CÁLCULO VECTORIAL

1. Vectores

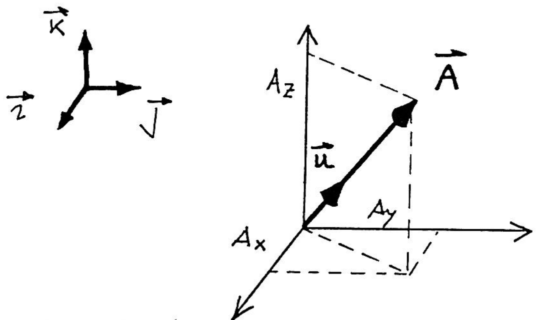
Un vector $\vec{A}$ viene dado en coordenadas cartesianas, respecto a un sistema de ejes rectangulares, por:

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

(A_x, A_y, A_z) son las componentes de $\vec{A}$.

Módulo de $\vec{A}$ $\Longrightarrow$ $|\vec{A}| \equiv A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

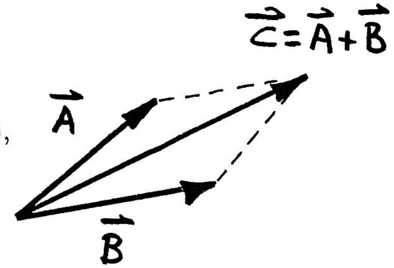
Vector unitario $\Longrightarrow$ $\vec{u} = \frac{1}{|\vec{A}|}\vec{A}$, $|\vec{u}| = 1$



2. Suma de vectores

Sean dos vectores, $\vec{A} = (A_x, A_y, A_z)$ y $\vec{B} = (B_x, B_y, B_z)$,

$\vec{C} = \vec{A}+\vec{B} = (A_x+B_x)\,\vec{i}+(A_y+B_y)\,\vec{j}+(A_z+B_z)\,\vec{k}$



3. Producto escalar

El producto escalar de $\vec{A}$ y $\vec{B}$ es un escalar (número) dado por:

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$ $\qquad$ ($\vec{A}\cdot\vec{A} = |\vec{A}|^2$)

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB}$ $\qquad$ (Si $\vec{A}\perp\vec{B} \longrightarrow \vec{A}\cdot\vec{B} = \vec{0}$)

θ_{AB} es el (menor) ángulo que forman los vectores $\vec{A}$ y $\vec{B}$.

4. Producto vectorial

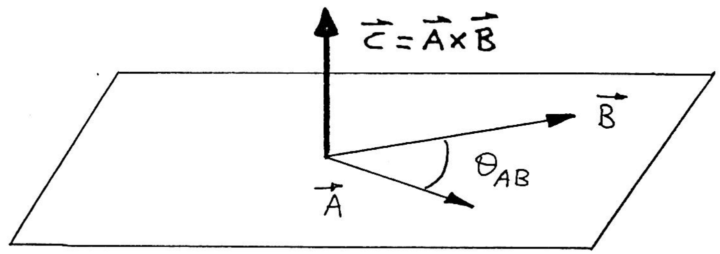
El producto vectorial de $\vec{A}$ y $\vec{B}$ es otro vector

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z-A_zB_y)\vec{i}+(A_zB_x-A_xB_z)\vec{j}+(A_xB_y-A_yB_x)\vec{k}$$

Módulo $\Longrightarrow$ $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\,\mathrm{sen}\theta_{AB}$ $\quad$ (Si $\vec{A}\parallel\vec{B} \longrightarrow \vec{A}\times\vec{B} = \vec{0}$)

Dirección $\Longrightarrow$ $\vec{C}\perp$ plano$(\vec{A},\vec{B})$

Sentido $\Longrightarrow$ regla de la mano derecha o regla del sacacorchos



TEOREMAS DE CÁLCULO VECTORIAL

Teorema de la divergencia: Si $\vec{A}$ es una magnitud vectorial función regular de las coordenadas, V es un volumen tridimensional (cuyo elemento infinitesimal es dV) y S es la superficie bidimensional que encierra a V (cuyo elemento infinitesimal es dS y cuyo vector normal externo es $\vec{n}$: $d\vec{S} = dS\,\vec{n}$):

$$\int_V \vec{\nabla}\cdot\vec{A}\,dV = \int_S \vec{A}\cdot d\vec{S}$$

Teorema de Stokes: Si S es una superficie abierta (cuyo elemento infinitesimal es $d\vec{S} = dS\,\vec{n}$) y su contorno es la línea cerrada C (que tiene como elemento infinitesimal $d\vec{l}$):

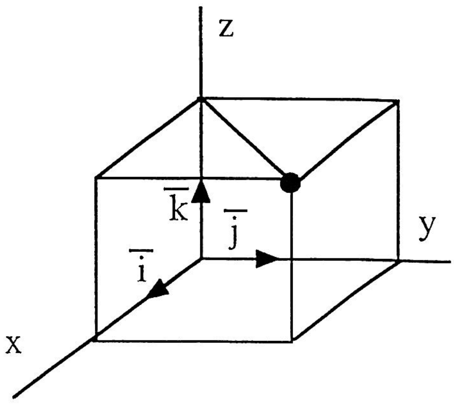
$$\int_S(\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{l}$$

(el sentido de $\vec{n}$ se define por la regla de la mano derecha, en relación al sentido de la integral de línea alrededor de C)

COORDENADAS CARTESIANAS

coordenadas **x, y, z**

vectores unitarios $\bar{\mathbf{i}}, \bar{\mathbf{j}}, \bar{\mathbf{k}}$ **fijos**



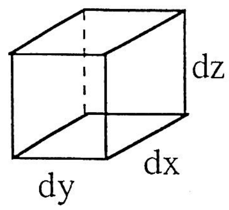
Vector posición $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$

Elementos de linea, superficie y volumen

$\mathbf{d}\bar{\mathbf{r}} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\mathbf{d}\bar{\mathbf{S}} = dy\,dz\,\bar{i} + dx\,dz\,\bar{j} + dx\,dy\,\bar{k}$

$\mathbf{dV} = dx\,dy\,dz$



Operadores

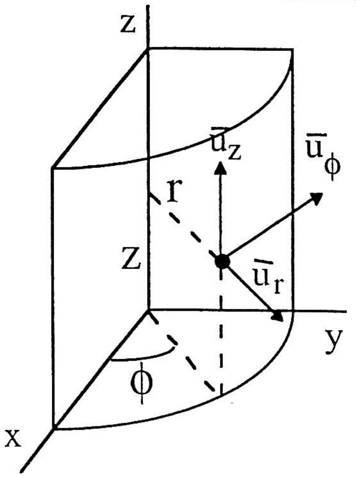
GRADIENTE $\bar{\nabla}\psi = \frac{\partial \psi}{\partial x}\bar{i} + \frac{\partial \psi}{\partial y}\bar{j} + \frac{\partial \psi}{\partial z}\bar{k}$

DIVERGENCIA $\bar{\nabla}\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

ROTACIONAL $\bar{\nabla}\times\bar{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\bar{i} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\bar{j} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\bar{k}$

LAPLACIANO $\Delta\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$

COORDENADAS CILÍNDRICAS



coordenadas **r, ϕ, z**

$$\begin{Bmatrix} \mathbf{x} = r\cos\phi \\ \mathbf{y} = r\sin\phi \\ \mathbf{z} = z \end{Bmatrix} \qquad \mathbf{r} = \sqrt{x^2+y^2}$$

$$\phi = tg^{-1}\frac{y}{x}; \qquad \mathbf{z} = z$$

vectores unitarios

$\bar{u}_z$ constante, $\bar{u}_r$ y $\bar{u}_\phi$ dependen del punto

$$\bar{u}_r = \cos\phi\,\bar{i} + \sin\phi\,\bar{j}$$

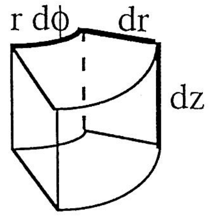
$$\bar{u}_\phi = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

$$\bar{u}_z = \bar{k}$$

Vector posición

$$\bar{r} = r\bar{u}_r + z\bar{u}_z;$$

Elementos de linea, superficie y volumen



$$\mathbf{d\bar{r}} = dr\bar{u}_r + rd\phi\bar{u}_\phi + dz\bar{u}_z$$

$$\mathbf{d\bar{S}} = rd\phi dz\,\bar{u}_r + drdz\,\bar{u}_\phi + rd\phi dr\,\bar{u}_z$$

$$\mathbf{dV} = rdrd\phi dz$$

Operadores

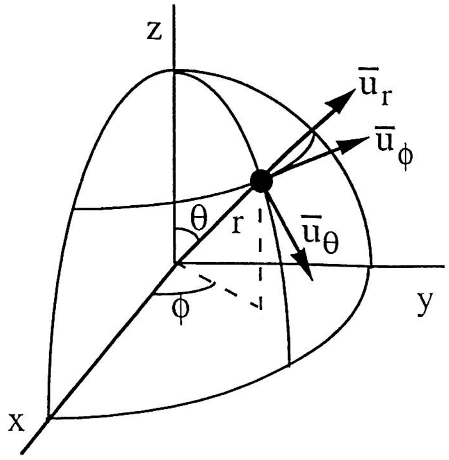
GRADIENTE $\quad \bar{\nabla}\psi = \frac{\partial\psi}{\partial r}\bar{u}_r + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\bar{u}_\phi + \frac{\partial\psi}{\partial z}\bar{u}_z$

DIVERGENCIA $\quad \bar{\nabla}\cdot\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$

ROTACIONAL $\quad \bar{\nabla}\times\bar{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\bar{u}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\bar{u}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\phi}\right)\bar{u}_z$

LAPLACIANO $\quad \Delta\psi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$

COORDENADAS ESFÉRICAS



coordenadas **r**, θ, ϕ

$$\begin{cases} \mathbf{x} = r\sin\theta\cos\phi \\ \mathbf{y} = r\sin\theta\sin\phi \\ \mathbf{z} = r\cos\theta \end{cases} \qquad \mathbf{r} = \sqrt{x^2+y^2+z^2} \qquad \theta = \cos^{-1}\frac{z}{r}; \quad \phi = tg^{-1}\frac{y}{x}$$

(ATENCIÓN: esta r es distinta a la de las coord. cilíndricas)

vectores unitarios

$\bar{\mathbf{u}}_r$, $\bar{\mathbf{u}}_\theta$ y $\bar{\mathbf{u}}_\phi$ dependen del punto

$$\bar{u}_r = \sin\theta\cos\phi\,\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\theta\,\bar{k}$$

$$\bar{u}_\theta = \cos\theta\cos\phi\,\bar{i} + \cos\theta\sin\phi\,\bar{j} - \sin\theta\,\bar{k}$$

$$\bar{u}_\phi = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

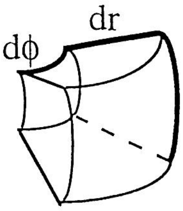
Vector de posición $\bar{\mathbf{r}} = r\bar{u}_r$

Elementos de linea, superficie y volumen

$$\mathbf{d\bar{r}} = dr\,\bar{u}_r + rd\theta\,\bar{u}_\theta + r\sin\theta\,d\phi\,\bar{u}_\phi$$

$$\mathbf{d\bar{S}} = r^2\sin\theta\,d\theta d\phi\,\bar{u}_r + r\sin\theta dr d\phi\,\bar{u}_\theta + r dr d\theta\,\bar{u}_\phi$$

$$\mathbf{dV} = r^2\sin\theta\,dr\,d\theta\,d\phi$$



Operadores

GRADIENTE $$\bar{\nabla}\psi = \frac{\partial\psi}{\partial r}\bar{u}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\bar{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\bar{u}_\phi$$

DIVERGENCIA $$\bar{\nabla}\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

ROTACIONAL $$\bar{\nabla}\times\bar{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right)\bar{u}_r + \frac{1}{r\sin\theta}\left(\frac{\partial A_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right)\bar{u}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\bar{u}_\phi$$

LAPLACIANO $$\Delta\psi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$