ON MANY-SORTED ALGEBRAIC CLOSURE OPERATORS

J. CLIMENT VIDAL AND J. SOLIVERES TUR

Abstract. A theorem of Birkhoff-Frink asserts that every algebraic closure operator on an ordinary set arises, from some algebraic structure on the set, as the operator that constructs the subalgebra generated by a subset. However, for many-sorted sets, i.e., indexed families of sets, such a theorem is not longer true without qualification. We characterize the corresponding many-sorted closure operators as precisely the uniform algebraic operators.

Some theorems of ordinary universal algebra can not be automatically generalized to many-sorted universal algebra, e.g., Matthiessen [5] proves that there exist many-sorted algebraic closure systems that can not be concretely represented as the set of subalgebras of a many-sorted algebra. As is well known, according to a representation theorem of Birkhoff and Frink [1], this is not so for the single-sorted algebraic closure systems.

In [2] it was obtained a concrete representation for the so-called many-sorted uniform 2-algebraic closure operators. However, as will be proved below, confirming a conjecture by A. Blass in his review of [2], the main result in [2] remains true if we delete from the above class of many-sorted operators the condition of 2-algebraicity. Therefore a many-sorted algebraic closure operator will be concretely representable as the set of subalgebras of a many-sorted algebra if it is uniform. We point out that the proof we offer follows substantially that in Grätzer [4] for the single-sorted case, but differs from it, among others things, by the use we have to make, on the one hand, of the concept of uniformity, missing in the single-sorted case, and, on the other hand, of the Axiom of Choice, because of the lack, in the many-sorted case, of a canonical choice in the definition of the many-sorted operations.

In what follows we use, for a set of sorts $S$ and an $S$-sorted signature $\Sigma$, the concept of many-sorted $\Sigma$-algebra and subalgebra in the standard meaning, see e.g., [3].

To begin with, as for ordinary algebras, also the set of subalgebras of a many-sorted algebra is an algebraic closure system.

Proposition 1. Let $A$ be a many-sorted $\Sigma$-algebra. Then the set of all subalgebras of $A$, denoted by $\text{Sub}(A)$, is an algebraic closure system on $A$, i.e., we have

1. $A \in \text{Sub}(A)$.
2. If $I$ is not empty and $(X^i)_{i \in I}$ is a family in $\text{Sub}(A)$, then $\bigcap_{i \in I} X^i$ is also in $\text{Sub}(A)$.
3. If $I$ is not empty and $(X^i)_{i \in I}$ is an upwards directed family in $\text{Sub}(A)$, then $\bigcup_{i \in I} X^i$ is also in $\text{Sub}(A)$.

However, as we will prove later on, in the many-sorted case the many-sorted algebraic closure operator canonically associated to the algebraic closure system of the subalgebras of a many-sorted algebra has an additional and characteristic property, that of being uniform.
Now we recall the concept of support of a sorted set and that of many-sorted algebraic closure operator on a sorted set, essentials to define that of many-sorted algebraic uniform closure operator.

**Definition 1.** Let $\mathcal{A}$ be an $S$-sorted set. Then the *support* of $\mathcal{A}$, denoted by $\text{supp}(\mathcal{A})$, is the subset $\{ s \in S \mid A_s \neq \emptyset \}$ of $S$.

**Definition 2.** Let $\mathcal{A}$ be an $S$-sorted set. A *many-sorted algebraic closure operator* on $\mathcal{A}$ is an operator $J$ on $\text{Sub}(\mathcal{A})$, the set of all $S$-sorted subsets of $\mathcal{A}$, such that, for every $X, Y \subseteq A$, satisfies:

1. $X \subseteq J(X)$, i.e., $J$ is extensive.
2. If $X \subseteq Y$, then $J(X) \subseteq J(Y)$, i.e., $J$ is isotone.
3. $J(J(X)) = J(X)$, i.e., $J$ is idempotent.
4. $J(X) = \bigcup_{F \in \text{Sub}(X)} J(F)$, i.e., $J$ is algebraic, where a part $F$ of $X$ is in $\text{Sub}(X)$, the set of *finite* $S$-sorted subsets of $X$, iff $\text{supp}(F)$ is finite and, for every $s \in \text{supp}(F)$, $F_s$ is finite.

A many-sorted algebraic closure operator $J$ on $\mathcal{A}$ is *uniform* iff, for $X, Y \subseteq A$, from $\text{supp}(X) = \text{supp}(Y)$, follows that $\text{supp}(J(X)) = \text{supp}(J(Y))$.

**Definition 3.** Let $\mathcal{A}$ be a many-sorted $\Sigma$-algebra. We denote by $\text{Sg}_{\mathcal{A}}$ the many-sorted algebraic closure operator on $\mathcal{A}$ canonically associated to the algebraic closure system $\text{Sub}(\mathcal{A})$. If $X \subseteq A$, $\text{Sg}_{\mathcal{A}}(X)$ is the *subalgebra of $\mathcal{A}$ generated by $X$*.

Next, as for ordinary algebras, we define for a many-sorted $\Sigma$-algebra $\mathcal{A}$ an operator on $\text{Sub}(\mathcal{A})$ that will allow us to obtain, for every subset of $\mathcal{A}$, by recursion, an $\mathcal{A}$-ascending chain of subsets of $\mathcal{A}$ from which, taking the union, we will obtain an equivalent, but more constructive, description of the subalgebra of $\mathcal{A}$ generated by a subset of $\mathcal{A}$. Moreover, we will make use of this alternative description to prove the uniformity of the operator $\text{Sg}_{\mathcal{A}}$ and also in the proof of the representation theorem.

**Definition 4.** Let $\mathcal{A} = (A, F)$ be a many-sorted $\Sigma$-algebra.

1. We denote by $E_{\mathcal{A}}$ the operator on $\text{Sub}(\mathcal{A})$ that assigns to an $S$-sorted subset $X$ of $\mathcal{A}$, $E_{\mathcal{A}}(X) = X \cup \{ \bigcup_{s \in S} F_s[S_{\sigma(s)}] \}_{s \in S}$, where, for $s \in S$, $\Sigma_s$ is the set of all $S$-sorted formal operations $\sigma$ such that the coarity of $\sigma$ is $s$ and for $\text{ar}(\sigma) = (s_j)_{j \in m} \in S^*$, the arity of $\sigma$, $X_{\text{ar}(\sigma)} = \prod_{j \in m} X_{s_j}$.
2. If $X \subseteq A$, then the family $(E^n_{\mathcal{A}}(X))_{n \in \mathbb{N}}$ in $\text{Sub}(\mathcal{A})$ is such that $E^0_{\mathcal{A}}(X) = X$ and $E^{n+1}_{\mathcal{A}}(X) = E_{\mathcal{A}}(E^n_{\mathcal{A}}(X))$, for $n \geq 0$.
3. We denote by $\text{Sg}_{\mathcal{A}}$ the operator on $\text{Sub}(\mathcal{A})$ that assigns to an $S$-sorted subset $X$ of $\mathcal{A}$, $\text{Sg}_{\mathcal{A}}(X) = \bigcup_{n \in \mathbb{N}} E^n_{\mathcal{A}}(X)$

**Proposition 2.** Let $\mathcal{A}$ be a many-sorted $\Sigma$-algebra and $X \subseteq A$. Then we have that $\text{Sg}_{\mathcal{A}}(X) = E^n_{\mathcal{A}}(X)$.

**Proof.** See [2]

**Proposition 3.** Let $\mathcal{A}$ be a many-sorted $\Sigma$-algebra and $X, Y \subseteq A$. Then we have that

1. If $\text{supp}(X) = \text{supp}(Y)$, then, for every $n \in \mathbb{N}$, $\text{supp}(E^n_{\mathcal{A}}(X)) = \text{supp}(E^n_{\mathcal{A}}(Y))$.
2. $\text{supp}(\text{Sg}_{\mathcal{A}}(X)) = \bigcup_{n \in \mathbb{N}} \text{supp}(E^n_{\mathcal{A}}(X))$.
3. If $\text{supp}(X) = \text{supp}(Y)$, then $\text{supp}(\text{Sg}_{\mathcal{A}}(X)) = \text{supp}(\text{Sg}_{\mathcal{A}}(Y))$.

Therefore the many-sorted algebraic closure operator $\text{Sg}_{\mathcal{A}}$ is uniform.

**Proof.** See [2]

Finally we prove the representation theorem for the many-sorted uniform algebraic closure operators, i.e., we prove that for an $S$-sorted set $A$ a many-sorted
algebraic closure operator \( J \) on \( \text{Sub}(A) \) has the form \( Sg_A \), for some \( S \)-sorted signature \( \Sigma \) and some many-sorted \( \Sigma \)-algebra \( A \) if \( J \) is uniform.

**Theorem 1.** Let \( J \) be a many-sorted algebraic closure operator on an \( S \)-sorted set \( A \). If \( J \) is uniform, then \( J = Sg_A \) for some \( S \)-sorted signature \( \Sigma \) and some many-sorted \( \Sigma \)-algebra \( A \).

**Proof.** Let \( \Sigma = (\Sigma_{w,s})_{(w,s) \in S^* \times S} \) be the \( S \)-sorted signature defined, for every \((w,s) \in S^* \times S\), as follows:

\[
\Sigma_{w,s} = \{ (X, b) \in \bigcup_{X \in \text{Sub}(A)} \{ X \times J(X)_s \} \mid \forall t \in S \left( \text{card}(X_t) = |w|_t \right) \},
\]

where for a sort \( s \in S \) and a word \( w : |w| \rightarrow S \) on \( S \), with \( |w| \) the length of \( w \), the number of occurrences of \( s \) in \( w \), denoted by \( |w|_s \), is \( \text{card}(\{ i \in |w| \mid w(i) = s \}) \).

We remark that for \((w,s) \in S^* \times S \) and \((X,b) \in \bigcup_{X \in \text{Sub}(A)} \{ X \times J(X)_s \}\) the following conditions are equivalent:

1. \((X,b) \in \Sigma_{w,s}\), i.e., for every \( t \in S \), \( \text{card}(X_t) = |w|_t \).
2. \( \text{supp}(X) = \text{Im}(w) \) and, for every \( t \in \text{supp}(X) \), \( \text{card}(X_t) = |w|_t \).
3. \( \text{supp}(X) = \text{Im}(w) \) and, for every \( t \in \text{supp}(X) \), \( \text{card}(X_t) = |w|_t \).

On the other hand, for the index set \( \Lambda = \bigcup_{Y \in \text{Sub}(A)} \{ Y \times \text{supp}(Y) \} \) and the \( \Lambda \)-indexed family \((Y_s)_{(Y,s) \in \Lambda} \) whose \((Y,s)\)-th coordinate is \( Y_s \), precisely the \( s \)-th coordinate of the \( S \)-sorted set \( Y \) of the index \((Y,s) \in \Lambda\), let \( f \) be a choice function for \((Y_s)_{(Y,s) \in \Lambda} \), i.e., an element of \( \prod_{Y \in \text{Sub}(A)} Y_s \). Moreover, for every \( w \in S^* \) and \( a \in \prod_{i \in |w|} A_{w(i)} \), let \( M^{w,a} = (M^{w,a}_s)_{s \in S} \) be the finite \( S \)-sorted subset of \( A \) defined as \( M^{w,a}_s \), i.e., \( \{ a_i \mid i \in w^{-1}[s] \} \), for every \( s \in S \).

Now, for \((w,s) \in S^* \times S \) and \((X,b) \in \Sigma_{w,s}\), let \( F_{X,b} \) be the many-sorted operation from \( \prod_{i \in |w|} A_{w(i)} \) into \( A_t \) that to an \( a \in \prod_{i \in |w|} A_{w(i)} \) assigns \( b \), if \( M^{w,a} = X \) and \( f(j(M^{w,a})) = s \), otherwise.

We will prove that the many-sorted \( \Sigma \)-algebra \( A = (A,F) \) is such that \( J = Sg_A \). But before that it is necessary to verify that the definition of the many-sorted operations is sound, i.e., that for every \((w,s) \in S^* \times S \), \((X,b) \in \Sigma_{w,s}\) and \( a \in \prod_{i \in |w|} A_{w(i)} \), \( s \in \text{supp}(J(M^{w,a})) \) and for this it is enough to prove that \( \text{supp}(M^{w,a}) = \text{supp}(X) \), because, by hypothesis, \( J \) is uniform and, by definition, \( b \in J(X)_s \).

Reciprocally, if \( t \in \text{supp}(X) \), \( |w|_t > 0 \), and there is an \( i \in |w| \) such that \( w(i) = t \), hence \( a_i \in A_t \), and from this we conclude that \( M^{w,a} \neq \emptyset \), i.e., that \( t \in \text{supp}(M^{w,a}) \).

Therefore, \( \text{supp}(M^{w,a}) = \text{supp}(X) \) and, by the uniformity of \( J \), \( \text{supp}(J(M^{w,a})) = \text{supp}(J(X)) \). But, by definition, \( b \in J(X)_s \), and \( s \in \text{supp}(J(M^{w,a})) \) and the definition is sound.

We prove now that, for every \( Y \subseteq A \), \( J(X) \subseteq Sg_A(X) \). Let \( X \) be an \( S \)-sorted subset of \( A \), \( s \in S \) and \( b \in J(X)_s \). Then, because \( J \) is algebraic, \( b \in J(Y)_s \) for some finite \( S \)-sorted subset \( Y \) of \( X \). From such an \( Y \) we will define a word \( w_Y \) in \( S \) and an element \( a_Y \) of \( \prod_{i \in |w_Y|} A_{w_Y(i)} \) such that

1. \( Y = M^{w_Y,a_Y} \),
2. \( (Y,b) \in \Sigma_{w_Y,s} \), i.e., \( b \in J(Y)_s \) and, for all \( t \in S \), \( \text{card}(Y_t) = |w_Y|_t \), and
3. \( a_Y \in \prod_{i \in |w_Y|} X_{w_Y(i)} \),

then, because \( F_{Y,b}(a_Y) = b \), we will be entitled to assert that \( b \in Sg_A(X)_s \).

But taking into account that \( Y \) is finite if \( \text{supp}(Y) \) is finite and, for every \( t \in \text{supp}(Y) \), \( Y_t \) is finite, let \( \{ s_\alpha \mid \alpha \in m \} \) be an enumeration of \( \text{supp}(Y) \) and, for every \( \alpha \in m \), let \( \{ y_{\alpha,i} \mid i \in p_\alpha \} \) be an enumeration of the nonempty \( s_\alpha \)-th coordinate, \( Y_{s_\alpha} \), of \( Y \). Then we define, on the one hand, the word \( w_Y \) as the
mapping from \(|w_Y| = \sum_{\alpha \in m} p_\alpha| into S such that, for every \(i \in |w_Y|\) and \(\alpha \in m\), \(w_Y(i) = s_\alpha\) if \(\sum_{\beta \in \alpha} p_\beta \leq i \leq \sum_{\beta \in \alpha+1} p_\beta - 1\) and, on the other hand, the element \(a_Y\) of \(\prod_{i \in |w_Y|} A_{w_Y(i)}\) as the mapping from \(|w_Y|\) into \(\bigcup_{i \in |w_Y|} A_{w_Y(i)}\) such that, for every \(i \in |w_Y|\) and \(\alpha \in m\), \(a_Y(i) = y_{\alpha,i-\sum_{\beta \in \alpha} p_\beta}\) if \(\sum_{\beta \in \alpha} p_\beta \leq i \leq \sum_{\beta \in \alpha+1} p_\beta - 1\). From these definitions follow (1), (2) and (3) above. Let us observe that (1) is a particular case of the fact that the mapping \(M\) from \(\bigcup_{w \in S} (\{w\} \times \prod_{i \in |w|} A_{w(i)})\) into \(\text{Sub}_b(A)\) that to a pair \((w,a)\) assigns \(M^{w,a}\) is surjective.

From the above and the definition of \(F_{Y,b}\) we can affirm that \(F_{Y,b}(a_Y) = b\), hence \(b \in \text{Sg}_A(X)\). Therefore \(J(X) \subseteq \text{Sg}_A(X)\).

Finally, we prove that, for every \(X \subseteq A\), \(\text{Sg}_A(X) \subseteq J(X)\). But for this, by the Proposition 2, it is enough to prove that, for every subset \(X\) of \(A\), we have that \(E_A(X) \subseteq J(X)\). Let \(s \in S\) be and \(c \in E_A(X)\). If \(c \in X_s\), then \(c \in J(X)_s\), because \(J\) is extensive. If \(c \notin X_s\), then, by the definition of \(E_A(X)\), there exists a word \(w \in S^*\), a many-sorted formal operation \((Y, b) \in \Sigma_{w,s}\) and an \(a \in \prod_{i \in |w|} X_{w(i)}\) such that \(F_{Y,b}(a) = c\). If \(M^{w,a} = Y\), then \(c = b\), hence \(c \in J(X)_s\), therefore, because \(M^{w,a} \subseteq X\), \(c \in J(X)_s\). If \(M^{w,a} \neq Y\), then \(F_{Y,b}(a) \in J(M^{w,a})_s\), but, because \(M^{w,a} \subseteq X\) and \(J\) is isotone, \(J(M^{w,a})_s\) is a subset of \(J(X)_s\), hence \(F_{Y,b}(a) \in J(X)_s\). Therefore \(E_A(X) \subseteq J(X)\).

From this last Theorem and the Proposition 3, we obtain

**Corollary 1.** Let \(J\) be a many-sorted algebraic closure operator on an \(S\)-sorted set \(A\). Then \(J = \text{Sg}_A\) for some many-sorted \(\Sigma\)-algebra \(A\) iff \(J\) is uniform.

**References**


Universidad de Valencia, Departamento de Lógica y Filosofía de la Ciencia, E-46010 Valencia, Spain

E-mail address: Juan.B.Climent@uv.es