Fractional wavelet transform

David Mendlovic, Zeev Zalevsky, David Mas, Javier García, and Carlos Ferreira

The wavelet transform, which has had a growing importance in signal and image processing, has been generalized by association with both the wavelet transform and the fractional Fourier transform. Possible implementations of the new transformation are in image compression, image transmission, transient signal processing, etc. Computer simulations demonstrate the abilities of the novel transform. Optical implementation of this transform is briefly discussed. © 1997 Optical Society of America

1. Introduction

The wavelet transform has been shown to be a successful tool for dealing with transient signals, data compression, bandwidth reduction, and time-dependent frequency analysis of short transient signals,[1] optical correlators,[2,3] sound analysis,[4] representation of the human retina, and representation of fractal aggregates.[5] The different wavelet components are scaled and shifted versions of a mother wavelet. Mathematically, the wavelet operation is equivalent to performing a Fourier transform of the input function, multiplying it by a differently scaled Fourier transforms of the wavelet mother function, and eventually performing an inverse Fourier transform.[2]

Commonly, the mother wavelet function $h(x)$ is a typical window function multiplied by a modulation term. The scaled and shifted function set generated is coined daughter wavelets $h_{ab}(x)$:

$$h_{ab}(x) = \frac{1}{\sqrt{a}} h\left(\frac{x-b}{a}\right), \tag{1}$$

where b is the shift amount, a is the scale parameter, and $\sqrt{a}$ is the normalization factor. A typical wavelet mother function is the Morlet wavelet function.[4] The definition of this function is

$$h(x) = 2\cos(2\pi f_0 x)\exp\left(-\frac{x^2}{2}\right), \tag{2}$$

and it's Fourier transform is

$$H(u) = 2\pi\{\exp[-2\pi^2(u-f_0)^2] + \exp[-2\pi^2(u+f_0)^2]\}. \tag{3}$$

One can see that this function is a real nonnegative and has a Gaussian ring shape.

A one-dimensional wavelet transform of a signal $f(x)$ is defined as[6]

$$W(a, b) = \int_{-\infty}^{\infty} f(x) h_{ab}{}^*(x)\mathrm{d}x. \tag{4}$$

Note that Eq. 4 has a type of correlation between the input signal $f(x)$ and the scaled and shifted mother wavelet function $h_{ab}(x)$. This fact is the basis for the optical implementation of this transform. Since each wavelet component is actually a differently scaled bandpass filter, the wavelet transform is a localized transformation and thus is efficient for processing transient signals. If the input is decomposed into several wavelet components and reconstructed back, the mean-square error between the original and reconstructed images is shown to be not too high, even when a restricted number of wavelet components is used. This property was successfully implemented for digital image compression and transmission.

The reconstruction of an image from its wavelet decomposition is achieved with

$$f(x) = \frac{1}{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{a^3} W(a, b) h\left(\frac{x-b}{a}\right)\mathrm{d}a\mathrm{d}b, \tag{5}$$

where C is a constant equal to

$$C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|H(u)|^2}{|u|}\mathrm{d}u. \tag{6}$$

D. Mendlovic and Z. Zalevsky are with the Faculty of Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel. D. Mas, J. García, and C. Ferreira are with the Departament Interuniversitari d'Optica, Universitat de Valencia, Calle Dr. Moliner 50, 46100 Burjassot, Spain.

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This property was successfully implemented for digital image compression and transmission. Although a high number of calculations is needed for performing those operations, acceleration of the computation may be achieved by use of optics.

In optical implementation with Fourier optics, it is quite easy to obtain a continuous b parameter. However, the scaling parameter a can be obtained only discretely. The wavelet transform having the continuous b and discrete a variables is termed the hybrid wavelet transform. Using the hybrid transform allows a complete reconstruction of the signal to be obtained. The reconstruction formula for the hybrid case is

$$f(x) = \frac{1}{C} \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} \frac{1}{2^{2n}} W(2^{2n}, b) h_{2^{2n},b}(x) \mathrm{d}b. \qquad (7)$$

Note that the scaling factor a was chosen to be powers of 2 to obtain fast computing algorithms.

In the one-dimensional hybrid wavelet transform a strip-structured filter is used, in which each strip corresponds to a different scaling parameter of the mother wavelet.[2,7,8] When a two-dimensional transform is to be obtained a problem arises regarding where to obtain the result that corresponds to each scaling factor. For solving this problem a multireference approach was used,[9] i.e., the Fourier domain was separated into rings corresponding to the different scalings of the wavelet function, with each ring containing a different grating that aimed the transform's result to different spatial positions in the output plane. The problem was that the differently scaled versions of the mother wavelets overlapped; thus, the scaled mother wavelets are approximated to rings to create the rings of the Fourier plane.

The above approximation is not too exact, and therefore a different approach based on replication of the input was suggested.[10] Here, the spectrum of the input function is replicated by use of a grating and then filtered by the differently scaled mother wavelets, which are located in different spatial positions according to the replications of the input's spectra. The disadvantage of this approach is that a large spatial region is needed to obtain the wavelet transform, even for several scales of the mother wavelet. A different approach for implementing the two-dimensional hybrid wavelet transform is to multiplex the different scales of the mother wavelet by different wavelengths.[11] Such an approach requires spatially coherent illumination that contains several wavelengths.

In this paper a novel approach for processing transient signals and compressing images is suggested. The new approach generalizes the conventional wavelet transform, with the generalization based on the fractional Fourier transform (FRT). The FRT is a new transformation that has been proven to have various important properties in the optical data-processing field.[12]

The Fourier transform of the fractional order p is defined in such a way that the conventional Fourier transform is a special case of this transform that has a fractional order of $p = 1$. An optical implementation of the FRT is provided (i) in terms of quadratic graded-index media[13,14] or (ii) in a setup that involves free-space propagation lens propagation plus free-space propagation, or lens propagation plus free-space propagation plus a second lens propagation.[15] Both definitions (i) and (ii) were proven to be equivalent in Ref. 16. Mathematically,

$$\{\mathscr{F}^p[u(x)]\}(x) = \int_{-\infty}^{\infty} B_p(x, x') u(x') \mathrm{d}x', \qquad (8)$$

where $B_p(x, x')$ is the kernel of the transformation and equals, according to the graded-index media definition,[13,14]

$$B_p(x, x') = \sqrt{2} \exp[-\pi(x^2 + x'^2)] \times \sum_{n=0}^{\infty} \frac{i^{-pn}}{2^n n!} H_n(\sqrt{2\pi}x) H_n(\sqrt{2\pi}x'), \qquad (9)$$

where H_n is a Hermite polynomial of order n, or, according to the bulk optics definition,[15]

$$B_p(x, x') = \frac{\exp\left[-i\left(\frac{\pi \operatorname{sgn}(\sin\phi)}{4} - \frac{\phi}{2}\right)\right]}{|\sin\phi|^{1/2} \exp\left[i\pi \frac{x^2 + x'^2}{\tan\phi} - 2i\pi \frac{xx'}{\sin\phi}\right]}, \qquad (10)$$

where $\phi = \pi p/2$.

Observation of Eqs. (8) and (10) reveals that the FRT is a localized transformation similar to the wavelet transform. In this paper, we define a new transformation, coined the fractional wavelet transform (FWT). The FWT adapts the localization of the signal, by use of the FRT, to the localization needed by the wavelet transformation. In this way, by control of the amount of localization in the FWT, the reconstruction (FWT followed by inverse FWT) mean-square error may be reduced. Thus, fewer wavelet components need to be stored to achieve the same reconstruction error. Note that, if the fractionalization parameter of the FWT is chosen to be 1, the FWT converges to be the conventional wavelet transformation.

2. Localization Condition of the Fractional Fourier Transform

Localization of the FRT may be estimated by use of Eqs. (8) and (10). For instance, for a given fractional order p, one may ask what is the x' coordinate in which the change of the kernel $B_p(0, x')$ is twice the maximal frequency of $u(x')$? Above this coordinate the contributions of the input $u(x')$ are negligible since, while calculating the integral of Eq. (8), one has a case of undersampling. Thus, if f_m denotes the maximal frequency of $u(x')$, one obtains

$$2f_m = \frac{x'}{\tan\phi}. \qquad (11)$$

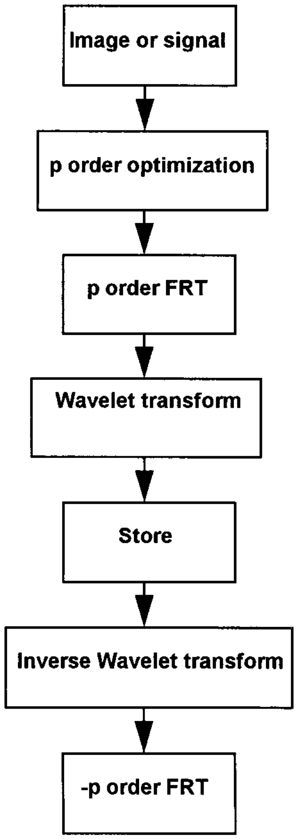

Fig. 1. Flowchart of the FWT.

Note that $x'/\tan\phi$ is the spatial frequency of the kernel $B_p(0, x')$ at the spatial location of x'. Thus, the width of the localization window is approximately

$$\Delta x = 2x' = 4f_m \tan\phi. \tag{12}$$

3. Fractional Wavelet Transform: Mathematical Definition

To adapt the localization existing in the FRT to the localization existing in the wavelet components, we suggest the following definition for the FWT: performing a FRT with the optimal fractional order p over the entire input signal and then performing the conventional wavelet decomposition. For reconstruction, one should use the conventional inverse wavelet transform and then carry out a FRT with the fractional order of $-p$ to return back to the plane of the input function. A flowchart of the FWT is illustrated in Fig. 1. The fractional order p of the FWT is determined in such a way that the mean-square error between the original input and the reconstructed input is minimal. Indeed, this optimization step may be long and be followed by many calculations. However, this stage should be done only once.

Mathematically, the FWT may be formulated as follows:

$$W^{(p)}(a, b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_p(x, x')f(x')h_{ab}^{*}(x)\mathrm{d}x'\mathrm{d}x, \tag{13}$$

where $W^{(p)}(a, b)$ is the FWT and B_p is defined by Eqs. (9) and (10). Note that, for $p = 1$, the FWT becomes the conventional wavelet transform.

The formula for backreconstructing the input is

$$f(x) = \frac{1}{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{a^3} W^{(p)}(a, b)B_{-p}(x, x') \times h\left(\frac{x'-b}{a}\right)\mathrm{d}a\mathrm{d}b\mathrm{d}x'. \tag{14}$$

And the hybrid FWT will then be

$$f(x) = \frac{1}{C}\sum_{n=-\infty}^{n=\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2^{2n}}W^{(p)}(2^{2n}, b) \times B_{-p}(x, x')h_{2^{2n},b}(x')\mathrm{d}x'\mathrm{d}b. \tag{15}$$

4. Computer Simulations

Several computer simulations were carried out to demonstrate briefly the performance of the new transformation. In Fig. 2(a) an input transient signal is illustrated. This signal contains a chirp structure that is inherently complicated to compress. We used one daughter wavelet function for the decomposition of the signal (except with a scaling factor of $a = 1$) and reconstructed it back. Figure 2(b) demonstrates the wavelet daughter function used (a Morlet function).

For reconstruction, the hybrid wavelet transform was used. After reconstruction, the error was calculated according to Eq. (16), below. With the conventional wavelet, when only one daughter function is used, the reconstruction error is 0.92. For obtaining the FWT, optimization of the selected FRT order was done by use of a trial-and-error algorithm. This led to a FRT of the order of 0.5, which finally provided a FWT reconstruction error of 0.68. This means there was an error reduction of approximately 27%. Note that this fractional order was the optimal order, in the sense of a minimal mean-square reconstruction error, for this input signal. This error is defined as

$$\epsilon = \int_{-\infty}^{\infty} |f(x) - f^{\tau}(x)|^2\mathrm{d}x, \tag{16}$$

where $f(x)$ is the input signal and $f^{\tau}(x)$ is the reconstructed signal. Figure 2(c) illustrates the reconstruction obtained with the FWT when only one wavelet component was used (one scaling factor). Figure 2(d) shows the reconstruction obtained with the conventional wavelet when only one component is used. In Fig. 2(e) one may see the reconstruction obtained with the conventional wavelet transform when five scaling factors were used (five daughter functions). Note that even then the obtained reconstruction error was 0.78 (larger than the error obtained with a single scaling factor of the FWT).

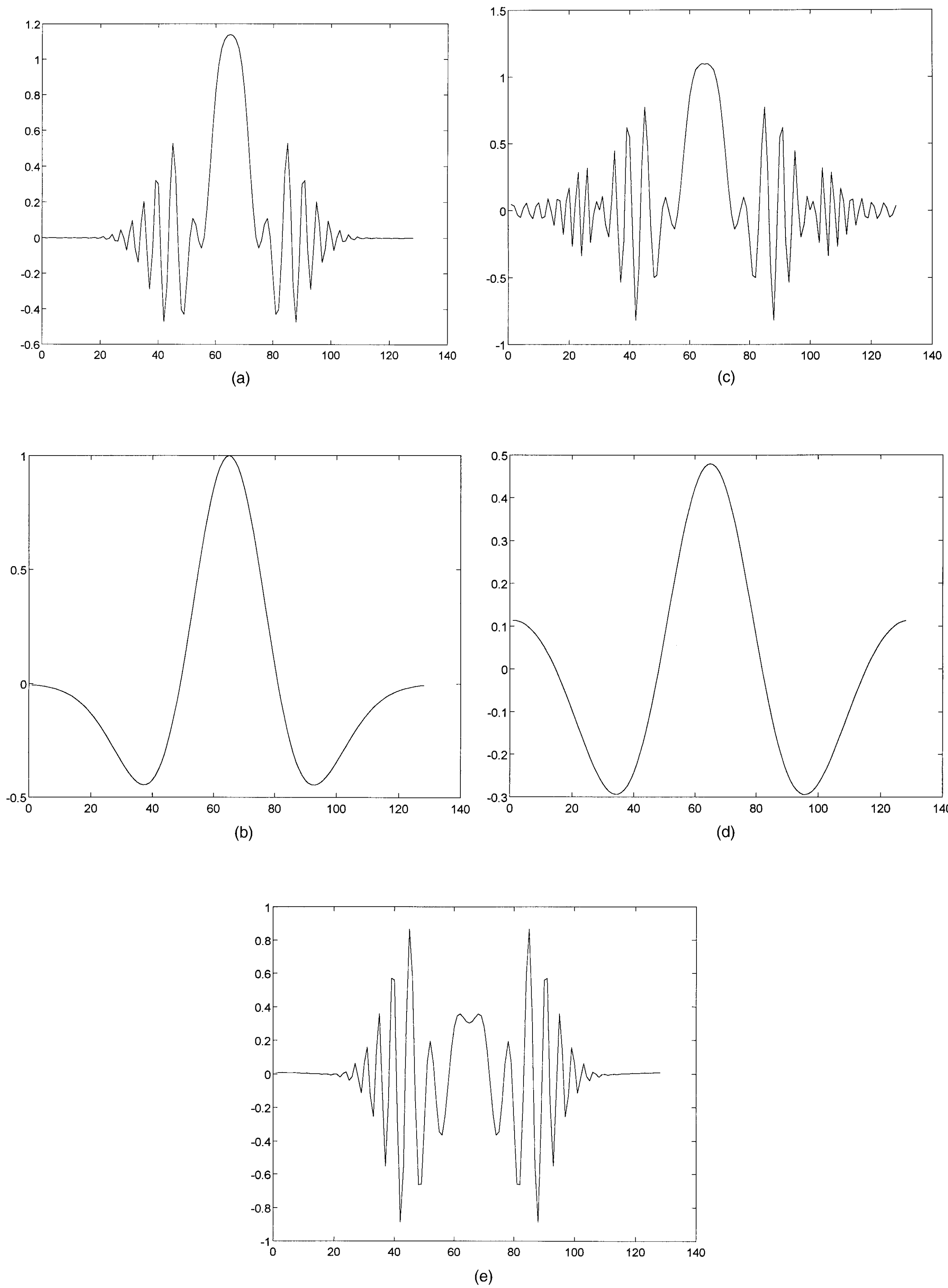

Fig. 2. (a) Input signal for the computer simulations. (b) Wavelet daughter function used in the simulations. (c) Reconstruction by use of the FWT with one daughter function. (d) Reconstruction by use of the conventional wavelet function with one daughter function. (e) Reconstruction by use of the conventional wavelet function with five daughter functions.

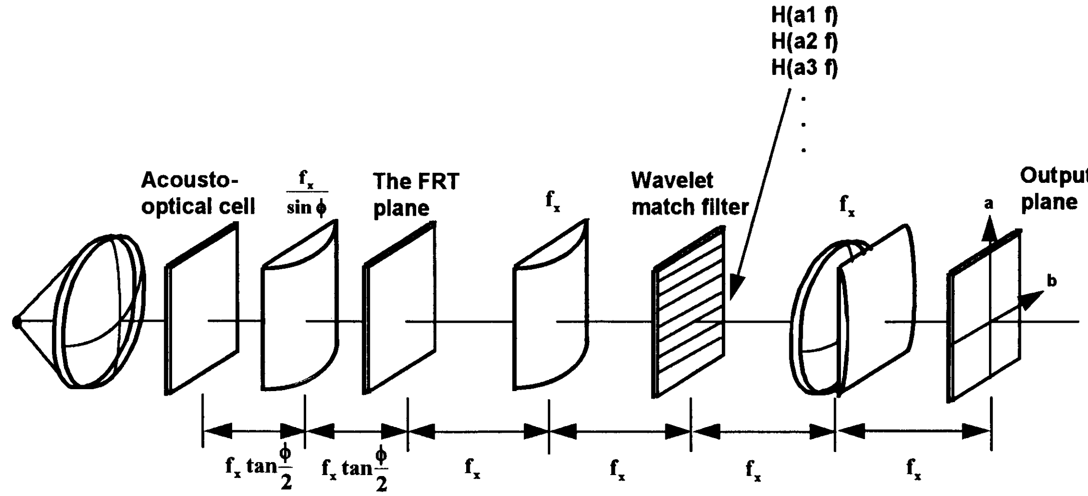

Fig. 3. Possible optical implementation of the FWT.

5. Optical Implementation

As indicated in Fig. 1, the amount of calculation required for performing the FWT with a reasonable number of pixels is high. This is due mainly to the optimization stage for finding the optimal fractional order.

A significant acceleration of the FWT calculation can be done if part or all of the stages are performed by use of optics. A possibility for optical implementation is illustrated in Fig. 3. This figure contains two parts. In the first part the temporal signal is fed into an acousto-optical cell that converts the temporal signal into a one-dimensional spatial signal. A one-dimensional FRT of the input pattern is performed according to the setup suggested in Ref. 15. Note that the FRT is obtained by bulk optics implementation. However, a graded-index fiber may be used instead.[13] The second part of the setup optically performs the wavelet transform of the one-dimensional signal. One accomplishes this by Fourier-transforming the spatial information, multiplying by a wavelet-matched filter for the one-dimensional signal, and then performing an inverse Fourier transform (according to the configuration suggested in Ref. 7).

The Fourier transformation is carried out by use of cylindrical lenses since the information is one dimensional. Spherical lenses are used to achieve imaging on the other (spatial) axis. The wavelet-matched filter contains several strips. Each strip of the filter represents a one-dimensional Fourier transform of the scaled mother wavelets and corresponds to a different scaling of the mother wavelet. The scaling parameter a varies along the vertical axis, as defined by the bank of strip filters. In the output plane one obtains a two-dimensional representation of a one-dimensional wavelet, with the horizontal axis as the continuous-shift parameter b and the vertical axis as the discrete-scale parameter a (Refs. 7 and 8). Note that, for two-dimensional input signals, the multiplexing approach may be applied to implement the two-dimensional FWT. Multiplexing may be spatial by means of Dammann gratings[10] or spectral by means of wavelength multiplexing.[11] In a similar manner, the inverse FWT may be optically implemented. This time the optical setup will contain the inverse wavelet transform first and then the FRT, with the fractional order of $-p$ (or $4 - p$) (Ref. 13).

6. Conclusions

In this paper a novel fractional transformation was defined and coined the fractional wavelet transform (FWT). This transform takes advantage of the localization existing in the FRT to improve the reconstruction performance of the wavelet transformation.

The FWT may be used for image compression. Computer simulations demonstrate the ability of this transform to provide a smaller reconstruction error when compared with the conventional wavelet transform. More work should be done to accelerate the fractional-order optimization step. The optical implementation of the proposed transformation is discussed, and it is indicated that most of the heavy calculations can be performed in a fast fashion by use of optics.

References

1. J. Caulfield and H. Szu, "Parallel discrete and continuous Wavelet transforms," Opt. Eng. **31,** 1835–1839 (1992).
2. H. Szu, Y. Sheng, and J. Chen, "Wavelet transform as a bank of matched filters," Appl. Opt. **31,** 3267–3277 (1992).
3. X. J. Lu, A. Katz, E. G. Kanterakis, and N. P. Caviris, "Joint transform correlation using wavelet transforms," Opt. Lett. **18,** 1700–1703 (1993).
4. R. K. Martinet, J. Morlet, and A. Grossmann, "Analysis of sound patterns through wavelet transforms," Int. J. Pattern Rec. Art. Intell. **1** (2), 273–302 (1987).
5. E. Freysz, B. Pouligny, F. Argoul, and A. Arneodo, "Optical wavelet transform of fractal aggregates," Phys. Rev. Lett. **64,** 7745–7748 (1990).
6. I. Daubechies, "The wavelet transform time–frequency localization and signal analysis," IEEE Trans. Inf. Theory **36,** 961–1005 (1990).

7. Y. Zhang, Y. Li, E. G. Kanterakis, A. Katz, X. J. Lu, R. Tolimieri, and N. P. Caviris, "Optical realization of the wavelet transform for a one-dimensional signal," Opt. Lett. **17,** 210–212 (1992).
8. Y. Sheng, D. Roberge, and H. Szu, "Optical wavelet transform," Opt. Eng. **31,** 1840–1845 (1992).
9. D. Mendlovic and N. Konforti, "Optical realization of the wavelet transform for two-dimensional objects," Appl. Opt. **32,** 6542–6546 (1993).
10. D. Mendlovic, I. Ouzieli, I. Kiryuschev, and E. Maron, "Two-dimensional wavelet transform achieved by computer-generated multireference matched filter and Dammann grating," Appl. Opt. **34,** 8213–8219 (1995).
11. J. García, Z. Zalevsky, and D. Mendlovic, "Two-dimensional wavelet transform by wavelength multiplexing," Appl. Opt. **35,** 7019–7024 (1996).
12. H. M. Ozaktas, B. Barshan, D. Mendlovic and L. Onural, "Convolution, filtering, and multiplexing in fractional Fourier domain and their relation to chirp and wavelet transforms," J. Opt. Soc. Am. A **11,** 547–559 (1994).
13. D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transformations and their optical implementation: part I," J. Opt. Soc. Am. A **10,** 1875–1881 (1993).
14. H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transformations and their optical implementation: part II," J. Opt. Soc. Am. A **10,** 2522–2531 (1993).
15. A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," J. Opt. Soc. Am. A **10,** 2181–2186 (1993).
16. D. Mendlovic, H. M. Ozaktas, and A. W. Lohmann, "Graded-index fibers, Wigner distribution functions, and the fractional Fourier transform," Appl. Opt. **33,** 6188–6193 (1994).