \implies \mathcal{B}_{t(i)}$ prop N and $\mathcal{B}_t$ not prop N, $\{t(i) : 1 \leqslant i \leqslant p\} \cap T = \{t\}$ and

$$\{\text{ext } t(p-1) \text{ in } T\} = t(p-1) \times \mathbb{N}.$$

□

Increasing sequences without property N

Proposition

Let $\mathcal{A}$ be an algebra of subsets of Ω and let $(\mathcal{B}_m)_m$ be an increasing sequence of subsets of $\mathcal{A}$ such that each $\mathcal{B}_m$ does not have property N and $\overline{\text{span}\{e_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$. There exists $n_0 \in \mathbb{N}$ such that for each $m \geqslant n_0$ there exists a deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset M_m of $ba(\mathcal{A})$ which is pointwise bounded in $\mathcal{B}_m$, i.e.,

$$\sup\{|\mu(C)| : \mu \in M_m\} < \infty \text{ for each } C \in \mathcal{B}_m.$$

In particular, this proposition holds if $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has property N.

Deep Ω-unbounded sets indexed by NV-trees

From the two preceding Propositions follows the next proposition.

Proposition

Let $\mathcal{B} := \{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in a set-algebra $\mathcal{A}$.
If $\mathcal{B}$ does not contain strands consisting of sets with property N then there exists an NV-tree T such that for each $t \in T$ there exists a deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset M_t of $ba(\mathcal{S})$ which is $\mathcal{B}_t$-pointwise bounded.

Outline

3. Web Nikodym property in $\mathcal{J}(K)$

$\mathcal{J}(K)$, $K := \Pi_{1\leqslant i\leqslant k}[a_i, b_i] \subset \mathbb{R}^k$

Definition

A bounded $B \subset \mathbb{R}^k$ is Jordan measurable whenever the Lebesgue measure of $\overline{B}\backslash \mathring{B}$ is zero.

$\{B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$, $M \subset ba(\mathcal{J}(K))$, deep B-unbounded absolutely convex.

Proposition

If $\alpha > 0$ and $\epsilon > 0$ $\exists C_1, C_1' \in \mathcal{J}(K)$, disjoints contained in B and $\mu \in M$ such that

1. $|\mu(C_1)| > \alpha$, $|\mu(C_1')| > \alpha$, $\Sigma_{1\leqslant j\leqslant r}\mu(Q_j) \leqslant 1$,
2. $C_1 \cup C_1' \subset \Pi_{1\leqslant i\leqslant k}[c_i, d_i] \subset K$, $\Pi_{1\leqslant i\leqslant k}(d_i - c_i) < \epsilon$
3. *M is deep C_1'-unbounded.*

$B \subset K := \Pi_{1 \leqslant i \leqslant k}[a_i, b_i] \subset \mathbb{R}^k$

Proof.

Let $(c_i^j)_{1 \leqslant j \leqslant s(i)}$ strictly increasing, $a_i = c_i^1$, $b_i = c_i^{s(i)}$ and

$$\Pi_{1 \leqslant i \leqslant k}(c_i^{j_i+1} - c_i^{j_i}) < \epsilon$$

Let $I_i^j := [c_i^{j-1}, c_i^j[$, with $1 < j < s(i)$, and $I_i^{s(i)} := [c_i^{s(i)-1}, c_i^{s(i)}]$, and
let $\{B_1, B_2, \ldots, B_m\}$ the non-void intersections of B with $I_1^{j_1} \times \ldots \times I_k^{j_k}$, $1 \leqslant j_i \leqslant s(i)$, $1 \leqslant i \leqslant k$.
There exists B_n, $1 \leqslant n \leqslant m$, with M deep B_n-unbounded. $\square$

$\{B_n \subset B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$

$M \subset ba(\mathcal{J}(K))$, deep B_n-unbd absco, $\mathcal{Q} := \{Q_1, \ldots, Q_r\}$

Proof.

$\sup\{|\mu(C)| : \mu \in rM \cap \mathcal{Q}^\circ, C \subset B_n, C \in \mathcal{A}\} = \infty$,
$\exists D_1 \subset B_n$, $D_1 \in \mathcal{J}(K)$, $\exists \lambda \in rM \cap \mathcal{Q}^\circ : |\lambda(D_1)| > r(1+\alpha)$.
$\mu = r^{-1}\lambda \in M$, $|\mu(B_n)| \leqslant r^{-1} \leqslant 1$ and
$\Sigma_{1 \leqslant j \leqslant r} \left|\mu(Q_j)\right| \leqslant r^{-1} r = 1$.
$|\mu(B_n \backslash D_1)| \geqslant |\mu(D_1)| - |\mu(B)| > 1 + \alpha - 1 = \alpha$.
M is deep D_1-unbded or $B_n \backslash D_1$-unbded. $\square$

$\{B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$, $M \subset ba(\mathcal{A})$

Corollary

If M deep B-unbd absco, $q \in \mathbb{N}$, $\alpha > 0$, $\epsilon > 0$, $\exists$ in B pairwise disjoint Jordan measurable subsets C_1, $C_2, \ldots, C_q$ and μ_1, $\mu_2, \ldots, \mu_q \in M$ such that:

1. $|\mu_i(C_i)| > \alpha$, $\Sigma_{1 \leqslant j \leqslant r} \mu_i(Q_j) \leqslant 1$, $i = 1, 2, \ldots, q$,
2. *M is deep C_q-unbounded*
3. $\cup_{1 \leqslant l \leqslant q} C_l \subset \Pi_{1 \leqslant i \leqslant k}[c_i, d_i] \subset K$, $\Pi_{1 \leqslant i \leqslant k}(d_i - c_i) < \epsilon$.

Proof:Exist disjoint subsets C_1, $C_1' \in \mathcal{J}(K)$, in B, and $\mu_1 \in M$:

1. $|\mu_1(C_1)| > \alpha$, $|\mu_1(C_1')| > \alpha$, $\Sigma_{1 \leqslant j \leqslant r} \mu_1(Q_j) \leqslant 1$,
2. M is deep C_1'-unbded
3. $C_1 \cup C_1' \subset \Pi_{1 \leqslant i \leqslant k}[c_i, d_i] \subset K$, $\Pi_{1 \leqslant i \leqslant k}(d_i - c_i) < \epsilon$.....

$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K)$, $M_t \subset ba(\mathcal{A})$, $t \in T$

Proposition

If T is an NV-tree, each M_t is deep B-unbd absco, $\alpha > 0$, $\epsilon > 0$ and $\{t^j : 1 \leqslant j \leqslant k\} \subset T$, there exists $\{B_1, B_1'\} \subset \mathcal{J}(K)$, $B_1 \cap B_1' = \emptyset$ and $B_1 \cup B_1' \subset B$, $\mu_1 \in M_{t^1}$, an NV-tree T_1 and $\{\Pi_{1 \leqslant i \leqslant k}[c_i^s, d_i^s], 1 \leqslant s \leqslant q\} \subset K$:

1. *$\{t^j : 1 \leqslant j \leqslant k\} \subset T_1 \subset T$ and M_t is deep B_1'-unbd, $\forall t \in T_1$.*
2. *$|\mu_1(B_1)| > \alpha$ and $\Sigma\{|\mu_1(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$.*
3. *$B_1 \cup B_1' \subset \cup_{1 \leqslant s \leqslant q} \Pi_{1 \leqslant i \leqslant k}[c_i^s, d_i^s] \subset K$, $\Sigma_{1 \leqslant s \leqslant q}(\Pi_{1 \leqslant i \leqslant k}(d_i^s - c_i^s)) < \epsilon$.*

$t^j := (t_1^j, t_2^j, \ldots, t_{p_j}^j)$, $1 \leqslant j \leqslant k$

Proof.

Let $q := 2 + \Sigma_{1\leqslant j\leqslant k} p_j$, $(c_i^j)_{1\leqslant j\leqslant s(i)} \uparrow$, $a_i = c_i^1$, $b_i = c_i^{s(i)}$, $1 \leqslant i \leqslant k$, $\Pi_{1\leqslant i\leqslant k}(c_i^{j_i+1} - c_i^{j_i}) < \epsilon/q$ for each $1 \leqslant j_i \leqslant s(i)$ and $1 \leqslant i \leqslant k$, $I_i^j := [c_i^{j-1}, c_i^j[$, with $1 < j < s(i)$, and $I_i^{s(i)} := [c_i^{s(i)-1}, c_i^{s(i)}]$.
Let $\{B_1, B_2, \ldots, B_m\}$ be the non-void intersections of B with the products $I_1^{j_1} \times \ldots \times I_k^{j_k}$.
$\exists B_n$, $1 \leqslant n \leqslant m$, such that M_{t^1} is deep B_n-unbounded.
Corollary applied to $\{B_n, Q_1, \ldots, Q_r\}$, M_{t^1}, α and ϵ/q provides a Jordan measurable partition $\{C_1, C_2, \cdots, C_q\}$ of B_n and $\{\lambda_1, \lambda_2, \cdots, \lambda_q\} \subset M_{t^1}$ such that:

$$|\lambda_k(C_k)| > \alpha \quad \text{and} \quad \Sigma_{1\leqslant i\leqslant r} |\lambda_k(Q_i)| \leqslant 1, \quad \text{for} \quad k = 1, 2, \ldots, q,$$

$\{D_i : 1 \leqslant i \leqslant q'\} := \{B_i : i \neq n\} \cup \{C_i : 1 \leqslant i \leqslant q\}$

Proof.

If $M \in ba(\mathcal{A})$ is deep B-unbd $\exists i_M \in \{1, 2, \dots, q'\} : M$ is deep D_{i_M}-unbd.
Whence if U is an NV-tree, M_u is deep B-unbd $\forall u \in U$ and

$$V_i := \{u \in U : M_u \text{ is deep } D_i\text{-}unbd\}$$

then $U = \cup_{1 \leqslant i \leqslant q'} V_i$. This implies that $\exists i_0$, with $1 \leqslant i_0 \leqslant q'$, such that V_{i_0} contains an NV-tree U_{i_0}. □

Deep D_i-unboundedness

Proof.

- $j \in \{1, 2, \ldots, k\} \Longrightarrow \exists i^j \in \{1, 2, \ldots, q'\} : M_{t^j}$ deep D_{i^j}-unbd.
- $\exists i_0 \in \{1, 2, \ldots, q'\}$ and an NV-tree $T_{i_0} \subset T : t \in T_{i_0}$, $M_t \Longrightarrow D_{i_0}$-unbounded.
- $t^j = (t_1^j, t_2^j, \ldots, t_{p_j}^j) \notin T_{i_0}$, $1 \leqslant j \leqslant k$, and $2 \leqslant m \leqslant p_j \Longrightarrow$

$$W_m^j := \{v \in \cup_s \mathbb{N}^s : t^j(m-1) \times v \in T\}$$

is an NV-tree. $M_{(t_1^j, t_2^j, \ldots, t_{m-1}^j) \times w}$ is deep B-unbd, $\forall w \in W_m^j$, whence $\exists i_m^j \in \{1, 2, \ldots, q'\}$ and an NV-tree $V_m^j \subset W_m^j$ such that

$$M_{(t_1^j, t_2^j, \ldots, t_{m-1}^j) \times w} \text{is deep } D_{i_m^j}\text{-}unbd\ \forall w \in V_m^j.$$

B_1 and μ_1

Proof.

Let D be the union of D_{i_0} and all D_{i^j} and $D_{i^j_m}$ obtained. The number of sets defining D is less or equal than $q-1$, hence $\exists C_h : D \subset B \backslash C_h$.

Let T_1 be the union of T_{i_0} and all $\{t^j\}$ and $\{(t^j_1, t^j_2, \ldots, t^j_{m-1}) \times V^j_m : 2 \leqslant m \leqslant p_j\}$ obtained. By construction T_1 has the increasing property ($\Longrightarrow NV$-set) and if $t \in T_1$ the set M_t is deep D-unbounded.

We get the proof with $B_1 := C_h$, $B'_1 := D$, $\mu_1 := \lambda_h$, because by construction

1. $|\mu_1(B_1)| > \alpha$, $\Sigma_{1 \leqslant i \leqslant r} |\mu_1(Q_i)| \leqslant 1$, and
2. $B_1 \cup B'_1 \subset \cup_{s \in J} \{\Pi_{1 \leqslant i \leqslant k} [c^s_i, d^s_i]\}$, $|J| \leqslant q$, whence $\Sigma_{s \in J}(\Pi_{1 \leqslant i \leqslant k}(d^s_i - c^s_i)) < (\epsilon/q)q = \epsilon$.

$\{B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$, $M_t \subset ba(\mathcal{A})$, $t \in T$

Corollary

If T is an NV-tree, each M_t is deep B-unbd absco, $\alpha > 0$, $\epsilon > 0$ and $\{t^j : 1 \leqslant j \leqslant k\} \subset T$, there exists in B a family of pairwise disjoint subsets $\{B_1, B_2, \ldots, B_k, B'\} \subset \mathcal{J}(K)$, k measures $\mu_j \in M_{t^j}$, $1 \leqslant j \leqslant k$, and an NV-tree T^* such that:

1. *$\{t^j : 1 \leqslant j \leqslant k\} \subset T^* \subset T$ and M_t is deep B'-unbounded for each $t \in T^*$.*
2. *$\left|\mu_j(B_j)\right| > \alpha$ and $\Sigma\{\left|\mu_j(Q_i)\right| : 1 \leqslant i \leqslant r\} \leqslant 1$, for $j = 1, 2, \ldots, k$.*
3. *$B' \subset \cup_{1 \leqslant s \leqslant q}\{\Pi_{1 \leqslant i \leqslant k}[c_i^s, d_i^s]\}$ of compact intervals contained in K, such that $\Sigma_{1 \leqslant s \leqslant q}(d_1^s - c_1^s) \times \ldots \times (d_k^s - c_k^s) < \epsilon$.*

Proof.

Firstly apply Proposition.
Then apply again Proposition with $\{t^2, t^3, \dots, t^k, t^1\}$ and $B := B'_1$. Finish with k repetitions. □

The diagonal order in $\mathbb{N}^2$

The elements of $\mathbb{N}^2$ ordered with the diagonal order give the sequence

$$((1,1),(1,2),(2,1),(1,3),(2,2),(3,1),\ldots)$$

We will need the sequence

$$(i_n)_n := (1,1,2,1,2,3,\ldots)$$

of its first components.

$\mathcal{J}(K)$ has property *wN*

Theorem

The algebra $\mathcal{J}(K)$ has property wN.

Proof.

Let us suppose that $\mathcal{J}(K)$ does not have property *wN*.
Then $\exists$ exists in $\mathcal{J}(K)$ an increasing web $\{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ without strands consisting of sets with Property N.
Whence $\exists$ an NV-tree T such that for each $t \in T$ there exists a deep K-unbd $\tau_s(\mathcal{J}(K))$-closed absco subset M_t of $ba(\mathcal{J}(K))$ which is $\mathcal{B}_t$-pointwise bounded. □

First induction

Proof.

By induction we determine an NV-tree $\{t^i : i \in \mathbb{N}\} \subset T$ and $(k_j \in \mathbb{N})_j \uparrow$ such that for each $(i, j) \in \mathbb{N}^2$ with $i \leqslant k_j$ there exists a set $B_{ij} \in \mathcal{J}(K)$ and $\mu_{ij} \in M_{t^i}$ that verify

1. $\Sigma_{s,v}\{\left|\mu_{ij}(B_{sv})\right| : s \leqslant k_v,\ 1 \leqslant v < j\}) < 1$,
2. $\left|\mu_{ij}(B_{ij})\right| > j$,
3. $B_{ij} \cap B_{i'j'} = \emptyset$ if $(i, j) \neq (i', j')$ and
4. $\cup\{B_{sv} : s \leqslant k_v,\ j < v\} \subset \cup_{1 \leqslant s \leqslant q_j}(\Pi_{1 \leqslant i \leqslant k}[c^s_{ij}, d^s_{ij}]) \subset K$
5. $\Sigma_{1 \leqslant s \leqslant q_j}(\Pi_{1 \leqslant i \leqslant k}(d^s_{ij} - c^s_{ij})) < 2^{-j}$, for each $j \in \mathbb{N}$.

□

First induction. Step I I

Proof.

Select $t^1 \in T$. Corollary with $B := \Omega$, $\alpha = 1$ and $\epsilon = 2^{-1}$ provides

- $B_{11}, B_1' \in \mathcal{J}(K)$, $\mu_{11} \in M_{t^1}$ and an NV-tree T_1 such that
- $|\mu_{11}(B_{11})| > 1$, $t^1 \in T_1 \subset T$,
- M_t is deep B_1'-unbd for each $t \in T_1$ and
- $B_1' \subset \cup_{1 \leqslant s \leqslant q_1} \{\Pi_{1 \leqslant i \leqslant k}[c_{i1}^s, d_{i1}^s]\} \subset K$, such that
- $\Sigma_{1 \leqslant s \leqslant q_1}(\Pi_{1 \leqslant i \leqslant k}(d_{i1}^s - c_{i1}^s)) < 2^{-1}$.

Define $k_1 := 1$ and $S^1 := \{t^1\}$. □

First induction. Induction hypothesis

Proof: Let us suppose that we have obtained:

- The natural numbers $k_1 < k_2 < k_3 < \ldots < k_n$,
- the NV-trees $T_1 \supset T_2 \supset T_3 \supset \ldots \supset T_n$,
- $S^j := \{t^i : i \leqslant k_j\} \subset T_j$, $1 \leqslant j \leqslant n$, S^j incr respect S^{j-1}, $1 < j \leqslant n$,
- $\{\mu_{iv} \in M_{t^i} : i \leqslant k_v, 1 \leqslant v \leqslant j\}$
- $\{B'_j, B_{iv} : i \leqslant k_v, 1 \leqslant v \leqslant j\} \subset \mathcal{J}(K)$, pairwise disjoint, $\forall j \leqslant n$,
- $\left|\mu_{ij}(B_{ij})\right| > j$ and $\Sigma_{s \leqslant k_v, 1 \leqslant v < j} \left|\mu_{ij}(B_{sv})\right| < 1$, $i \leqslant k_j$ and $j \leqslant n$,
- $j \leqslant n$ and $t \in T_j \Longrightarrow M_t$ is deep B'_j-unbd,
- $\cup_{s \leqslant k_v, j < v \leqslant n} B_{sv} \subset B'_j \subset \cup_{1 \leqslant s \leqslant q_j} \Pi_{1 \leqslant i \leqslant k} [c^s_{ij}, d^s_{ij}] \subset K$,
- $\Sigma_{1 \leqslant s \leqslant q_j} (\Pi_{1 \leqslant i \leqslant k} (d^s_{ij} - c^s_{ij}]) < 2^{-j}$.

First induction. End

Proof.

Select $S_{n+1} := \{t^{k_n+1}, \ldots, t^{k_{n+1}}\} \subset T_n \setminus \{t^i : i \leqslant k_n\}$ incr respect S^n. Apply Corollary to $\{B'_n, B_{sv} : s \leqslant k_v, 1 \leqslant v \leqslant n\}$, T_n, the finite subset $S^{n+1} := \{t^i : i \leqslant k_{n+1}\}$ of T_n, $\alpha = n+1$ and $\epsilon = 2^{-n-1}$ to obtain:

- $\{B'_{n+1}, B_{in+1} : 1 \leqslant i \leqslant k_{n+1}\} \subset \mathcal{J}(K)$, pairwise disjoint $\subset B'_n$,
- $\{\mu_{in+1} \in M_{t^i}, 1 \leqslant i \leqslant k_{n+1}\}$, $|\mu_{in+1}(B_{in+1})| > n+1$, $\Sigma_{s \leqslant k_v, 1 \leqslant v \leqslant n} |\mu_{in+1}(B_{sv})| < 1$,
- $S^{n+1} \subset T_{n+1}$(incr tree)$\subset T_n$, M_t deep B'_{n+1}-unbd $\forall t \in T_{n+1}$,
- $B'_{n+1} \subset \cup_{1 \leqslant s \leqslant q_{n+1}} \{\Pi_{1 \leqslant i \leqslant k}[c^s_{i,n+1}, d^s_{i,n+1}]\} \subset K$
- $\Sigma_{1 \leqslant s \leqslant q_{n+1}} (\Pi_{1 \leqslant i \leqslant k}(d^s_{i,n+1} - c^s_{i,n+1}]) < 2^{-n-1}$.

□

A Jordan measurable set

Proof.

If $H_j \subset \{1, 2, \ldots, k_j\}$, $j \in \mathbb{N}$, then $B := \cup\{B_{ij} : i \in H_j, j \in \mathbb{N}\}$ is Jordan measurable.
In fact: For each $j_0 \in \mathbb{N}$ we have

$$\overline{B} \backslash \mathring{B} \subset \{\cup_{j \leqslant j_0} (\overline{B}_j \backslash \mathring{B}_j)\} \cup \overline{B'_{j_0}},$$

hence $\overline{B} \backslash \mathring{B}$ is a subset of
$\{\cup_{j \leqslant j_0} (\overline{B}_j \backslash \mathring{B}_j)\} \cup \{\cup_{1 \leqslant s \leqslant q_{j_0}} \Pi_{1 \leqslant i \leqslant k} [c^s_{ij_0}, d^s_{ij_0}]\}$, which Lebesgue measure is $\leqslant 0 + 2^{-j_0}$. Therefore $\overline{B} \backslash \mathring{B}$ is a Lebesgue measurable set with Lebesgue measure zero. □

Second induction and basic idea

Proof.

With a new easy induction we obtain a strictly increasing subset $J := \{j_1, j_2, \ldots, j_n, \ldots\}$ of $\mathbb{N}$ such that for each $n \in \mathbb{N}$ we have that

$$\left|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})\right| < 1.$$

Note that $\left|\mu_{i_n j_n}\right|(\Omega) < s_n \in \mathbb{N}$, $\{N_u, 1 \leqslant u \leqslant s_n\}$ partition of $J_n \subset \mathbb{N} \backslash \{1, 2, \ldots, j_n\}$ in s_n infinite subsets and $B_u := \cup_{s \leqslant k_v, v \in N_u} B_{sv}$, then

$$\Sigma\{\left|\mu_{i_n j_n}\right|(B_u) : 1 \leqslant u \leqslant s_n\} < s_n$$

implies $\exists u'$, with $1 \leqslant u' \leqslant s_n$, such that

$$\left|\mu_{i_n j_n}\right|(B_{u'}) < 1.$$

Second induction. End

Proof.

Let $j_{n+1} \in N_{u'}$ and the induction is finish.
Then $\cup_{m>n} B_{i_m j_m} \subset B_{u'}$ implies that

$$\left|\mu_{i_n j_n}\right| (\cup_{m>n} B_{i_m j_m}) < 1,$$

whence the Jordan measurable set $\cup_{m>n} B_{i_m j_m}$ verifies that

$$\left|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})\right| < 1.$$

Recall that $(B_{i_n j_n}, \mu_{i_n j_n})_n$ verifies for that:

$$\Sigma\{\left|\mu_{i_n j_n}(B_{i_m j_m})\right| : \ m < n\}) < 1,$$

$$\left|\mu_{i_n j_n}(B_{i_n j_n})\right| > j_n.$$

The contradiction

Proof.

S^{n+1} increasing respect $S^n \Longrightarrow \{t^i : i \in \mathbb{N}\}(\subset T)$ is an NV-tree$\Longrightarrow \cup\{\mathcal{B}_{t^i} : i \in \mathbb{N}\} = \mathcal{J}(K) \ni H := \cup_{s\in\mathbb{N}} B_{i_s j_s} \Longrightarrow \exists r \in \mathbb{N} : H \in \mathcal{B}_{t^r}$.

$(n_p)_p \uparrow$, $i_{n_p} = r \Longrightarrow \{\mu_{i_{n_p} j_{n_p}} : p \subset \mathbb{N}\} \subset M_{t^r}$ ($\mathcal{B}_{t^r}$-pointwise bounded)$\Longrightarrow$ $\sup\{\left|\mu_{i_{n_p} j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty$

in contradiction with:

$$\left|\mu_{i_{n_p} j_{n_p}}(\cup_{m<n_p} B_{i_m j_m})\right| < 1$$

$$\mu_{i_{n_p} j_{n_p}}(B_{i_{n_p} j_{n_p}}) > j_{n_p} > n_p \quad \text{and}$$

$$\left|\mu_{i_{n_p} j_{n_p}}(\cup_{n_p<m} B_{i_m j_m})\right| < 1$$

References I

- Diestel, J.: Sequences and Series in Banach Spaces. Springer, New York (1984)
- Diestel, J., Uhl, J.J.: Vector Measures. Mathematical Surveys, Number **15.** American Mathematical Society, Providence (1977)
- Dieudonné, J.: Sur la convergence de suites de measures de Radon. An. Acad. Brasi. Ciên. **23**, 277–282 (1951)
- Kakol, J., López-Pellicer, M.: On Valdivia strong version of Nikodym boundedness property. Preprint

References II

- López-Alfonso, S. On Schachermayer and Valdivia results in algebras of Jordan measurable sets. RACSAM DOI 10.1007/s13398-015-0267-x, Published on line: 22 December 2015.
- López-Alfonso, S., Mas, J., Moll, S.: Nikodym boundedness property and webs in σ-algebras. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. (2015) DOI 10.1007/s13398-015-0260-4
- Nikodym, O.M.: Sur les familles bornées de fonctions parfaitement additives d'ensembles abstrait. Monatsh. Math. U. Phys. **40**, 418–426 (1933)

References III

- Schachermayer, W.: On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras. Dissertationes Math. (Rozprawy Mat.) **214**, 33 pp., (1982)
- Valdivia, M.: On Nikodym boundedness property. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. **107**, 355–372 (2013)