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Ray matrix analysis of anamorphic fractional Fourier systems

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Abstract
In this work we extend the application of the ray matrix approach to analyse anamorphic fractional Fourier systems, i.e., fractional Fourier optical systems where the fractional power is different for two orthogonal directions. The application of the ray matrix approach allows for easily obtaining the properties of the optical system, and it is therefore a powerful tool to design and simplify complicated systems. For simplicity we consider fractional Fourier systems with real orders and systems without apertures. We start by presenting the analysis of some previously reported anamorphic Fourier and fractional Fourier systems, and we end by proposing a simple optical system with tunable anamorphic fractional orders that can be varied continuously without changing the input and output planes.

Keywords: anamorphic systems, fractional Fourier transform, ray matrices

1. Introduction

Anamorphic optical systems are very well known and widely used for several applications, including metrology [1], read-out optical disc systems [2], laser diode beam shaping [3], or laser mode transformations [4, 5]. In optical data processing, the so-called astigmatic processor [6] was employed as a device for one-dimensional (1D) Fourier transforming and imaging in two mutually perpendicular directions. But was Szoplik and co-workers who for the first time proposed an anamorphic two-dimensional (2D) Fourier transformer composed of crossed cylindrical lenses of different focal lengths [7], working under parallel beam illumination. In their paper, Szoplik and co-workers proposed two systems. The first, composed of only two lenses, provided a non-exact Fourier transform; the second, composed of four cylindrical lenses, was designed to provide an exact anamorphic Fourier transform. Later on, the association in cascade of two such Fourier transformers permitted the obtention of an anamorphic coherent 2D optical processor [8].

Based on these articles, several papers were published in order to extend the performance of such systems. Using spherical wave illumination, the simplest 2D anamorphic Fourier transformer becomes more flexible than one working under parallel illumination and, under given conditions, is able to provide an exact Fourier transform [9]. Different combinations of these systems allowed their applications in optical processing, as for instance in pseudocolouring [10], for improving the angular discrimination in the Fourier plane [11] or for building anamorphic multiple matched filters [12, 13].

More recently, with the development of optical fractional Fourier transform (FRFT) processors [14], the combination of FRFT and anamorphic systems resulted in different fractional orders along the two main axes of an optical system [15]. This capability considerably extends the number of applications of the FRFT systems. For instance they can be applied to the space-variant simultaneous detection of several objects by the use of multiple anamorphic fractional Fourier transform filters [16], or for optical encryption in holographic memories [17]. In addition, the use of the FRFT tool to analyse propagation in optical resonators [18, 19] makes the

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anamorphic FRFT a very interesting tool to study laser mode converters based on cylindrical lenses [4, 5].

The key elements in all these anamorphic systems are anamorphic lenses. Refractive [7] or diffractive [20] cylindrical lenses have been employed to build anamorphic optical processors. Recently, the advances in optical technology has permitted the use of programmable lenses to build adaptive anamorphic optical processors [21, 22].

On the other hand, it is well known that the use of simple matrix algebra has been successfully applied in the study of several optical topics [23]. These matrix methods are very useful for numerical analysis when a large number of elements are considered, but also for theoretical derivation of optical properties. In particular, the relationship between diffraction in paraxial systems and ray matrices [24] permits one to simplify complex diffraction problems in beam propagation [25, 26], and Fourier optics [27]. The ray matrix approach was applied to describe anamorphic lenses in [28, 29]. It has been also employed to analyse FRFT rotationally symmetric systems [30]. To our knowledge, the use of ray matrices to deal with an anamorphic FRFT is limited to systems with independent behaviour in two orthogonal (x–y) directions [31], which can be described in a simpler way by a pair of standard ray matrices. The use of extended 4×4 ray matrices is required when the action of the anamorphic system is not x–y independent. Thus, taking into account all these previous works, we propose to extend the use of the ray matrix formalism to study anamorphic Fourier and FRFT optical systems, where the x–y action is not independent.

Therefore, the outline of the paper is as follows. For simplicity, here we only consider fractional Fourier systems with real orders and systems without apertures, and therefore all matrix elements are real-valued [25]. In section 2 we start by reviewing previous studies on symmetrical systems and their application to FRFT systems. In section 3, we extend the ray matrix approach to anamorphic systems by using 4×4 ray matrices. We use an alternative notation compared to previous studies in anamorphic systems, which presents the advantage of simplifying orthogonal anamorphic systems. We find the conditions of the ray matrix to obtain an optical system that performs an optical anamorphic 2D Fourier transform and use this formalism to review earlier proposed anamorphic systems. In section 4 we apply the formalism to the analysis of FRFT anamorphic systems providing different fractional orders in two orthogonal directions. Finally, in section 5, and on the basis of the previously developed formalism, we propose a simple system that allows a tunable anamorphic FRFT without changing the input and output planes.

2. Ray matrices and fractional Fourier transform in rotationally symmetric systems

The standard 2×2 ray matrix formalism applies to regular centred rotationally symmetric geometrical optical systems working under the paraxial approximation. The rays are assumed to travel only within a single plane, so that the formalism is applicable to systems with planar geometry and to meridional rays [32]. A ray crossing a plane $z = z_0$ perpendicular to the optical axis is described with two components, the height $r(z_0)$ and the angle $\sigma(z_0)$ at which it crosses the plane (figure 1(a)). Since the paraxial approximation indicates that the ray travels close to the z-axis, σ follows the small angle approximation and can be considered as the slope of the ray $\sigma \cong \tan(\sigma) = \mathrm{d}r/\mathrm{d}z$. Here r is the coordinate in a plane perpendicular to the optical z-axis, as indicated in figure 1(b).

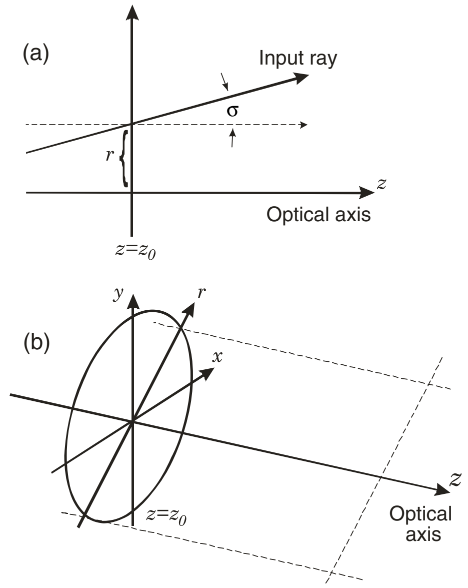

Figure 1. Coordinates for the ray matrix formalism. Input rays in rotationally symmetric systems are characterized by the height r and the angle σ they cross with respect to the optical axis.

An optical system changes the position and the angle of the ray. An input ray with coordinates (r_1, σ_1) at the input plane is changed to an output ray with coordinates (r_2, σ_2) at the output plane. In the paraxial approximation, the relations between these coordinates are linear, and therefore they can be related in the form of an $ABCD$ matrix $\mathbf{M}$, as

$$\begin{pmatrix} r_2 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ \sigma_1 \end{pmatrix}. \tag{1}$$

In this section we review some very well known properties of the ray matrix $\mathbf{M}$ that will be exploited in the rest of the paper. We use bold font notation to indicate the standard 2×2 ray matrix.

We consider the two simplest basic elements to build bulk optical systems: a free space propagation and a refractive thin lens. The ray matrix $\mathbf{D}$ describing a free space propagation of a distance d in a homogeneous medium is given by

$$\mathbf{D}(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \tag{2}$$

while the refraction of a ray by a spherical refractive thin lens, with back focal length f', is given by the ray matrix $\mathbf{L}$

$$\mathbf{L}(f') = \begin{pmatrix} 1 & 0 \\ -1/f' & 1 \end{pmatrix}. \tag{3}$$

An optical system providing an exact Fourier transform (FT) between the input and output planes is obtained when the A and D parameters of the ray matrix vanish [27]. For simplicity we

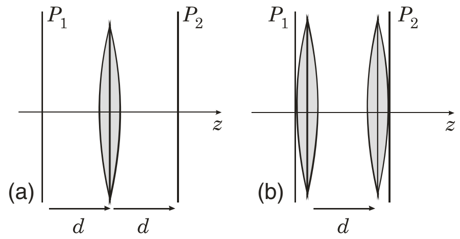

Figure 2. Two Lohmann fractional Fourier transform systems: (a) propagation–lens–propagation, (b) lens–propagation–lens.

consider standard optical systems where the refractive index of the incident and the final media are equal. Therefore the determinant of the ray matrix is equal to one, and the ray matrix describing an exact Fourier optical system is given by

$$\mathbf{F}(f) = \begin{pmatrix} 0 & f' \\ -1/f' & 0 \end{pmatrix}, \tag{4}$$

where f' represents the back focal length of the Fourier transform system. The scale of the Fourier transform is given by the relations $u = x_2/\lambda f'$ and $v = y_2/\lambda f'$, where (x_2, y_2) are the spatial coordinates at the output plane, (u, v) are the spatial frequencies, and λ is the wavelength of the light.

The generalization of the Fourier transform operation to fractional orders, the so-called fractional Fourier transform (FRFT), has received intensive attention during the last 15 years. The ray matrix formalism has been also successfully applied to the description of FRFT optical systems [30]. The ray matrix describing an FRFT system has the property of having equal A and D parameters, and it can be written as

$$\mathbf{P}(p, b) = \begin{pmatrix} \cos(\phi) & b\sin(\phi) \\ -\frac{1}{b}\sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{5}$$

where b is a scaling factor that depends on the specific optical system, and ϕ is an angle that determines the fractional order p of the Fourier transform through the relation $\phi = p\pi/2$. Figure 2 shows two bulk FRFT optical systems proposed by Lohmann [33], following a propagation–lens–propagation scheme (system I) and a lens–propagation–lens scheme (system II). The ray matrices for these two systems are respectively

$$\mathbf{P}_{\mathrm{I}} = \begin{pmatrix} 1 - \frac{d}{f'} & d\left(2 - \frac{d}{f'}\right) \\ -\frac{1}{f'} & 1 - \frac{d}{f'} \end{pmatrix}, \tag{6a}$$

$$\mathbf{P}_{\mathrm{II}} = \begin{pmatrix} 1 - \frac{d}{f'} & d \\ -\frac{1}{f'}\left(2 - \frac{d}{f'}\right) & 1 - \frac{d}{f'} \end{pmatrix}. \tag{6b}$$

For both systems the fractional order of the FRFT is determined by the ratio between the focal length f' and the distance d through the same relation

$$\cos(\phi) = \cos\left(p\frac{\pi}{2}\right) = 1 - \frac{d}{f'}. \tag{7}$$

However, the scaling factors b_{I} and b_{II} are different in each system (the scaling factor b in equation (5) corresponds to the parameter f_1 in [33]). Some simple calculations lead to the following scaling factors for the two Lohmann systems:

$$b_{\mathrm{I}} = f'\sqrt{\frac{d}{f'}\left(2 - \frac{d}{f'}\right)}, \tag{8a}$$

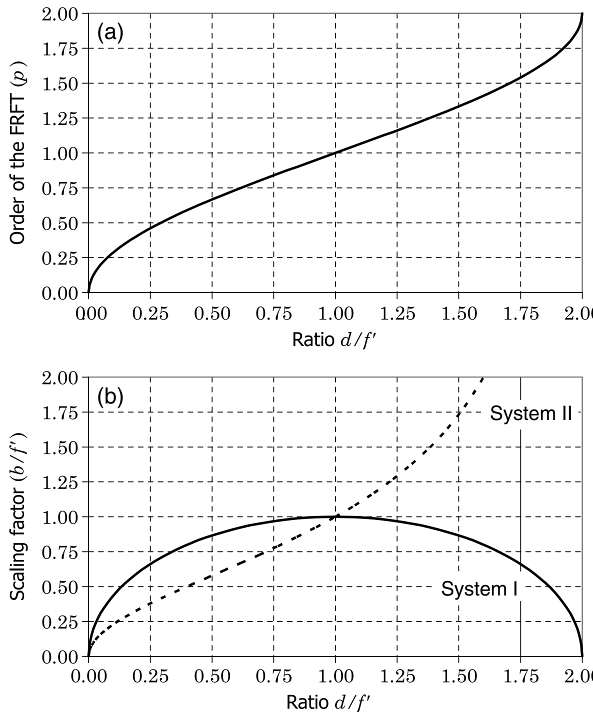

Figure 3. Evolution of the fractional order p and the scaling factor b as a function of the ratio d/f' in the two FRFT Lohmann systems in figure 2.

$$b_{\mathrm{II}} = \frac{f'}{\sqrt{\frac{2}{d/f'} - 1}}. \tag{8b}$$

The two FRFT systems become exact Fourier transform systems when $d = f'$.

In figure 3 we show the evolution of the fractional order p and the scaling factors b_{I} and b_{II} as a function of the ratio d/f', in the range [0, 2] (this range corresponds to real values of the fractional order p). It is noticeable that the scaling factors only coincide for the trivial solution $d/f' = 0$ and for the exact Fourier transform systems $d/f' = 1$. It is also noticeable that in the limit $d/f' = 2$ the scaling factor b_{II} diverges to infinity, while b_{I} becomes zero.

The importance of these scaling factors arises from the additive property of two FRFT systems in cascade, which states that the fractional orders must be added. This property is valid only if the scaling factor of the two consecutive FRFT systems is the same [34, 35]. The ray matrix formalism easily demonstrates this property. Let us consider two FRFT systems described with ray matrices given by equation (4). The order and the scaling factors of the two FRFT systems are (p_1, b_1) and (p_2, b_2) respectively. The concatenation of these two FRFT systems leads to the ray matrix multiplication, i.e.,

$$\mathbf{P}(p_2, b_2) \cdot \mathbf{P}(p_1, b_1) = \begin{pmatrix} \cos\phi_1\cos\phi_2 - \frac{b_2}{b_1}\sin\phi_1\sin\phi_2 & b_1\sin\phi_1\cos\phi_2 + b_2\cos\phi_1\sin\phi_2 \\ -\frac{\cos\phi_1\sin\phi_2}{b_2} - \frac{\sin\phi_1\cos\phi_2}{b_1} & \cos\phi_1\cos\phi_2 - \frac{b_1}{b_2}\sin\phi_1\sin\phi_2 \end{pmatrix}, \tag{9}$$

where $\phi_i = p_i\pi/2$, $i = 1, 2$. In general, if $b_1 \neq b_2$, equation (9) does not correspond to an FRFT system, since the A and D parameters of the ray matrix are not equal. Only when $b_1 = b_2$ does equation (9) adopt the form of equation (5), where the fractional angle $\phi = \phi_1+\phi_2$, and therefore the FRFT orders add, $p = p_1 + p_2$.

This matrix formalism can be generalized to describe skewed paraxial rays in circularly symmetric systems, and to astigmatic systems with the use of 4×4 matrices. In the next section we extend the previous FT and FRFT systems to anamorphic systems by the use of cylindrical lenses. For the analysis of such anamorphic systems we use the 4×4 matrix formalism.

3. Extension to 4 × 4 matrices for anamorphic systems

When the action in x and y coordinates is different but independent, the anamorphic system can be described with two independent standard 2×2 ray matrices, one describing the action in the x direction and one describing the action in the y direction. However, when the action in x–y coordinates is not independent, a 4×4 matrix formalism is required. In this case, the ray coordinates are described using a four-component column vector: the height x and the angle σ_x at which the ray crosses a plane in the horizontal direction, and equivalent parameters in the vertical direction, y and σ_y respectively. This decomposition leads to 4×4 ray matrices, like those proposed by Arsenault and Macukow [28, 29]. In this work we use an alternative notation introduced by Siegman [36], where the first and third components of the vector are the heights in the x and y directions, while the second and fourth components refer to the angles. The connection between input and output ray coordinates, i.e., the extension of equation (1), is provided through a 4×4 matrix $\hat{\mathbf{M}}$ as

$$\begin{pmatrix} x_2 \\ \sigma_{2x} \\ y_2 \\ \sigma_{2y} \end{pmatrix} = \hat{\mathbf{M}} \cdot \begin{pmatrix} x_1 \\ \sigma_{1x} \\ y_1 \\ \sigma_{1y} \end{pmatrix}. \qquad (10)$$

We add the upper triangular symbol to indicate that the matrices are 4×4. When the action of the anamorphic system is independent in the x and y directions, the 4×4 ray matrix of the system can be written as

$$\hat{\mathbf{M}} = \begin{pmatrix} \mathbf{M}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_y \end{pmatrix} \qquad (11)$$

where $\mathbf{M}_x$ and $\mathbf{M}_y$ are the standard 2×2 ray matrices corresponding to the systems in the x and y directions, and $\mathbf{0}$ represents a null 2×2 matrix defined as

$$\mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad (12)$$

In the following, in order to simplify the equations, we will also employ the 2×2 identity matrix:

$$\mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (13)$$

Equation (11) shows clearly the independence of the x and y action of the anamorphic system though the cancellation of the anti-diagonal 2×2 submatrices of $\hat{\mathbf{M}}$. For instance, it is straightforward to derive the matrix for a free propagation in a homogeneous medium, which now takes the form

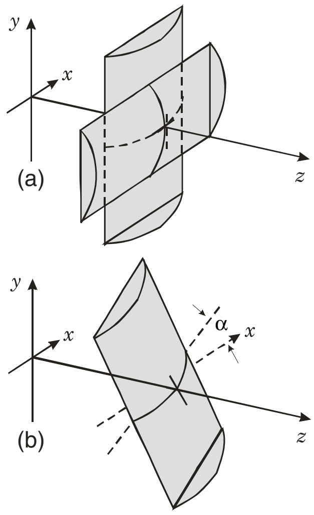

Figure 4. (a) Anamorphic lens with curvatures along the x and y directions. (b) Cylindrical lens with arbitrary orientation.

$$\hat{\mathbf{D}}(d) = \begin{pmatrix} \mathbf{D}(d) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}(d) \end{pmatrix}. \qquad (14)$$

Now the four components in the matrix in equation (14) are 2×2 matrices given in equations (2) and (12) respectively.

Figure 4(a) shows an anamorphic thin lens, with different focal lengths f'_x and f'_y along the x and y directions. Again, since the action in the x and y directions is independent, the anti-diagonal submatrices vanish and the 4×4 ray matrix is given by

$$\hat{\mathbf{L}}_{xy}(f'_x, f'_y) = \begin{pmatrix} \mathbf{L}(f'_x) & \mathbf{0} \\ \mathbf{0} & \mathbf{L}(f'_y) \end{pmatrix}. \qquad (15)$$

These two simple examples evidence that the use of 4×4 matrices is not required in x–y independent systems, since they can be reduced to two 2×2 matrices.

Cylindrical lenses are particular cases of anamorphic lenses, when there is no curvature in the x or y directions. The corresponding ray matrices are

$$\hat{\mathbf{L}}_{xy}(f', \infty) = \begin{pmatrix} \mathbf{L}(f') & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \equiv \hat{\mathbf{L}}_0(f'), \qquad (16a)$$

$$\hat{\mathbf{L}}_{xy}(\infty, f') = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}(f') \end{pmatrix} \equiv \hat{\mathbf{L}}_{90}(f'), \qquad (16b)$$

where we use the notation $\hat{\mathbf{L}}_\alpha(f')$ to denote a cylindrical lens, α being the relative angle between the direction of the lens curvature and the x direction.

The simplest example which is not x–y independent is a cylindrical lens with an arbitrary orientation α (figure 4(b)). In this case the same lens is affecting both the x and y directions, and the ray matrix is obtained by an in-plane rotation of the cylindrical lens given by

$$\hat{\mathbf{L}}_\alpha(f') = \hat{\mathbf{R}}(-\alpha) \cdot \hat{\mathbf{L}}_0(f') \cdot \hat{\mathbf{R}}(+\alpha), \qquad (17)$$

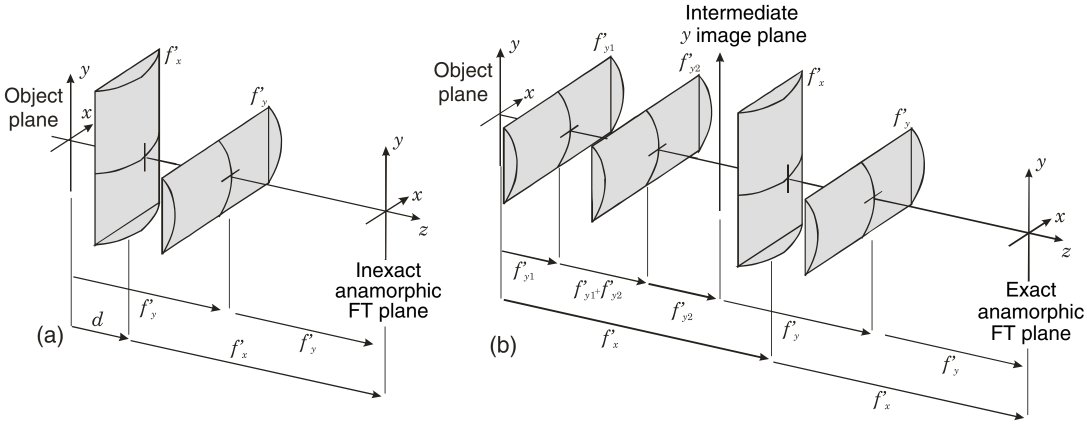

Figure 5. Inexact (a) and exact (b) anamorphic Fourier transformers.

where $\hat{\mathbf{L}}_0(f')$ is the ray matrix in equation (16*a*), and $\hat{\mathbf{R}}(\alpha)$ is a rotation-type matrix defined as

$$\hat{\mathbf{R}}(\alpha) \equiv \begin{pmatrix} \cos(\alpha)\cdot\mathbf{1} & \sin(\alpha)\cdot\mathbf{1} \\ -\sin(\alpha)\cdot\mathbf{1} & \cos(\alpha)\cdot\mathbf{1} \end{pmatrix}. \tag{18}$$

Note that this is a 4×4 matrix since the sine and cosine terms multiply the 2×2 identity matrix $\mathbf{1}$ in equation (13). The result of the matrix multiplication in equation (17) leads to the ray matrix

$$\hat{\mathbf{L}}_\alpha(f') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f'}\cdot\cos^2(\alpha) & 1 & -\frac{1}{f'}\cdot\sin(\alpha)\cdot\cos(\alpha) & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{f'}\cdot\sin(\alpha)\cdot\cos(\alpha) & 0 & -\frac{1}{f'}\cdot\sin^2(\alpha) & 1 \end{pmatrix}. \tag{19}$$

Note that now the off-diagonal 2×2 matrices do not vanish. For $\alpha = 0°$ and $90°$ the ray matrices in equations (16*a*) and (16*b*) are recovered.

In the next section we use these matrices to analyse and design anamorphic lens systems to produce anamorphic Fourier and fractional Fourier transformations.

4. Anamorphic Fourier and fractional Fourier transformers

In this section we deal with the design of 2D Fourier and fractional transformers and their analysis based on the above ray matrix formalism. We start by considering the two anamorphic Fourier transformers proposed by Szoplik *et al* [7]. Then we analyse an orthogonal anamorphic fractional Fourier transformer.

4.1. Inexact anamorphic Fourier transform system

The first system was based on using two crossed cylindrical lenses with different focal length (figure 5(a)). 1D Fourier transforms are obtained in orthogonal directions despite the rear focal planes of both lenses coinciding. A redistribution of information is obtained at the common focal plane, depending on the degree of *anamorphism*, i.e., depending on the ratio of the focal lengths. The object distance can only match one of the two focal distances, and therefore a quadratic phase factor appears in the Fourier plane oriented along the coordinate that does not match the Fourier condition.

The analysis of this situation is straightforward with the ray matrix formalism. Since the lenses are orthogonal, we can treat the system as two standard 2×2 matrices. A simple calculation leads to the two following matrices from the object to the Fourier plane, for the x and y directions:

$$\mathbf{S}_x^{\mathbf{In}} = \begin{pmatrix} 0 & f'_x \\ -1/f'_x & 1-d/f'_x \end{pmatrix}, \tag{20a}$$

$$\mathbf{S}_y^{\mathbf{In}} = \mathbf{F}(f'_x) = \begin{pmatrix} 0 & f'_x \\ -1/f'_x & 0 \end{pmatrix}. \tag{20b}$$

Equation (20*b*) reveals the exact Fourier transform relation for the y coordinate. However, since $D \neq 0$ in equation (20*a*), a quadratic phase factor appears in the x direction. This phase factor multiplies the Fourier transform in this direction, and can be easily calculated from the matrix since it is proportional to the ratio between the parameters D and B of the matrix [27], being equal to

$$g(x) = \exp\left[\mathrm{j}\frac{\pi x^2}{\lambda f'_x}\left(1-\frac{d}{f'_x}\right)\right] = \exp\left[\mathrm{j}\frac{2\pi x^2}{\lambda f'_x}(1-c)\right], \tag{21}$$

where we used that $d = 2f'_y - f'_x$ and we employ the *anamorphism factor* (c) of the Fourier transform, defined as the ratio of the focal lengths [7]:

$$c = \frac{f'_y}{f'_x}. \tag{22}$$

4.2. Exact anamorphic Fourier transform system

The second anamorphic optical system proposed in [8] performs an exact anamorphic Fourier transform. Figure 5(b) shows a scheme of this optical system, which uses four cylindrical lenses, three of them active in the y direction, and one active in the x direction. Since all lenses are orthogonal, again the system can be analysed with two independent standard 2×2 ray matrices for the x and y directions.

In this case the optical system for the x direction is equivalent to the $2f$ Fourier transformer and the ray matrix $\mathbf{S}_x^{\mathbf{Ex}}$ describing this system is equivalent to equation (4) with the focal length f'_x. However, for the y direction the system is composed of three $2f$ Fourier transformers in cascade. The

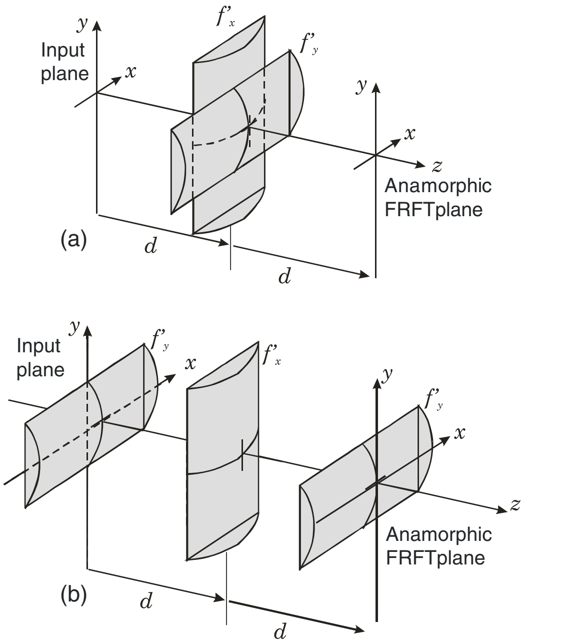

Figure 6. Anamorphic FRFT transform system in the x–y directions.

two first lenses, with focal lengths f'_{y1} and f'_{y2}, compose the two first $2f$ Fourier transformers. The ray matrix for this system is given by the product of two Fourier transform matrices, i.e.,

$$\mathbf{F}(f'_{y2})\cdot\mathbf{F}(f'_{y1}) = \begin{pmatrix} -f'_{y2}/f'_{y1} & 0 \\ 0 & -f'_{y1}/f'_{y2} \end{pmatrix} \equiv \begin{pmatrix} m_y & 0 \\ 0 & 1/m_y \end{pmatrix}. \quad (23)$$

This ray matrix corresponds to a perfect imaging with magnification $m_y = -f'_{y2}/f'_{y1}$. Since both lenses are considered convergent, this magnification is negative and the image is inverted. The final y active lens provides an exact Fourier transform of this intermediate image. In order to obtain the anamorphic Fourier transform, it is necessary that the Fourier transforms in the x and y directions appear in the same plane, which happens provided the condition $f'_x = f'_{y1} + f'_{y2} + f'_y$ holds. The ray matrix describing the transition from the object to the anamorphic Fourier transform plane in the y direction is therefore given by

$$\mathbf{S}_y^{\mathrm{Ex}} = \mathbf{F}(f'_y)\cdot\mathbf{F}(f'_{y2})\cdot\mathbf{F}(f'_{y1}) = \begin{pmatrix} 0 & f'_y \\ -1/f'_y & 0 \end{pmatrix} \times \begin{pmatrix} m_y & 0 \\ 0 & 1/m_y \end{pmatrix} = \begin{pmatrix} 0 & f'_y/m_y \\ -m_y f'_y & 0 \end{pmatrix}. \quad (24)$$

Viewed from the anamorphic fractional Fourier transform point of view, an FRFT of order 1 is obtained in the x direction, while an FRFT of order 3 is obtained in the y direction. The anamorphic Fourier transform is obtained since the focal lengths applied in each direction are different. In this system the anamorphism factor is now given by

$$c = \frac{f'_y/m_y}{f'_x} = -\frac{f'_{y1}}{f'_{y2}}\frac{f'_y}{f'_x}. \quad (25)$$

4.3. Anamorphic fractional Fourier transform system with orthogonal lenses

The extension of the FRFT symmetrical systems to anamorphic FRFT is straightforward, with the generalization of the Lohmann-type systems described in figure 3 to cylindrical systems. Figure 6 shows to possible extensions. The simplest case is shown in figure 6(a) where the Lohmann type I system is used, but with an anamorphic lens. In figure 6(b) the anamorphic FRFT employs the two Lohmann-type systems: type I for the x direction, and type II for the y direction. For both cases it is straightforward to obtain the ray matrix for the system, which is

$$\hat{\mathbf{M}}_a = \begin{pmatrix} \mathbf{P}_{\mathrm{I}}(p_x) & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathrm{I}}(p_y) \end{pmatrix}, \quad (26a)$$

$$\hat{\mathbf{M}}_b = \begin{pmatrix} \mathbf{P}_{\mathrm{I}}(p_x) & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathrm{II}}(p_y) \end{pmatrix}, \quad (26b)$$

$\mathbf{P}(p)$ being the FRFT 2×2 matrix in equation (5). In both systems, and taking into account equation (7), the fractional orders in the x and y directions are equivalent, and they are given by

$$p_x = \frac{2}{\pi}\arccos\left(1-\frac{d}{f'_x}\right) \qquad p_y = \frac{2}{\pi}\arccos\left(1-\frac{d}{f'_y}\right). \quad (27)$$

However, according to the discussion at the end of section 2, systems in figures 6(a) and (b), although yielding equal values of the fractional orders p_x and p_y, provide different scaling factors in the y direction.

5. Tunable anamorphic fractional Fourier transform system with non-orthogonal doublet

The previous systems allow for obtaining anamorphic FRFTs. In these systems, tuning the fractional order requires a change in the distance d or on the focal lengths f'_x and f'_y. It is desirable to design optical systems that can produce FRFTs with tunable fractional orders, without changing the object and FRFT planes. A diffractive lens displayed onto a liquid crystal display has been demonstrated to be a useful tool to obtain this kind of tuning [37]. Here we exploit the formalism developed in section 3 to design an anamorphic FRFT system with tunable fractional orders, maintaining the input and output planes fixed, and using standard glass cylindrical lenses.

The key element for this optical system is the cylindrical non-orthogonal doublet shown in figure 7. Let us consider that the first cylindrical lens has focal length f'_a, and it is active along the x direction. The second cylindrical lens has focal length f'_b, and it is active along a direction with an angle α with respect to the x axis (we consider α in the range from 0° to 90°). The 4×4 ray matrix describing this doublet is given by the following product:

$$\hat{\mathbf{M}} = \hat{\mathbf{R}}(-\alpha)\cdot\hat{\mathbf{L}}_0\left(f'_b\right)\cdot\hat{\mathbf{R}}(+\alpha)\cdot\hat{\mathbf{L}}_0\left(f'_a\right), \quad (28)$$

where matrices $\hat{\mathbf{R}}$ and $\hat{\mathbf{L}}_0$ are given by equation (17) and equation (16a) respectively. The result of this multiplication

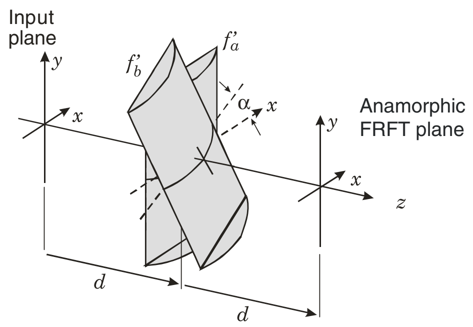

Figure 7. Cylindrical non-orthogonal doublet, and its application to perform an anamorphic fractional Fourier transformer with tunable fractional orders.

is

$$\hat{\mathbf{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f'_a} - \frac{\cos^2(\alpha)}{f'_b} & 1 & -\frac{\sin\alpha\cos\alpha}{f'_b} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\sin\alpha\cos\alpha}{f'_b} & 0 & -\frac{\sin^2(\alpha)}{f'_b} & 1 \end{pmatrix}. \tag{29}$$

In general the off-diagonal submatrices are non-zero, showing that the x and y directions are not independent. This non-orthogonal doublet is equivalent to a virtual orthogonal doublet rotated with respect to the coordinate system [29]. The focal lengths f'_x and f'_y and orientation φ of the equivalent orthogonal doublet in terms of the real non-orthogonal doublet are

$$\frac{1}{f'_x} = \frac{1}{2f'_a} + \frac{1}{2f'_b} + \frac{1}{2}\sqrt{\frac{1}{f'_a} + \frac{1}{f'_b} + \frac{2\cos(2\alpha)}{f'_a f'_b}} \tag{30a}$$

$$\frac{1}{f'_y} = \frac{1}{2f'_a} + \frac{1}{2f'_b} - \frac{1}{2}\sqrt{\frac{1}{f'_a} + \frac{1}{f'_b} + \frac{2\cos(2\alpha)}{f'_a f'_b}} \tag{30b}$$

and

$$\tan(2\varphi) = \frac{f'_a \sin(2\alpha)}{f'_b + f'_a \cos(2\alpha)}. \tag{31}$$

The focal lengths of the equivalent orthogonal doublet range from f'_a and f'_b when the cylindrical lenses are orthogonal ($\alpha = 90°$) to infinite and $f'_a f'_b/(f'_a + f'_b)$ when they are parallel ($\alpha = 0$). In the case where the two cylindrical lenses have equal focal length, $f'_a = f'_b = f'$, these equations reduce to

$$\frac{1}{f'_x} = \frac{1 + \cos\alpha}{f'} \tag{32a}$$

$$\frac{1}{f'_y} = \frac{1 - \cos\alpha}{f'} \tag{32b}$$

and

$$\tan(2\varphi) = \frac{\sin(2\alpha)}{1 + \cos(2\alpha)} = \tan(\alpha). \tag{33}$$

In this situation the rotation angle of the equivalent orthogonal doublet goes as $\varphi = \alpha/2$. The optical power is doubled in the x direction when the two lenses are parallel, being zero in the y direction. When the two lenses are orthogonal ($\alpha = 90°$), the focal length is equal to f' in both directions.

Thus, rotating the relative angle α between the cylindrical lenses permits one to tune the two focal lengths of the equivalent orthogonal doublet. Therefore, this simple system permits obtaining anamorphic fractional Fourier transforms with different fractional orders, without having to move the input or the output planes. For that purpose we choose the FRFT system I proposed by Lohmann (propagation–lens–propagation), with this anamorphic doublet in between (figure 7). Figure 8 shows some calculated results corresponding to this anamorphic fractional Fourier transformer. We select the propagation distance d equal to the focal distance f'_b of the second cylindrical lens. We show two cases, when $f'_a = f'_b$ and when $f'_a = 2f'_b$. Figures 8(a) and (c) show the evolution of the orientation φ of the equivalent orthogonal doublet as a function of the relative angle α between the two cylindrical lenses. The angle φ changes continuously and therefore the equivalent doublet rotates with α according to figures 8(a) and (c). The two focal lengths f'_x and f'_y of the equivalent doublet also change with α according to equations (30), and their action is produced along angles φ and $\varphi + 90$. Therefore, for each angle α two different fractional orders p_x and p_y are obtained in orthogonal orientations at angles φ and $\varphi + 90°$. For each value of the propagation distance d there is a tunable range of the fractional Fourier orders, the rotation angle α being the tuning parameter.

The system does not provide a complete set of independent anamorphic fractional orders, as for instance could be obtained with a programmable anamorphic liquid crystal lens [36]. In addition, the object should be rotated by the angle φ to be aligned with the two anamorphic axes of the equivalent doublet (or alternatively the complete doublet should be rotated in order to maintain the equivalent orthogonal doublet fixed). However, its simplicity, and the fact that it does not require programmable lenses, can make it interesting for applications requiring a certain degree of tunability of anamorphic fractional orders, without moving the input and output planes. The selection of the distance d and the focal lengths f'_a and f'_b of the cylindrical lenses defines the degree of tunability of the anamorphic fractional orders provided by the system.

6. Conclusions

In summary, we have presented the extension of the ray matrix formalism to anamorphic systems, in its application to Fourier and fractional Fourier optics. We have employed an alternative notation to previous works on ray matrices for anamorphic Fourier optics, which helps to identify orthogonal systems. We have applied this formalism to easily derive the properties of anamorphic Fourier transformers proposed in the literature and we have extended it to the application to anamorphic fractional Fourier transformers.

Finally we have used this powerful tool to analyse a tunable fractional optical Fourier transform system that produces anamorphic fractional orders in two orthogonal directions as a function of the relative angle of two cylindrical lenses of a compound doublet. The system maintains the input and output planes fixed. The two fractional orders change with the relative orientation of the cylindrical lenses.

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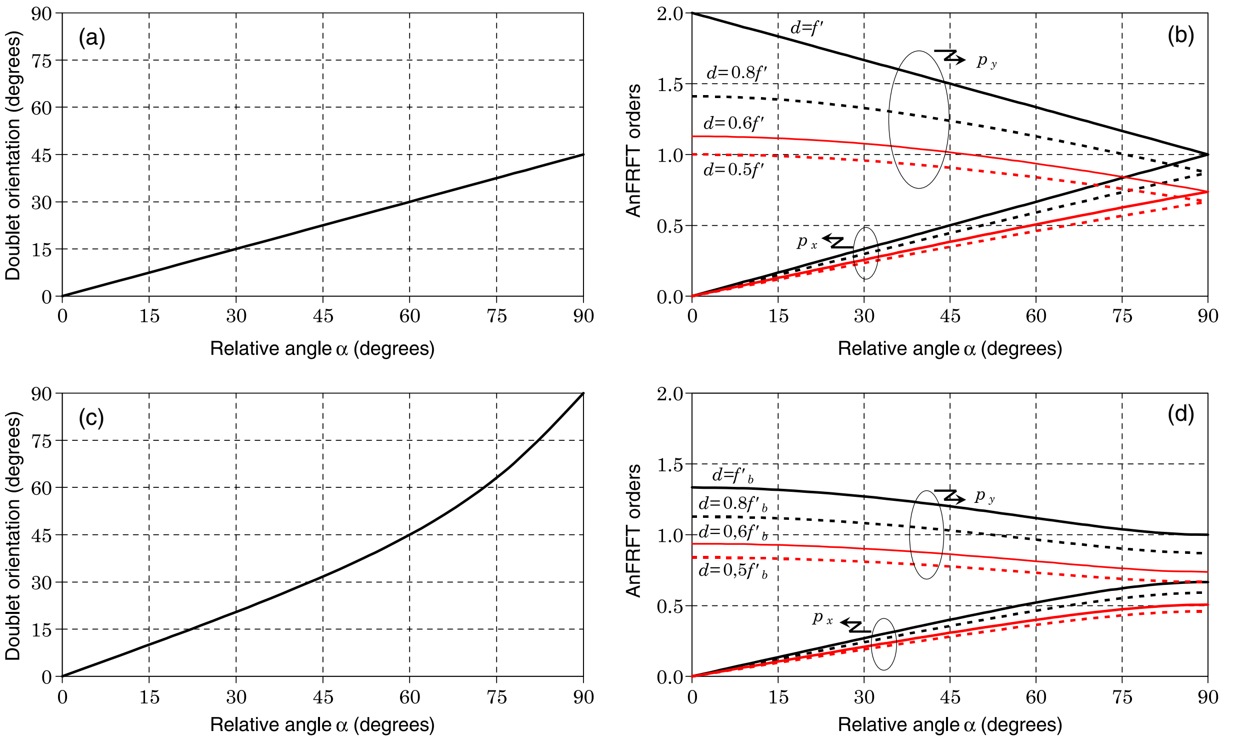

Figure 8. Orientation φ of the equivalent anamorphic doublet (a)–(c) and anamorphic fractional orders (AnFRFT orders) p_x and p_y (b)–(d), as a function of the relative angle α between the two cylindrical lenses. (a) and (b): $f_a' = f_b' \equiv f'$. (c) and (d): $f_a' = 2f_b'$. Fractional orders are calculated for propagation distances $d = f'$, $d = 0.8f'$, $d = 0.6f'$ and $d = 0.5f'$.

(This figure is in colour only in the electronic version)

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Appendix

This appendix is devoted to obtain the transformation rule between previously reported 4×4 matrices proposed by Macukow and Arsenault in [29], and the 4×4 ray matrices we are using in this work. The difference comes from a different ordering in the components of the ray coordinates column vector in equation (10). We have selected a different ordering because it provides a very simple visualization when x and y coordinates are independent, since the off-diagonal 2×2 submatrices cancel. The 2×2 diagonal matrices correspond to the standard ray matrices for symmetrical systems. For completeness, here we present how to transfer anamorphic ray matrices from our system to the one in [29], where the order of the ray components in a column was $(x, y, \sigma_x, \sigma_y)$. The two columns are related through a 4×4 matrix as

$$\begin{pmatrix} x \\ y \\ \sigma_x \\ \sigma_y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ \sigma_x \\ y \\ \sigma_y \end{pmatrix}. \tag{A.1}$$

The 4×4 matrix in the previous equation, which we denote from now on as $\hat{\Pi}$, has the property that $\hat{\Pi}^{-1} = \hat{\Pi}$. Therefore, the transformation rule from a 4×4 matrix $\hat{\mathbf{M}}'$ using the formalism in [29], and the equivalent 4×4 matrix $\hat{\mathbf{M}}$ using the convention we follow here is

$$\hat{\mathbf{M}}' = \hat{\Pi} \cdot \mathbf{M} \cdot \hat{\Pi}. \tag{A.2}$$

As two simple examples we consider the free space propagation and the cylindrical lens active in x direction from equations (14) and (16*a*) respectively. The application of the transformation rule in equation (A.2) gives the following results:

$$\hat{\mathbf{D}}'(d) = \hat{\Pi} \cdot \begin{pmatrix} \mathbf{D}(d) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}(d) \end{pmatrix} \cdot \hat{\Pi} = \begin{pmatrix} 1 & 0 & d & 0 \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.3}$$

and

$$\hat{\mathbf{L}}_0'(f') = \hat{\Pi} \cdot \begin{pmatrix} \mathbf{L}(f') & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \cdot \hat{\Pi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/f' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.4}$$

Although these two examples are x–y independent propagations, the corresponding 4×4 matrices do not evidence this property as clearly as it does the notation we are selecting.

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