In this note, we show that the proof of [EGA Théorème III.3.2.1] can be slightly modified to avoid spectral sequences. The statement of the theorem is as follows:

*Let $Y$ be a locally Noetherian scheme and $f : X \longrightarrow Y$ a proper morphism. For each coherent $\mathcal{O}_X$-module $\mathcal{F}$, the $\mathcal{O}_Y$-modules $R^q f_*(\mathcal{F})$ are coherent for $q \geq 0$.*

The previous results [EGA Théorème III.3.1.2, Corollaire III.3.1.3] reduce the problem to show the following fact:

(1) *For each irreducible closed subset $Z$ of $X$, with generic point $z$, there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_z \neq 0$ and $R^q f_*(\mathcal{F})$ is coherent for $q \geq 0$.*

To prove (1), we can suppose that $Z = X$ is integral, i.e., it suffices to prove:

(2) *If $X$ is integral with generic point $x$, there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \neq 0$ and $R^q f_*(\mathcal{F})$ is coherent for $q \geq 0$.*

To prove that (2) $\Rightarrow$ (1), we consider $Z$ as reduced (and hence integral) closed subscheme of $X$ and take the corresponding closed immersion $j : Z \longrightarrow X$. Since $f \circ j$ is a proper morphism, we know that there exists a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $\mathcal{G}_z \neq 0$ and $R^q(f \circ j)_*(\mathcal{G})$ is coherent for $q \geq 0$. Now, the $\mathcal{O}_X$-module $\mathcal{F} = j_*(\mathcal{G})$ is coherent by [L, 5.1.14 d] and it satisfies (1). The fact that $R^q f_*(\mathcal{F})$ is coherent is a consequence of the equality $R^q(f \circ j)_*(\mathcal{G}) = R^q f_*(j_*(\mathcal{G}))$, which can be proved by using that $j_*$ is an exact functor and the same argument we use for proving (3) below.

Now let us prove (2): By Chow's lemma, there exists a projective and surjective morphism $g : X' \longrightarrow X$ such that $f \circ g : X' \longrightarrow Y$ is also projective. Let $\mathcal{O}_{X'}(1)$ be a very ample sheaf on $X'$ with respect to $g$. By [H, III.8.8] there exists an integer $n \geq 1$ such that $\mathcal{F} = g_*(\mathcal{O}_{X'}(n))$ is a coherent $\mathcal{O}_X$-module, the natural map $g^* g_*(\mathcal{O}_{X'}(n)) \longrightarrow \mathcal{O}_{X'}(n)$ is surjective and $R^q g_*(\mathcal{O}_{X'}(n)) = 0$ for $q \geq 1$.

The surjectivity of $g^*\mathcal{F} \longrightarrow \mathcal{F}$ implies that $\mathcal{F}_x \neq 0$. Now it suffices to prove that

$$R^q f_*(\mathcal{F}) = R^q(f \circ g)_*(\mathcal{O}_{X'}(n)), \qquad (3)$$

since the $\mathcal{O}_Y$-modules on the right are coherent by [H, III.8.8]. (This is the only step of the proof where spectral sequences are used in [EGA].)

We start from an injective resolution of $\mathcal{O}_{X'}(n)$. By definition of derived functors, the fact that $R^q g_*(\mathcal{O}_{X'}(n)) = 0$ means that the sequence remains exact if we apply to it the functor $g_*$, and so, we get a resolution of $\mathcal{F}$, which is obviously flasque. Hence, it can be used to calculate the right hand side of (3). (See [H, III.8.3 and III.1.2A].) So, if we apply the functor $f_*$ and take the cohomology groups, we are calculating both sides of (3), and this proves (2).

**References**


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