An introduction to Hamilton and Perelman's work on the conjectures of Poincaré and Thurston

Notes of the Talks given by Alexander Borisenko in Valencia, May-August 2004

written by

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PREFACE

These notes contain the lectures given by A. Borisenko in a Seminar on the Perelman's work on conjectures of Poincaré and Thurston held in the Department of Geometry and Topology of the University of Valencia along the months of May, June, July and August 2004, .

The written lectures correspond to the "spoken" ones only in a broad sense. Sometimes, the order of the lectures has been changed following the convenience of the writers.

The lectures were first written by A. Borisenko when preparing his talks. Based on these notes and the lectures, and with the references at hand, E. Cabezas-Rivas and V. Miquel wrote a first version. This was corrected by the speaker, then the second and third author modified the first version, and, then, new corrections and writings until the authors arrived to an agreement.

We think it is impossible to give a complete account of the topics of these notes in only 150 pages. The paper is directed to lay people in the main subject of it (Ricci flow) and also in the topological background of the problem. Then, the general philosophy of these lectures is to begin with the more elementary facts, give some details on them (sometimes many details), and introduce to the more advanced topics, with a decreasing exposition of details. However, this is not being done in a linear way, the difference between elementary and advanced depends on the background of the three writers which is clearly not uniform. However, we hope that these notes can serve as an introduction to the Geometrization Conjecture and its solution as it has been useful to us.

Acknowledgments

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INTRODUCTION

This work pretends to be a quick introduction to the Ricci flow theory (created by Richard Hamilton) and its relation with the Geometrization Conjecture (stated by William P. Thurston), with the purpose of providing to the reader the tools necessary to understand the works [67], [68] and [69] by Grisha Perelman.

In general, the present survey is organized as we described next. We dedicate the three first sections to the Geometrization Conjecture, whereas in §4 and §5 we give some prerequirements and basic concepts from the Ricci flow theory. On the other hand, section 6 summarizes the fundamental steps of the Hamilton-Perelman's proof of the Geometrization Conjecture. After this overview, from §7 to §11, we detail the main techniques needed for the Hamilton-Ricci flow. In order to be able to understand the most difficult part of the aforementioned proof, it is necessary to introduce a notion of convergence in metric spaces and the concept of Alexandrov space; to these definitions along with the prerequirements for their understanding, we dedicate §12 and §13. Section 14 gathers the methods introduced by Hamilton for the analysis of the singularities developed by the Ricci flow. It is precisely in this step of his proof where Hamilton found technical problems for whose overcoming it will be necessary to wait for the appearance of the famous works by Perelman, whose fundamental results take shelter in the three last sections.

We will detach with some more detail the sections of the present work, with the goal to facilitate its reading.

Sections 1-3. In the first section we give the topological prerrequirements to understand the statement of the Geometrization Conjecture and the way it includes the Poincaré Conjecture as a particular case. Roughly speaking, the Thurston's Geometrization conjecture claims that each closed 3-manifold can be decomposed into geometric pieces (see definition 1.7). The pieces are classified in 8 possible model geometries and their quotients, and therefore the problem of the classification of closed 3-manifolds would be essentially solved with the proof of the aforementioned conjecture. We dedicate §2 to the description of the eight model geometries, and in §3 we give details about Thurston's proof of such classification.

Sections 4-5. These two are devoted to some prerrequisites and elementary definitions for the Ricci flow, which constitutes both the most successful tool developed to make an attempt of proof for the Geometrization Conjecture, and a rich theory of independent interest.

As the Ricci flow can be seen as a heat type equation for a Riemannian metric on a given manifold, in §4 we review some fundamental facts on the heat equation (with a compilation philosophy, this is, without including proofs) both on the Euclidean space and on a Riemannian manifold.

On the other hand, in §5 we introduce the notion of harmonic map in order to clarify the statement of a result by Eells and Sampson which says that any map between Riemannian manifolds (with some conditions on the target manifold) can be deformed into an harmonic map. We state this result because it was the inspiration for Hamilton to develop his Ricci flow.

Moreover, section 5 gathers the definition of Ricci flow (which is roughly a method of deforming a Riemannian metric on a given manifold), its relation with the heat equation, a first example and the introduction of a remarkable equivalent equation: the normalized Ricci flow.

Section 6. This section pretends to summarize both the rest of the contents in the present work, and the Hamilton-Perelman's proof of the Geometrization Conjecture. We begin gathering the most outstanding of the ample task made by Hamilton in his Ricci flow theory: short time existence theorem, long time existence and asymptotic behaviour of the 2-dimensional Ricci flow, a first achievement in dimension 3 (which implies the proof of the Geometrization Conjecture for 3-manifolds with $\mathrm{Ric} \geq 0$) and a theorem referring to the 4-dimensional Ricci flow. Here it is also introduced an important classification of the Ricci flow solutions in two big groups and remarkable results by Hamilton for each group. The section is finished with a list of the main steps done by Perelman in order to overcome the technical difficulties and complete the Hamilton's proof of the Geometrization Conjecture. We also show how to combine such steps in order to perform the aforementioned proof.

Section 7. We dedicate this section to the maximum principles, which are the fundamental tool used by Hamilton to find out how the geometric quantities associated to the manifold will evolve under the Ricci flow if the initial condition is given.

Section 8. Here we sketch the main steps of the proof of the long time existence theorem for solutions of the Ricci flow. This result claims that the solution of the Ricci flow exists on a time interval $[0, \infty)$ unless the curvature explodes in magnitude. If this is the case, we say that the flow encounters a singularity. In this section it is also introduced the fundamental key which makes possible the proof of the long time existence (and it is of independent interest within this theory): the Shi's estimates for the derivatives of the curvature.

Section 9. In this part, we undertook the study of the existence of solutions for the 2-dimensional Ricci flow and its asymptotic behavior. For it, we have to distinguish three cases: scalar curvature (R) positive, null and negative. Consequently, a great part of §9 is dedicated to the analysis of the evolution of the scalar curvature. Here it is also introduced the concept of Ricci soliton, which will be of enormous importance and constant presence as much in the works of Hamilton like in those of Perelman.

Sections 10-11. The cases $R > 0$ and $R = 0$ for the 2-dimensional Ricci flow appear in Section 9, but its proof is postponed until §10, where it is introduced a fundamental tool: the Harnarck inequalities. They are estimates that control the oscillation of the positive solutions of certain equations. As they are similar to the original estimate by Li-Yau for positive solutions of the heat equation, §10 shall begin introducing this one.

Hamilton extensively developed this type of inequalities for several equations of geometric evolution. In particular, in 1988 he adapts the Li-Yau inequality to a 2-dimensional Ricci flow (this one is indeed the result which appears in §10.2). In 1993, he obtains an estimation in arbitrary dimension, to which we dedicate the full section 11.

Sections 12-13. These two sections try to constitute a survey of results (without detailing their proofs) coming from the theory of length spaces and whose knowledge is essential for anyone who tries to understand in depth the works by Perelman. We have divided the survey in two parts clearly differentiated: the first one (in section 12) focuses on the notion of Alexandrov space, and the second one (last part of §12 and full section 13) gathers different possible definitions of convergence in a metric space (uniform, Lipschitz and Gromov-Hausdorff convergence).

Section 14. In this section we deal with the first stage of Hamilton's program to prove the Geometrization Conjecture: the understanding of the geometry in the proximities of the points where the curvature is going to become infinite in a singular time. The technique to study that regions is called *dilation about singularities* and consists on rescaling the flow more and more as we approach to the singularity and hope that, if the rescalement is enough to keep the curvature under some control, then we can pass to a limit which is a Ricci flow and gives us any information about what happens near the singularity.

To do so, we first have to specify the meaning of convergence for a sequence of flows (or even manifolds). Then we need a compactness theorem which allows us to take limits. To these issues related with the convergence we dedicate the two last subsections of §14. It is important to emphasize that a point of obstruction of Hamilton's program is the existence of these limits (he was not able to prove it in general), but this is a question completely solved by Perelman.

Section 15. In the last part of the present work (§15-17), we give a brief and quick survey of Perelman's papers [67] and [68]. In §15, we expose the main techniques and tools introduced by Perelman to study the Ricci flow. Such techniques suppose a great advance in the understanding of the Ricci flow's nature and the resolution of several conceptual open questions about the Ricci flow, for instance, the way to see it as a gradient flow or the complete proof of the non-existence of periodic solutions (module diffeomorphisms and homotheties) not trivial (that is, which are not Ricci solitons). We end §15 with the result which is considered as the first great contribution by Perelman to Hamilton's program to solve the Geometrization Conjecture: the No Local Collapsing Theorem.

Section 16. This section begins with the development of the most relevant technique introduced by Perelman which leads to a complete classification of the 3-dimensional singularity models (that is, the solutions of the Ricci flow obtained as limit of solutions by dilation about singularities). Such method is known as reduced volume technique and an essential requirement to understand it is the theory of $\mathcal{L}$-geometry which consists of remaking all the theory about geodesics and Jacobi fields in Riemannian geometry, but applied to certain families of metrics depending on time. We summarize this in the first subsection of §16.

The reduced volume technique also allows to prove a weak version (but it is enough for the proof of the Geometrization Conjecture) of the No Local Collapsing Theorem. Here this version is reported in §16.3. As last subsection, we study the properties and the classification of the κ-solutions, which are a special type of Ricci flow solutions having the same characteristics as the 3-dimensional singularity models. This section is finished with the statement of the so-called Canonical Neighbothood Theorem, result considered as the main achievement of [67].

Section 17. This part is a very quick overview of [68] which (in agreement with its own author) is technically complex but contains no new idea with respect to [67]. Here we give a description of the manifold evolving under the Ricci flow at the first singular time. This information will be the key which allows the performance of the metric surgery procedure (here it is only described qualitatively since, because of its complexity, to give more details would move away of the objective of our overview). Finally, we mention the Perelman's result about the asymptotic behaviour of the flow together with an auxiliary result by Shioya and Yamaguchi which (if correct) would allow to finish the proof of the Geometrization Conjecture.

1. THE TOPOLOGY SETTING

1.1. The Poincaré conjecture

For simplicity, unless otherwise stated, all the manifolds we shall consider here will be oriented and closed (compact without boundary).

In the XIX Century, the topology of manifolds of dimension 2 was very well known, mainly due to the works of *Poincaré* and *Koebe*. Their classification is given in the following version of uniformization theorem.

Theorem 1.1 (of uniformization) *Every oriented and closed surface S has a Riemannian metric with Gauss curvature $+1$, 0, or -1. That is, it is possible giving to S a geometric structure modeled on H^2, E^2 or S^2 respectively (called the standard space forms).*

As a consequence of this theorem and the classification of Riemannian surfaces of constant sectional curvature, it follows that every surface S is the quotient of the hyperbolic plane H^2, the euclidean plane E^2 or the 2-sphere S^2 by a discrete subgroup of isometries acting freely on it.

Moreover, the geometry and the topology of the surface are related by the Gauss-Bonnet formula $2\,\pi\,\chi(M) = \int_M K\,dV$, where $\chi(M)$ is the Euler characteristic of M, which is related with the genus g of M by $\chi(M) = 2-2g$, and every oriented closed surface has a genus g and can be described as a sphere with g handles glued to it, where a handle is a cylinder $I \times S^1$. It is remarkable that it follows from these results that we have only one topology for curvature 1 (genus 1), one for curvature 0 (genus 0), and infinite topologies (genus ≤ -1) for curvature -1, i. e., negative curvature is infinitely richer in topology than nonnegative.

Then, we have a topological classification of surfaces by their genus, and also a description of them using geometry.

To find a similar classification and description of 3-manifolds is much harder. In fact, it is impossible for dimension ≥ 4. But the classification of 3-manifolds has been largely pursued along last century, and the first problem related with it is the Poincaré conjecture.

In 1900, Henri Poincaré (1854-1912) made the following claim. *If a closed 3-dimensional manifold has the homology of the sphere S^3, then it is necessarily homeomorphic to S^3.*

However, within four years, he had developed the concept of "fundamental group", and hence the machinery needed to disprove this statement.

In 1904, he presented a counterexample, the Poincaré icosaedral manifold, which can be described as the quotient $SO(3)/I_{60}$. Here $SO(3)$ is the group of rotations of the Euclidean 3-space, and I_{60} is the subgroup consisting of those rotations which carry a regular icosahedron onto itself. This manifold has the homology of the 3-sphere, but its fundamental group is of order 120. He concluded the discussion by asking:

> *If a closed* 3*-dimensional manifold has trivial fundamental group, must it be homeomorphic to the 3-sphere?*

The *the Poincaré's Conjecture* says that the answer is "yes". It has turned out to be an extraordinarily difficult question, much harder than the corresponding question in dimension five or more (answered by Smale in 1966) and in dimension 4, answered by Freedman in 1982. Moreover, Poincaré's conjecture become one of the key stumbling blocks in the effort to classify 3-dimensional manifolds.

1.2. Some examples of 3-dimensional manifolds

Usually, to solve a question, like Poincaré's conjecture, where some concrete space is characterized by a special property, it is helpful to know other spaces which, although they do not satisfy this property, are similar to the desired space from other viewpoints. Moreover, the knowdlege of these spaces is also useful to go on a more ambitious program: the problem of classification of this kind of spaces. The knowdlege of many examples of closed 3-manifolds has also been useful in the way to the solution of Poincaré conjecture and the statement of the Thurston conjecture on the classification of 3-manifolds. Here we shall give a few examples of 3-manifolds, conducted more by the beauty of them than by the way they can give some insight on the classification problem, which is far beyond of our scope.

Example 1 (Gieseking, 1912) We shall begin by other construction which help us to understand the example.

We start with a regular tetrahedron without its vertices. First, we take an axis through one of the vertices an cutting orthogonally the opposite face C_0. Let C_1, C_2 and C_3 be the other faces of the tetrahedron; let us identify them by a rotation of angle $2\dfrac{\pi}{3}$ around the axis we have just defined.

Now, let us repeat the process taking an axes through other side and its opposite vertex. After this, the 4 sides of the original tetrahedron will be identified. Then we obtain a manifold without boundary which is closed except for a finite number of points missing (those coming from the vertices).

Then we ask if this topological construction is differentiable, in the sense that at each point there is a well defined tangent space. To answer, we pay attention at the points where problems could arise: the edges.

Let us observe that the 6 edges of the original tetrahedron are now identified. Then, after gluing, there are 6 pieces of faces with a common edge. Since they fill a 3-space, the angle formed at this edge must be of 2π radians if the tangent vector space is well defined. But each dihedral angle of the tetrahedron is of $\pi/3$, then, the total angle after gluing will be of $6\times\frac{\pi}{3}=2\pi$. Then the quotient manifold we have just constructed is differentiable.

In the Gieseking example the construction is similar, but now the starting manifold is a regular 3-simplex with vertices in the points of infinity of the Poincaré's hyperbolic ball. In this case the resulting space is a hyperbolic 3-manifold non-compact, but it is complete and with finite volume. Moreover, it is not orientable.

Example 2 (Poincaré dodecahedral space, 1904) In a dodecahedron every face has an opposite one. Then we make the construction of this space following the steps:

(1) Let us take two opposite faces. Let us do a translation of one parallel to the other until they are in the same plane. We do this for every two opposite faces and, in each case we obtain the same configuration.

(2) Then, for every couple of faces in the same plane, do a rotation of an angle of $2\pi/10$ radians around the center of these faces (both have the center at the same point). Then, after translation and rotation, let us identify every face with its opposite one.

In the resulting quotient space, the old edges become glued from 3 to 3 and the old vertices from 4 to 4.

Since the dodecahedron has 30 edges, 20 vertices and 12 faces, after the identifications we obtain:

- 10 groups of 3 edges identified,
- 6 groups of 2 faces identified ,
- 5 groups of 4 vertices identified.

The quotient space is locally euclidean, then it is a topological 3-manifold, that we shall call $VT1$. But it is not a differentiable manifold. In fact, the dihedral angles of the dodecahedron are of approximately 116 degrees and the sum of the three dihedral angles which met an edge is $\approx 116\times 3 = 348 < 360$, then, we have singularities in all points of the edges.

We are going to avoid the singularity by enlarging the measure of the dihedral angles. For it, the trick will be to consider the dodecahedron inside S^3. Now, the edges of the dodecahedron will be geodesics of S^3, and the faces totally geodesic surfaces.

We begin with a small dodecahedron with the center in the North Pole. Since, for small distances, Euclidean geometry is a good approximation of any other Riemannian geometry, the measure of the angles of this small dodecahedron will be near to 116 degrees. Now, we grow the dodecahedron in S^3 by increasing uniformly the distance of the vertices to the center of the dodecahedron, keeping the dodecahedron regular. As the dodecahedron increases, also the dihedral angles increase. When the distance of the vertices to the North Pole approaches $\pi/2$, the dodecahedron approaches the equator of S^3, itself becomes similar to the equator, and the dihedral angles approach 180 degrees.

Since, in the above process, the measure of the dihedral angles is continuous as a function of the distance from the vertices to the North Pole, and this measure varies between 116 and 180 degrees as the distance varies between 0 and $\pi/2$, there will be a dodecahedron with dihedral angles of 120 degrees. The manifod obtained from this spherical dodecahedron by the above gluing process will be differentiable, and topologically equivalent to $VT1$.

Example 3 (Seifert-Weber dodecahedral space, 1933) We start with a dodecahedron as in the previous example, and we made the same process of gluing, with one difference, this time the rotation before identifying opposite faces will be of $3 \times 2\pi/10$ radians. Now, after identification we obtain

- 6 groups of 5 edges identified.
- All the 20 vertices are identified.

Since we have 5 edges identified, we need dihedral angles of $360/5 = 72$ degrees to avoid singularities at the edges. But we recall that we have mentioned before that the dihedral angles of a dodecahedron is around 116. Then, to obtain a differentiable quotient, we need the opposite operation that we did before: to decrease the angles. For it, we shall consider now a dodecahedron in the hyperbolic space H^3. Using the Poincaré ball as a model for H^3, we shall put the dodecahedron with its center at the center of the ball O. As above, if we begin with a very small dodecahedron, the dihedral angles will be approximately of 116 degrees.

The dihedral angle is a continuous function of the hyperbolic distance of the vertices to O. Then, if we enlarge the dodecahedron by increasing uniformly the distance of its vertices to O, the dihedral angle will change continuously and, when distance goes to ∞, the faces of the dodecahedron have their vertices at the sphere of the infinity, and the dihedral angle between two such faces is 60 degrees. Then the dihedral angle varies continuously from ≈ 116 to 60, then, at some distance from the vertices to O, it has the needed value 72. If we do the indicated identifications on the hyperbolic dodecahedron with this dihedral angle, we obtain a differentiable manifold which is the dodecahedrical space of Seifert and Weber.

Remark 1 *There are many difficult points that we have hidden in the above examples, they have more complications than those we have described here. However, there are two simple things which we have learned here, although the examples are not completely described. The first one is that, if we match faces of a polyhedron, we will produce differentiable singularities (no tangent plane) along the axes if the sum of dihedral angles between the faces matched is different from 2π. The second is that we can smooth these singularities by considering the corresponding polyhedron in the sphere or in the hyperbolic space.*

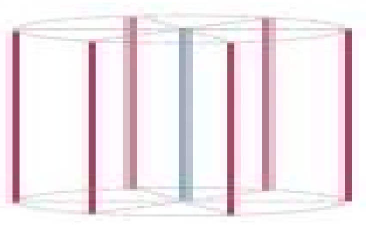
Example 4 (Seifert fibrations) This is not, in fact, an example, but a family of examples, which play an important role in subsequent developments. These fibrations where introduced by Herbert Seifert in 1933.

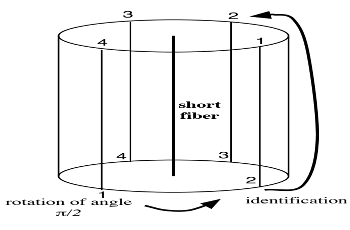
A Seifert fibration is a 3-manifold fibered by circles which are the orbits of a circle action which is free except on at most finitely many fibers, which are called "short" fibers

We recall that an action of a circle on a manifold M is a map $(x,t) \mapsto x^t$ from $M \times \mathbb{R}/\mathbb{Z}$ to M satisfying the usual conditions that $x^0 = x$ and $x^{s+t} = (x^s)^t$. In a Seifert fibration we require that each fiber $x^{\mathbb{R}/\mathbb{Z}}$ should be a circle, and that the action of $\mathbb{R}/\mathbb{Z}$ should be free except on at most finitely many of these fibers.

We recall also that an action of a group G on a set X is called free if $xg \neq x$ for all $x \in X$ and $g \in G - \{e\}$.

Each "short fiber" of a Seifert fibration has a neigbourhood which is a model Seifert fibering. A model Seifert fibering of $S^1 \times D^2$ is a decomposition of $S^1 \times D^2$ into disjoint circles, called fibers, constructed as follows. Starting with $[0,1] \times D^2$ decomposed into the segments $[0,1] \times \{x\}$, identify the disks $\{0\} \times D^2$ and $\{1\} \times D^2$ via the rotation of angle $2\pi p/q$, where p and q are relatively prime integral numbers. The segment $[0,1] \times \{0\}$ then becomes a fiber $S^1 \times \{0\}$ (the short fiber), while every other fiber in the quotient $S^1 \times D^2/\sim$ is a S^1 made from q segments $[0,1] \times \{x\}$.





Model Seifert fibering before the identification. At the center the short fiber

Then, a Seifert fibering of a 3*-manifold* M *can also be defined as a decomposition of* M *into disjoint circles, the fibers, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to a neighborhood of a fiber in some model Seifert fibering of* $S^1 \times D^2$*. A Seifert manifold is one which possesses a Seifert fibering.*

Seifert manifolds where classified along the thirties.

Example 5 (graph manifolds) *In 1967, Friedhelm Waldhausen introduced and analyzed the class of graph manifolds. By definition, these are manifolds which can be split by disjoint embedded tori into pieces (see the torus decomposition Theorem 1.4 for more precise explanation of the meaning of this splitting), each of which is a circle bundle over a surface. Seifert manifolds are a particular case of graph manifolds.*

1.3. The Sphere (or Prime) Decomposition

The first two main steps in the way to classification of 3-manifolds are the Sphere Decomposition Theorem and the Torus Decomposition Theorem.

Helmut Kneser (1898-1973) carried out the first step. Although he stated his definitions and theorems in the category of PL-manifolds, we shall declare it in the category of topological manifolds, because, in dimension 3, TOP=PL=DIFF, that is, each topological 3-manifold admits a unique PL and a unique smooth structure.

In order to state this result, first we need some definitions

Definition 1.1 (connected sum) *If a closed* 3*-manifold* M *contains an embedded sphere* S^2 *separating* M *into two components, we can split* M *along this* S^2 *into manifolds* M_1 *and* M_2 *with boundary* S^2*. We can then fill in these boundary spheres with* 3*-balls to produce two closed manifolds* N_1 *and* N_2*. One says that* M *is the connected sum of* N_1 *and* N_2*, and one writes* $M = N_1 \sharp N_2$*.*

This splitting operation is commutative and associative.

One rather trivial possibility for the decomposition of M as a connected sum is $M = M \sharp S^3$. If this is the only way to decompose M as a connected sum, M is called *irreducible or prime.* This property is equivalent to the following

Definition 1.2 (prime manifolds) *A closed 3-manifold is called prime if every separating embedded 2-sphere bounds a 3-ball.*

A related definition is the following:

Definition 1.3 (irreducible manifolds) *A closed 3-manifold is called irreducible if every embedded 2-sphere bounds a 3-ball.*

There are three main classes of prime manifolds:

Type I With finite fundamental group. All the known examples are the *spherical* 3-manifolds, of the form $M = S^3/\Gamma$, where Γ is a finite subgroup of $SO(4)$ acting freely on S^3 by rotations. Thus $\Gamma = \pi_1(M)$. The spherical manifolds with Γ cyclic are called *lens spaces.* It is an old conjecture that spherical 3-manifolds are the only closed 3-manifolds with finite fundamental group (It is, in fact, Poincaré conjecture).

Type II With infinite cyclic fundamental group. There is only one prime 3-manifold satisfying this condition: $S^1 \times S^2$. *This is also the only orientable 3-manifold that is prime but not irreducible.* It is also the only prime orientable 3-manifold with non-trivial π_2.

Type III With infinite noncyclic fundamental group. These are $K(\pi, 1)$ manifolds (also called aspherical), i.e., manifolds with contractible universal cover. Any irreducible 3-manifold M, with π_1 infinite is a $K(\pi, 1)$.

Suppose that we start with a 3-manifold M which is connected and not prime. Then we can decompose $M = N_1 \sharp N_2$, where no N_i is a sphere. Now, either each N_i is irreducible, or we can iterate this procedure. The theorem of Kneser (1929) states that this procedure always stops after a finite number of steps, yielding a manifold M such that each connected component of M is irreducible. In fact, if we keep careful track of orientations and the number n of non-separating cuts, then the original connected manifold M can be recovered as the connected sum of the components of M, together with n copies of the "handle" $S^1 \times S^2$. With more precision

Theorem 1.2 (Sphere (or Prime) Decomposition - cf. [57], [63]-) *Let M be a orientable closed 3-manifold. Then M admits a finite connected sum decomposition*

$$M = (K_1 \sharp ... \sharp K_p) \sharp (L_1 \sharp ... \sharp L_q) \sharp (\sharp_1^r S^2 \times S^1). \tag{1.1}$$

The K and L factors here are closed and irreducible 3-manifolds. The K factors have infinite fundamental group and are aspherical 3-manifolds (are of type III), while the L factors have finite fundamental group and have universal cover a homotopy 3-sphere (are of Type I).

Since $M \sharp S^3 = M$, we assume no L factor is S^3 unless $M = L = S^3$. The factors in (1.1) are unique up to permutation and are obtained from M by performing surgery on a collection of essential, i.e. topologically nontrivial, 2-spheres in M (replacing regions $S^2 \times S^1$ by two copies of B^3); see Figure 1 for a schematic representation.

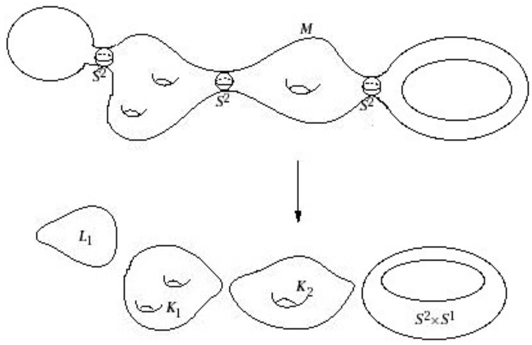


Figure 1

It is worth emphasizing that the sphere decomposition is perhaps the simplest topological procedure that is performed in understanding the topology of 3-manifolds. In contrast, in dealing with the geometry and analysis of metrics on 3-manifolds, this procedure is the most difficult to perform or understand[1]. It is more convenient the following

1.4. The Torus Decomposition

Before stating the torus decomposition theorem, we introduce several definitions. Let N be an irreducible manifold (possibly with border) and let S be a compact, oriented (which is the same that two-sided) surface embedded in N (and thus having trivial normal bundle).

Definition 1.4 *The above surface S is incompressible if, for every closed disc D embedded in N with $D \cap S \subset \partial D$, the curve ∂D is contractible in S. If S is not incompressible, it is compressible.*

Proposition 1.3 *The surface S is incompressible if and only if the inclusion map induces an injection $\pi_1(S) \longrightarrow \pi_1(N)$ of fundamental groups.*

Definition 1.5 *A 3-manifold N is called sufficiently large if it contains an incompressible surface.*

A 3-manifold N is called a Haken manifold if it contains an incompressible surface of genus $g \geq 1$.

Incompressible tori play the central role in the torus decomposition of a 3- manifold, just as spheres do in the prime decomposition. Note, however, that when one cuts a 3-manifold along an incompressible torus, there is

[1] In fact, this is like an Uniformization Theorem, that says that every orientable closed 2-manifold can be written as the connected sum of one sphere and a finite number of tori. Prime decomposition Theorem states that every 3-manifold can be written as a connected sum of one sphere and a finite number of prime 3-manifolds, and reduces the problem to classify the prime 3-manifolds

no canonical way to cap off the boundary components thus created, as is the case for spheres. For any toral boundary component, there are many ways to glue in a solid torus, corresponding to the automorphisms of T^2; typically, the topological type of the resulting manifold depends on the choice. Thus, when a 3-manifold is split along incompressible tori, one leaves the compact manifolds with toral boundary fixed. This leads to another definition:

Definition 1.6 *A closed* 3*-manifold* N *is torus-irreducible if it has no incompressible tori.*

Theorem 1.4 (Torus Decomposition - cf. [53], [54]-) *Let* M *be a closed, oriented, irreducible* 3*-manifold. Then there is a finite collection (possibly empty) of disjoint incompressible tori* $T_i^2 \subset M$ *that separate* M *into a finite collection of compact* 3*-manifolds with toral boundary, each of which is either torus-irreducible or Seifert fibered. A minimal such collection (with respect to cardinality) is unique up to isotopy.*

Of course, it is possible that the collection of incompressible tori is empty. In this case, M *is itself a closed irreducible 3-manifold that is either Seifert fibered or torus-irreducible.*

Joining The Sphere and the torus decomposition theorems, we see that we can obtain any closed orientable 3-manifold by gluing torus-irreducible and Seifert fibered manifolds. Then, the main question for the classification of orientable closed 3-manifolds is to understand the structure of torus-irreducible and Seifert fibered manifolds.

The Geometrization Conjecture (GC) of Thurston asserts that the torus irreducible and Seifert fibered components of a closed, oriented, irreducible 3- manifold admit canonical geometric structures. We shall be back on GC at the end of this lecture. Now, we are going to explain what do we mean by "geometric structure"

1.5. Geometric structures and the Geometrization Conjecture

Definition 1.7 *A simply connected geometric structure or a model geometry is a pair* (G, X) *where* X *is a manifold and* G *is a Lie group of diffeomorphisms of* X *satisfying*

1. X *is connected and simply connected,*
2. G *acts transitively on* X[2]*, with compact identity component* H_0 *of the isotropy group,*
3. G *is not contained in any larger group of diffeomorphisms of* X *with compact isotropy group, and*

[2] G acts transitively on X if for every $x, y \in X$ there is a $g \in G$ satisfying $xg = y$

4. there is at least one compact manifold modelled on (G, X). That is, there is a compact manifold M and a discrete subgroup Γ of G satisfying $M = X/\Gamma$ (this is equivalent to say that G is unimodular).

Each $M = X/\Gamma$ of condition 4 is also called a model geometry. In other words, a compact manifold M^n has a geometric structure or is a model geometry if the universal covering X of M has a simply connected geometric structure.

Condition 2 means that the space X admits a homogeneous Riemannian metric invariant by G, $X = G/H_0$, and it is complete. Condition 3 says that no Riemannian metric invariant by G is also invariant by any larger group. The point of conditions 3 and the second part of 2 is to avoid redundancy.

In dimension 2 the classification is easy and well known:

Theorem 1.5 *There are precisely three two-dimensional model geometries: Spherical, Euclidean and Hyperbolic*

Proof. Since G acts transitively on X, it follows that any G-invariant Riemannian metric on X has constant Gaussian curvature. When a metric is multiplied by k, the Gaussian curvature is multiplied by k^2, so we can find a metric whose curvature is either 0, 1 or -1. It is a standard fact from Riemannian Geometry that the only simply connected Riemannian 2-manifolds with constant sectional curvature 0, 1 or -1 are E^2, S^2, H^2. $\square$

Conjecture 1.1 The Thurston Geometrization Conjecture. *Let M be a closed, oriented 3-manifold. Then each component of the sphere and torus decomposition admits a geometric structure. In other words, any prime closed 3-manifold is either geometric or its simple pieces are geometric.*

From the point of view of Riemannian geometry, the Thurston conjecture essentially asserts the existence of a "best possible" metric on an arbitrary closed 3-manifold.

The geometrization conjecture gives a complete and effective classification of all closed 3-manifolds, closely resembling in many respects the classification of surfaces. More precisely, it reduces the classification to that of geometric 3-manifolds.

Thurston showed that there are 8 simply connected geometries G/H in dimension 3 which admit compact quotients.[3]: We shall give the details

[3]The Thurston classification is essentially a (non completely proved) special case of the much older (rigurously proved) Bianchi classification of homogeneous space-time metrics arising in general relativity; cf. [4] for further remarks on the dictionary relating these classifications. Another complete and modern proof of the classification can be found in Sekigawa's [77] and [78]

of these geometries in the next chapter. The classification of the compact quotients of these 8 models is well known, except for the case of hyperbolic 3-manifolds, which remains an active area of research.

The Geometrization Conjecture includes the following important special cases:

Hyperbolization Conjecture. If M is prime, $\pi_1(M)$ is infinite and M is atoroidal (that is, $\pi_1(M)$ has no subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z} = \pi_1(T^2)$), then M is hyperbolic, that is, admits a hyperbolic metric.

Elliptization or spherical Poincaré Conjecture. If $\pi_1(M)$ is finite, then M is spherical , that is, admits a metric of constant positive curvature. In particular, any closed 3-manifold with trivial fundamental group must be homeomorphic to S^3 (Poincaré conjecture).

In fact, these are the only remaining open cases of the Geometrization Conjecture:

• If M has a nontrivial torus decomposition (equivalently, if M contains an incompressible torus), then in particular M is Haken. Thurston ([82], [83], [84]) has proved the conjecture for Haken manifolds.

• If M has no incompressible tori, recent work on the Seifert fibered space conjecture (cf.cf. [33], [10]) implies that M is either Seifert fibered or atoroidal. It is known that Seifert fibered spaces have geometric structures.

In the remaining cases, M is atoroidal, and so satisfies the hypotheses of either the elliptization or the hyperbolization conjectures.

As an illustration of the power of *the Thurston Conjecture*, let us see how it *implies the Poincaré Conjecture*. If M is a simply connected 3-manifold, then the sphere decomposition (1.1) implies that M must be an L factor. The geometrization conjecture implies that L is geometric, and so $L = S^3/\Gamma$. Hence, $M = L = S^3$.

More remarks

It is well known (cf. [75]) that the same 3-manifold cannot have geometric structures modeled on two distinct geometries.

Of course, it is not true that the geometric structure itself, that is, the homogeneous metric, is unique in general. In this respect, we recall:

Theorem 1.6 (Mostow Rigidity, cf. [65], [71]) *Let N be a 3-manifold carrying a complete hyperbolic metric of finite volume. Then the hyperbolic metric is unique, up to isometry. Further, if N and N' are 3-manifolds with isomorphic fundamental groups, and if N and N' carry complete hyperbolic metrics of finite volume, then N and N' are dffeomorphic.*

In particular, invariants of the hyperbolic metric such as the volume and the spectrum are topological invariants of the 3-manifold.

There is a similar rigidity for spherical 3-manifolds, in the sense that any metric of curvature $+1$ on the manifold is unique, up to isometry (cf.[89]). The fundamental group in this case does not determine the topological type of the manifold. There are further topological invariants, such as the Reidemeister torsion. The other six geometries are typically not rigid, but have moduli closely related to the moduli of constant curvature metrics on surfaces.

2. THE EIGHT 3-DIMENSIONAL MODEL GEOMETRIES

2.1. Generalities and classes

In terms of Riemannian Geometry, a closed manifold M of dimension n has a *geometric structure* if it admits a complete locally homogeneous metric. This means the universal covering space X of M has a complete homogeneous metric. That is, M has a complete metric such that there is a group of isometries of X which acts transitively on X with compact isotropy group H, and $X = G/H$. We recall also that, in order to avoid redundancy, we added the following restriction: "G is a maximal group of isometries of X".

On the other hand, since M is closed and X is the universal covering of M, $M = X/\Gamma$, where Γ is the deck transformation group of the covering $X \longrightarrow M$, which will be a discrete subgroup of G.

Then to look for simply connected geometric structures we have to look for simply connected complete homogeneous spaces X with compact identity component of the isotropy group and having a compact quotient by a discrete subgroup of the group of isometries of X.

Thurston proved that there are eight simply connected geometries in dimension 3 which admit compact quotients. It is a remarkable fact that in each of the eight 3-dimensional model geometries (X, G), X is isometric to a Lie group with a left invariant metric[4]. In this lecture we shall describe the eight simply connected model geometries. In the next one we shall give an outline (with some details) of the classification theorem.

First we have the three constant curvature geometries, all of them with isotropy group ($\equiv H_0$) $SO(3)$. The remaining five geometries are products or twisted products with the two dimensional geometries. Concretely, it is possible to divide the geometric structures into three categories.

- Constant (sectional) curvature geometries ($H_0 = SO(3)$)
  - Spherical($\equiv$ geometry of curvature 1)$\begin{cases} X = S^3 \\ G = SO(4) \end{cases}$

[4]Let G be a Lie group. A Riemannian metric on G is said to be left-invariant if it is invariant under all left translations: $L_p^* g = g$ for every $p \in G$.

– Hyperbolic($\equiv$ geometry of curvature -1) $\begin{cases} X = H^3 \\ G = PSL(2,\mathbb{C}) \end{cases}$

– Euclidean ($\equiv$ geometry of curvature 0) $\begin{cases} X = E^3 \\ G = \mathbb{R}^3 \times SO(3) \end{cases}$

- Product geometries ($H_0 = SO(2)$)

– Geometry of $H^2 \times \mathbb{R}$ $\begin{cases} X = H^2 \times \mathbb{R} \\ G = \text{orientation preserving subgroup of} \\ \qquad IsomH^2 \times IsomE^1 \end{cases}$

– Geometry of $S^2 \times \mathbb{R}$ $\begin{cases} X = S^2 \times \mathbb{R} \\ G = \text{orientation preserving subgroup of} \\ \qquad SO(3) \times IsomE^1 \end{cases}$

- Twisted product geometries

– Geometry of $\widetilde{SL(2,\mathbb{R})}$ $\begin{cases} X = \text{the universal cover of the} \\ \qquad \text{unit sphere bundle of } H^2 \\ G = \widetilde{SL}(2,\mathbb{R}) \times \mathbb{R} \\ H_0 = SO(2) \end{cases}$

– *Nil* geometry $\begin{cases} X = \text{3-dimensional Heisenberg group} \\ G = \text{semidirect product of } X \text{ with } S^1, \\ \qquad \text{acting by rotations on the quotient of } X \\ \qquad \text{by its center} \\ H_0 = SO(2) \end{cases}$

– Sol geometry $\begin{cases} X = \text{3-dimensional solvable Lie group} \\ G = \text{extension of } X \text{ by an automorphism group} \\ \qquad \text{of order eight} \\ H_0 = \{e\} \end{cases}$

Here we enumerate some properties of the above geometries.
- The first five geometries are symmetric spaces[5]
- Only the spherical geometry is compact.
- The constant curvature geometries are isotropic, that is, they look the same in every direction. In particular, the isometry groups of these spaces act transitively on their orthonormal frame bundles.
- The twisted product geometries are the least isotropic ones and are modelled on unimodular Lie groups.

2.2. Contact structures

[5]A Riemannian manifold (M,g) is called symmetric (or a Riemannian symmetric space) if for each $x \in M$ there exists an isometry f_x of (M,g) such that $f_x(x) = x$ and $(f_x)_{*x} = -Id_{T_xM}$. The isometry f_x is called the symmetry around x.

As we shall see in the proof of the Thurston theorem, twisted product geometries (X, G) have the following characteristic property: X admits a foliation $\mathcal{F}$ by lines such that $X/\mathcal{F}$ has constant Gaussian curvature and $(T\mathcal{F})^{\perp}$ is not involutive. The last condition is equivalent to say that X admits a contact structure. We try to understand in this section what is the meaning of this.

First, let us revisit some fundamental concepts.

• Let M^n be a differentiable manifold. A c-dimensional (tangent) distribution (or c-plane field) is an application $\mathcal{D}$ such that

$$m \in M \longmapsto \mathcal{D}_m \subset T_m M,$$

where $\mathcal{D}_m$ is a vector subspace of dimension c. Moreover, $\mathcal{D}$ acts in a differentiable way, that is, $\forall x \in M$ there exist a neighborhood $\mathcal{U}$ of m and differentiable vector fields $X_1, \ldots, X_c \in \mathfrak{X}(\mathcal{U})$ satisfying $\mathcal{D}_m = \langle X_{1m}, \ldots, X_{cm} \rangle \equiv span(X_{1m}, \ldots, X_{cm})\ \forall m \in \mathcal{U}$.

• A distribution is said to be involutive if, given any pair of local sections of $\mathcal{D}$[6], their Lie bracket [7] is also a section of $\mathcal{D}$. Briefly,

$$\mathcal{D} \text{ is involutive if } X, Y \in \mathcal{D} \text{ implies } [X, Y] \in \mathcal{D}.$$

• From the Frobenius theorem, an involutive distribution is equivalent to a integrable one[8].

Definition 2.1 *Let M be a differentiable 3-manifold. A codimension* 1 *distribution on M is called a contact structure on M if it is not involutive on M.*

We can also define the concept of contact structure in a dual way. Indeed, let ω be a one-form on a 3-dimensional manifold M; if there exist $X \in \mathfrak{X}(M)$ such that $\omega(X) = 0$ everywhere, then $\forall p \in M$ $\mathrm{Ker}\omega_p := \{X \in T_pM / \omega_p(X) = 0\}$ is a non-empty vectorial subspace of T_pM. Therefore the map

$$p \in M \longmapsto \mathcal{D}_p := \mathrm{Ker}\omega_p \subset T_pM$$

defines a 2-distribution on M, which we shall denote by $\mathcal{D} \equiv \mathrm{Ker}\omega$. So we have defined a distribution as the kernel of a one-form.

---

[6]A vector field X defined on an open subset of M is said to be a local section of a distribution $\mathcal{D}$ (and this is denoted by $X \in \mathcal{D}$) if $X_p \in \mathcal{D}_p$ for each p.

[7]Given $X, Y \in \mathfrak{X}(M)$, their Lie bracket (denoted as $[X, Y]$) is another vector field on M with coordinates

$$[X, Y]^i = \frac{\partial X^i}{\partial u^j} Y^j - \frac{\partial Y^i}{\partial u^k} X^k.$$

[8]Recall that an n-distribution $\mathcal{D}$ over M is called integrable if there exists a foliation $\mathcal{F}$ of dimension n over M such that $\mathcal{D} = T\mathcal{F}$ (i.e. $\mathcal{D}$ is tangent to a foliation).

Now we can rewrite the previous definitions in this new language of one-forms.

• A local section of $\mathcal{D}$ is a vector field X on an open subset of M such that $\omega_p(X) = 0\, \forall p \in M$.

• $\mathcal{D}$ is involutive if $\forall X, Y \in \mathcal{D}$ (i.e. such that $\omega(X) = 0$, $\omega(Y) = 0$), we have $\omega([X, Y]) = 0$ at every point.

Finally, we can rephrase the Definition 2.1 in terms of ω.

Definition 2.2 *Given a linear form ω on a 3-manifold M. the 2-plane field $\tau = \mathrm{Ker}\omega$ is called a contact structure if there exist $X, Y \in \tau$ such that $\omega([X, Y]) \neq 0$ at any point.*

We have another equivalent definition provided by the Frobenius theorem: $\mathcal{D}$ is involutive if and only if $d\omega \wedge \omega \equiv 0$ and τ is a contact structure iff $d\omega \wedge \omega \neq 0$ at any point.

Examples 2.2.1 *Consider E^3 and $\omega = dz - xdy$, then*

$$\omega \wedge d\omega = (dz - xdy) \wedge (-dx \wedge dy) = -dx \wedge dy \wedge dz \neq 0$$

Thus Kerω *is a contact structure on E^3.*

Examples 2.2.2 *Consider a Riemannian manifold M^n with metric $ds^2 = g_{ij} dx^i\, dx^j$. Let now TM^n be the tangent bundle with the Sasaki metric, which is expressed in local coordinates as*

$$d\sigma^2_{(x,\xi)} = g_{ij} dx^i\, dx^j + g_{ij} (\nabla\xi)^i\, (\nabla\xi)^j, \tag{2.1}$$

where

$$(\nabla\xi)^i = d\xi^i + \Gamma^i_{jk} \xi^j dx^k. \tag{2.2}$$

We shall denote by G_{ij} the components of the Sasaki metric.

In particular, we take $M = H^2$ with metric $ds^2 = dx_1^2 + \cosh^2 x_1 dx_2^2$. let us compute the expression for the covariant derivative associated to this metric.

First we calculate the Christoffel symbols for the Levi-Civita connection:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

Considering that the matrix of the metric and its inverse have the following form: $g = \begin{pmatrix} 1 & 0 \\ 0 & \cosh^2 x_1 \end{pmatrix}$ and $g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\cosh^2 x_1} \end{pmatrix}$, the result obtained for the Chistoffel symbols is

$$\Gamma^1_{22} = -\cosh x_1 \sinh x_1, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \tanh x_1$$

and the remaining symbols are equal to zero.

Substituting these in (2.2)*, we reach*

$$(\nabla\xi)^1 = (d\xi)^1 + \Gamma^1_{jk}\xi^j dx^k = (d\xi)^1 + \Gamma^1_{22}\xi^2 dx^2 = (d\xi)^1 - \cosh x_1 \sinh x_1 \xi^2 dx^2 \tag{2.3}$$

$$\begin{aligned}(\nabla\xi)^2 &= (d\xi)^2 + \Gamma^2_{jk}\xi^j dx^k = (d\xi)^2 + \Gamma^2_{12}(\xi^1 dx^2 + \xi^2 dx^1)\\ &= (d\xi)^2 + \tanh x_1(\xi^1 dx^2 + \xi^2 dx^1)\end{aligned} \tag{2.4}$$

Consider $\xi \in T_1H^2$*, then* $|\xi| = 1$ *or, equivalently,* $g_{ij}\xi^i\xi^j = 1$ *and, in our particular case,*

$$(\xi^1)^2 + \cosh^2 x_1 (\xi^2)^2 = 1 \longrightarrow (\xi^2)^2 = \frac{1-(\xi^1)^2}{\cosh^2 x_1} \tag{2.5}$$

Taking derivatives in the equality $\langle\xi,\xi\rangle = 1$*, we get* $\langle\nabla\xi,\xi\rangle = 0$ *or, in local coordiantes,* $g_{ij}(\nabla\xi)^i\xi^j = 0$*. Thus, substituting the concrete values of the* H^2*-metric, we get*

$$(\nabla\xi)^1\xi^1 + \cosh^2 x_1 (\nabla\xi)^2\xi^2 = 0 \longrightarrow (\nabla\xi)^2 = -\frac{\xi^1(\nabla\xi)^1}{\xi^2\cosh^2 x_1} \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}
g_{ij}(\nabla\xi)^i(\nabla\xi)^j &= ((d\xi)^1 - \cosh x_1 \sinh x_1 \xi^2 dx^2)^2\\
&\qquad + \cosh^2 x_1 \frac{(\xi^1)^2((d\xi)^1 - \sinh x_1 \cosh x_1 \xi^2 dx^2)^2}{(\xi^2)^2\cosh^4 x_1}\\
&= ((d\xi)^1 - \cosh x_1 \sinh x_1 \xi^2 dx^2)^2 \left(1 + \frac{(\xi^1)^2}{(\xi^2)^2\cosh^2 x_1}\right)\\
&= ((d\xi)^1 - \cosh x_1 \sinh x_1 \frac{\sqrt{1-(\xi^1)^2}}{\cosh x_1} dx^2)^2 \left(1 + \frac{(\xi^1)^2}{\left(\frac{1-(\xi^1)^2}{\cosh^2 x_1}\right)\cosh^2 x_1}\right)\\
&= ((d\xi)^1 - \sinh x_1 \sqrt{1-(\xi^1)^2} dx^2)^2 \left(1 + \frac{(\xi^1)^2}{1-(\xi^1)^2}\right)\\
&= ((d\xi)^1 - \sinh x_1 \sqrt{1-(\xi^1)^2} dx^2)^2 \left(\frac{1}{1-(\xi^1)^2}\right)
\end{aligned}$$

And so, substituting the above equality in (2.1)*, we obtain*

$$d\sigma^2 = (dx_1)^2 + \cosh^2 x_1 (dx_2)^2 + ((d\xi)^1 - \sinh x_1 \sqrt{1-(\xi^1)^2} dx^2)^2 \left(\frac{1}{1-(\xi^1)^2}\right) \tag{2.7}$$

Consider a 2-distribution orthogonal to V, that is, this distribution is defined by the vectors which satisfy the equation

$$G_{ij}V^i dy^j = 0, \quad \textit{where } y = (y^1, y^2, y^3) = (x^1, x^2, \xi^1)$$

Then, for $V = (0, 0, 1)$, we have

$$G_{13}dx^1 + G_{23}dx^2 + G_{33}d\xi^1 = 0 \tag{2.8}$$

From (2.7), we have

$$G_{13} = 0, \quad G_{23} = \frac{-2\sinh x_1}{\sqrt{1-(\xi^1)^2}}, \quad G_{33} = \frac{1}{1-(\xi^1)^2}$$

Substituting this in (2.8), yields

$$-\frac{2\sinh x_1}{\sqrt{1-(\xi^1)^2}}dx^2 + \frac{1}{1-(\xi^1)^2}d\xi^1 = 0 \longrightarrow -2\sinh x_1\sqrt{1-(\xi^1)^2}dx^2 + d\xi^1 = 0$$

So, the distribution can be defined as the kernel of the linear form

$$\omega = -2\sinh x_1\sqrt{1-(\xi^1)^2}dx^2 + d\xi^1$$

If we compute

$$d\omega = -2\sqrt{1-(\xi^1)^2}\cosh x_1 dx^1 \wedge dx^2 + \frac{\xi^1}{\sqrt{1-(\xi^1)^2}}\sinh x_1 d\xi^1 \wedge dx^2,$$

$$\omega \wedge d\omega = -2\sqrt{1-(\xi^1)^2}\cosh x_1 dx^1 \wedge dx^2 \wedge d\xi^1$$

and notice that $\omega \wedge d\omega \neq 0$, we can conclude that on T_1H^2 with the induced Sasaki metric, ω is a contact structure.

Theorem 2.1 *Locally any contact structure on a 3-manifold has the form $\omega = dz - xdy$.*

2.3. Twisted product geometries

In this section we shall explain the less known geometries which have appeared in the classification.

2.3.1. The geometry of $\widetilde{\boldsymbol{SL}}(2,\mathbb{R})$

Algebraically, the special linear group $X \equiv SL(2,\mathbb{R})$ is the 3-dimensional Lie group of all 2×2 real matrices with determinant 1. $\widetilde{SL}(2,\mathbb{R})$ denotes its universal covering and is also a Lie group; so it admits a metric invariant under left multiplication.

Topologically, $\widetilde{SL}(2,\mathbb{R})$ is homeomorphic to $H^2\times\mathbb{R}$. However, they are not isometric.

To study the metric structure of $\widetilde{SL}(2,\mathbb{R})$ we use the model. Let T_1H^2 be the unit tangent bundle of H^2, consisting of all tangent vectors of length 1 of H^2. Consider in T_1H^2 the induced Sasaki metric. This metric makes T_1H^2 a homogeneous Riemannian manifold and, moreover, the space T_1H^2 has the homotopy type of a circle. Its universal covering is the model for $\widetilde{SL}(2,\mathbb{R})$. Since the Riemannian manifold T_1H^2 is homogeneous, so is its universal covering.

As T_1H^2 is a circle bundle over H^2, we see that $\widetilde{SL}(2,\mathbb{R})$ is naturally a line bundle over H^2. We call the fibres of this bundle vertical. The horizontal plane field($\equiv$ distribution of 2-planes orthogonal to the vertical fibers) on T_1H^2 gives a plane field on $\widetilde{SL}(2,\mathbb{R})$ which we call again horizontal. As the projection map is an isometry, this plane field is non-integrable. This shows that $\widetilde{SL}(2,\mathbb{R})$ is not isometric to $H^2\times\mathbb{R}$ by any isomorphism preserving fibers.

The isometry group of $\widetilde{SL}(2,\mathbb{R})$ preserves this bundle structure and is 4-dimensional. However, $Isom\widetilde{SL}(2,\mathbb{R})$ has only two components, both orientation preserving. In particular, $\widetilde{SL}(2,\mathbb{R})$ admits no orientation reversing isometry. In other words, for $\widetilde{SL}(2,\mathbb{R})$ (and also for Nil), the contact structure determines an orientation of the geometry which cannot be reversed.

2.3.2. Nil geometry

Algebraically, X is the 3-dimensional Heisenberg group, which consists of all 3×3 real upper triangular matrices of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, endowed with the usual matrix multiplication. This group is nilpotent [9]. In fact, it is the only 3-dimensional nilpotent but non abelian connected and simply connected Lie group; this explains the term Nil geometry. For now on, we shall use the notation $X \equiv Nil$.

Topologically, Nil is diffeomorphic to $\mathbb{R}^3$ under the map

$$Nil \ni \gamma = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \longmapsto (a, b, c) \in \mathbb{R}^3$$

Under this identification, left multiplication by γ corresponds to the map

$$L_\gamma(x, y, z) = (x + a, y + b, z + ay + c) \tag{2.9}$$

In other words, from this point of view, $\mathbb{R}^3$ now has the multiplication

$$(x_0, y_0, z_0)(x, y, z) = (x + x_0, y + y_0, z + z_0 + x_0 y)$$

It is possible to check this formula doing the matrix multiplication.

It is easy to prove that Nil is a Lie group, so it admits a metric invariant under left multiplication; it is defined in the following way: we shall take $ds^2 = dx^2 + dy^2 + dz^2$ at $(0, 0, 0)$ -the unit of the Heisenberg group- and then extend ds^2 at all other points of X as a left invariant metric. The result is

$$ds^2 = dx^2 + dy^2 + (dz - x dy)^2.$$

Next we check that ds^2 is invariant under left multiplication.

$$\begin{aligned} L_\gamma^*(ds^2) &= d(L_\gamma^* x)^2 + d(L_\gamma^* y)^2 + (d(L_\gamma^* z) - L_\gamma^* y \, d(L_\gamma^* y))^2 \\ &= d(x \circ L_\gamma)^2 + d(y \circ L_\gamma)^2 + (d(z \circ L_\gamma) - (x \circ L_\gamma) d(y \circ L_\gamma))^2 \\ &= d(x + a)^2 + d(y + b)^2 + (d(z + ay + c) - (x + a) d(y + b))^2 \\ &= dx^2 + dy^2 + (dz + a dy - x dy - a dy)^2 = ds^2 \end{aligned}$$

Moreover, as a Riemannian manifold, X is homogeneous. With this metric, Nil is a line bundle over the Euclidean plane E^2. The fibres of Nil

[9]Let $Z(G)$ the center of a group G. The upper central series

$$Z_0 \subset Z_1 \subset \ldots \subset Z_i \subset \ldots$$

of G is defined inductively by $Z_0 = \{e\}$ (where e is the identity of G) and $Z_i / Z_{i-1} = Z(G/Z_{i-1})$. G is said to be nilpotent (of class c) if $Z_c = G$. Thus a group is abelian if and only if it is nilpotent of class 1.

are called vertical, and the orthogonal plane field is called horizontal. Nil can't be thought of as the universal covering T_1E^2, which is isometric to $E^2 \times S^1$ and so its universal covering is isometric to E^3. In addition, it is possible to prove that Nil is not isometric to E^3.

If we identify S^1 with the interval $[0, 2\pi]$ with the ends identified then a point $\theta \in S^1$ acts on Nil by $\mathcal{A} : S^1 \times Nil \to Nil$ such that

$$(\theta, (x, y, z)) \overset{\mathcal{A}}{\longmapsto} (\theta x, \theta y, \theta z), \text{ where}$$
$$\begin{cases} \theta x = x\cos\theta - y\sin\theta \\ \theta y = x\sin\theta + y\cos\theta \\ \theta z = z + \frac{1}{2}\sin\theta\left(\cos\theta(x^2 - y^2) - 2\sin\theta xy\right) \end{cases}$$

$\mathcal{A}$ is an action of S^1 on Nil which is a group of automorphisms of Nil preserving the above metric. In fact,

$$\begin{aligned}
\mathcal{A}_\theta^*(ds^2) &= d(\theta x)^2 + d(\theta y)^2 + (d(\theta x) - (\theta x)d(\theta y))^2 = d(x\cos\theta - y\sin\theta)^2 \\
&\quad + d(x\sin\theta + y\cos\theta)^2 + \left(d\left(z + \frac{1}{2}\sin\theta(\cos\theta(x^2 - y^2) - 2\sin\theta xy\right)\right. \\
&\quad \left. - \ (x\cos\theta - y\sin\theta)d(x\sin\theta + y\cos\theta)\right)^2 \\
&= dx^2 + dy^2 + \big(dz - \sin\theta\cos\theta\, y\, dy + \sin\theta\cos\theta\, x\, dx - \sin^2\theta\, x\, dy \\
&\quad - \sin^2\theta\, y\, dx + \sin\theta\cos\theta\, y\, dy - \cos^2\theta\, x\, dy + \sin^2\theta\, y\, dx - \sin\theta\cos\theta\big)^2 \\
&= dx^2 + dy^2 + (dz - xdy)^2
\end{aligned}$$

The action $\mathcal{A}$ also preserves the bundle structure of Nil and induces a rotation on E^2 fixing the origin. Moreover, the isometry group of Nil is generated by Nil and this circle action and is 4-dimensional.

A nice example of a manifold with a geometric structure modelled on Nil is obtained by taking the quotient of Nil by the subgroup Γ of Nil consisting of all matrices in Nil with integer entries. This manifold is a circle bundle over the torus with orientable total space.

On the other hand, closed 3-manifolds M modelled on Nil admit a contact structure. Indeed, in this case M has a natural Seifert fibration and it is possible to prove that the orthogonal plane field to this foliation of M by circles is not integrable.

2.3.3. Sol geometry

It is the only maximal geometry with trivial isotropy group; because of that, this is the geometry with the least symmetry. Algebraically, $X \equiv Sol$

may be regarded as the solvable[10] Poincar-Lorentz group $E(1,1)$, which consists of rigid motions of the Minkowski 2-space [11]. This Lie group is a semidirect product[12] of subgroups isomorphic to $\mathbb{R}\oplus\mathbb{R}$ and to $\mathbb{R}$, where each $t\in\mathbb{R}$ acts on $\mathbb{R}\oplus\mathbb{R}$ by $(t,(x,y))\mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot\begin{pmatrix} x \\ y\end{pmatrix} = (e^t x, e^{-t}y)$.

Topologically, we can identify Sol with $\mathbb{R}^3$ so that the multiplication is given by

$$(x_0,y_0,z_0)\cdot(x,y,z) = (x+e^{-z_0}x_0, y+e^{z}y_0, z+z_0)$$

Clearly $(0,0,0)$ is the identity and the xy-plane is a normal subgroup isomorphic to $\mathbb{R}^2$. In fact,

$$\begin{aligned}(x,y,z)^{-1}\cdot(a,b,0)\cdot(x,y,z) &= (-xe^{-z},-ye^{z},-z)\cdot(a,b,0)\cdot(x,y,z)\\ &= (a-xe^{-z}, b-ye^{z},-z)\cdot(x,y,z) = (*,*,0)\end{aligned}$$

Metrically, Sol is just $\mathbb{R}^3$ but endowed with the left invariant Riemannian metric which at (x,y,z) is

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2. \tag{2.10}$$

Another equivalent approach to the algebraic definition of Sol will be given along the proof of Thurston theorem. In this way, we can say that Sol is the unimodular Lie group completely determined by

$$[e_1,e_2]=0, \quad [e_2,e_3]=e_1, \quad [e_3,e_1]=-e_2;$$

where $\{e_1,e_2,e_3\}$ is an orthonormal basis of eigenvectors of the Lie algebra of X. As the generators e_1,e_2 commute, Sol contains a copy of $\mathbb{R}^2$ which

[10]A Lie group G is said to be solvable if it is connected and its Lie algebra is solvable. A Lie algebra $\mathfrak{g}$ is solvable when its Lie algebra commutator series, or derived series, $\mathfrak{g}^k$ vanishes for some k. The commutator series of a Lie algebra $\mathfrak{g}$ is the sequence of subalgebras recursively defined by

$$\mathfrak{g}^{k+1} := [\mathfrak{g}^k,\mathfrak{g}^k],$$

with $\mathfrak{g}^0=\mathfrak{g}$. The notation $[\mathfrak{a},\mathfrak{b}]$ means the linear span of elements of the form $[A,B]$, when $A\in\mathfrak{a}$ and $B\in\mathfrak{b}$. Since nilpotent Lie algebras are also solvable, any nilpotent Lie group is a solvable Lie group.

[11]It is a $1+1$ dimensional spacetime provided with the flat metric dt^2-dx^2.

[12]Let G and H two Lie groups and consider a homomorphism from G to the abstract group of automorphisms of H, that is, $\rho : G \longrightarrow Aut(H)$. The semidirect product $H\times_\rho G$ of H and G with respect to ρ is the product manifold $H\times G$ endowed with the Lie group estructure given by

$$(h,g)\cdot(h',g') = (h\rho(g)h', gg') \quad (h,g)^{-1} = (\rho(g^{-1})h^{-1}, g^{-1}) \quad \forall h,h'\in M \text{ and } g,g'\in G.$$

is a normal subgroup and the quotient group is $\mathbb{R}$. The group is therefore a semidirect product of $\mathbb{R}^2$ with $\mathbb{R}$ and determined by the action of the one-parameter group generated by e_3 on $\mathbb{R}^2$. The derivative of this action at the identity is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By rotating e_1, e_2 45º we can make this action diagonal, with derivative at $t = 0$ equal to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore, the transformations in this basis are of the form

$$t \longmapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

The metric (2.10) is preserved by the group G of transformations of X of the form

$$(x, y, z) \mapsto (\varepsilon e^{-c}x + a, \varepsilon' e^c y + b, z + c) \text{ or } (\varepsilon e^{-c}y + a, \varepsilon' e^c x + b, z + c),$$

where $a, b, c \in \mathbb{R}$ and $\varepsilon, \varepsilon' = \pm 1$.

The full isometry group of Sol has eight components and the identity component is Sol acting on itself by left multiplication. The stabilizer of the origin is isomorphic to the diedral group of order eight and consists of the linear maps of $\mathbb{R}^3$ given by $(x, y, z) \mapsto (\pm x, \pm y, z)$ and $(x, y, z) \mapsto (\pm y, \pm x, -z)$.

The surfaces $z =$ constant form a 2-dimensional foliation of Sol which is preserved by $Isom(Sol)$. The induced metric on the plane $z =$ constant of $\mathbb{R}^3$ makes this plane isometric to E^2. Thus any 3-manifold M with a geometric structure modelled on Sol inherits a natural 2-dimensional foliation and the leaves must be planes, annuli, Moebius bands, tori or Klein bottles.

3. THE THURSTON THEOREM ON THE 8 GEOMETRIC STRUCTURES

3.1. Preliminares

We want to obtain all simply connected 3-dimensional homogeneous Riemannian manifolds having compact quotient by a discrete subgroup of isometries. This property leads to the following definitions and properties:

Definition 3.1 *A Lie group G is called unimodular if it has a bi-invariant volume form (with more precision, if the left invariant Haar measure is also right invariant).*

Proposition 3.1 *If a Lie group G has a compact quotient by a discrete subgroup Γ, then it is unimodular.*

Idea of the proof. . Let D be a fundamental domain of G under the action of Γ, that is, a compact set $D \subset G$ satisfying that $\cup_{\gamma\in\Gamma}\gamma D = G$ and $\mu(\gamma D \cap \gamma' D) = 0$ if $\gamma \neq \gamma'$, where μ is the left invariant Haar measure. If E is another fundamental domain and μ is left invariant, then $\mu(E) = \sum_{\gamma\in\Gamma}\mu(\gamma D \cap E) = \sum_{\gamma\in\Gamma}\mu(D\cap\gamma^{-1}E) = \mu(D)$. But, if D is a fundamental domain and $g \in G$, then Dg is also a fundamental domain. From the above equality we have that $\mu(D) = \mu(Dg)$, then μ is also *right invariant*. From this, by a "standard" argument it follows the right invariance of the volume form ω associated to the measure μ. □

Proposition 3.2 *A connected n-dimensional Lie group G is unimodular if and only if the linear transformation* ad$X$ *of its associated Lie algebra $\mathfrak{g}$ has trace zero for every $X \in \mathfrak{g}$.*

Proof. Each $g \in G$ determines a G-automorphism $\alpha_g : h \mapsto ghg^{-1}$. The differential of this map at $e \in G$ determines an automorphism $\mathrm{Ad}g := \alpha_{g*e}$ of its Lie algebra $\mathfrak{g}$, that is, a representation $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$; $g \mapsto \mathrm{Ad}g$.

The group G is unimodular if and only if the volume form ω is bi-invariant, $\alpha_g^*\omega_{ghg^{-1}} = \omega_h$. If $e_1, ..., e_n$ is a basis of $\mathfrak{g}$, we have $(\alpha_g^*\omega_e)(e_1, ..., e_n) = \omega_e(\mathrm{Ad}g\; e_1, ..., \mathrm{Ad}g\; e_n) = \det(\mathrm{Ad}g)\omega_e(e_1, ..., e_n)$, then, for every $g \in G$, $\omega_e = \alpha_g^*\omega_e$ if and only if $\det\mathrm{Ad}g = 1$. But, since ω is left-invariant (even when G is not unimodular), L_h commutes with R_g and $\alpha_g = L_g \circ R_{g^{-1}}$, $\omega_h = \alpha_g^*\omega_{ghg^{-1}}$ iff $\omega_h = R_{g^{-1}}^*L_g^*\omega_{ghg^{-1}}$ iff $\omega_h = R_{g^{-1}}^*\omega_{hg^{-1}}$ iff $L_{*h}\omega_e = R_{g^{-1}}^*L_h^*\omega_{g^{-1}}$ iff $\omega_e = R_{g^{-1}}^*\omega_{g^{-1}}$ iff $\omega_e = R_{g^{-1}}^*L_g^*\omega_e = \alpha_g^*\omega_e$. Then G is unimodular if and only if $\det\mathrm{Ad}g = 1$ for every $g \in G$.

The morphism $\mathrm{ad} : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$ is defined by $\mathrm{ad}_v = \mathrm{Ad}\exp v_{*0}$ and it satisfies $\mathrm{ad}_v(u) = [v, u]$. Moreover, the following commutation rule holds: $\mathrm{Ad}\exp v = e^{\mathrm{ad}_v}$, where the second is the classical exponential of an element of $gl(\mathfrak{g})$ defined by $e^\ell = \sum \frac{\ell^n}{n!}$ which has the property that $\det(e^\ell) = e^{\mathrm{tr}\ell}$. From this property we have that, for every $v \in \mathfrak{g}$, the property $1 = \det\mathrm{Ad}\exp v = \det\, e^{\mathrm{ad}_v} = e^{\mathrm{tr}\,(\mathrm{ad}_v)}$ is equivalent to $\mathrm{tr}\,(\mathrm{ad}_v) = 0$, and this finishes the proof of the proposition. □

If G is 3-dimensional with the canonical left invariant metric $\langle,\rangle$ and left invariant volume form, we have on its Lie algebra $\mathfrak{g}$ a cross vector product "$\times$" defined by the orientation ω and the scalar product $\langle,\rangle$. Then we define a linear map $L : \mathfrak{g} \longrightarrow \mathfrak{g}$ by the equation $L(u \times v) = [u, v]$. Then

Proposition 3.3 *G is unimodular if and only if L is selfadjoint.*

Proof. Let e_1, e_2, e_3 be an oriented orthonormal basis of $\mathfrak{g}$. We have

$$\begin{aligned}
\mathrm{tr}\,(\mathrm{ad}_{e_i}) &= \langle [e_i, e_1], e_1\rangle + \langle [e_i, e_2], e_2\rangle + \langle [e_i, e_3], e_3\rangle \\
&= \langle L(e_i \times e_1), e_1\rangle + \langle L(e_i \times e_2), e_2\rangle + \langle L(e_i \times e_3), e_3\rangle \\
&= \begin{cases} \langle Le_3, e_2\rangle - \langle Le_2, e_3\rangle & \text{if } i = 1 \\ -\langle Le_3, e_1\rangle + \langle Le_1, e_3\rangle & \text{if } i = 2\,, \\ \langle Le_2, e_1\rangle - \langle Le_1, e_2\rangle & \text{if } i = 3 \end{cases} \quad \text{then,}
\end{aligned}$$

$$\mathrm{tr}\,(\mathrm{ad}_{e_i}) = 0 \text{ for every } i \text{ if and only if } L \text{ is self-adjoint.} \tag{3.1}$$

But, since ad_v is $\mathbb{R}$-linear respect to v (as follows from $\mathrm{ad}_v u = [v, u]$), from (3.1) we have the proposition. ☐

In the proof of the classification Theorem, we shall use the following lemmas:

Lemma 3.3.1 *Let $\varphi : V \longrightarrow V$ be an isomorphism of a 3-dimensional euclidean vector space, and $v \in V$ such that $\varphi(v) = v$. If φ satisfies that, for every orientation preserving isometry (also called rotation) R of V having v as a fixed point there is another rotation R' with v as a fixed point satisfying $\varphi \circ R = R' \circ \varphi$, then φ is the identity over* $\mathrm{span}\{v\}$ *and the composition of an isometry and a homothecy on $v^\perp$.*

Proof. The first statement is obvious, we have only to show that $\varphi|_{v^\perp}$ is an isometry composed with a homothecy. In fact:

1) $\varphi(v^\perp) = v^\perp$, because if $\langle u, v\rangle = 0$, then, for every rotation R of V satisfying $Rv = v$, we have $\langle \varphi u, v\rangle = \langle R'\varphi u, v\rangle = \langle \varphi R u, v\rangle$. Taking R as a rotation of axis v and angle π, $Ru = -u$, then $\langle \varphi u, v\rangle = -\langle \varphi u, v\rangle$, then $\langle \varphi u, v\rangle = 0$, as claimed.

2) Let $u \in v^\perp$ with $|u| = 1$. For any other vector $w \in v^\perp$ with $|w| = 1$, there is a rotation R fixing v such that $w = Ru$. Then

$$\langle \varphi w, \varphi w\rangle = \langle \varphi R u, \varphi R u\rangle = \langle R'\varphi u, R'\varphi u\rangle = \langle \varphi u, \varphi u\rangle\,.$$

Then, if $\lambda^2 = \langle \varphi u, \varphi u\rangle$, the map $\dfrac{1}{\lambda}\varphi|_{v^\perp}$ is an isomorphism preserving the norm, then it is an isometry, and $\varphi|_{v^\perp}$ is λ times an isometry. ☐

Lemma 3.3.2 *The flow of a G-invariant vector field in a homogeneous space G/H admitting a compact quotient preserves the volume.*

Lemma 3.3.3 *The integral curves of a G-invariant unit Killing vector field*[13] *on a G-homogeneous Riemannian manifold define a* 1*-dimensional regular*[14] *foliation.*

3.2. The theorem and its proof

In this lecture we shall describe and prove the classification Theorem of the simply connected geometric structures of dimension 3. We shall follow more or less the ideas of Thurston, and, like his proof[15], the following one will be intuitive but not completely rigurous.

Theorem 3.4 (Thurston) *Any maximal simply connected* 3*-dimensional geometry which admits a compact quotient is equivalent to one of the geometries $(X, Isom(X))$, where X is one of the eight spaces described in the previous lecture: S^3, H^3, $\mathbb{R}^3$, $X = H^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $\widetilde{SL}(2, \mathbb{R})$, Nil, or Sol.*

Proof. Since G is the isometries group of X, for every $g \in G$ and every $x \in X$, g_{*x} is an isometry from T_xX to $T_{gx}X$, then, if h is in the isotropy subgroup of G at $x \in X$, h_{*x} must be an isometry of $T_xX \equiv \mathbb{R}^3$, then the map $i_x : H_x \longrightarrow O(3) \ / \ h \mapsto h_{*x}$ is an homomorphism whose image is a subgroup of $O(3)$, and the image of the connected component H_{xe} of H_x containing the identity is a subgroup of $SO(3)$. Then, for every $x \in X$, there are only three possibilities for $i_x(H_{xe})$. It must be equal to $SO(3)$, $SO(2)$ or the trivial group.

The group $i_x(H_x)$ does not depend (up to an isomorphism) on the $x \in X$, because the isotropy groups at points x and gx (for $g \in G$) are related by $H_{gx} = gH_xg^{-1}$, then, all the elements in H_{gx} are of the form ghg^{-1} for $h \in H_x$, then $i_{gx}(ghg^{-1}) = (ghg^{-1})_{*gx}$

a) If $i_x(H_{xe}) = SO(3)$, then for every $x \in X$ and every two planes $\pi, \sigma \subset T_xM$ there is an element $h \in H_{xe}$ such that $h_{*x}\pi = \sigma$. Since h is an isometry of X and an isometry preserves the sectional curvature, both planes have the same sectional curvature, then the sectional curvature is pointwise constant and, by Schur Lemma, it is constant, but it is well known that the unique simply connected spaces of constant sectional curvature are E^3, S^3 and H^3, whose isometry groups are also well known.

[13] A Killing field is a vector field X on a Riemannian manifold (M, g) such that $\mathcal{L}_X g = 0$ (where $\mathcal{L}$ denotes the Lie derivative). Its associated flow ϕ_t is a family of isometries.

[14] Roughly speaking, a foliation on a manifold is a partition of this manifold into submanifolds, called leaves. Here, a foliation is called regular if its leaves are regular submanifolds. In a regular foliation, the space of its leaves has a natural structure of differentiable manifold, called the quotient manifold of the foliation.

[15] Recall the remark above about who really proved this theorem.

b) If $i_x(H_{xe}) = SO(2)$, at each point $x \in X$, H_{xe} acts as the group of oriented rotations around an axis, and the orientation of the rotations (all are the same because $i_x(H_{xe}) = SO(2)$) allows us to select a direction of the axis, that is, a unit vector v_x. This defines a unit vector field v on M. This vector field can also be defined by $x \mapsto v_x$, where v_x is the unique unit vector at T_xX such that, for every $h \in H_x$, $h_{*x}v_x = v_x$ and $\{u, h_{*x}u, v_x\}$ is a positively oriented basis, for every u in T_xX.

This vector field v is G-invariant. In fact, we know that the isotropy groups at points x and gx (for $g \in G$) are related by $H_{gx} = gH_xg^{-1}$, then, all the elements in H_{gx} are of the form ghg^{-1} for $h \in H_x$. Hence, in order to check that $g_{*x}v_x = v_{gx}$ we have only to compute $(ghg^{-1})_{*gx}g_{*x}v_x = g_{*hx}h_{*x}v_x = g_{*x}v_x$. Then the flow ϕ_t of v commutes with the action of G, then, for every $h \in H_x$, $h\phi_t(x) = \phi_t(h(x)) = \phi_t(x)$, that is, $H_{\phi_t(x)} = H_x$.

Moreover, for every t, let $g_t \in G$ such that $g_t(\phi_t(x)) = x$. Since v is G-invariant, $(g_t \circ \phi_t)_{*x}v_x = v_x$. Moreover, if $h \in H_x$, $(g_t \circ \phi_t)_{*x} \circ h_{*x} = g_{t*\phi_t(x)} \circ h_{*\phi_t(x)} \circ \phi_{t*x} = g_{t*\phi_t(x)} \circ h_{*\phi_t(x)} \circ {g_t^{-1}}_{*x} \circ g_{t*x} \circ \phi_{t*x} = (g_thg_t^{-1})_{*x} \circ (g_t \circ \phi_t)_{*x}$, and $g_thg_t^{-1} \in g_tH_{\phi_t(x)}g_t^{-1} = H_x$. Then

$$\begin{gathered}(g_t \circ \phi_t)_{*x} \text{ is an isomorphism from } T_xX \text{ onto } T_xX \text{ satisfying that} \\ (g_t \circ \phi_t)_{*x}v_x = v_x \text{ and } (g_thg_t^{-1})_{*x} \circ (g_t \circ \phi_t)_{*x} = (g_t \circ \phi_t)_{*x}h_{*x} \\ \text{for every } h \in H_x\end{gathered} \tag{3.2}$$

From (3.2) and Lemma 3.3.1, it follows that $g_{t*\phi_t(x)} \circ \phi_{t*x}$ is the identity on v_x and a homothecy on $v_x^\perp$ for every $x \in X$. Since g_t is an isometry, it follows that ϕ_t has also this behaviour. But, from Lemma 3.3.2, ϕ_t must preserve the volume, then it must be an isometry. Then the vector field v is a Killing vector field, and it follows from Lemma 3.3.3 that, if $\mathcal{F}$ is the foliation defined by its integral curves, the quotient $X/\mathcal{F}$ is a manifold, and, since ϕ_t are isometries, we can define a metric on $X/\mathcal{F}$ such that the quotient map $\pi : X \longrightarrow X/\mathcal{F}$ be a Riemannian submersion. Moreover, since the action of G commutes with ϕ_t and the leafs of $\mathcal{F}$ are $\{\phi_t(x),\ t \in \mathbb{R}\}$, the image of a leaf of $\mathcal{F}$ by G is a leaf, then the transitive action of G on X by isometries induces a transitive action of G on $X/\mathcal{F}$ by isometries, then $X/\mathcal{F}$ has constant sectional curvature, and it is simply connected because X is, then $X/\mathcal{F}$ must be E^2, S^2 or H^2, and X is a bundle on these spaces with fiber $\mathbb{R}$ or S^1. Now, we have two possibilities:

b1) The distribution orthogonal to $\mathcal{F}$ is integrable (or equivalently, by Frobenius Theorem, $\omega \wedge d\omega = 0$, where $\omega = v^\flat$[16]. The inverse of $\flat$ will be

[16] Here $\flat$ denotes the vector bundle isomorphism $\flat : TX \longrightarrow T^*X;\ T_xX \ni v \mapsto v^\flat : T_xM \to \mathbb{R}; v^\flat(u) = \langle v, u\rangle$

denoted by $\sharp$). Then the Riemannian submersion is locally a Riemannian product, and, since $X/\mathcal{F}$ is simply connected, it is a product. In this case, since X is simply connected, the fiber can be only $\mathbb{R}$. Then $X = S^2 \times \mathbb{R}$, or $X = H^2 \times \mathbb{R}$ or $X = E^2 \times \mathbb{R} = E^3$. The last case (which repeats a space obtained in case a)) is ruled out because in this case $i_x(H_x)$ is not the largest group of isometries, then also G is not. (We understand now why the hypothesis 3 in Definition 1.7 is to avoid redundancy).

b2) The distribution orthogonal to $\mathcal{F}$ is not integrable (or equivalently, $\omega \wedge d\omega \neq 0$). This is what we called a contact structure on X in lecture 2. It is known that contact structures having left invariant dual on a simply connected 3-manifold fibering by lines $\mathbb{R}$ or circles S^1 on S^2, H^2 or E^2, and with the 1-form ω the dual of the unit vector tangent to the fibers are:

- If the basis is S^2, the only possibility is the Hopf fibration $\pi : S^3 \longrightarrow S^2$, which again is ruled out because in this case the group G of isometries of S^3 is not maximal.

-If the basis is E^2, the only possibility is the Nil geometry, and $\omega = dz - x\ dy$ is the 1-form $v^\flat$.

-If the basis is H^2, then $X = \widetilde{SL}(2, \mathbb{R})$

c) If $i_x(H_{xe})$ is the trivial group, then $X = G'/\{e\}$, where G' is the identity component of G, and G' acts on G' as a group of isometries, then X is a 3-dimensional group G' with left invariant metric, and having a compact quotient by a discrete subgroup. Then G' is unimodular, and the map L defined in the preliminares of this lecture is selfadjoint. Let λ_1, λ_2, λ_3 be the eigenvalues of L, according to the computations of Proposition 3.3 we have

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3,$$

and there is the following classification of Lie groups of dimension 3 associated to the Lie algebras, with the above multiplication rule, according to the signs of λ_1, λ_2 and λ_3,

Signs of λ_1, λ_2, λ_3	Associated Lie group	type of group
$+ + +$	$SU(2)$ or $O(1,2)$	compact and simple
$+ + -$	$SL(2, \mathbb{R})$ or $SO(3)$	non compact and simple
$+ + 0$	$E(2)$	solvable
$+ - 0$	$E(1,1) = Sol$	solvable
$+ 0\ 0$	Nil	nilpotent
$0\ 0\ 0$	$\mathbb{R}^3$	commutative

The unique case where G is the maximal group of isometries is Sol. Then, Sol, with a left invariant metric, is the unique geometric structure in this case. $\square$

3.3. On the uniqueness of the geometric structure of a topological three-manifold

Note that in six of the eight three dimensional geometries, all but S^3 and $S^2 \times \mathbb{R}$, the total space X is homeomorphic to $\mathbb{R}^3$. So it is not obvious that such geometries are all different. Moreover, in the case of $H^2 \times \mathbb{R}$, $\widetilde{SL}(2, \mathbb{R})$ and Nil it is not trivial that they are maximal. As a consequence of this, the following result implies at once that none of the eight geometries is a subgeometry of any other. Hence all eight are maximal and no two are equivalent.

Theorem 3.5 *If M is a closed 3-manifold which admits a geometric structure modelled on one of the eight geometries, then the geometry involved is unique.*

Next we state two theorems which classify those closed 3-manifolds admitting a non-hyperbolic geometric structure.

Theorem 3.6 *Let M be a closed 3-manifold. M admits a geometric structure modelled on one of S^3, E^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL}(2, \mathbb{R})$ or Nil if and only if M is a Seifert fibre space.*

Remark 3.6.1 *It is possible to classify the 3-manifolds which admit one of the six geometries above in two groups, corresponding to whether the S^1 bundle is trivial or not.*

- *3-manifolds with product geometries $H^2 \times \mathbb{R}$, E^3 or $S^2 \times \mathbb{R}$ are, up to finite covers, trivial circle bundles over oriented surfaces of genus g, where $g \geq 2$, $g = 1$, and $g = 0$, respectively.*
- *3-manifolds with the twisted product geometries $\widetilde{SL}(2, \mathbb{R})$, Nil, and S^3 are, up to finite covers, nontrivial circle bundles over surfaces of genus $g \geq 2$, $g = 1$, and $g = 0$, respectively.*

Theorem 3.7 *Let M be a closed 3-manifold. M is a geometric manifold modelled on Sol if and only if M is finitely covered by a torus bundle over S^1 with holonomy[17] given by a hyperbolic automorphism of T^2, that is, an element of $SL(2, \mathbb{Z})$ with distinct real eigenvalues.*

[17] Let (M, g) be a Riemannian manifold and fix a base point $p \in M$. Let us denote by $P(c)$ the parallel translation along a C^∞ loop c based at p. The holonomy group of

On the other hand, note that manifolds modelled on Sol are graph manifolds, that is, they may be split by incompresssible tori into a union of Seifert fibered spaces. Thus, seven of the eight geometric 3-manifolds are topologically S^1-fibered over surfaces or T^2-fibered over S^1. Since most 3-manifolds do not admit such fibrations, the hyperbolic geometry is by far the most interesting one.

4. SOME ELEMENTARY FACTS ABOUT HEAT EQUATION

In this chapter we shall review, very quickly, the essential facts of heat equation on $\mathbb{R}^n$ and on Riemannian manifolds[18]. The motivation for this review is that heat eqution is the model of diffusion equations, and the Ricci flow is a diffusion equation. There are two main facts which seem to be in the mind of R. Hamilton when he invented Ricci flow to prove Poincaré Conjecture. The first one is that heat flow smoothes functions, you may begin with a non smooth temperature distribution and, after heat flows, the temperature function becomes smooth. The second is that heat flow tends to the homogenization of the temperature: as time goes on, the temperature goes more similar in all the points of the space where heat flows. Technically this is given by the last theorem that we recall in this lecture: when time goes to infinity, then the solution of heat equation tends to be harmonic.

4.1. The heat equation in the euclidean space

The heat equation in the euclidean space $\mathbb{R}^n$ is

$$\Delta u + \frac{\partial u}{\partial t} = 0, \text{ where } \Delta := -\sum_{i=1}^{n} \frac{\partial^2}{\partial (x^i)^2} \tag{4.1}$$

It is a parabolic equation. It is not preserved under the change $t \mapsto -t$ (it is not time reversible), and it is preserved under the transformation $x \mapsto a\ x,\quad t \mapsto a^2\ t$, which leaves $\dfrac{|x|^2}{t}$ invariant. Then, it is not surprising that $\dfrac{|x|^2}{t}$ appears frequently in connection with the heat equation.

(M, g) at p is defined as the following subgroup of the orthogonal transformation group $O(T_pM)$:

$$H(p) := \{P(c) : c \text{ is a piecewise } C^\infty \text{ loop based at } p\}.$$

For more details, see [76] p.121.

[18][12] is a good reference for more details on the heat equation.

Now, let us give some ideas about the solution of (4.1). For $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, let us try a solution of (4.1) of the form

$$u(x,t) = e^{i(\lambda t + \langle x, \xi \rangle)}. \tag{4.2}$$

By substitution in (4.1), we obtain that (4.2) is a solution of heat equation when $i\lambda = -|\xi|^2$, i.e., $e^{i\langle x,\xi\rangle - |\xi|^2 t}$ is a solution of (4.1).

All the solutions of the heat equation can be obtained from the so-called *fundamental solution* or *the heat kernel*, which is a function $K : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-|x-y|^2/4t} \tag{4.3}$$

and satisfies the properties:

i) $\left(\Delta + \dfrac{\partial}{\partial t}\right) K(x,y,t) = 0$ for $t > 0$,

ii) $K(x,y,t) > 0$ for $t > 0$

iii) $\displaystyle\int_{\mathbb{R}^n} K(x,y,t)dy = 1$ for $x \in \mathbb{R}^n$, $t > 0$,

iv) For any $\delta > 0$, we have $\displaystyle\lim_{t\to 0^+} \int_{|y-x|>\delta} K(x,y,t)\ dy = 0$ uniformly for $x \in \mathbb{R}^n$.

v) For any C^∞ function f on $\mathbb{R}^n$, we have $\displaystyle\lim_{t\to 0^+} \int_{\mathbb{R}^n} K(x,y,t) f(y)\ dy = f(x)$.

Properties ii), iv) and v) are expressed saying that $\lim_{t\to 0^+} K(x,y,t) = \delta_x$, the delta of Dirac at x.

The "pure initial value problem" for the heat equation consists on finding a solution $u(x,t)$ of (4.1) for $x \in \mathbb{R}^n$, $t > 0$ and satisfying $u(x,0) = f(x)$, where we require u to be C^∞ on $\mathbb{R}^n \times]0,\infty[$ and C^0 on $\mathbb{R}^n \times [0,\infty[$. For this problem, we have the following existence and uniqueness theorem

Theorem 4.1 *Let f be a function on $\mathbb{R}^n$ continuous and bounded. If $u(x,t)$ is a real function continuous on $\mathbb{R}^n \times [0,T[$, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x^i \partial x^j}$ exist and are continuous on $\mathbb{R}^n \times]0,T[$, and $u(x,t) \geq 0$ on $\mathbb{R}^n \times]0,T[$, then the function*

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t) f(y) dy \tag{4.4}$$

is the unique solution of (4.1) satisfying the condition $u(x,0) = f(x)$. Moreover it is analytic on $\mathbb{R}^n \times]0,T[$.

The proof of the theorem follows from the basic properties of $K(x,y,t)$ stated above.

The following maximum principle is extremely useful to prove uniqueness and regularity theorems.

Theorem 4.2 *Let ω be an open bounded set of $\mathbb{R}^n$, $T \in \mathbb{R}^+$. Let Ω be the cylinder $\Omega = \{(x,t);\ x \in \omega,\ 0 < t < T\}$. Its boundary is the union of the next two disjoint portions: $\partial'\Omega = \partial\omega \times [0,T] \cup \omega \times \{0\}$ and $\partial''\Omega = \omega \times \{T\}$. If $u(x,t)$ is a real function continuous on $\overline{\Omega}$, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x^i \partial x^j}$ exist and are continuous on Ω, and they satisfy*

$$\frac{\partial u}{\partial t} + \Delta u \leq 0, \tag{4.5}$$

then

$$\max\{u(x,t);\ (x,t) \in \overline{\Omega}\} = \max\{u(x,t);\ (x,t) \in \partial'\Omega\} \tag{4.6}$$

Proof. First, let us suppose that $\dfrac{\partial u}{\partial t} + \Delta u < 0$. For $0 < \epsilon < T$, let us denote $\Omega_\epsilon = \{(x,t);\ x \in \omega,\ 0 < t < T - \epsilon\}$. Since $u \in C^0(\overline{\Omega_\epsilon})$, there is a point $(x',t') \in \overline{\Omega_\epsilon}$ where $u(x',t') = \max\{u(x,t);\ (x,t) \in \overline{\Omega_\epsilon}\}$.

If $(x',t') \in \Omega_\epsilon$, then $\dfrac{\partial u}{\partial t}(x',t') = 0$ and $\Delta u(x',t') \geq 0$, which is a contradiction with the hypothesis $\dfrac{\partial u}{\partial t} + \Delta u < 0$.

If $(x',t') \in \partial''\Omega_\epsilon$, then $\dfrac{\partial u}{\partial t}(x',t') \geq 0$ and $\Delta u(x',t') \geq 0$, which again is a contradiction with the hypothesis $\dfrac{\partial u}{\partial t} + \Delta u < 0$.

Then

$$\max\{u(x,t);\ (x,t) \in \overline{\Omega_\epsilon}\} = \max\{u(x,t);\ (x,t) \in \partial'\Omega_\epsilon\} \leq \max\{u(x,t);\ (x,t) \in \partial'\Omega\}.$$

Since every point of $\overline{\Omega}$ with $t < T$ belongs to some $\overline{\Omega}_\epsilon$, and u is continuous on $\overline{\Omega}$, (4.6) follows.

Now, let us consider the general case $\dfrac{\partial u}{\partial t} + \Delta u \leq 0$, let us introduce the function $v(x,t) = u(x,t) - kt$, where k is a positive constant. Then $\dfrac{\partial v}{\partial t} + \Delta v = \dfrac{\partial u}{\partial t} + \Delta u - k < 0$, and we can apply the above proved maximum principle to the function v and we obtain $\max\{u(x,t);\ (x,t) \in \overline{\Omega}\} = \max\{v(x,t) + kt;\ (x,t) \in \overline{\Omega}\} \leq \max\{v(x,t);\ (x,t) \in \overline{\Omega}\} + kT \leq \max\{v(x,t);\ (x,t) \in \partial'\Omega\} + kT \leq \max\{u(x,t);\ (x,t) \in \partial'\Omega\} + kT$, and we obtain (4.6) by letting $k \to 0$. □

This maximum principle immediately yields an uniqueness theorem:

Theorem 4.3 *If $u(x,t)$ is a real function continuous on $\overline{\Omega}$, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x^i \partial x^j}$ exist and are continuous on Ω, then $u(x,t)$ is determined uniquely in $\overline{\Omega}$ by the values of $\dfrac{\partial u}{\partial t} + \Delta u$ in Ω and of u on $\partial'\Omega$.*

Proof. Let u, v two functions with the same value on $\partial'\Omega$ and the same value of $\dfrac{\partial u}{\partial t} + \Delta u$ and $\dfrac{\partial v}{\partial t} + \Delta v$ on Ω. The uniqueness theorem follows from applying Theorem 4.2 to $u - v$. □

For the heat equation on a domain of $\mathbb{R}^n$, we have the following analogous of Theorem 4.1:

Theorem 4.4 *The solution of the heat equation in a domain ω of $\mathbb{R}^n$ is*

$$u(\xi, T) = \int_\omega K(x, \xi, T) u(x, 0)\ dx \\ + \int_0^T \int_{\partial\omega} \left(K(x, \xi, T - t) \frac{\partial u}{\partial \vec{n}}(x, t) - u(x, t) \frac{\partial K}{\partial \vec{n}}(x, \xi, T - t) \right)\ dx\ dt.$$

Many regularity properties of a solution of the heat equation in a bounded region follow from this formula.

In general, initial data alone do not determine the solution of the heat equation uniquely. Some additional information on u is required. In the above theorems we needed to know u on $\partial\omega \times [0, T]$. In the Theorem 4.1 less conditions are needed.

4.2. The heat equation in a Riemannian manifold

4.2.1. Some previous definitions

For a C^k real function f on a Riemannian manifold M ($k \geq 1$), the gradient of f is defined by $\langle \mathrm{grad} f, \xi \rangle = \xi(f)$ for every $\xi \in TM$. In other words, $\mathrm{grad} f = (df)^\sharp$.

For every vector field X on M, its divergence $\mathrm{div} X$ is defined by

$$\mathrm{div} X = -\mathrm{tr}(\xi \mapsto \nabla_\xi X) :$$

The laplacian of a C^k-function $f : M \longrightarrow \mathbb{R}$ ($k \geq 2$) is defined by

$$\Delta f = \mathrm{div} \mathrm{grad} f.$$

Another usual definition for the laplacian is $\Delta f = -\mathrm{tr}\nabla^2 f$, where $\nabla^2 f$ is a symmetric tensor field of type $(0,2)$, called Hessian of f or second covariant derivative of f, defined by

$$\nabla^2 f(X,Y) := \langle \nabla_X(\mathrm{grad} f), Y\rangle = XYf - (\nabla_X Y)f$$

In other words, $\nabla^2 f = \nabla(df)$. In fact,

$$\nabla(df)(X,Y) = Xdf(Y) - df(\nabla_X Y) = XYf - (\nabla_X Y)f = \nabla^2 f(X,Y)$$

Locally, we have $\nabla^2_{ij} f(\partial_i,\partial_j) = \partial_i\partial_j f - (\nabla_{\partial_i}\partial_j)f = \nabla_i\partial_j f$.

Next we show that the two definitions given for the laplacian are equivalent. By definition of divergence, we have $\Delta f = \mathrm{div}\mathrm{grad} f = -\mathrm{tr}(X \mapsto \nabla_X \mathrm{grad} f)$. Then it is sufficient to prove that

$$\underbrace{\mathrm{tr}(X \mapsto \nabla_X \mathrm{grad} f)}_{(1)} = \mathrm{tr}\nabla^2 f = \underbrace{\mathrm{tr}\nabla df}_{(2)}.$$

Taking an orthonormal basis and doing computations, we get

$$(1) = \sum_i \langle \nabla_{e_i}\mathrm{grad} f, e_i\rangle = \sum_i \left(e_i\langle \mathrm{grad} f, e_i\rangle - \langle \mathrm{grad} f, \nabla_{e_i} e_i\rangle\right)$$

$$(2) = (\nabla_{e_i} df)(e_i) = e_i df(e_i) - df(\nabla_{e_i} e_i)$$

And the equality holds using $\mathrm{grad} f = (df)^\sharp$.

In local coordinates, we have the following expressions:

$$\begin{aligned}
\mathrm{grad} f &= g^{kj}\partial_j f\ \partial_k,\\
\mathrm{div} X &= -\frac{1}{\sqrt{g}}(X^j\sqrt{g}),\\
\Delta f &= -\frac{1}{\sqrt{g}}\partial_j(\sqrt{g}\ g^{jk}\partial_k f),
\end{aligned}$$

and, in geodesic normal coordinates around $p \in M$,

$$\Delta f(p) = -\sum_i \frac{\partial^2 f}{\partial(x^i)^2}(p).$$

In fact, $\mathrm{grad} f = (df)^\sharp = g^{jk}(df)_j\partial_k = g^{jk}\partial_j f\partial_k$. On the other hand,

$$\begin{aligned}
\mathrm{div} X &= -g^{ij}\langle \nabla_{\partial_i} X, \partial_j\rangle = -g^{ij}\left\langle \nabla_{\partial_i}(X^k\partial_k), \partial_j\right\rangle\\
&= -g^{ij}\left\langle (\nabla_{\partial_i} X^k)\partial_k, \partial_j\right\rangle - g^{ij}\left\langle X^k(\nabla_{\partial_i}\partial_k), \partial_j\right\rangle =\\
&= -g^{ij}g_{kj}\partial_i X^k - g^{ij}X^k\Gamma^l_{ik}g_{lj} = -\partial_k X^k - \Gamma^i_{ik}X^k = (\nabla_{\partial_k} X)^k
\end{aligned}$$

This is also an often used local expression for the divergence. We continue doing computations[19]

$$\begin{aligned}\mathrm{div}X &= -\partial_k X^k - \Gamma^i_{ik} X^k = -\partial_k X^k - \frac{1}{2} g^{il} \left(\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}\right) X^k \\ &= -\partial_k X^k - \frac{1}{2} g^{il} \partial_k g_{il} X^k = -\partial_k X^k - \frac{\partial_k(\sqrt{g})}{\sqrt{g}} X^k = -\frac{1}{\sqrt{g}} \partial_k (X^k \sqrt{g})\end{aligned}$$

In fact,

$$\frac{\partial_k(\sqrt{g})}{\sqrt{g}} = \frac{\partial_k(\sqrt{\det(g_{ij})}}{\sqrt{\det(g_{ij})}} = \frac{1}{\sqrt{\det(g_{ij})}} \frac{1}{2} g^{ij} \partial_k g_{ij} \sqrt{\det(g_{ij}} = \frac{1}{2} g^{ij} \partial_k g_{jk}$$

Using the obtained expression for the divergence, it is straightforward to compute an expression in local coordinates for the laplacian.

$$\Delta f = \mathrm{div}(\mathrm{grad} f) = -\frac{1}{\sqrt{g}} \partial_j \left((\mathrm{grad} f)^j \sqrt{g} \right) = -\frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g}\; g^{jk}\; \partial_k f \right) \quad (4.7)$$

4.2.2. The heat equation

The heat equation on a riemannian manifold is defined by the same expression (4.1) that in the euclidean case. Here is also a heat kernel or fundamental solution of the heat equation. It plays the same role that in the euclidean case, but has not an expression so simple than with zero curvature. However, we have the following nice relations with the spectrum of δ (which, of course, also hold in the similar cases -when there exist- in the euclidean space).

If M is a closed (compact with no boundary) Riemannian manifold, the eigenvalue problem $\Delta F = \lambda f$ has solution given by a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 < .... \uparrow +\infty$$

such that the eigenspace corresponding to each eigenvalue is finite dimensional. If we write the sequence of eigenvalues with repetition (each one as many times as the dimension of its eigenspace) $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ... \uparrow +\infty$, and ϕ_j denotes an eigenfunction with eigenvalue λ_j satisfying

[19]Note that, in the calculations, we use the general formula $\dfrac{\partial}{\partial t}\det g = g^{ij} \dfrac{\partial g_{ij}}{\partial t} \det g$.

$\int_M \phi_j^2(x)dx = 1$, then

$$p(x, y, t) = \sum_j e^{-\lambda_j t}\phi_j(x)\phi_j(y)$$

is a fundamental solution of the heat equation on M, and a solution of it satisfying $u(x, 0) = f(x)$ is given by

$$u(x, t) = \int_M p(x, y, t)f(y)dy.$$

Analogous facts are true for compact manifolds with boundary if we add Neumann or Dirichlet conditions on the boundary.

The following result is also interesting:

Proposition 4.5 *Let $u(x, t)$ be a solution of the heat equation on M. Then $\int_M u(x, t)dx$ is a constant function of t, and $\int_M u^2(x, t)dx$ is a decreasing function of t.*

In general (even when M is non compact), a fundamental solution of the heat equation is a function $p(x, y, t) : M \times M \times]0, \infty[\longrightarrow \mathbb{R}$ which is C^2 with respect to x, y, C^1 with respect to t and which satisfies

i) $\left(\Delta_x + \frac{\partial}{\partial t}\right) p(x, y, t) = 0$ for $t > 0$, (where Δ_x means the Laplacian respect to the variable x),

ii) $\lim_{t\to 0^+} p(\ , y, t) = \delta_y$, whre δ_y is the Dirac delta function.

For all bounded continuous function f on M, the solution of the heat equation on M satisfying $u(x, 0) = f(x)$ is given by

$$u(x, t) = \int_M p(x, y, t)f(y)dy.$$

For the inhomogeneous heat equation $\Delta u + \frac{\partial u}{\partial t} = F$, the corresponding solution is given by

$$u(x, t) = \int_M p(x, y, t)\ f(y)dy + \int_0^t \int_M p(x, y, t - \tau)F(y, \tau)\ dy\ dt.$$

The following result is specially interesting in relation with the ideas that Heat flow can give about Ricci flow.

Theorem 4.6 *For any $f \in L^2(M)$, the function $W_t(x)$ defined by*

$$W_t(x) = \int_M p(x, y, t) f(y) dy$$

converges uniformly, as $t \to +\infty$, to a harmonic function on M.

When M is compact, this limit is constant, because in a closed manifold, all the harmonic functions are constant.

5. INTRODUCTION TO HARMONIC MAPS AND RICCI FLOW

5.1. Harmonic maps

Definition 5.1 *Given a map $f : (M, g) \longrightarrow (M', g')$ between two Riemannian manifolds, the energy density of f is defined by $e(f) = \langle g, f^* g' \rangle$, where $\langle , \rangle$ is the product in the bundle $T^{(0,2)}M$ of 2-covariant tensors on M induced by g.*

If M is compact, the energy $E(f)$ of f is defined by the integral $E(f) := \int_M e(f)(x)\ dx$, where dx is the volume element of M.

An harmonic map $f : M \longrightarrow M'$ is a map f which is a critical point for the functional E defined on $C^\infty(M, M')$.

If $\{e_i\}_{i=1}^n$ is a local orthonormal frame of (M, g), the energy density has the expression

$$e(f) = \sum_{i,j=1}^{n} \delta_{ij} g'(f_* e_i, f_* e_j) = \sum_i g'(f_* e_i, f_* e_i), \tag{5.1}$$

and, in a coordinate system $(x^1, ..., x^n)$ for M and $(y^1, ..., y^m)$ for M',

$$e(f) = \sum_{i,j=1}^{n} \sum_{a,b=1}^{m} g^{ij} g'_{ab} \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j}. \tag{5.2}$$

Remark 5.1.1 *From the above definition it follows that also $e(f) = ||df||^2$, where $|| \ ||$ is the norm in $TM \otimes f^* TM'$ induced by the metrics g and g'.*

When $M = \mathbb{R}^2$ and $M' = \mathbb{R}$, then $e(f) = \left(\frac{\partial f}{\partial x^1} \right)^2 + \left(\frac{\partial f}{\partial x^2} \right)^2$ and $E(f) = \int_{\mathbb{R}^2} \left(\frac{\partial f}{\partial x^1} \right)^2 + \left(\frac{\partial f}{\partial x^2} \right)^2 dx^1\ dx^2$, which is the well known Dirichlet energy.

We can obtain a more practical definition of harmonic map by writing the Euler Lagrange equations corresponding to the functional $E(f)$, that is, by obtaining the differential equations which characterize the critical points f of the functional E. In order to do it, we consider a curve $F(t,x)$ in $C^\infty(M,M')$ satisfying $F(0,x) = f(x)$ for every $x \in M$. Now, let us compute. We use the notation $F_t(x) = F(t,x)$

$$\begin{aligned}\frac{dE(F(t,x))}{dt}(0) &= \int_M \frac{\partial e(F(t,x))}{\partial t}(0)\; dx = \int_M \sum_i \frac{\partial}{\partial t}|_{t=0} g'(F_{t*}e_i, F_{t*}e_i)\; dx \\ &= 2\int_M \sum_i g'(\nabla'_{f_*(e_i)}X, f_*(e_i))\; dx = 2\int_M g'(X,\tau(f))\; dx,\end{aligned}$$

where X is the vector field on M' along f defined by $X_{f(x)} = \dfrac{\partial F_(t,x)}{\partial t}|_{t=0}$ and $\tau(f)$ is the tension of f, which is defined as the vector field on M' along f satisfying

$$\tau(f) = -\sum_i \nabla'_{f_*(e_i)} f_*(e_i) + f_* \nabla_{e_i} e_i,$$

that is, $\tau(f)$ is the trace of the tensor $-\overline{\nabla}\; f_* \in T^*M \otimes TM'$, where $\overline{\nabla}$ is the covariant derivative operator on $T^*M \otimes TM$ induced by the covariant derivatives ∇ and ∇' (that is $(\overline{\nabla}_Z L)(Y) = \nabla'_{LZ}(LY) - L\nabla_Z Y)$.

From this formula and the definition of harmonic map, it follows that a map $f : M \longrightarrow M'$, where M is compact, is harmonic iff $e(f) = 0$. We extend this property as the definition of harmonic maps when M is not compact.

One has the following nice property of regularity of harmonic maps:

Theorem 5.1 *Every C^2 map $f : M \longrightarrow M'$ satisfying $e(f) = 0$ is C^∞. Moreover, if M and M' are both analytic, then f is also analytic.*

Harmonic maps include a big range of interesting maps that we all know. For instance,

i) If $\dim(M) = 1$ (that is, $M = \mathbb{R}$ or $M = S^1$, f is harmonic if it is a well parametrized geodesic.

ii) If $M' = \mathbb{R}$, f is harmonic if it is an harmonic map.

iii) If $f : M \longrightarrow M'$ is an isometric immersion and f is harmonic, then it is minimal.

iv) If $f : M \longrightarrow M'$ is a riemannian submersion and f is harmonic, then the fibers $f^{-1}(y)$ are minimal.

As usual, when we have a definition, there arise an immediate problem: the existence of the defined object. Eells and Sampson proved that, with some hypotheses on M', it is possible to deform any map into a harmonic map. With more precision,

Theorem 5.2 ([32]) *Suppose that M is a closed Riemannian manifold, and that M' is a Riemannian manifold with non-positive sectional curvature. Then, for any $u_0 \in C^\infty(M, M')$ there is a unique, global and smooth solution $u : M \times [0, \infty] \longrightarrow M'$ of the equation*

$$\frac{\partial f}{\partial t} + \tau(f) = 0; \qquad u(x, 0) = u_0(x),$$

such that, when $t \to \infty$ suitably, converges smoothly to a harmonic map $u_\infty \in C^\infty(M, M')$ homotopic to u_0.

The idea of writing a diffusion equation related to the tension and obtaining harmonic maps as the limits when $t \to \infty$ of the solutions of the diffusion equation, and proving the existence and uniqueness of the solutions by using maximum principles, all like in heat equation and harmonic functions, was taken by R. Hamilton for different purposes, as we shall explain below.

5.2. Ricci flow

Let (M, g) be a Riemannian manifold. We shall denote by R_{ij} its Ricci curvature, by $R = \sum_{i,j} g^{ij} R_{ij}$ its scalar curvature and by $d\mu$ the Riemannian volume element of (M, g). By Λ we shall denote a constant. We consider, on the family of metrics on M, the functional $SH(g) = \int_M (R - 2\Lambda)\, dx$, called the Hilbert functional. The Euler Lagrange equations for this functional are

$$-2R_{ij} + \frac{2}{n} R g_{ij} = 0,$$

i.e., the critical points of $SH(g)$ are Einstein metrics.

Then, following Eells and Sampson method on harmonic maps, the natural way to obtain a deformation of a metric to an Einstein one is to consider the diffusion equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \frac{2}{n} R g_{ij},$$

but this equation has no solution, even for a short time, because it implies a backward equation in R. Then R. Hamilton had the idea of introducing the equations system

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{5.3}$$

which is called the *Ricci flow*. It is easier to understand that (5.3) is similar to the heat equation if we write the Taylor expansion of the metric tensor in normal coordinates around $p \in M$, it is given (cf. [35] p. 193) by

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{iajb}(p)x^a x^b + O(|x|^3),$$

then, at the point p, (that is, up to higher order terms)

$$\Delta g_{ij} = -g^{ab}\frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} = \frac{1}{3}R_{ij},$$

and (5.3) can be approximated by $\dfrac{\partial g_{ij}}{\partial t} = -6\Delta g_{ij}$, which looks like the heat equation.

Let us look at the solution of (5.3) in some simple cases:

If the initial metric g is a flat metric, the unnormalized Ricci flow becomes $\dfrac{\partial g_{ij}}{\partial t} = 0$, whose solution is $g = \mathit{constant}$, the metric is stationary, does not change.

If, at $t = 0$, the curvature of g is constant$= \dfrac{1}{r^2}$, at this time we have $R_{ij} = \dfrac{n-1}{r^2}g_{ij}$ and $\dfrac{\partial g_{ij}}{\partial t} = -2\dfrac{n-1}{r^2}g_{ij}$ at $t = 0$. Let us try with a solution such that the sectional curvature at time t is $K(t) = \dfrac{1}{r(t)^2}$. A metric with this curvature has the form $g(t) = r(t)^2 g_{S^{n-1}}$, where $g_{S^{n-1}}$ is the metric of the sphere of sectional curvature 1. Then the Ricci flow has the expression

$$\frac{\partial r^2}{\partial t} g_{S^{n-1}ij} = -2\frac{n-1}{r(t)^2}r(t)^2 g_{S^{n-1}ij}, \text{ i.e. } 2r(t)r'(t) = -2(n-1),$$

whose solution, with initial condition $g(0) = r^2\ g_{S^{n-1}}$, is

$$r^2(t) = r^2 - 2(n-1)t,$$

which shrinks to a point when $t \to T = \dfrac{r^2}{2(n-1)}$.

Any Einstein metric of positive scalar curvature behaves in the same way: the manifold shrinks to a point homothetically as t approaches some finite time T while the curvature becames infinite like $\dfrac{1}{T-t}$

If, at $t = 0$, the curvature of g is constant$= -\frac{1}{r^2}$, similar arguments to the previous case give $r^2(t) = r^2 + 2(n-1)t$, then there is no collapse, but the curvature tends to zero: $K(t) = -\frac{1}{r^2(t)} = -\frac{1}{r^2 + 2(n-1)t} \to 0$ as $t \uparrow \infty$. With more generality, if we start with an Einstein metric with negative scalar curvature, the metric will expand homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t}$.

Product metric If we take a product metric on a product manifold $M \times N$ to start, the metric will remain a product metric under Ricci flow, and the metric of each factor will evolve independently of the other factor. Thus, on $S^2 \times S^1$, S^2 will collapse to a point and S^1 will remain fix (because it is flat), then the entire manifold will collapse to a circle S^1. On a product $S^2 \times S^2$ with different radii, the sphere of smaller radius collapses faster, and shrinks to a point, whereas the other is still a sphere, then the product collapses to a S^2. If the radius of both spheres are the same, then they remain equal along all the flow, and the whole product will shrink to a point in a finite time.

Quotient metric If M/Γ is a quotient of a Riemannian manifold M by a group of isometries at the starting time $t = 0$, it will remain so under the Ricci flow. This is because the Ricci flow preserves the isometry group. Since the Ricci flow is invariant under the full diffeomorphism group, any isometry in the initial metric will persist as an isometry in each subsequent metric.

5.3. The normalized Ricci flow

It seems natural that one way to avoid collapse is to add the condition that volume be preserved along the evolution. To preserve the volume we need to change a little bit the equation for the Ricci flow. In order to see how we have to change it, let us consider the riemannian volume form $d\mu = \sqrt{\det(g_{ij})}dx^1 ... dx^n$. Let us define the *average scalar curvature* (constant respect to M, but a function of t)

$$r = \frac{\int_M R \, d\mu}{\int_M d\mu}, \tag{5.4}$$

To see which equation we need to preserve the volume of M, let us compute the derivative of the volume element

$$\begin{aligned}\frac{\partial}{\partial t} d\mu = \frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\ dx^1...dx^n = \frac{1}{2} g^{ij}\frac{\partial g_{ij}}{\partial t}\ \sqrt{\det(g_{ij})}\ dx^1...dx^n \\ = \frac{1}{2} g^{ij}\frac{\partial g_{ij}}{\partial t}\ d\mu = (r-R)\ d\mu,\end{aligned} \tag{5.5}$$

which gives, for the volume,

$$\frac{\partial}{\partial t}\int_M d\mu = \int_M \frac{\partial}{\partial t} d\mu = \int_M (r-R)\ d\mu = 0 \ \text{ if } \ \frac{\partial g_{ij}}{\partial t} = \frac{2}{n} r\ g_{ij} - 2\ R_{ij}.$$

Then, the (curve of metrics) solution of the evolution equation

$$\frac{\partial g_{ij}}{\partial t} = \frac{2}{n} r\ g_{ij} - 2\ R_{ij}, \tag{5.6}$$

called *normalized Ricci flow*, preserves the volume of the manifold.

The equations (5.3) and (5.6) differ only by a change of scale in space and a change of parametrization in time, as the following proposition shows:

Proposition 5.3 *Let g_t be a family of metrics on M depending smoothly on t. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a real function defined by the condition*

$$\psi^{\frac{n}{2}}(t) \int_M d\mu_t = 1, \tag{5.7}$$

where $d\mu_t$ is the volume element defined by g_t. Then g_t is a solution of (5.3) *respect to t if and only if*

$$\widetilde{g}_t = \psi(t) g_t$$

is a solution of (5.6) *with respect to*

$$\widetilde{t} = \int_0^t \psi(s)\, ds.$$

Proof. The volume element $d\tilde{\mu}_t$ associated to $\tilde{g}_t$ is related to $d\mu_t$ by

$$d\tilde{\mu}_t = [\psi(t)]^{n/2}\, d\mu_t.$$

Then, from the definition of $\psi(t)$ we obtain

$$Volume_{\tilde{g}_t}(M) = 1.$$

Moreover, the scalar and Ricci curvatures R_{g_t}, Ric_{g_t} and $R_{\tilde{g}_t}$, $Ric_{\tilde{g}_t}$ of the homothetic metrics g_t and $\tilde{g}_t$ are related by

$$R_{\tilde{g}_t} = \frac{1}{\psi(t)} R_{g_t}, \quad Ric_{\tilde{g}_t} = Ric_{g_t} \tag{5.8}$$

Then we have, for the average scalar curvature $\tilde{r}$ of $\tilde{g}_t$,

$$\tilde{r} = (Volume_{\tilde{g}_t}(M))^{-1} \int_M R_{\tilde{g}_t}\, d\tilde{\mu}_t = [\psi(t)]^{\frac{n}{2}-1} \int_M R_{g_t}\, d\tilde{\mu}_t \tag{5.9}$$

From (5.7) we obtain $\int_M dV_{g_t} = [\psi(t)]^{-n/2}$, and, taking the derivative respect to t, and using (5.9),

$$-\frac{n}{2}\frac{\psi'(t)}{\psi(t)^2} = [\psi(t)]^{\frac{n}{2}-1} \frac{1}{2} \int_M \mathrm{tr}_{g_t} \frac{\partial g_t}{\partial t}\, d\mu_t = -\, [\psi(t)]^{\frac{n}{2}-1} \int_M R_{g_t}\, d\mu_t = -\tilde{r}. \tag{5.10}$$

Now, taking the derivative of $\tilde{g}_t$ respect to $\tilde{t}$, we have that, if g_t satisfies (5.3), then

$$\begin{aligned} \frac{\partial \tilde{g}_t}{\partial \tilde{t}} &= \frac{1}{\psi(t)} \frac{\partial \tilde{g}}{\partial t} = \frac{1}{\psi(t)} \left(\psi'(t) g_t + \psi(t) \frac{\partial g_t}{\partial t} \right) \\ &= \frac{1}{\psi(t)} \left(\psi'(t) g_t + \psi(t) \frac{\partial g_t}{\partial t} \right) = \frac{2}{n}\, \tilde{r}\, \tilde{g}_t - 2\, Ric_{\tilde{g}_t}, \end{aligned}$$

i.e., $\tilde{g}_{\tilde{t}}$ satisfies (5.6). The reciprocal is proved in the same way. □

For the sphere, the normalized equation gives g constant, whereas the unnormalized equation shrinks to a point

6. AN OVERVIEW OF THE FUNDAMENTAL STEPS IN THE HAMILTON-PERELMAN PROOF OF GEOMETRIZATION CONJECTURE

6.1. The existence of solutions

In the last lecture we introduced the Ricci flow equation (5.3) and the normalized Ricci flow (5.6) which appears when we take some dilatation in order to have constant volume.

The first question for such an equation is the *short time existence solution.* The corresponding theorem was obtained by Hamilton in 1982 ([38]), using the Nash-Moser theorem, because the Ricci flow is only a weakly parabolic (and not parabolic) equation. In 1983, De Turck ([25]) found a more elementary proof by reducing to a parabolic system and applying standard theory.

Theorem 6.1 *Given a compact Riemannian manifold* (M, g_0)*, there is an* $\epsilon > 0$ *such that the equation* $\dfrac{\partial g_{ij}}{\partial t} = -2R_{ij}$ *has a unique smooth solution* $g(t)$ *in* $[0, \epsilon[$ *satisfying* $g(0) = g_0$*.*

For the 2-dimensional case the equation was completely solved by R. Hamilton in 1988 (cf. [47]) for M compact and the Euler characteristic $\chi < 0$, or $\chi > 0$ and the Gaussian curvature $K > 0$. The general case $\chi > 0$ was solved by B. Chow in 1991 (cf. [18]).

Theorem 6.2 ([47]) *If M is a closed surface, then for any initial metric* g_0 *on M, the solution to the normalized Ricci flow*

$$\frac{\partial g_{ij}}{\partial t} = (r - R)g \tag{6.1}$$

with $g(0) = g_0$ *exists for all time and has constant area. Moreover*

i) If the Euler characteristic χ *of M is negative, then the solution metric* $g(t)$ *converges to a smooth constant negative curvature metric as* $t \to \infty$*.*

ii) If $\chi = 0$ *then* $g_\infty = \lim_{t\to\infty} g(t)$ *is a smooth metric of curvature zero.*

iii) If $\chi > 0$ *and* $R = 2K > 0$ *(being R the scalar curvature of the initial metric* g_0*), then* $g_\infty = \lim_{t\to\infty} g(t)$ *is a smooth constant positive curvature metric.*

Remark 6.2.1 *1. In Theorem 6.2 we stated iii) with the condition* $R > 0$ *for the initial metric. The general case was proved by B. Chow in 1991. Later J. Bartz, M. Struwe and B. Ye (cf. [5]) gave a new proof of this result using the Alexandrov reflection method. Nevertheless, we shall prove only Hamilton result because in this case appear the ideas useful in higher dimensions.*

2. It seems that, as consequences of this theorem we obtain new proofs of many classical results, such as the uniformization theorem for Riemann surfaces and the topological classification of surfaces. However, this is not yet the case, because in the proofs of the above theorem for $\chi > 0$ *and no hypothesis on R, the uniformization theorem is used at some point of the proof. It is still an open problem to find a proof of Theorem 6.2 independent of the uniformization theorem.*

In dimension 3, Hamilton proved in 1982 that

Theorem 6.3 ([38]) *If M is a closed Riemannian* 3*-manifold with* $\mathrm{Ric} > 0$*, the solution of the normalized Ricci flow exists for all the time, and $g(t)$ converges exponencially , as $t \to \infty$, to a metric of constant positive sectional curvature.*

An obvious corollary of this theorem is

Corollary 6.4 *If M is compact with $Ric > 0$, then M is homeomorphic to S^3/Γ, where Γ is a discrete group of isometries of S^3*

Theorem 6.5 *If M is orientable, compact and $Ric \geq 0$, the solution $g(t)$ of the normalized Ricci flow exists for all the time, and $g(t)$ converges, as $t \to \infty$, to the metric of the quotient by a discrete group of isometries of one of the following Riemannian manifolds with its standard metric: S^3, $S^2 \times S^1$ or $\mathbb{R}^3$.*

Dimension 3 is very special for the connections between Ricci flow and sectional curvature because in this dimension the Ricci curvature tensor determines all the curvature tensor. In this dimension, the positivity of the Ricci curvature is preserved under the Ricci flow. This is not longer true in higher dimensions. Even the positivity of sectional curvature is not preserved under Ricci flow in dimension > 3. However, the stronger condition of positive curvature operator is preserved under Ricci flow.

In general, the curvature tensor Rm defines, for every $x \in M$, a selfadjoint map

$$Rm_x : \wedge^2 T_x M \longrightarrow \wedge^2 T_x M \text{ defined by } \langle Rm_x X \wedge Y, Z \wedge W \rangle = R_{XYZW}.$$

As usual, we say that this operator is positive if all its eigenvalues are positive.

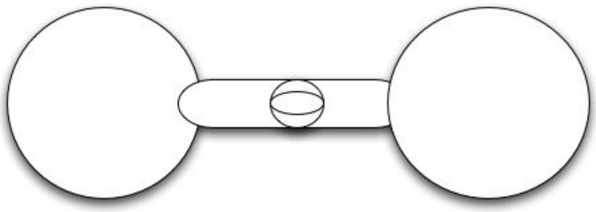
In 1986, R. Hamilton proved that

Theorem 6.6 *If M is a compact* 4*-dimensional manifold with positive curvature operator, the solution $g(t)$ of the normalized Ricci flow exists for all the time, and $g(t)$ converges, as $t \to \infty$, in C^∞ to the metric of the standard S^4, or $\mathbb{R}P^4$.*

If the curvature operator of M is non-negative, the solution $g(t)$ of the normalized Ricci flow exists for all the time, and $g(t)$ converges, as $t \to \infty$, to the metric of the quotient by a discrete group of fixed point free isometries of one of the following Riemannian manifolds with its standard metric: S^4, $\mathbb{C}P^2$, $S^3 \times \mathbb{R}$, $S^2 \times S^2$, $S^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$.

There are also many results on Kähler manifolds.

As we have seen, in dimension 2 the solution exists for all time. In dimension 3 the difficulties arise: for certain initial metrics, even normalized Ricci flow develops singularities in finite time. A typical example is something like a dumbbell



In the central part of the picture we may have positive sectional curvature in some directions and negative in others, in such a way that, the positive curvature is bigger in modulus and the metric collapses, whereas left and right parts of the pictures become non collapsed and at a distance which is increasing when we approach the collapsing time.

Before going on, we need to give a precise definition of singularity.

Definition 6.1 *A solution (M, g) of Ricci flow on an interval $[0, T]$ is maximal if it cannot be extended for time $t > T$.*

i) If $T = \infty$ and $\sup_{M\times[0,T]} |Rm| < \infty$, the maximal solution is called a non-singular solution.

ii) If $T \leq \infty$ and $\sup_{M\times[0,T]} |Rm| = \infty$, the maximal solution is called a singular solution and that a singularity occurs at time T.

Many times (as we have seen for spheres in the previous lecture), there are singular solutions of the unnormalized Ricci flow which can be removed going to the normalized Ricci flow, where we obtain nonsingular solutions. An example is given by the following table:

normalized Ricci flow *non-singular solution*	*unnormalized Ricci flow* *singular solution*
(N1) $\dim M = 2$, any initial metric	(U1) $\dim M = 2$, $\chi > 0$, desingularizes with (N1)
(N2) $\dim M = 3$, $\mathrm{Ric} > 0$	(U2) $\dim M = n$, $R > 0$, desingularizes with (N2), (N3), (N4)
(N3) $\dim M = 4$, $Rm > 0$	
(N4) $\dim M = n$, sufficiently pointwise pinched sectional curvature	(U4) M^n, Kähler metric with Chern class $c_1(M) > 0$.
(N5) $\dim M = 3$, with locally homogeneous metric	(U5) Locally homogeneous metrics on M^3 of class $SU(2)$ or $S^2 \times \mathbb{R}$: desingularizes with (N5)
In all these cases we have convergence to an Einstein metric as $t \to \infty$	

6.2. The possible solutions and singularities

The next biggest steps in the work of Hamilton to the solution of geometrization conjecture are (1999 -with corrections added later- and 1995 respectively):

Theorem 6.7 *Let M be a compact Riemannian 3-manifold which admits (something stronger than) a non-singular solution of the normalized Ricci flow. Then it can be decomposed into the geometric pieces of Thurston conjecture. With more precision:*

If $g(t)$ is a solution in M of the normalized Ricci flow which exists for all time and satisfies that the normalized curvature $t \cdot Rm(x,t)$ is bounded as $t \to \infty$, then M is one of the following manifolds:

a Seifert fibered manifold,

a quotient S^3/Γ,

a flat manifold,

a hyperbolic manifold

the union along incompressible tori of finite volume hyperbolic manifolds and Seifert fibered spaces.

Theorem 6.8 *Let M be a compact Riemannian 3-manifold which admits a solution of the Ricci flow which develops a singularity at time T. Then there is a sequence of dilations*[20] *of the solution which converges to a quotient by isometries of one of the following manifolds:*

i) S^3, *(a topological space form)*

[20] we shall explain later the precise dilation which are used

ii) $S^2 \times \mathbb{R}$, *(a neck)*

iii) $\Sigma \times \mathbb{R}$, *where* Σ *is a cigar solution, with metric* $ds^2 = \dfrac{dx^2 + dy^2}{1 + x^2 + y^2}$

We would like to show that, after a finite number of surgeries on the singularities, the solution becomes a non-singular solution.

6.3. The Perelman's work

Let us suppose that M is an oriented closed 3-manifold. We can distinguish 5 steps in the Perelman's work of the claimed proof on Geometrization conjecture:

(1) Only cases i) and ii) are possible in the above Theorem 6.8 of "classification" of singularities by Hamilton (the cigars solutions are not possible)

(2) Surgery. Cut the above singularities appearing in each blow ups at finite times T_i and glue a spherical cap with bounded curvature at each cutting. In the resulting manifold, throw away the components of positive Ricci curvature. Then we have a new manifold M_{sur}, and continue the Ricci flow on it.

(3) There are only locally finitely many blow ups times T_i.

(4) For t big enough, there are no singularities, and, at such big time, M_{sur} splits (along incompressible tori) in two parts $M_{sur} = M_{\text{thick}} \cup M_{\text{thin}}$, where M_{thick} is the union of points with injectivity radius $\geq \epsilon$ and[21] M_{thin} is the union of points with injectivity radius $< \epsilon$ for some appropriate ϵ, such that M_{thick} has a finite volume complete hyperbolic metric.

(5) M_{thin} is a graph manifold (recall its definition in Example 5).

Step 1 is carried out in the first preprint of Perelman, and has been completely checked by Kleiner and Lott. Steps 2 to 4 are the contents of the second preprint, and step 5 is proved in a paper by Shioya and Yamaguchi, except for one case which is contained in the third preprint of Perelman, which also contains an independent proof of the Poincaré's conjecture.

Notice that the above steps would imply the proof of the Geometrization Conjecture. In fact, in an instant t' large enough, the surgery procedure (which is efectively performable because the possible singularities have a simple and well known topological form -as is claimed in (1)-, and also because by (2) we have a sufficient control of the surgery times) allows to decompose the original manifold in primes as

$$M \cong P_1 \sharp \ldots \sharp P_r \sharp (S^3/\Gamma_1) \sharp \ldots \sharp (S^3/\Gamma_k) \sharp S^1 \times S^2 \sharp \ldots \sharp S^1 \times S^2. \qquad (6.2)$$

[21] Let M be a complete Riemannian manifold. For $p \in M$, we define the injectivity radius at p as

$$inj(p) := \sup\{r > 0 : \exp_p \big|_{B_r(p)} \text{ is a diffeomorphism}\}.$$

Moreover, (4) assures that each irreducible factor P_i admits a torus decomposition of the following form:

$$P_i \cong H_i \cup G_i, \tag{6.3}$$

where H_i is a hyperbolic piece and G_i a graph manifold (both possibly disconnected). Observe that the last claim corresponds to the step 5 above.

It is important to remark that Perelman's results do not prove that there will be no more singularities in times greater than t'. Instead of this, it is shown that if this is the case, such singularities would appear in the G_i-part (or in M_{thin} following the notation of (4)), and therefore there is no problem since it is well known that every graph manifold admits a geometric decomposition.

In conclusion, from (6.2), (6.3) and the above remarks, it follows that M can be topologically decomposed in geometric pieces, and this is exactly the statement of the Geometrization Conjecture.

7. THE MAXIMUM PRINCIPLE

7.1. Prerequisites about Lipschitz functions

In order to give an idea of the proof of the strong maximum principle, we need to make some remarks on functions which are not quite differentiable.

Definition 7.1 *A function $f : [a, b] \to \mathbb{R}$ is said to be a Lipschitz function (or to satisfy a Lipschitz condition) on $[a, b]$ if there is a constant $C > 0$ such that*

$$|f(x) - f(y)| \leq C|x - y| \quad \text{for all } x \text{ and } y \text{ in } [a, b]. \tag{7.1}$$

The smallest number C for which (7.1) holds is called the Lipschitz constant.

Remark 7.1.1 *As an immediate consequence of the Mean Value theorem, if a real valued function has a bounded derivative on an interval, then it satisfies a Lipschitz condition there.*

On the other hand, a function may satisfy a Lipschitz condition on an interval and not be differentiable at certain points. A function whose graph consists of several connected straight line segments (like, for example, $f(x) = |x|$) illustrates this.

Definition 7.2 *(Superior and inferior limit of a real function). Let f be a real (or extended real) valued function defined for all x in an interval containing y, we define*

$$\overline{\lim}_{x\to y} f(x) \equiv \limsup_{x\to y} f(x) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x) = \lim_{\delta\to 0} \sup_{|x-y|<\delta} f(x)$$

$$\overline{\lim}_{x\to y^+} f(x) \equiv \limsup_{x\to y^+} f(x) = \inf_{\delta>0} \sup_{0<x-y<\delta} f(x) = \lim_{\delta\to 0} \sup_{0<x-y<\delta} f(x)$$

$$\underline{\lim}_{x\to y} f(x) \equiv \liminf_{x\to y} f(x) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x) = \lim_{\delta\to 0} \inf_{|x-y|<\delta} f(x)$$

$$\underline{\lim}_{x\to y^+} f(x) \equiv \liminf_{x\to y} f(x) = \sup_{\delta>0} \inf_{0<x-y<\delta} f(x) = \lim_{\delta\to 0} \inf_{0<x-y<\delta} f(x)$$

Next we shall consider some generalizations of the usual derivative which have the advantage of applying to functions that are not necessarily differentiable in the usual sense (such as Lipschitz functions).

Definition 7.3 *Let $f : \mathbb{R} \to \mathbb{R}$. The upper right and lower right Dini derivatives*[22] *of f at $t \in \mathbb{R}$ are, respectively, defined by*

$$D^+ f(t) = \limsup_{h\to 0^+} \frac{f(t+h) - f(t)}{h}$$

$$D_+ f(t) = \liminf_{h\to 0^+} \frac{f(t+h) - f(t)}{h}$$

Remark 7.3.1 *1. The Dini derivatives always exist (finite or infinite) for any function f, and $D^+ f(x) \geq D_+ f(x)$.*[23]

2. If f is a differentiable function the two derivatives defined above are identical, finite numbers and coincide with the usual derivative.

Lemma 7.1 *Let $f : [a, b] \to \mathbb{R}$ be a Lipschitz function such that $f(a) \leq 0$ and $D^+ f(t) \leq 0$ when $f \geq 0$ for $a \leq t \leq b$, then $f(b) \leq 0$.*

Proof. Without loss of generality, we can suppose $a = 0$. We shall show $f(t) \leq \varepsilon t$ for any $\varepsilon > 0$ so, taking limits when $\varepsilon \to 0$, this will mean that the function is always non positive.

If $f(0) < 0$, by continuity, there exists a neighbourhood of 0 in $\mathbb{R}$ (that is, an interval containing 0) where $\varepsilon t \geq f(t)$.

If $f(0) = 0$ since, by hypothesis, $\limsup_{h\to 0^+} \frac{f(h) - f(0)}{h} \leq 0$, there must be some interval $0 \leq t < \delta$ on which $\varepsilon t \geq f(t)$.

In both cases, let $0 \leq t < c$ be the largest such interval with $c \leq b$. Then by continuity $f(t) \leq \varepsilon t$ on the closed interval $0 \leq t \leq c$.

Assume $c < b$ (if $c = b$ we have finished), then there are two possibilities:

[22] For our purposes, we only need these definitions. However, it is possible to define, analogously, the upper left and lower left Dini derivatives of f at $t \in \mathbb{R}$, by the following formulas

$$D^- f(t) = \limsup_{h\to 0^-} \frac{f(t+h) - f(t)}{h}$$

$$D_- f(t) = \liminf_{h\to 0^-} \frac{f(t+h) - f(t)}{h}$$

These are the four Dini derivatives, introduced by Ulisse Dini (1845-1918) in *Fondamenti per la teorica della funzioni di variabili reali* (1878).

[23] A proof of this fact can be seen in [11] p. 160. Moreover, the analogous property for lower Dini derivatives is $D^- f(x) \geq D_- f(x)$.

i) $f(c) < 0$: by continuity, $f(t) \leq \varepsilon t$ for some $t > c$, against the hypothesis that [0,c] is the largest interval on which $f(t) \leq \varepsilon t$.

ii) $f(c) = 0$: since $\limsup_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$, we can find $\tilde{\delta} > 0$ with $f(t) \leq \varepsilon t$ on $0 \leq t \leq c + \tilde{\delta}$, which is, again, a contradiction.

Therefore, $c = b$.

Thus we have proved $f(t) \leq \varepsilon t$ on $0 \leq t \leq b$ for any $\varepsilon > 0$, so when $\varepsilon \to 0$ we obtain, in particular, $f(b) \leq 0$. □

Remark 7.1.1 *Without the hypothesis $f(a) \leq 0$, we can repeat the above proof, with the obvious modifications, to show that $f(t) \leq f(a)$ for $t \in [a, b]$.*

Corollary 7.2 *If $f(a) \geq 0$ and $D_+ f(t) \geq 0$ for $a \leq t \leq b$, then $f(b) \geq 0$.*

Corollary 7.3 *If $f(a) \leq 0$ and $D^+ f(t) \leq cf$ for some $c \in \mathbb{R}$ and for $a \leq t \leq b$, then $f(b) \leq 0$.*

Proof. Take $g = e^{-ct} f$, then

$$\begin{aligned}
D^+ g(t) &= \limsup_{s \to t^+} \frac{e^{-cs} f(s) - e^{-ct} f(t)}{s - t} \\
&= \limsup_{s \to t^+} \left(\frac{e^{-cs} f(s) - e^{-ct} f(s) + e^{-ct} f(s) - e^{-ct} f(t)}{s - t} \right) \\
&= \lim_{\delta \to 0} \sup_{0 < s - t < \delta} \left(f(s) \frac{e^{-cs} - e^{-ct}}{s - t} + e^{-ct} \frac{f(s) - f(t)}{s - t} \right) \\
&\leq \lim_{\delta \to 0} \left[\sup_{0 < s - t < \delta} \left(f(s) \frac{e^{-cs} - e^{-ct}}{s - t} \right) + \sup_{0 < s - t < \delta} \left(e^{-ct} \frac{f(s) - f(t)}{s - t} \right) \right] \\
&= \limsup_{s \to t^+} \left(f(s) \frac{e^{-cs} - e^{-ct}}{s - t} \right) + e^{-ct} \limsup_{s \to t^+} \left(\frac{f(s) - f(t)}{s - t} \right) \\
&= \limsup_{s \to t^+} \left(f(s) \frac{e^{-cs} - e^{-ct}}{s - t} \right) + e^{-ct} D^+ f(t)
\end{aligned}$$

It is well known that the product of continuous functions is also a continuous function. So the upper limit in the first addend above is actually a limit, and the limit of a product is the product of the limits. In the second addend, we can apply the hypothesis of the corollary about $D^+ f(t)$. Therefore, we have

$$D^+ g(t) \leq f(t) (e^{-cs})'_{s=t} + e^{-ct} c f(t) = -f(t) c e^{-ct} + e^{-ct} c f(t) = 0$$

As a result of applying lemma 7.1 to g, we have $g(b) \leq 0$; but $g(b) = e^{-cb} f(b)$, so $f(b) \leq 0$. □

Corollary 7.4 *If $f(a) \leq g(a)$ and $D^+(f)(t) \leq D_+ g(t)$ for $a \leq t \leq b$, then $f(b) \leq g(b)$.*

Proof. We choose $h := f - g$, then

$$\begin{aligned}
D^+h(t) &= \limsup_{s\to t^+} \frac{h(s)-h(t)}{s-t} = \limsup_{s\to t^+} \frac{(f-g)(s)-(f-g)(t)}{s-t} \\
&= \limsup_{s\to t^+} \left(\frac{f(s)-f(t)}{s-t} - \frac{g(s)-g(t)}{s-t} \right) \\
&= \lim_{\delta\to 0} \left[\sup_{0<s-t<\delta} \left(\frac{f(s)-f(t)}{s-t} - \frac{g(s)-g(t)}{s-t} \right) \right] \\
&\le \lim_{\delta\to 0} \left[\sup_{0<s-t<\delta} \left(\frac{f(s)-f(t)}{s-t} \right) - \inf_{0<s-t<\delta} \left(\frac{g(s)-g(t)}{s-t} \right) \right] \\
&= \limsup_{s\to t^+} \frac{f(s)-f(t)}{s-t} - \liminf_{s\to t^+} \frac{g(s)-g(t)}{s-t} \\
&= D^+f(t) - D_+g(t) \le 0
\end{aligned}$$

for $a \le t \le b$ and $h(a) = f(a) - g(a) \le 0$; so, by Lemma 7.1, $h(b) = f(b) - g(b) \le 0$, that is, $f(b) \le g(b)$. □

Lemma 7.5 *Let $g(t, y)$ be a smooth function of $t \in \mathbb{R}$ and $y \in \mathbb{R}^k$. Let $f(t) := \sup_{y\in Y} g(t, y)$, where $Y \subset \mathbb{R}^k$ is a compact set. Then f is a Lipschitz function and its upper right derivative satisfies*

$$D^+f(t) \le \sup_{y\in Y(t)} \frac{\partial}{\partial t} g(t, y),$$

being $Y(t) = \{y \in Y : f(t) = g(t, y)\}$.

Proof. Choose an arbitrary $t_0 \in \mathbb{R}$ and a sequence $\{t_j\}_{j=1}^{\infty}$ decreasing to t_0 for which $\lim_{t_j\to t_0} \frac{f(t_j)-f(t_0)}{t_j-t_0}$ equals the lim sup.

Since Y is a compact set, the maximum is attained; so for each index j we can take $y_j \in Y$ such that $f(t_j) = g(t_j, y_j)$. Therefore, $\{y_j\}$ is a sequence in Y and, because of the compactness of Y, there is a subsequence convergent to some $y_0 \in Y$. We can assume (for simplicity of the notation) $y_j \to y_0$.

Taking limits in $f(t_j) = g(t_j, y_j)$ and using the continuity of f and g, we have $f(t_0) = g(t_0, y_0)$; so $y_0 \in Y(t_0)$.

By definition of f, $g(t_0, y_*) \le g(t_0, y_0)$ $\forall y_* \in Y$; then

$$f(t_j) - f(t_0) = g(t_j, y_j) - g(t_0, y_0) \le g(t_j, y_j) - g(t_0, y_j)$$

Dividing by $t_j - t_0$ and using the Mean Value Theorem, we obtain

$$\frac{f(t_j) - f(t_0)}{t_j - t_0} \le \frac{g(t_j, y_j) - g(t_0, y_j)}{t_j - t_0} = \frac{\partial}{\partial t} g(T_j, y_j)$$

with $t_0 < T_j < t_j$. Taking limits when $t_j \to t_0$, we have

$$\lim_{t_j\to t_0} \frac{f(t_j) - f(t_0)}{t_j - t_0} \le \frac{\partial}{\partial t} g(t_0, y_j) \ \le \sup_{y\in Y(t)} \frac{\partial}{\partial t} g(t, y)$$

Since t_0 is arbitrary, we have proved the estimate on the upper right derivative of f. Moreover, since $Y(t)$ is a compact set, this supremum is attained and the above inequality shows that f has bounded first derivative and so it is a Lipschitz function (although, in general, it is not differentiable). □

Next, we state the analog result for lower right derivatives.

Lemma 7.6 *Let $g(t, y)$ be a smooth function of $t \in \mathbb{R}$ and $y \in \mathbb{R}^k$. Let $h(t) := \inf_{y \in Y} g(t, y)$, where $Y \subset \mathbb{R}^k$ is a compact set. Then h is a Lipschitz function and its lower right derivative satisfies*

$$D_+ h(t) \geq \sup_{y \in Y(t)} \frac{\partial}{\partial t} g(t, y),$$

being $Y(t) = \{y \in Y : h(t) = g(t, y)\}$.

7.2. The maximum principle for systems

Roughly speaking, the maximum principle says that if the solution of a parabolic system of equations lies in a convex set at a starting time 0, it will remain in it for every time $t > 0$. The basic idea of the maximum principle of R. Hamilton which we shall discuss here was suggested by Moe Hirch. We note that this idea is a generalization of Lyapunov functions, and the proof is analogous to the Lyapunov proof of stability.

Let (M, g) be a compact Riemannian manifold and let $f = (f^1, \ldots, f^k) : M \to \mathbb{R}^k$ be a system of k functions on M. Let $U \subset \mathbb{R}^k$ be an open subset and let $\phi : U \to \mathbb{R}^k$ be a smooth vector field on U. We let f, g and ϕ depend on time also.

Consider the non linear heat equation

$$\frac{\partial f}{\partial t} = -\Delta_g f + \phi \circ f \tag{7.2}$$

with $f(0) = f_0$, and suppose that it has a solution for some time interval $0 \leq t \leq T$. We want to know the behaviour of the solution f of (7.2) when t changes; in particular, consider $X \subset U$ a closed convex subset containing f_0 and ask when the solution remains in X.

To answer this question we first study the system of ordinary differential equations (ODE)

$$\frac{df}{dt} = \phi \circ f \tag{7.3}$$

(which we shall call the ODE associated to the PDE (7.2)) and ask the same question.

Before dealing with this, we need some previous definitions.

Definition 7.4 [24] *We define the tangent cone $T_z(X)$ to a closed convex set $X \subset \mathbb{R}^k$ at a point $z \in \partial X$ as the smallest closed convex cone with vertex at z which*

[24] You can see a more general definition of tangent cone in [76] p. 171.

contains X. It is the intersection of all the closed half spaces containing X with z on the boundary of the half space.[25]

Definition 7.5 *We say that a linear function $l : \mathbb{R}^k \to \mathbb{R}$ is a support function for $X \subset \mathbb{R}^k$ at $z \in \partial X$ (and write $l \in S_z X$) if*

1. $|l| = 1$
2. $l(z) \geq l(x)$ *for all* $x \in X$ *(i.e.* $l(z - x) \geq 0$*).*

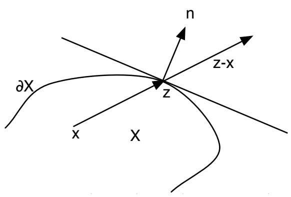
Remark 7.5.1 *If $z \in \partial X$ and $X \subset \mathbb{R}^k$ is a convex set, always exists some plane (not unique in general) at z that leaves the set aside (this is called a supporting plane). Each of these planes has a well defined outward unit normal vector n at z. For our purposes we can identify each supporting plane with its outward unit normal vector.*

The $\flat$ and $\sharp$ operators give an $1:1$ correspondence between support functions and unit vectors normal to a supporting plane. In fact, we can check that $n^\flat$ satisfies the conditions of the previous definition.

For any $x \in X$ we have

$$n^\flat(z - x) = \langle z - x, n \rangle = |z - x||n| \cos \angle(z - x, n) \geq 0 \leftrightarrow \cos \angle(z - x, n) \geq 0$$

And this inequality is true because $z - x$ makes an acute angle with the unit normal as we can see in the next figure.



On the other hand, $|n^\flat|^2 = \left\langle n^\flat, n^\flat \right\rangle = \left\langle (n^\flat)^\sharp, (n^\flat)^\sharp \right\rangle = \langle n, n \rangle = |n|^2 = 1.$ *Therefore we have shown that $n^\flat \in S_z X$.*

Remark 7.5.2 *The following equivalence is quite intuitive*

$$x \in T_z(X) \text{ iff } \angle(x - z, n) \geq \frac{\pi}{2} \quad \text{for all unit vector } n \text{ normal to a supporting plane at } z$$

Since the right side is equivalent to $\cos \angle(x - z, n) \leq 0$ for all n, from the viewpoint of the support functions, we can write

$$x \in T_z(X) \quad \text{if and only if} \quad l(x - z) \leq 0 \qquad \text{for every } l \in S_z X$$

[25] Some authors define the tangent cone as the boundary of our definition.

Lemma 7.7 *The solutions of the ODE* (7.3) *which are in the closed convex set* $X \subset \mathbb{R}^k$ *at* $t = 0$ *will remain in* X *if and only if* $\phi(z) \in T_z X$ *for every* $z \in \partial X$

Proof. Because of the last remark, it suffices to prove that the solution to (7.3) remains in X if and only if $l(\phi(z)) \leq 0$ for every $l \in S_z X$ and $z \in \partial X$.

Suppose first that $l(\phi(z)) > 0$ for some $z \in \partial X$, then taking images in the equation (7.3)

$$\frac{d}{dt} l(f)(0) = l\left(\frac{df}{dt}\right)(0) = l(\phi(f(0))) > 0, \quad \text{if } f(0) = z$$

so $l(f)$ is increasing at 0, in particular, for some small t

$$l(f(t)) > l(f(0)) \text{ iff } l(f(t) - f(0)) > 0 \text{ iff } f(t) \notin T_{f(0)} X$$

and, since $f(0) \in \partial X$, we conclude that f cannot remain in X.

To see the converse, let us suppose that $l(\phi(x)) \leq 0$ for every $l \in S_x X$ and $x \in \partial X$. First note that we may assume X compact. If it is not we can modify the vector field ϕ by multiplying by a bump function[26]. The paths of solution are unchanged where the bump function is 1.

We set, for every $z \in \mathbb{R}^k$,

$$s(z) := d(z, X) = \sup\{l(z - x) : x \in \partial X, l \in S_x X\}. \tag{7.4}$$

$l(z - x) = \langle n, z - x \rangle = |z - x| \cos\angle(z - x, n)$ gives the distance from z to the supporting hyperplane Π_l of X defined by l, $X = \cap_{l \in S_z X, z \in \partial X} H_l$, where H_l is the half space defined by Π_l, and $d(z, X) = d(z, \bigcap H_l) = \sup d(z, H_l) = \sup l(z - x)$.

Moreover, from the convexity of X, given $z \in \mathbb{R}^k$, there is a unique $x_0 \in \partial X$ such that $d(z, X) = l_1(z - x_0)$ (where l_1 is a linear function of length 1 with gradient in the direction of $z - x_0$).

By definition of distance, $s(z) = 0$ if $z \in X$.

Note that $Y = \{(x, l) : x \in \partial X, l \in S_x X\} \subset \mathbb{R}^k \times \mathbb{R}^k$ is a compact set.

We are now in position to use Lemma 7.5 and conclude

$$D^+ s(f(t)) \leq \sup_{(x,l) \in Y(t)} \frac{\partial}{\partial t} l(f(t) - x)$$

where $Y(t) = \{(x, l) \in Y : s(f(t)) = l(f(t) - x)\}$.

Note that, if x_0 is the unique $x_0 \in \partial X$ such that $d(f(t), X) = l_1(f(t) - x_0)$, then

$$(x_0, l_1) \in Y(t) \quad \text{and} \quad l_1(f(t) - x_0) = |f(t) - x_0|. \tag{7.5}$$

On the other hand, since ϕ is smooth and X is a compact set, ϕ has bounded first derivative and therefore it is a Lipschitz function. Using this fact, there exists some constant C such that

$$|\phi(z) - \phi(y)| < C|z - y| \qquad \forall z, y \in X$$

[26] $\{\varphi : \mathbb{R}^k \to \mathbb{R}/\varphi \geq 0, \varphi_{B_\rho(x_0)} = 1, \varphi_{\mathbb{R}^k - B_R(x_0)} = 0, R > \rho\}$, and we can take $X \bigcap B_R(x_0)$ instead of X.

Then, since $l(\phi(x_0)) \leq 0$ by hypothesis, using (7.5) and $|l| = 1$, we have

$$\begin{aligned} D^+s(f(t)) &\leq \sup_{(x,l)\in Y(t)} \frac{\partial}{\partial t} l(f(t)-x) = \sup_{(x,l)\in Y(t)} l(\frac{df}{dt})(t) = \sup_{(x,l)\in Y(t)} l(\phi(f(t))) \\ &\leq \sup_{(x,l)\in Y(t)} l(\phi(f(t))) \; - \; l(\phi(x_0)) \\ &\leq \sup_{(x,l)\in Y(t)} l(\phi(f(t)) - \phi(x_0)) \leq C\ |f(t) - x_0| = C\ s(f(t)) \end{aligned}$$

Finally, we have obtained $D^+s(f(t)) \leq C\ s(f(t))$ and $s(f(0)) = 0$ so, applying corollary 7.3 ,we conclude $s(f(t)) = 0$, and this means $f(t) \in X$. □

Theorem 7.8 *If the solution of the ODE* (7.3) *with* $f(0) \in X$ *stays in* X*, then the solution of the PDE* (7.2) *with* $f(0) \in X$ *stays in* X

Proof. As before we may suppose that X is compact. Again, let $s(z) = d(z, X)$, $z \in \mathbb{R}^k$. Given a solution $f : M \times \mathbb{R} \to \mathbb{R}^k$ of (7.2), we define

$$s(t) := \sup_{x\in M} s(f(x,t)), \tag{7.6}$$

so, by (7.4), we have

$$s(t) = \sup_{(x,q,l)\in Y} l(f(x,t) - q), \tag{7.7}$$

being $Y = \{(x, q, l) : x \in M, q \in \partial X, l \in S_qX\}$ a compact set. So, Lemma 7.5 assures that

$$D^+s(t) \leq \sup_{(x,q,l)\in Y(t)} \frac{\partial}{\partial t} l(f(x,t) - q)$$

where $Y(t) = \{(x, q, l) \in Y : l(f(x, t) - q) = s(t)\}$ From this definition of $Y(t)$ and (7.7), it follows that if $(x, q, l) \in Y(t)$, then $\max_{(x,q,l)\in Y} l(f(x, t) - q) = l(f(x, t) - q)$. Since l is a linear function independent of t, we have

$$\begin{aligned} D^+s(t) &\leq \sup_{(x,q,l)\in Y(t)} \frac{\partial}{\partial t} l(f(x,t) - q) = \sup_{(x,q,l)\in Y(t)} l\left(\frac{\partial f(x,t)}{\partial t}\right) \\ &= \sup_{(x,q,l)\in Y(t)} \{-l(\Delta_g f(x,t)) + l(\phi(f(x,t)))\} \end{aligned} \tag{7.8}$$

Note that the last equality is true because f is a solution of the PDE (7.2).

By definition of $Y(t)$, $l(f(x, t))$ has its maximum at x; so

$$l(-\Delta_g f) = -\Delta_g l(f) \leq 0. \tag{7.9}$$

On the other hand, by hypothesis, the solution of the ODE (7.3) stays in X and we have proved in Lemma 7.7 that this means $l(\phi(z)) \leq 0$ for every $l \in S_zX$ and $z \in \partial X$. Then, in particular, $l(\phi(q)) \leq 0$. So we get

$$\begin{aligned} l(\phi(f(x,t))) &\leq l(\phi(f(x,t))) - l(\phi(q)) \\ &= l(\phi(f(x,t)) - \phi(q)) \leq |l||\phi(f(x,t)) - \phi(q)| \\ &\leq c|f(x,t) - q| = c\, l(f(x,t) - q) \end{aligned}$$

where c is the Lipschitz constant of ϕ and the last equality follows by the definition of $Y(t)$.

Finally, by substitution of the above inequality and (7.9) in (7.8), we obtain $D^+ s(t) \leq c\, s(t)$, and, since $f(0) \in X$, $s(t) = 0$. Then applying Corollary 7.3, we conclude that $s(t) = \sup_{x \in M} s(f(x,t)) = 0$ for all time in which the solution is defined. But this shows that $f(x,t)$ remains in X. □

This theorem admits a generalization for vector bundles. Since the main ideas of the proof are contained in the previous lemmas, we just give the statement of the theorem.

Theorem 7.9 (Bundle formulation of maximum principle) *Let V be a vector bundle over a compact Riemannian manifold (M, g) and h a fixed metric on V. Consider ∇ a connection on V compatible with h, both possibly varying in time. Let ϕ be a vector field on V tangent to the fibers [27]. Suppose that X is a closed subset of V such that it is convex on each fiber and invariant respect to ∇-parallel translation at all times. Assume that the solution of the ODE $\dfrac{df}{dt} = \phi(f)$ (where f is a section of V depending smoothly on t) remains all time in X. Then the solutions of $\dfrac{\partial f}{\partial t} = -\Delta_g f + \phi(f)$ also remain inside X.*

7.3. Maximum principle for functions

The following maximum principle will be used very often in the next lectures

Theorem 7.10 (scalar maximum principle) *Let M be a compact manifold, and let g_t a 1-parametric family of smooth metrics on M depending smoothly on t. Let X_t be a family of smooth vector fields on M depending smoothly on t. Let us consider the partial differential inequations*

$$\frac{\partial f_t}{\partial t} \geq -\Delta_{g_t} f_t + \langle \mathrm{grad}_t f_t, X_t \rangle + \phi \circ f_t, \tag{7.10}$$

and the associated ordinary differential equation

$$\frac{dh(t)}{dt} = \phi \circ h \text{ with } h(0) = \min\{f_0(x);\ x \in M\}, \tag{7.11}$$

then $f_x(t) \geq h(t)$ for every t in an interval $[0, T]$ where there is a solution of (7.10) *and* (7.11). *One has the analog result for reversing inequalities.*

[27] A vector field ϕ on V tangent to the fibers is a C^r map $\phi : V \to V$ satisfying that $\pi \circ \phi = \pi$, for the vector bundle $\pi : V \to M$.

8. LONG TIME EXISTENCE

Theorem 6.1 implies that there is a maximal interval $[0, T[$ on which the solution of the Ricci flow exists. If $T < \infty$, it is important to understand the reason why solution stops at that time. Next theorem says that the only obstacle to the long time existence of the flow is the curvature tensor becoming unbounded

Theorem 8.1 (long time existence) *If g_0 is a smooth metric on a compact n-dimensional manifold M, the unnormalized Ricci flow with $g(0) = g_0$ has a unique solution $g(t)$ on a maximal time interval $0 \leq t \leq T \leq \infty$. Moreover, if $T < \infty$, then*

$$\lim_{t \nearrow T} \left(\sup_{x \in M} |Rm(x, t)| \right) = \infty$$

where Rm is the Riemannian curvature tensor.

8.1. Steps of the proof

1. *Short time existence* asserts that there is a unique smooth solution on a maximal time interval $0 \leq t < T$ for $T \leq \infty$.

2. Suppose $T < \infty$ and the norm of the Riemannian curvature tensor (that is, $|Rm|$) is bounded as t tends to T. Next show that all the space-time derivatives $\frac{\partial}{\partial t} \nabla^k Rm$ are also bounded when $t \to T$.

3. Prove that the metric g and all its ordinary derivatives in a local coordinate chart remain bounded, and g remains bounded away from zero below.

4. As a consequence of step 3, the metric g_t at time t converges to a smooth limit metric g_T as $t \to T$.

5. Applying *short time existence* with $g(0) = g_T$ we can continue the solution past T which is a contradiction, because T was maximal by election.

8.2. Results used along the proof

Fundamental in steps 1 and 5, is the *short time existence* Theorem 6.1.

Theorem 8.2 *Given a compact Riemannian manifold (M, g_0), there is an $\epsilon > 0$ such that the equation $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ has a unique smooth solution $g(t)$ in $]0, \epsilon[$ satisfying $g(0) = g_0$.*

On the other hand, step 2 includes the proof of the following theorem about derivative estimates of the curvature.

Theorem 8.3 *Let (M, g_t) be a solution of the Ricci flow on a compact n-dimensional manifold. Then for every $k \in \mathbb{N}$, there exists a constant C_k, depending only on k and n, such that if*

$$|Rm(x, t)|_g \leq \rho \qquad \textit{for all } x \in M \textit{ and } t \in [0, \frac{1}{\rho}],$$

then

$$|\nabla^k Rm(x,t)|_g \leq \frac{C_k \rho}{t^{k/2}} \quad \textit{for all } x \in M \ \textit{ and } t \in [0, \frac{1}{\rho}]$$

Notation 8.1 *From now on, given any tensor T, we shall write $C_{ij}^{k\ell}T$ to indicate the contraction, in T, of the i-th argument with the k-th argument, and that of the j-th argument with the ℓ-th argument.*

Sketch of the proof.

• *Step 1.* Establish the evolution formula for the components of the Riemannian curvature tensor.

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + 2(B_{abcd} + B_{acbd} - B_{abdc} - B_{adbc}) \tag{8.1}$$

where $B = C_{24}^{68} Rm \otimes Rm)$ (i.e., using a local orthonormal frame, $B_{abcd} = R_{aebf} R_{cedf}$ Next write the formula in the following way:

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm \tag{8.2}$$

where $*$ is defined by:

$$\begin{aligned}(Rm * Rm)(X,Y,Z,W) = 2(&C_{24}^{68} Rm \otimes Rm(X,Y,Z,W)\\ &+ C_{24}^{68} Rm \otimes Rm(X,Z,Y,W)\\ &- C_{24}^{68} Rm \otimes Rm(X,Y,W,Z)\\ &- C_{24}^{68} Rm \otimes Rm(X,W,Y,Z))\end{aligned}$$

• *Step 2.* Proof of the case k=1.
Using (8.2) we obtain the inequality

$$\frac{\partial}{\partial t} |Rm|^2 \leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3$$

Applying (8.2) to ∇Rm, we reach

$$\frac{\partial}{\partial t} \nabla Rm = \Delta \nabla Rm + Rm * \nabla Rm$$

and this leads to the formula

$$\frac{\partial}{\partial t} |\nabla Rm|^2 \leq \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + C|Rm||\nabla Rm|^2$$

Next define $F := t|\nabla Rm|^2 + A|\nabla Rm|^2$, being A a constant. Next find upper bounds for this function F and its time derivatives. This helps us to get an upper bound for $|\nabla Rm|$.

• *Step 3.* Proof of the general case.

- Use (8.2) to find an evolution equation of $\nabla^k Rm$.

- Suppose by induction that $|\nabla^k Rm| \leq C_k M/t^{k/2}$. Next we have to prove the same for $k+1$.
- Obtain upper bounds on $\frac{\partial}{\partial t}|\nabla^k Rm|^2$ and $\frac{\partial}{\partial t}|\nabla^{k+1} Rm|^2$.
- Define $F_k := t|\nabla^{k+1} Rm|^2 + A_k|\nabla^k Rm|^2$ and find upper bounds for $\frac{\partial}{\partial t}F_k$ and F_k. Finally, this gives the estimate on $|\nabla^{k+1} Rm|$ we were looking for. □

Remark 8.3.1 *Notice that the derivative estimates deteriorate as t goes to 0, but this is the best we can do without further assumptions on the initial metric.*

To complete step 2 only remains to prove the estimation for time derivatives.

Corollary 8.4 *There exists constants $C_{j,k}$ such that if the curvature is bounded $|Rm| \leq \rho$ then the space-time derivatives are bounded*

$$|\frac{\partial}{\partial t}^j \nabla^k Rm| \leq \frac{\rho\, C_{j,k}}{t^{j+(k/2)}}$$

Next we state the results about the bounds of the metric corresponding to step 3.

Proposition 8.5 *Let $(M, g(t))$ be a solution of the Ricci flow on a compact n-dimensional manifold with a fixed background metric $\bar{g}$ and connection $\bar{\nabla}$. If there exists $\rho > 0$ such that*

$$|Rm(x,t)|_g \leq \rho \qquad \text{for all } x \in M \text{ and } t \in [0,T),$$

then there exists for every $k \in \mathbb{N}$ a constant C_k depending on k, n, K, T, g_0, and the pair $(\bar{g}, \bar{\nabla})$ such that

$$|\bar{\nabla}^k g(x,t)|_{\bar{g}} \leq C_k \qquad \text{for all } x \in M \text{ and } t \in [0,T).$$

Corollary 8.6 *Let $(M^n, g(t))$ be a solution of the Ricci flow. If there exists a constant K such that $|Ric| \leq K$ on the time interval $[0,T]$, then*

$$e^{-2KT} g(x,0) \leq g(x,t) \leq e^{2KT} g(x,0)$$

for all $x \in M$ and $t \in [0,T]$.

In particular, last result shows that we obtain positive definite metrics under the Ricci flow.

8.2.1. Some more comments

On the other hand, there are some situations in which singularities develop at finite time as shows the following lemma.

Lemma 8.7 *Let* (M, g_t), $0 \le t \le T$ *be a solution of the unnormalized Ricci flow on a compact n-dimensional manifold. If there are* $t_o \ge 0$ *and* $\rho > 0$ *such that*

$$\inf_{x \in M} R(x, t_0) = \rho,$$

then g_t *becomes singular in finite time.*

Idea of the proof. The lemma follows by using the evolution equation of scalar curvature under the Ricci flow, namely,

$$\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2,$$

and applying the weak maximum principle. □

8.3. Case $\dim(M) = 3$

3-dimensional case was the first problem discussed by Hamilton. Consider now the normalized Ricci flow equation for $n = 3$

$$\frac{\partial}{\partial t}\widetilde{g}_{ij} = \frac{2}{3}\widetilde{r}\,\widetilde{g}_{ij} - 2\tilde{R}_{ij}$$

Then it is possible to prove the following result.

Lemma 8.8 *If* $(M, g(t))$ *is a solution of the normalized Ricci flow on a closed 3-manifold of initially positive Ricci curvature, then there exist positive constants* $C < \infty$ *and* $\delta > 0$ *such that*

$$\left| \widetilde{Ric} - \frac{1}{3}\widetilde{R}\widetilde{g} \right| \le Ce^{-\delta t}$$

for all positive time.

Last lemma allows us to prove that all derivatives of the curvature decay exponentially.

Theorem 8.9 *For every* $n > 0$ *we have* $\max_M |\Delta^n \widetilde{Ric}| \le Ce^{-\delta t}$ *for some constants* $C < \infty$ *and* $\delta > 0$ *depending on* n.

In short, last theorem means that $g(t)$ converges exponentially fast in every C^m norm to a smooth Einstein metric g_∞ [28]. Then, since $\dim(M) = 3$, the metric g_∞ has constant sectional curvature.

[28] Recall that a metric is Einstein if the Ricci tensor is proportional to the metric, that is,

$$R_{ij} = \frac{1}{n} R g_{ij}$$

When $n \ge 3$, this implies the scalar curvature R is constant.

9. RICCI FLOW ON SURFACES (I)

9.1. The normalized Ricci flow on surfaces

Recall the normalized Ricci flow equation

$$\frac{\partial g_{ij}}{\partial t} = \frac{2}{n} r \; g_{ij} - 2 \; R_{ij} \tag{9.1}$$

When $n = 2$ the scalar curvature R and the Gaussian curvature K are related by $R = 2K$, and the Ricci curvature is given by [29]

$$R_{ij} = K g_{ij} = \frac{R}{2} g_{ij} \tag{9.2}$$

By substitution of (9.2) into the normalized Ricci flow equation (9.1) we obtain the following equation for the metric

$$\frac{\partial g_{ij}}{\partial t} = (r - R) g_{ij} \tag{9.3}$$

Notice that the change in the metric is pointwise a multiple of the metric, so the conformal structure is preserved. The term r in the equation is added to keep the area of the surface constant (cf. the comments around (5.6)).

The integral of R over a surface M gives the Euler characteristic $\chi(M)$ by the Gauss-Bonnet formula $\displaystyle\int R \, d\mu = 4\pi\chi(M)$ and, as a consequence, on a surface we see that r is constant even respect to t. Indeed

$$r = \frac{4\pi\chi(M)}{A}, \quad A \text{ being the area of } M$$

The equation (9.3) makes perfectly good sense in higher dimension, but differs from the Ricci flow. It is in fact the gradient flow for the *Yamabe problem*, where we fix the conformal structure and the volume and try to minimize the mean scalar curvature r.

When the metric g_{ij} evolves, so does its scalar curvature. The equation for its evolution is

Proposition 9.1 *when the metric g evolves following the normalized Ricci flow, the scalar curvature R associated to g satisfies the equation:*

$$\frac{\partial R}{\partial t} = -\Delta_g R + R(R - r) \tag{9.4}$$

[29] Although the Ricci curvature is always a multiple of the metric on a 2-manifold, this does not imply that R is constant (see [59] p.125). Thus the notion of an Einstein metric makes sense only when $n \geq 3$.

Proof. Using isothermal coordinates [30], the metric of a surface can be written locally as

$$ds^2 = \Lambda(du_1^2 + du_2^2), \qquad \Lambda \equiv \Lambda(u_1, u_2) \tag{9.5}$$

(that is, we can always consider the metric to be conformal to the standard euclidean metric). In these new coordinates, (9.3) can be expressed as

$$\frac{\partial \Lambda}{\partial t} = (r - R)\Lambda \qquad \text{i.e.} \qquad \frac{\partial \ln \Lambda}{\partial t} = r - R \tag{9.6}$$

On the other hand, it is possible to check (substituting $g_{11} = g_{22} = \Lambda, g_{12} = g_{21} = 0$ in the formula for the Gaussian curvature obtained in the Gauss Egregium Theorem) the following formula

$$K = \frac{1}{2}\frac{\Delta_E \ln \Lambda}{\Lambda}, \qquad \text{so} \qquad R = \frac{\Delta_E \ln \Lambda}{\Lambda} \tag{9.7}$$

where $\Delta_E = -(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2})$ denotes the standard euclidean laplacian.

In dimension 2, in isothermal coordinates, we have $\sqrt{g} = \Lambda$, $g^{11} = g^{22} = \frac{1}{\Lambda}$ and, by substitution in formula (4.7) for the laplacian, we obtain

$$\Delta_g f = \frac{\Delta_E f}{\Lambda} \tag{9.8}$$

We are already in position to deduce the expression (9.4). Derivating (9.7) respect to t

$$\frac{\partial R}{\partial t} = \frac{\Delta_E(\frac{\partial}{\partial t} ln\Lambda)}{\Lambda} - \frac{1}{\Lambda^2}\frac{\partial \Lambda}{\partial t}\Delta_E ln\Lambda \tag{9.9}$$

From (9.6)

$$\begin{aligned}
\frac{\partial R}{\partial t} &= \frac{\Delta_E(r - R)}{\Lambda} - \frac{1}{\Lambda}\frac{\partial ln\Lambda}{\partial t}\Delta_E ln\Lambda \\
&= \frac{\Delta_E(r - R)}{\Lambda} - (r - R)\frac{\Delta_E ln\Lambda}{\Lambda} \\
&= \frac{\Delta_E(r - R)}{\Lambda} - (r - R)R
\end{aligned}$$

[30]The proof of the existence of isothermal coordinates systems for any regular surface can be found in Bers, L. Riemann Surfaces. New York University, Institute of Mathematical Sciences. New York, 1957-1958 pp.15-35.

where the last equality follows from (9.7). So, using (9.8), we obtain (9.4). □

Note that (9.4) is a non linear parabolic equation. An equation of this type is called of reaction-diffusion and it appears frequently in mathematical ecology, chemistry, etc. The laplacian term in (9.4) is causing the diffusion of R. If the equation only contained this term, then the evolution would be the heat equation and R would tend to a constant as t tends to ∞. The quadratic terms in R represent the reaction terms, if the equation only contained this term, then the solution would blow up in finite time for any initial data satisfying $R(0) > \max\{r, 0\}$. How the scalar curvature behaves under the normalized Ricci flow depends on weather the diffusion or the reaction term dominates.

From now on, when there is no possibility of confusion, we shall use the notation $\Delta \equiv \Delta_g$.

9.2. The reaction equation

Fix a point $x \in M$ and consider the ordinary differential equation associated to (9.4)

$$\frac{d}{dt}R = R(R - r). \tag{9.10}$$

9.2.1. Analytic solution

The first question is to determine the growth of the solution of this equation. With this aim, we are going to solve it in an explicit way and with initial datum $R(0) = R_0$. We denote $R' := \dfrac{dR}{dt}$.

When $r \neq 0$ and $R_0 \neq 0$, we do the change of variable $z = \dfrac{1}{R}$ (so $z' = -\frac{1}{R^2}R'$ and $z(0) = \frac{1}{R_0}$). Dividing equation (9.10) by R^2, we obtain $\dfrac{R'}{R^2} = 1 - \dfrac{r}{R}$. Applying the change of variable, the result is $z' = rz - 1$. So we have now a first order Cauchy problem which can be solved using

Lagrange formula[31]

$$\begin{aligned} z(t) &= e^{\int_0^t r\,du}\left(\frac{1}{R_0} + \int_0^t e^{-\int_0^u r\,dw}(-1)du\right) = e^{rt}\left(\frac{1}{R_0} + \int_0^t -e^{-ru}\,du\right) \\ &= e^{rt}\left(\frac{1}{R_0} + \left(\frac{e^{-ru}}{r}\right)_0^t\right) = e^{rt}\left(\frac{1}{R_0} + \frac{1}{r}(e^{-rt}-1)\right) \\ &= \frac{e^{rt}}{r}\left(\frac{r}{R_0} + (e^{-rt}-1)\right) = \frac{1}{r}\left(e^{rt}\left(\frac{r}{R_0}-1\right)+1\right) \end{aligned}$$

Therefore, the solution of equation (9.10) when $r \neq 0$ and $R_0 \neq 0$ can be written as

$$R(t) = \frac{r}{1-(1-\frac{r}{R_0})e^{rt}} \tag{9.11}$$

If $r = 0$, we have the equation $R' = R^2$. Using the same change of variable as above, we obtain $z' = -1$; integrating both sides of this equality, $z(t) = -t + C$, where C is a constant of integration which can be determinated using the initial condition. In fact, $C = z(0) = \frac{1}{R_0}$. So $z(t) = -t + \frac{1}{R_0} = \frac{-R_0 t + 1}{R_0}$. Substituting this into the formula of the change of variable $R = \frac{1}{z}$, we arrive to the solution of equation (9.10) when $r = 0$

$$R(t) = \frac{R_0}{1 - R_0 t} \tag{9.12}$$

If $R_0 = 0$ we are going to solve $z' = rz - 1$ using the Lagrange formula with undefined integrals[32]. After rearranging the constants of integration obtained by applying this formula, the result is $z(t) = \frac{re^{rt}C + K}{r}$ and so $R(t) = \frac{r}{re^{rt}C + K}$. Using the initial datum, we have $0 = R(0) = \frac{r}{rC + K}$ and this implies $r = 0$. Therefore, substituting into the formula obtained for $R(t)$, we obtain

$$R(t) \equiv 0 \qquad \text{when } R_0 = 0$$

[31] In general, an ODE of the form $y' = a(x)y + b(x)$ with initial data $y(x_0) = y_0$ is called first order Cauchy problem. For solving it, we use the Lagrange formula

$$y(x) = e^{\int_{x_0}^x a(u)du}(y_0 + \int_{x_0}^x e^{-\int_{x_0}^u a(w)dw} b(u)du)$$

[32] $y(x) = e^{\int a(u)\,du}(c + \int e^{-\int a(u)du} b(u)\,du)$

Looking at the solution of equation (9.10), we conclude that, for all values of r, the solution blows up in finite time when $R_0 > \max\{r, 0\}$. In fact, if $r = 0$,

$$\lim_{t\to T} \frac{R_0}{1 - R_0 t} = \infty \quad \text{if and only if} \quad T = \frac{1}{R_0}$$

Since $R_0 > 0$, T is a finite positive number.

If $r \neq 0$,

$$\lim_{t\to T} \frac{r}{1 - (1 - \frac{r}{R_0})e^{rt}} = \infty \quad \text{if and only if} \quad T = \frac{1}{r}\ln(1 - \frac{r}{R_0})$$

Since $R_0 > \max\{r, 0\}$, T is finite and greater than zero.

Hence we cannot obtain an upper bound for the curvature under the normalized Ricci flow by directly solving equation (9.10).

On the other hand, the ODE behaves much better when $R_0 < \min\{r, 0\}$, in which case we have:

$$R(t) \geq R_0$$

In fact, if $r \neq 0$ then

$$R(t) = \frac{rR_0}{R_0 - R_0 e^{rt} + re^{rt}} \geq R_0 \text{ iff } A(t) := \frac{r}{R_0 - R_0 e^{rt} + re^{rt}} \leq 1$$

Here we distinguish two possibilities:

• $r > 0$. In this case we write $A(t) = \dfrac{r}{re^{rt} + |R_0|(1 - e^{rt})}$. Note that $r > 0$ implies e^{rt} is an increasing function and, moreover, it is equal to 1 when $t = 0$, so $e^{rt} \geq 1$ and thus $re^{rt} + |R_0|(1 - e^{rt}) \geq r + |R_0|(1 - e^{rt}) \geq r$. Because of this remark, we can conlude $A(t) \leq 1$.

• $r < 0$. Now we have

$$\begin{aligned} A(t) &= \frac{-r}{-re^{rt} - R_0(1 - e^{rt})} = \frac{-r}{[r(1 - e^{rt}) - r] - R_0(1 - e^{rt})} \\ &= \frac{-r}{-r + (r - R_0)(1 - e^{rt})}. \end{aligned}$$

At $t = 0$, $e^{rt} = 1$ and e^{rt} is a decreasing function of t because $r < 0$. Therefore $1 - e^{rt} \geq 0$. On the other hand, by election of R_0, we have $r - R_0 > 0$. As a consequence of these two facts, $-r + (r - R_0)(1 - e^{rt}) \geq -r$ and thus $A(t) \leq 1$.

It remains to study what happens when $r = 0$. In this situation the solution of the reaction equation is

$$R(t) = \frac{R_0}{1 - R_0 t}$$

Now we can write the following chain of equivalences:

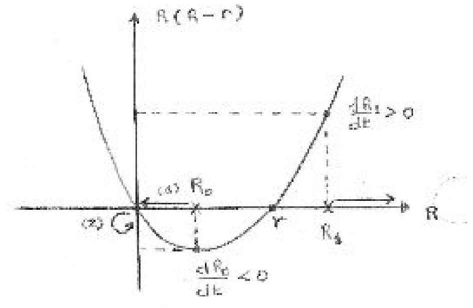
$$R(t) \geq R_0 \quad \text{iff} \quad \frac{1}{1 - R_0 t} \leq 1 \quad \text{iff} \quad - R_0 t \geq 0$$

and the last equality is true by the election of R_0.

Since r is the mean value of the scalar curvature, we cannot have $R(0) \leq \min\{r, 0\}$ nor $R(0) \geq \max\{r, 0\}$ at all points of M. Then we need to consider also points with $R(0)$ between 0 and r. For it, now we do an heuristic-qualitative study of equation (9.10) which includes these cases.

9.2.2. Heuristic study

Let us study how varies $\frac{d}{dt}R$ in function of R in the equation (9.10) looking at the graphic of $R(R - r)$ as a function of R.



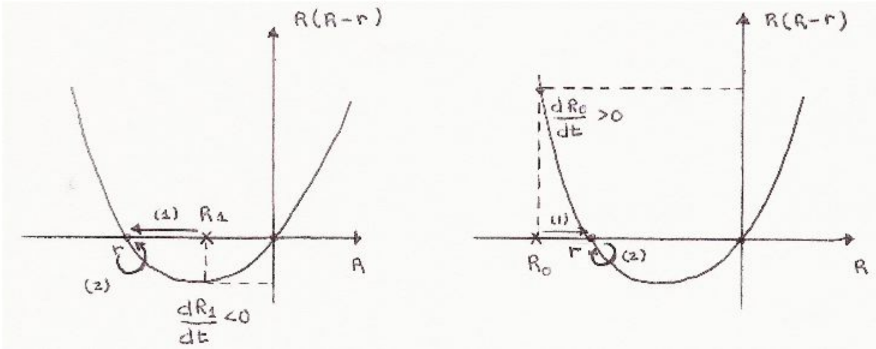
• Case $R(0) \geq 0$ at every $x \in M$ (which implies $r \geq 0$)

At the points $x \in M$ where $R(0) = R_0 < r$ as in the figure, we see $\frac{dR_0}{dt} < 0$, so R decreases. After the value zero, $\frac{dR}{dt} > 0$; then R increases, but now the derivative is again negative and therefore R have to increase. In short, it seems that the value of the scalar curvature tends to zero even if we take R_0 as close to r as we want.

Now, let us consider the points in M where $R(0) = R_1 > r$. In this case the derivative is always positive and R increases all the time, so R goes to ∞.

In conclusion, $R = r$ is a fixed point for the reaction equation (i.e. a solution of it) that is repulsive in the sense that, if we take an initial value for the scalar curvature in a neighbourhood of r, at any case R moves away from r in the course of time.

• Case $R(0) \leq 0$ at every $x \in M$ (which implies $r \leq 0$)



First, let us consider the points where $R(0) = R_0 \leq r$, then $\dfrac{dR_0}{dt} > 0$ and, therefore, R increases until r. After that, the derivative is negative, so R decreases; but beyond r we have $\dfrac{dR}{dt} > 0$ and R would increase again. Then R tends to r.

At the points where $R(0) = R_1 > r$ the situation is similar: $\dfrac{dR_1}{dt} < 0$ so R_1 decreases and after arriving to r, it begins to increase since $\dfrac{dR}{dt} > 0$; but on the right of r the derivative is again negative, so R decreases. Finally, in this case R tends to r too.

In short, when $R(0) < 0$, r is an attractive fixed point of the reaction equation.

9.2.3. Conclusion

Both from the precise solutions of (9.10) and the above heuristic arguments, we conclude that, for (9.10), if R is nonnegative (resp. nonpositive) at the beginning, it remains so for all time.

9.3. The complete evolution equation of R

Applying any of the maximum principles stated in Theorems 7.8 or 7.10, we obtain from the conclusion 9.2.3 for equation (9.10) that

Theorem 9.2 *In the normalized Ricci flow for surfaces, if, at every point in M, $R \geq 0$ at the start, it remains so for all time. Likewise if, at every point in M, $R \leq 0$ at the start it remains so for all time. Thus both positive and negative curvature are preserved for surfaces.*

For negative scalar curvature and if we have restrictions on R, we can tell more.

Theorem 9.3 *If the scalar curvature of a Riemannian manifold satisfies $-c \leq R \leq -\varepsilon < 0$ at a starting time $t = 0$ (for some $c > 0$, $\varepsilon > 0$), then it* *remains so and* $re^{-\varepsilon t} \leq r - R \leq ce^{rt}$*. So, taking limits when* t *goes to* ∞*,* R *tends to* r *exponentially.*

Proof. First assertion follows from the maximum principle. Next we write equation (9.4) specifying all the variables involved

$$\frac{\partial R(x,t)}{\partial t} = -\Delta_{g(x,t)} R(x,t) + (R(x,t) - r)R(x,t) \tag{9.13}$$

where $x \in M$ and $t \in \mathbb{R}^+ \bigcup \{0\}$.

We define $R_{max}(t) = \sup_{x \in M} R(x,t)$. Since M is a compact manifold, this supremum is actually a maximum. Then, at the points x where $R(x,t) = R_{max}(t)$,

$$-\Delta_{g(t)} R(x,t) \leq 0, \tag{9.14}$$

then, it follows from (9.13) and lemma 7.5 that R_{max} is a Lipschitz function and its upper right derivative satisfies

$$D^+ R_{max}(t) \leq \sup_{x \in M(t)} \frac{\partial R}{\partial t}(x,t) \leq \sup_{x \in M(t)} (R(x,t) - r)R(x,t),$$

where $M(t) = \{x \in M : R(x,t) = R_{max}(t)\}$ and the last equality follows from (9.13) and (9.14). So, by definition of $M(t)$,

$$D^+ R_{max} \leq R_{max}(R_{max} - r) \leq -\varepsilon(R_{max} - r)$$

Using the same rule of derivation as in the proof of Corollary 7.3, we get

$$\begin{aligned} D^+((R_{max} - r)e^{\varepsilon t}) &\leq \varepsilon e^{\varepsilon t}(R_{max} - r) + e^{\varepsilon t} D^+(R_{max} - r) \\ &\leq \varepsilon e^{\varepsilon t}(R_{max} - r) - \varepsilon e^{\varepsilon t}(R_{max} - r) = 0 \end{aligned}$$

Arguing as in Lemma 7.1 with $f(t) = R_{max}(t) - r$, we obtain

$$(R_{max}(t) - r)e^{\varepsilon t} \leq R_{max}(0) - r$$

Therefore

$$R - r \leq R_{max} - r \leq (R_{max}(0) - r)e^{-\varepsilon t},$$

that is,

$$r - R \geq (r - R_{max}(0))e^{-\varepsilon t} \geq re^{-\varepsilon t} \tag{9.15}$$

The last inequality is true because R_{max} is negative.

Analogously, using a result similar to lemma 7.5 for the lower right derivative, and having account that, at the points x where $R(x,t) = R_{min}(t)$, $-\Delta_{g(t)}R(x,t) \geq 0$, we obtain from (9.13) that

$$D_+R_{min} \geq R_{min}(R_{min} - r) \geq r(R_{min} - r),$$

being $R_{min}(t) = \min_{x\in M} R(x,t)$, because $R_{min} - r \leq 0$.

On the other hand,

$$\begin{aligned} D_+((R_{min} - r)e^{-rt}) &\geq -re^{-rt}(R_{min} - r) + e^{-rt}D_+(R_{min} - r) \\ &\geq -re^{-rt}(R_{min} - r) + re^{-rt}(R_{min} - r) = 0 \end{aligned}$$

Then, using Corollary 7.2, we obtain

$$(R_{min} - r)e^{-rt} \geq R_{min}(0) - r \geq R_{min}(0),$$

because $r < 0$, which is a consequence of $R < 0$

In short,

$$r - R \leq r - R_{min} \leq -R_{min}(0)e^{rt} \leq ce^{rt} \tag{9.16}$$

Note that the last inequality follows from $R_{min} \geq -c$.

The theorem follows from (9.15) and (9.16). □

From Theorem 9.3 and the long time existence theorem, we obtain

Corollary 9.4 *On a compact surface, if $R < 0$ then the solution exists for all time and converges exponentially to a metric of constant negative curvature.*

For positive curvature, the situation is much worse, because $R = r$ is a repulsive fixed point for the ODE

$$\frac{dR}{dt} = R^2 - rR$$

and hence the reaction term in (9.10) is fighting strongly against the diffusion term (cf. the pictures in section 9.2.2).

9.4. Ricci solitons

To improve the results when $R_0 > 0$ (and also when $r < 0$ but with $R_0 > 0$ at some points) we need better methods. With this aim, we will introduce a function that will help us to find estimations of the curvature. First we shall introduce the concept of Ricci soliton, trying to understand why it is natural the introduction of such a function.

Definition 9.1 *A solution $g(t)$ of the Ricci flow is called a (steady) Ricci soliton if there exists a 1-parameter family $\{\varphi_t\}$ of diffeomorphisms of M such that*

$$g(t) = \varphi_t^* g(0) \tag{9.17}$$

Remark 9.1.1 *Equation (9.17) means that we obtain no new metric under the Ricci flow because the solution metric $g(t)$ is the same as the initial metric $g(0)$ with a change of coordinates φ_t.*

Differentiating equation (9.17) with respect of time (which is not a direct computation, because φ_t is not necessarily a 1-parametric local group, see [37]), we have[33]

$$\frac{dg}{dt} = \mathcal{L}_V g$$

where V_t is the family of vector fields generated by φ_t[34].

From the normalized Ricci flow equation on surfaces, we have

$$(r - R)g_{ij} = \nabla_i V_j + \nabla_j V_i$$

where $\nabla_i V_j := dx^k(\nabla_{\partial_i} V)g_{kj}$. In fact[35]

$$\begin{aligned}
(\mathcal{L}_V g)_{ij} &= V g_{ij} - g([V, \partial_i], \partial_j) - g(\partial_i, [V, \partial_j]) \\
&= V g_{ij} - g(\nabla_V \partial_i, \partial_j) + g(\nabla_{\partial_i} V, \partial_j) - g(\partial_i, \nabla_V \partial_j) + g(\partial_i, \nabla_{\partial_j} V) \\
&= g(\nabla_{\partial_i} V, \partial_j) + g(\partial_i, \nabla_{\partial_j} V)
\end{aligned}$$

If we take $V = -\mathrm{grad} f$, where f is some function which depends on time, we obtain the expression

$$(R - r)g_{ij} = 2\nabla^2_{ij} f$$

[33]In general, given a covariant tensor field τ on M, we define the *Lie derivative* of τ with respect to a smooth vector field X on M by

$$(\mathcal{L}_X \tau)_p := \frac{\partial}{\partial t}|_{t=0}(\theta_t^* \tau)_p = \lim_{t \to 0} \frac{\theta_t^* \tau_{\theta_t(p)} - \tau_p}{t},$$

where θ_t is the local 1-parametric group of diffeomorphisms associated to X

[34]$V_t(\varphi_t(p)) = \frac{\partial}{\partial s}\big|_{s=t}(\varphi_s(p))$.

[35]In general, if X is a smooth vector field and σ is a smooth covariant tensor field, then $\mathcal{L}_X \sigma$ can be computed by the following expression:

$$(\mathcal{L}_X \sigma)(Y_1, \ldots, Y_k) = X(\sigma(Y_1, \ldots, Y_k)) - \sigma([X, Y_1], Y_2, \ldots, Y_k) - \ldots - \sigma(Y_1, \ldots, Y_{k-1}, [X, Y_k])$$

This follows from elementary properties of the Lie derivative as we can see in [60] p. 475.

In fact,

$$\begin{aligned}\nabla_i(\mathrm{grad} f)_j + \nabla_j(\mathrm{grad} f)_i &= g(\nabla_{\partial_i}\mathrm{grad} f, \partial_j) + g(\nabla_{\partial_j}\mathrm{grad} f, \partial_i)\\ &= \partial_i(g(\mathrm{grad} f, \partial_j)) - g(\mathrm{grad} f, \nabla_{\partial_i}\partial_j)\\ &\quad + \partial_j(g(\mathrm{grad} f, \partial_i)) - g(\mathrm{grad} f, \nabla_{\partial_j}\partial_i)\\ &= \partial_i\partial_j f + \partial_j\partial_i f - (\nabla_{\partial_i}\partial_j)f - (\nabla_{\partial_j}\partial_i)f\\ &= 2\partial_i\partial_j f - 2(\nabla_{\partial_i}\partial_j)f = 2\nabla^2_{ij} f\end{aligned}$$

In this case the solution $g(t)$ is called a *gradient Ricci soliton.* Taking traces in the above equation, we arrive to the result

$$-\Delta f = R - r \tag{9.18}$$

Examples 9.1.1 *In two dimensions, the complete metric on the xy plane given by*

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

is a gradient Ricci soliton of positive curvature with the metric flowing in along the conformal vector field $V = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$. *This Ricci soliton is called the cigar soliton, because it is asymptotic to a flat cylinder at infinity and has maximal curvature at the origin.*

Theorem 9.5 *On a compact surface there are no soliton solutions other than constant curvature.*

9.5. Potential of the curvature

We can always solve equation (9.18) because $R - r$ has mean value zero, and the solution is unique up to a constant, so we can make f have mean value zero, that is

$$\frac{\int_M f \; d\mu}{\int_M d\mu} = 0. \tag{9.19}$$

Note that the solution is well defined even if g is not a gradient Ricci soliton. We are now in position to give next definition.

Definition 9.2 *The potential f of the curvature is the solution of the equation*

$$-\Delta f = R - r \tag{9.20}$$

with mean value zero[36].

Lemma 9.6 *Under the normalized Ricci flow, the potential of the curvature satisfies*

$$\frac{\partial}{\partial t} f = -\Delta f + rf \tag{9.21}$$

Proof. Taking isothermal coordinates, we have $g_{ij} = \Lambda\delta_{ij} \equiv \Lambda(du_1^2 + du_2^2)$.

Substituting (9.20) in (9.8), we obtain

$$\frac{\Delta_E f}{\Lambda} = r - R \tag{9.22}$$

Differentiating both sides of (9.22) respect of t,

$$\frac{\Delta_E\left(\dfrac{\partial}{\partial t} f\right)}{\Lambda} - \frac{1}{\Lambda^2}\frac{\partial \Lambda}{\partial t}\Delta_E f = -\frac{\partial R}{\partial t} = \Delta_g R - R(R - r), \tag{9.23}$$

where the last equality follows from the equation of evolution of the scalar curvature under the Ricci flow (9.4).

We can rewrite (9.23) in the following way

$$\begin{aligned}
\Delta_g\left(\frac{\partial f}{\partial t}\right) &= \frac{\partial \ln \Lambda}{\partial t}\frac{\Delta_E f}{\Lambda} + \Delta_g R - R(-\Delta_g f) \qquad \text{(from (9.8), (9.20))} \\
&= (r - R)\Delta_g f + \Delta_g R + R\Delta_g f \qquad \text{(from (9.6), (9.8))} \\
&= r\Delta_g f - \Delta_g \Delta_g f \qquad \text{(taking laplacians in (9.20))}
\end{aligned}$$

So $$\Delta_g\left(\frac{\partial f}{\partial t} - rf + \Delta_g f\right) = 0.$$

Since harmonic functions are constant on compact surfaces, we can conclude

$$\frac{\partial f}{\partial t} = -\Delta_g f + rf - b \tag{9.24}$$

for some b which is constant over space and a function only of time. By integration of (9.24) along M, and having account of (9.19), we obtain $b = 0$ □

The next step in our discussion is to introduce a new function h and a 2-covariant tensor $\mathbb{M}$.

[36]This condition, i.e., the condition that f must satisfy (9.19) does not seem to be necessary. Only R. Hamilton requires this condition in this definition

Definition 9.3 *We let*

$$h = -\Delta f + |df|^2 \tag{9.25}$$

and

$$\mathbb{M} = \nabla^2 f + \frac{1}{2}\Delta f g \tag{9.26}$$

which is the trace-free part of the second covariant derivative of f.

Lemma 9.7 *(Ricci identity). Let X, Y, Z be vector fields on (M, g), and α be a differential one-form. Then*

$$((\nabla_X \nabla_Y - \nabla_Y \nabla_X)\alpha)(Z) = \alpha(R(X, Y)Z)$$

Proof. If we take X, Y vector fields such that $\nabla_X Y\big|_p = \nabla_Y X\big|_p = 0$, we get

$$\nabla_X \nabla_Y \alpha(Z) = \nabla^2_{XY}\alpha(Z) = \nabla^2\alpha(X, Y, Z)$$

From the definition of curvature, we have

$$\begin{aligned} ((\nabla_X \nabla_Y - \nabla_Y \nabla_X)\alpha)(Z) &= \alpha(R(X, Y)Z) = \left\langle (\nabla_X \nabla_Y - \nabla_Y \nabla_X)\alpha^\sharp, Z \right\rangle \\ &= R(Y, X, \alpha^\sharp, Z) = R(X, Y, Z, \alpha^\sharp) = \alpha(R(X, Y)Z). \end{aligned}$$

□

Lemma 9.8 *(Bochner's formula). For any smooth function on (M,g), we have*

$$-\frac{1}{2}\Delta|\mathrm{grad} f|^2 = |\nabla^2 f|^2 - \langle \mathrm{grad} f, \mathrm{grad}(\Delta f)\rangle + Ric(\mathrm{grad} f, \mathrm{grad} f) \tag{9.27}$$

Proof. Choose an orthonormal basis $\{e_i\}$ of T_pM and extend it to a radial parallel orthonormal frame around p. By these elections, we have

$$\nabla_W e_i = 0 \quad \text{for every vector } W \in T_pM. \tag{9.28}$$

Moreover, using the symmetry of the Levi-Civita connection,

$$[e_i, e_j](p) = (\nabla_{e_i} e_j - \nabla_{e_j} e_i)(p) = 0 \tag{9.29}$$

Take X, Y, Z three arbitrary elements of the aforementioned frame.

Using Ricci identity with $\alpha = df$, we obtain

$$\underbrace{\nabla^2 df(X,Y,Z)}_{1} - \underbrace{\nabla^2 df(Y,X,Z)}_{2} = \underbrace{df(R(X,Y)Z)}_{3} \tag{9.30}$$

On the other hand, notice that

$$\nabla^2 df(X,Y,Z) = \nabla^2 df(X,Z,Y) \tag{9.31}$$

In fact, since df is closed [37],

$$0 = d(df)(Z,Y) = Zdf(Y) - Ydf(Z) - df([Z,Y]) = ZYf - YZf$$

So $ZYf = YZf$. Then,

$$\nabla df(Y,Z) = Ydf(Z) - df(\nabla_Y Z) = YZf = ZYf = \nabla df(Z,Y)$$

Moreover,

$$\begin{aligned}\nabla(\nabla df)(X,Y,Z) &= X(\nabla df(Y,Z)) - \nabla df(\nabla_X Y, Z) - \nabla df(Y, \nabla_X Z)\\ &= X(\nabla df(Y,Z)) = X(\nabla df(Z,Y))\\ &= \nabla(\nabla df)(X,Z,Y)\end{aligned}$$

Tracing with respect to X and Z in (9.30), we have

$$\mathrm{tr}_{XZ}(1) = \mathrm{tr}_{XZ}[\nabla^2 df(X,Y,Z)] = \mathrm{tr}_{XZ}[\nabla^2 df(X,Z,Y)] = (\mathrm{tr}\nabla^2 df)(Y)$$

$$\begin{aligned}\mathrm{tr}_{XZ}(2) = \mathrm{tr}_{XZ}[\nabla^2 df(Y,X,Z)] &= \sum_X \nabla^2 df(Y,X,X) = \sum_X Y(\nabla df(X,X))\\ &= Y[\mathrm{tr}(\nabla^2 df)] = Y(-\Delta f) = -(d\Delta f)(Y) = -\langle Y, \mathrm{grad} f\rangle\end{aligned}$$

$$\begin{aligned}\mathrm{tr}_{XZ}(3) = \mathrm{tr}_{XZ}[df(R(X,Y)Z)] &= \sum_X \langle R(X,Y)X, \mathrm{grad} f\rangle\\ &= \sum_X R(X,Y,X,\mathrm{grad} f) = \sum_X R(Y,X,\mathrm{grad} f, X) = Ric(Y, \mathrm{grad} f)\end{aligned}$$

So, we have obtained next formula

$$(\mathrm{tr}\nabla^2 df)(Y) = -\langle Y, \mathrm{grad} f\rangle + Ric(Y, \mathrm{grad} f) \tag{9.32}$$

[37] Recall the general formula $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$

On the other side, $\nabla_Y|\mathrm{grad}f|^2 = 2\langle\nabla_Y \mathrm{grad}f, \mathrm{grad}f\rangle$

$$\begin{aligned}\nabla^2_{XY}|\mathrm{grad}f|^2 &= \nabla_X\nabla_Y|\mathrm{grad}f|^2 - (\nabla_X Y)(|\mathrm{grad}f|^2)\\ &= \nabla_X\nabla_Y|\mathrm{grad}f|^2 = \nabla_X(2\langle\nabla_Y\mathrm{grad}f, \mathrm{grad}f\rangle)\\ &= 2\langle\nabla_X\nabla_Y\mathrm{grad}f, \mathrm{grad}f\rangle + 2\langle\nabla_Y\mathrm{grad}f, \nabla_X\mathrm{grad}f\rangle\end{aligned}$$

Tracing last equality and dividing by 2, we arrive to

$$-\frac{1}{2}\Delta|\mathrm{grad}f|^2 = (\mathrm{tr}\nabla^2 df)(\mathrm{grad}f) + |\nabla^2 f|^2 \tag{9.33}$$

In fact,

$$|\nabla^2 f|^2 = \sum_{X,Y}(\nabla^2 f(X,Y))^2 = \sum_{X,Y}\langle\nabla_X\mathrm{grad}f, Y\rangle^2 = \sum_X\langle\nabla_X\mathrm{grad}f, \nabla_X\mathrm{grad}f\rangle$$

$$\begin{aligned}\mathrm{tr}_{XY}\left\langle\nabla^2_{XY}\mathrm{grad}f, \mathrm{grad}f\right\rangle &= \mathrm{tr}_{XY}\left((\nabla^2_{XY}df)(\mathrm{grad}f)\right)\\ &= (\mathrm{tr}\nabla^2 df)(\mathrm{grad}f)\end{aligned}$$

Finally, taking $Y = \mathrm{grad}f$ in (9.30) and adding the resulting equality to (9.33) we reach the desired formula. □

Remark 2 *Very often from now on we shall use the fact that, in dimension 2,* $Ric = \dfrac{R}{2}g$.

Lemma 9.9 *Under the normalized Ricci flow, the evolution equation of h has the form:*

$$\frac{\partial h}{\partial t} = -\Delta h - 2|\mathbb{M}|^2 + rh \tag{9.34}$$

Proof. Using the definition of h and the equation (9.4) for the evolution of R under the Ricci flow, we begin the computation of the evolution equation of h.

$$\begin{aligned}\frac{\partial h}{\partial t} &= \frac{\partial}{\partial t}(-\Delta f + |df|^2) = \frac{\partial}{\partial t}(-\Delta f) + \frac{\partial}{\partial t}(|df|^2)\\ &= \frac{\partial}{\partial t}(R-r) + \frac{\partial}{\partial t}(|df|^2) = \frac{\partial R}{\partial t} + \frac{\partial}{\partial t}(|df|^2) = -\Delta R + R(R-r) + \frac{\partial}{\partial t}(|df|^2)\\ &= -\Delta(R-r) + (R-r)^2 + r(R-r) + \frac{\partial}{\partial t}(|df|^2)\\ &= \Delta\Delta f + (\Delta f)^2 - r\Delta f + \frac{\partial}{\partial t}(|df|^2)\end{aligned} \tag{9.35}$$

Note that in the last equality we have used the definition of the potential of the curvature.

Using isothermal coordinates, we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}(|df|^2) &= \frac{\partial}{\partial t}\left(\frac{1}{\Lambda}\sum_i(\partial_i f)^2\right) = -\frac{1}{\Lambda^2}\frac{\partial\Lambda}{\partial t}\sum_i(\partial_i f)^2 + \frac{1}{\Lambda}\sum_i 2(\partial_i f)\frac{\partial}{\partial t}(\partial_i f)\\
&= -\frac{1}{\Lambda^2}(r-R)\Lambda\sum_i(\partial_i f)^2 + \frac{2}{\Lambda}\sum_i(\partial_i f)\partial_i(\frac{\partial f}{\partial t})\\
&= (R-r)\frac{1}{\Lambda}\sum_i(\partial_i f)^2 + \frac{2}{\Lambda}\sum_i(\partial_i f)\partial_i(-\Delta f + rf - b)\\
&\qquad\text{[where we have used the evolution equation for } f]\\
&= (R-r)|df|^2 - \frac{2}{\Lambda}\sum_i(\partial_i f)\partial_i(\Delta f) + 2r\frac{1}{\Lambda}\sum_i(\partial_i f)^2\\
&= (R-r)|df|^2 - 2\left\langle \mathrm{grad} f, \mathrm{grad}(\Delta f)\right\rangle + 2r|df|^2\\
&= (R+r)|df|^2 - \Delta|df|^2 - 2|\nabla^2 f|^2 - 2Ric(\mathrm{grad} f, \mathrm{grad} f)\\
&\qquad\text{[by Bochner's formula]}\\
&= (R+r)|df|^2 - \Delta|df|^2 - 2|\nabla^2 f|^2 - R|\mathrm{grad} f|^2\\
&= r|df|^2 - \Delta|df|^2 - 2|\nabla^2 f|^2
\end{aligned}$$

Substituting in (9.35) and rearranging the formula, we get

$$\frac{\partial h}{\partial t} = \underbrace{\Delta\Delta f - \Delta|df|^2}_{-\Delta h} + \underbrace{r(-\Delta f) + r|df|^2}_{rh} - 2|\nabla^2 f|^2 + (\Delta f)^2 \tag{9.36}$$

On the other hand, we compute the norm of $\mathbb{M}$

$$\begin{aligned}
|\mathbb{M}|^2 &= |\nabla^2 f|^2 + \frac{1}{4}(\Delta f)^2|g|^2 + 2\frac{1}{2}\Delta f\left\langle \nabla^2 f,\ g\right\rangle \\
&= |\nabla^2 f|^2 + \frac{1}{2}(\Delta f)^2 - (\Delta f)^2 = |\nabla^2 f|^2 - \frac{1}{2}(\Delta f)^2
\end{aligned}\tag{9.37}$$

and (9.34) follows from (9.36) and (9.37) □

Corollary 9.10 *If $h \le C$ at the start, then $h \le C\ e^{rt}$ for all time. In particular, $R \le Ce^{rt} + r$.*

Proof. Let us consider the ODE associated to (9.34), that is,

$$\frac{dh}{dt} = -2|\mathbb{M}|^2 + rh$$

with initial condition $h(0) \leq C$.

Because of the nonpositivity of the first addend on the right, the next inequality comes true

$$\frac{dh}{dt} \leq rh$$

Solving it, we obtain $h \leq Ke^{rt}$ and, using the initial datum, we get $C = h(0) = K$. So

$$h \leq Ce^{rt}$$

Now, the scalar maximum principle (Theorem 7.10) assures that last inequality remains true when h is the solution of the PDE (9.34).

Moreover, substituting the definition of the potential f (i.e. equation (9.17)) in (9.25), we have

$$h = -\Delta f + |df|^2 = R - r + |df|^2$$

Therefore, $R = h - |df|^2 + r \leq h + r \leq Ce^{rt} + r$. □

Remark 9.10.1 *As a consequence of the last corollary, we have obtained an upper bound for R. But it tends to infinity as $t \to \infty$ in the case $r > 0$.*

We are going to find a bound from below with the aid of the maximum principle. In order to do that, we study the ODE (9.10) associated to (9.4) for R_{min} instead of R like we did in section 9.2. We can conclude the following:

- If $r \geq 0$ and $R_{min}(0) < 0$, then R_{min} increases.
- If $r \leq 0$ and $R_{min}(0) \leq r$, then R_{min} increases.

We summarize the conclusions obtained in the following theorem.

Theorem 9.11 *For any initial metric on a compact surface, there is a constant C with*

$$-C \leq R \leq Ce^{rt} + r$$

Then, applying the long time existence theorem, we obtain

Corollary 9.12 *For any initial metric on a compact surface, the Ricci flow equation has a solution for all time. Moreover, if $r \leq 0$, then the scalar curvature R remains bounded both above and below.*

Theorem 9.13 *On a compact surface with $r < 0$, for any initial metric the solution exists for all time and converges to a metric with constant negative curvature*

Thus we have proved (up to many details, mainly on the convergence of the metric) the theorem of Hamilton in the case $r < 0$.

10. RICCI FLOW ON SURFACES (II). HARNACK INEQUALITY

In this lecture we will introduce Hamilton's Harnack inequality for the scalar curvature under the normalized Ricci flow. This construction was inspired in the Li-Yau Harnack inequality for the heat equation on a Riemannian manifold.

10.1. Classical Harnack inequality

Theorem 10.1 *Let (M, g) be a compact n-dimensional Riemannian manifold with non-negative Ricci curvature. Let f be a positive solution of the heat equation*

$$\frac{\partial f}{\partial t} = -\Delta f, \quad \text{for } 0 < t < T.$$

Then for any two points (ξ_1, t_1), $(\xi_2, t_2) \in M \times]0, T[$ with $t_1 < t_2$, we have

$$t_1^{\frac{n}{2}} f(\xi_1, t_1) \leq e^{\frac{\Psi}{4}} t_2^{\frac{n}{2}} f(\xi_2, t_2), \tag{10.1}$$

where $\Psi = \dfrac{d(\xi_1, \xi_2)^2}{t_2 - t_1}$ and $d(\cdot, \cdot)$ denotes the distance in (M, g) .

Proof. Introduce $L := \ln f$, then

$$\frac{\partial L}{\partial t} = -\Delta L + |dL|^2 \tag{10.2}$$

In fact, using the expression (4.7) for the laplacian, we have

$$\begin{aligned}
\Delta L &= -\frac{1}{\sqrt{g}} \partial_j (\sqrt{g}\ g^{jk} \partial_k L) = -\frac{1}{\sqrt{g}}\ \partial_j \left(\sqrt{g}\ g^{jk} \frac{\partial_k f}{f} \right) \\
&= -\frac{1}{\sqrt{g}} \partial_j (\sqrt{g}\ g^{jk} \partial_k f) \frac{1}{f} - \frac{1}{\sqrt{g}} (\sqrt{g}\ g^{jk} \partial_k f) \left(-\frac{\partial_j f}{f^2} \right) \\
&= \frac{\Delta f}{f} + g^{jk} \frac{\partial_k f}{f} \frac{\partial_j f}{f} = \frac{\Delta f}{f} + g^{jk} \partial_k L\ \partial_j L = \frac{\Delta f}{f} + |dL|^2
\end{aligned} \tag{10.3}$$

Derivating respect of t in the definition of L, we get

$$\begin{aligned}\frac{\partial L}{\partial t} &= \frac{1}{f}\frac{\partial f}{\partial t} = -\frac{\Delta f}{f} \qquad \text{[by the heat equation]}\\ &= -\Delta L + |dL|^2 \qquad \text{[from (10.3)]}\end{aligned}$$

Next define

$$Q := \frac{\partial L}{\partial t} - |dL|^2 = -\Delta L, \tag{10.4}$$

so we have

$$\frac{\partial L}{\partial t} = Q + |dL|^2. \tag{10.5}$$

and, using Bochner's formula, let us compute

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \frac{\partial}{\partial t}(-\Delta L) = -\Delta\left(\frac{\partial L}{\partial t}\right) \underset{(10.5)}{=} -\Delta Q - \Delta(|dL|^2)\\ &\underset{(10.4)}{=} -\Delta Q - 2\langle \mathrm{grad}(\Delta L), \mathrm{grad}L\rangle + 2|\nabla^2 L|^2 + 2Ric(\mathrm{grad}L, \mathrm{grad}L)\\ &= -\Delta Q + 2\langle \mathrm{grad}Q, \mathrm{grad}L\rangle + 2|\nabla^2 L|^2 + 2Ric(\mathrm{grad}L, \mathrm{grad}L) \quad (10.6)\end{aligned}$$

By hypothesis, we have $Ric(\mathrm{grad}L, \mathrm{grad}L) \geq 0$. In order to find a lower bound for $\dfrac{\partial Q}{\partial t}$ we are going to use the well known inequality between the square of the norm and the trace of a symmetric tensor:

$$|\nabla^2 L|^2 \geq \frac{1}{n}(\mathrm{tr}\nabla^2 L)^2 = \frac{1}{n}(-\Delta L)^2 = \frac{1}{n}Q^2$$

By these remarks together with $|\nabla^2 L| \geq 0$, we return to (10.6) and write

$$\frac{\partial Q}{\partial t} \geq -\Delta Q + 2\langle \mathrm{grad}L, \mathrm{grad}Q\rangle + \frac{2}{n}Q^2 \tag{10.7}$$

By the scalar maximum principle 7.10, we obtain from (10.7) the inequality

$$Q(x,t) \geq Q_{min}(t) \geq \varphi(t), \tag{10.8}$$

where $\varphi(t)$ is the solution of the ODE

$$\frac{d\varphi}{dt} = \frac{2}{n}\varphi^2 \qquad \text{with } \varphi(0) = Q_{min}(0). \tag{10.9}$$

Since M is compact, $\displaystyle\int_M \Delta L = 0$, then, if f is not constant at the start, $Q = -\Delta L$ must be negative at some point at the start, then $\varphi(0) = Q_{min}(0) < 0$. From (10.8) and (10.9), we obtain

$$\varphi(t) \geq -\frac{n}{2t - \frac{n}{\varphi(0)}} \geq -\frac{n}{2t} \quad \text{and} \quad Q \geq -\frac{n}{2t}. \tag{10.10}$$

Because M is a compact manifold, it is also complete [38]. So it follows that there exists a minimal geodesic joining any pair of points in the manifold. Hence we can take a minimal geodesic parametrized by time t joining ξ_1 and ξ_2, that is, $\gamma : [t_1, t_2] \to M$, such that $\gamma(t_1) = \xi_1$, $\gamma(t_2) = \xi_2$ and $d(\xi_1, \xi_2) = L(\gamma|_{[t_1,t_2]})$.

Denoting $\gamma^i \equiv \gamma^i(t)$ the component functions of the geodesic, we compute (using the chain rule)

$$\begin{aligned}
\frac{dL}{dt}(\gamma(t), t) &= \frac{\partial L}{\partial t}(\gamma(t), t) + \frac{\partial L}{\partial x^i}\frac{d\gamma^i(t)}{dt} = \frac{\partial L}{\partial t}(\gamma(t), t) + dL(\gamma'(t)) \\
&= \frac{\partial L}{\partial t}(\gamma(t), t) + \langle \mathrm{grad} L, \gamma'(t)\rangle \\
&\geq -\frac{n}{2t} + |dL|^2 + \langle \mathrm{grad} L, \gamma'(t)\rangle \qquad \text{[using (10.5) and (10.10)]} \\
&\geq -\frac{n}{2t} + |dL|^2 - |\langle \mathrm{grad} L, \gamma'(t)\rangle| \\
&\geq -\frac{n}{2t} + |dL|^2 - |\mathrm{grad} L||\gamma'(t)| \qquad \text{[by Cauchy-Schwarz]} \\
&= -\frac{n}{2t} + \left(|dL| - \frac{1}{2}\left|\frac{d\gamma}{dt}\right|\right)^2 - \frac{1}{4}\left|\frac{d\gamma}{dt}\right|^2 \geq -\frac{n}{2t} - \frac{1}{4}\left|\frac{d\gamma}{dt}\right|^2.
\end{aligned} \tag{10.11}$$

Integrating along the geodesic and using the fundamental theorem of calculus,

$$\begin{aligned}
L(\gamma(t_2), t_2) - L(\gamma(t_1), t_1) &= \int_{t_1}^{t_2} \frac{dL}{dt}(\gamma(t), t)\, dt \geq -\int_{t_1}^{t_2} \frac{n}{2t}\, dt - \frac{1}{4}\int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}\right|^2 dt \\
&= -\frac{n}{2}\ln\left(\frac{t_2}{t_1}\right) - \frac{1}{4}\int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}\right|^2 dt.
\end{aligned} \tag{10.12}$$

On the other hand, it is a well-known fact that all Riemannian geodesics are constant speed curves [39]. So we have

$$d(\xi_1, \xi_2) = L(\gamma|_{[t_1,t_2]}) := \int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}\right| dt = \left|\frac{d\gamma}{dt}\right|(t_2 - t_1) \to \left|\frac{d\gamma}{dt}\right| = \frac{d(\xi_1, \xi_2)}{t_2 - t_1}.$$

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[38] From a corollary of Hopf-Rinow theorem. See [28] p. 149

[39] We say γ is *constant speed* if $|\gamma'(t)| \equiv \left|\frac{d\gamma}{dt}\right|$ is independent of t. For more details, see [59] p.70.

In conclusion, we get

$$\int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}\right|^2 dt = \int_{t_1}^{t_2} \frac{d(\xi_1,\xi_2)^2}{(t_2-t_1)^2}\, dt = \frac{d(\xi_1,\xi_2)^2}{(t_2-t_1)^2}\int_{t_1}^{t_2} dt = \frac{d(\xi_1,\xi_2)^2}{t_2-t_1} = \Psi.$$

Substituting this in (10.12), we reach

$$L(\gamma(t_2),t_2) - L(\gamma(t_1),t_1) \geq -\frac{n}{2}\ln\left(\frac{t_2}{t_1}\right) - \frac{\Psi}{4},$$

and the definition of L given at the beginning of the proof yields have

$$\ln\frac{f(\gamma(t_2),t_2)}{f(\gamma(t_1),t_1)} \geq -\frac{n}{2}\ln\left(\frac{t_2}{t_1}\right) - \frac{\Psi}{4},$$

and, taking exponentials,

$$\frac{f(\xi_2,t_2)}{f(\xi_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-\frac{n}{2}} e^{-\frac{\Psi}{4}},$$

which gives (10.1). □

10.2. Hamilton's Harnack inequality

In this section we are going to adapt last theorem for the Ricci flow on a surface. The main difference is that the metric is changing in this case; so we need a new definition for Ψ.

In the following sections we shall consider the case with $R > 0$ initially.

Definition 10.1 *On a manifold with a Riemannian metric g_t depending on time t, we define*

$$\Psi(\xi_1,t_1,\xi_2,t_2) = \inf\left\{\int_{t_1}^{t_2}\left|\frac{d\gamma}{dt}\right|^2_{g_t} dt;\ \gamma:[t_1,t_2]\longrightarrow M,\ \gamma(t_1)=\xi_1,\ \gamma(t_2)=\xi_2\right\}.$$

Remark 10.1.1 *a) When the metric is fixed, $\Psi(\xi_1,t_1,\xi_2,t_2) = \dfrac{d(\xi_1,\xi_2)^2}{t_2-t_1}$.*

b) Let γ and G be two fixed metrics independents of t, with associated distances δ and D respectively. If $\gamma \leq g_t \leq G$, then

$$\frac{\delta(\xi_1,\xi_2)^2}{t_2-t_1} \leq \Psi(\xi_1,t_1,\xi_2,t_2) \leq \frac{D(\xi_1,\xi_2)^2}{t_2-t_1}.$$

Theorem 10.2 *Suppose we have a solution of the Ricci flow equation on a compact surface with $R > 0$ for $0 < t \leq T$. Then for any two points (ξ_1, t_1) and (ξ_2, t_2) in space-time with $0 < t_1 < t_2 \leq T$, we have*

$$(e^{rt_1} - 1)R(\xi_1, t_1) \leq e^{\frac{\Psi}{4}}(e^{rt_2} - 1)R(\xi_2, t_2) \tag{10.13}$$

Proof. Let $L := \ln R$. Using the equation (9.4) for the evolution of the scalar curvature under the Ricci flow, we compute the evolution of L

$$\frac{\partial L}{\partial t} = \frac{1}{R}\frac{\partial R}{\partial t} = -\frac{\Delta R}{R} + R - r \tag{10.14}$$

On the other hand, imitating formula (10.3) with R instead of f we also have

$$-\frac{\Delta R}{R} = -\Delta L + |dL|^2 \tag{10.15}$$

Substituting last expression in (10.14), the result is

$$\frac{\partial L}{\partial t} = -\Delta L + |dL|^2 + R - r$$

Now define

$$Q := \frac{\partial L}{\partial t} - |dL|^2 = -\Delta L + R - r \tag{10.16}$$

Next we want to compute the equation for the evolution of Q.

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}(-\Delta L + R - r) = -\frac{\partial}{\partial t}(\Delta L) + \frac{\partial R}{\partial t} \tag{10.17}$$

Taking isothermal coordinates, we can develop the first addend in (10.15) in the following way:

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta L) &\underset{(9.8)}{=} \frac{\partial}{\partial t}\left(\frac{\Delta_E L}{\Lambda}\right) = \frac{1}{\Lambda}\Delta_E\left(\frac{\partial L}{\partial t}\right) - \frac{\Delta_E L}{\Lambda^2}\frac{\partial \Lambda}{\partial t} \\
&= \Delta\left(\frac{\partial L}{\partial t}\right) - \frac{\Delta_E L}{\Lambda}\frac{\partial \ln \Lambda}{\partial t} \\
&\underset{(9.6)}{=} \Delta\left(\frac{\partial L}{\partial t}\right) - \Delta L(r - R) = \Delta\left(\frac{\partial L}{\partial t}\right) + (R - r)\Delta L \\
&\underset{(10.16)}{=} \Delta(Q + |dL|^2) + (R - r)\Delta L = \Delta Q + \Delta|dL|^2 + (R - r)\Delta L
\end{aligned}$$

Substituting this in (10.17) and replacing the second addend by $-\Delta R + R(R-r)$, we have

$$\frac{\partial Q}{\partial t} = -\Delta Q - \Delta|dL|^2 - (R-r)\Delta L - \Delta R + R(R-r) \tag{10.18}$$

Applying Bochner's formula, using $Ric = (R/2)g$ and (10.16), we get

$$\begin{aligned} -\Delta|dL|^2 &= 2|\nabla^2 L|^2 - 2\,\langle \mathrm{grad}L, \mathrm{grad}(\Delta L)\rangle + 2Ric(\mathrm{grad}L, \mathrm{grad}L) \\ &= 2|\nabla^2 L|^2 + 2\,\langle \mathrm{grad}L, \mathrm{grad}(Q-(R-r))\rangle + R\,|\mathrm{grad}L|^2 \\ &= 2|\nabla^2 L|^2 + 2\,\langle dL, dQ\rangle - 2\,\langle dL, dR\rangle + R\,|dL|^2 \\ &= 2|\nabla^2 L|^2 + 2\,\langle dL, dQ\rangle - 2\,\langle dL, RdL\rangle + R\,|dL|^2 \\ &= 2|\nabla^2 L|^2 + 2\,\langle dL, dQ\rangle - R\,|dL|^2 \end{aligned}$$

Substituting the last equality in (10.18),

$$\frac{\partial Q}{\partial t} = -\Delta Q + 2|\nabla^2 L|^2 + 2\,\langle dL, dQ\rangle - R|dL|^2 - (R-r)\Delta L - \Delta R + R(R-r) \tag{10.19}$$

On the other hand, using (10.15) and (10.16),

$$\begin{aligned} -R|dL|^2 - (R-r)\Delta L - \Delta R &= -R|dL|^2 - (R-r)\Delta L - R\Delta L + R|dL|^2 = \\ &= -2(R-r)\Delta L - r\Delta L = -2(R-r)\Delta L + rQ - r(R-r) \end{aligned}$$

So, returning to (10.19), we reach

$$\begin{aligned} \frac{\partial Q}{\partial t} &= -\Delta Q + 2|\nabla^2 L|^2 + 2\,\langle dL, dQ\rangle - 2(R-r)\Delta L + rQ\,\underbrace{-r(R-r) + R(R-r)}_{-rR+r^2+R^2-rR} \\ &= -\Delta Q + 2|\nabla^2 L|^2 + 2\,\langle dL, dQ\rangle + \underbrace{(\Delta L)^2 - 2(R-r)\Delta L + (R-r)^2}_{Q^2} - (\Delta L)^2 + rQ \\ &= -\Delta Q + 2|\nabla^2 L|^2 - (\Delta L)^2 + 2\,\langle dL, dQ\rangle + Q^2 + rQ \end{aligned}$$

where $2|\nabla^2 L|^2 - (\Delta L)^2 \geq 0$ because we are working in dimension 2. Taking this into account, we have found a lower bound for $\dfrac{\partial Q}{\partial t}$:

$$\frac{\partial Q}{\partial t} \geq -\Delta Q + 2\,\langle dL, dQ\rangle + Q^2 + rQ \tag{10.20}$$

Arguing in the same way that when we obtained (10.10) from (10.7), from (10.20) we get

$$Q \geq -\frac{re^{rt}}{e^{rt}-1}$$

Now choose any path σ joining ξ_1 and ξ_2 and parametrized by time t for $t_1 \le t \le t_2$, and compute using the chain rule

$$\frac{dL}{dt}(\sigma(t),t) = \frac{\partial L}{\partial t}(\sigma(t),t) + \frac{\partial L}{\partial x^i}\frac{d\sigma^i(t)}{dt} \underset{(10.16)}{=} Q + |dL|^2 + \frac{\partial L}{\partial x^i}\frac{d\sigma^i(t)}{dt}.$$

Integrating between t_1 and t_2,

$$L(\sigma(t_2),t_2) - L(\sigma(t_1),t_1) = \int_{t_1}^{t_2} \frac{dL}{dt}(t,\sigma(t))\,dt$$
$$\ge \int_{t_1}^{t_2} \left\{ |dL|^2 - \frac{re^{rt}}{e^{rt}-1} + \frac{\partial L}{\partial x^i}\frac{d\sigma^i(t)}{dt} \right\} dt \ge -\ln\frac{e^{rt_2}-1}{e^{rt_1}-1} - \frac{1}{4}\int_{t_1}^{t_2} \left|\frac{d\sigma}{dt}\right|^2 dt,$$

where where the last inequality follows from computations similar to that used to obtain (10.11). Now the theorem follows by exponentiation, having account of the definition of Ψ. $\square$

10.3. Entropy

Definition 10.2 *If $R(\cdot,0) > 0$ for a metric g_t under the normalized Ricci flow, we define the entropy N by:*

$$N(g_t) = \int_M R_{g_t} \ln R_{g_t}\, d\mu_t \qquad (10.21)$$

It is well defined, because Theorem 9.2 assures that $R(.,t) > 0$ for $t > 0$.

Remark 10.2.1 *This quantity is called entropy because it resembles other quantities (also called entropy in Physics) which are the integral of a positive function times its logarithm.*

Theorem 10.3 *For the Ricci flow on a compact surface with $R > 0$ at the starting time, the entropy is decreasing.*

Proof. From (5.5) and (9.4) we obtain

$$\frac{\partial}{\partial t}(R\,d\mu) = (-\Delta R + R(R-r))\,d\mu + R(r-R)\,d\mu = -\Delta R\,d\mu, \qquad (10.22)$$

then

$$\begin{aligned}\frac{dN}{dt} &= \int_M \frac{\partial}{\partial t}(\ln R)\, R\, d\mu + \int_M \ln R\, \frac{\partial}{\partial t}(R\, d\mu)\\ &= \int_M (-\Delta R + R(R-r))\, d\mu - \int_M \ln R\, \Delta R\, d\mu\\ &= \int_M (R(R-r) - \ln R\, \Delta R)\, d\mu\\ &= \int_M \left(R(R-r) - \frac{|\mathrm{grad} R|^2}{R}\right) d\mu, \qquad (10.23)\end{aligned}$$

because $\displaystyle\int_M \ln R\, \Delta R\, d\mu = \int_M \langle d\ln R,\ dR\rangle\, d\mu = \int_M \frac{|\mathrm{grad}R|^2}{R}\, d\mu.$

On the other hand, since $\displaystyle\int_M (R-r)r\, d\mu = 0,$

$$\int_M R\,(R-r)\, d\mu = \int_M R(R-r)\, d\mu - \int_M (R-r)r\, d\mu = \int_M (R-r)^2\, d\mu \qquad (10.24)$$

$$\begin{aligned}&= \int_M (-\Delta f)^2\, d\mu = \int_M \langle df,\ d\Delta f\rangle\, d\mu\\ &= \int_M \left(|\nabla^2 f|^2 + Ric(\mathrm{grad} f, \mathrm{grad} f)\right)\, d\mu \text{ [by the Bochner formula (9.27)]}\\ &= \int_M \left(|\nabla^2 f|^2 + \frac{R}{2}\, |\mathrm{grad} f|^2\right)\, d\mu. \qquad (10.25)\end{aligned}$$

Moreover, from (9.37),

$$\begin{aligned}\int_M |\mathbb{M}|^2\, d\mu &= \int_M \left(|\nabla^2 f|^2 - \frac{1}{2}(\Delta f)^2\right)\, d\mu\\ &\underset{(10.25)}{=} \int_M R(R-r)\, d\mu - \int_M \frac{R}{2}|\mathrm{grad} f|^2\, d\mu - \frac{1}{2}\int_M (R-r)^2\, d\mu\\ &\underset{(10.24)}{=} \frac{1}{2}\int_M \left(R\,(R-r) - R\,|\mathrm{grad} f|^2\right)\, d\mu. \qquad (10.26)\end{aligned}$$

Now, we compute the expression

$$\begin{aligned}\int_M \frac{|\mathrm{grad}R + R\,\mathrm{grad} f|^2}{R}\, d\mu &= \int_M \frac{|\mathrm{grad}R|^2 + 2R\,\langle \mathrm{grad}R, \mathrm{grad} f\rangle + R^2|\mathrm{grad} f|^2}{R}\, d\mu\\ &= \int_M \frac{|\mathrm{grad}R|^2}{R} d\mu + \int_M R\,|\mathrm{grad} f|^2\, d\mu + 2\int_M \langle \mathrm{grad} f, \mathrm{grad}R\rangle\, d\mu\\ &= \int_M \frac{|\mathrm{grad}R|^2}{R} d\mu + \int_M R\,|\mathrm{grad} f|^2\, d\mu - 2\int_M R\,(R-r)\, d\mu, \quad (10.27)\end{aligned}$$

since $\int_M \langle \mathrm{grad} f, \mathrm{grad} R \rangle \; d\mu = \int_M R\Delta f \; d\mu = -\int_M R \; (R-r) \; d\mu$. Multiplying (10.26) by 2, and adding with (10.27), we obtain

$$2\int_M |\mathbb{M}|^2 \; d\mu + \int_M \frac{|\mathrm{grad} R + R \; \mathrm{grad} f|^2}{R} \; d\mu = -\int_M \left(R \; (R-r) - \frac{|\mathrm{grad} R|^2}{R} \right) \; d\mu \tag{10.28}$$

From (10.28) and (10.23), we obtain

$$\frac{dN}{dt} = -2\int_M |\mathbb{M}|^2 \; d\mu - \int_M \frac{|\mathrm{grad} R + R \; \mathrm{grad} f|^2}{R} \; d\mu \tag{10.29}$$

which is negative because $R > 0$, then N is decreasing. □

10.4. Bounds on $R(t)$ when $R > 0$

In this section, we shall combine the Harnack inequality and the entropy estimate to conclude that R is bounded.

Theorem 10.4 *If we have a solution for the normalized Ricci flow with $R > 0$ on a compact surface, then there are constants $c > 0$ and $C < \infty$ with $c \leq R \leq C$ for all time.*

Proof. At time τ, pick a point ξ where the curvature R is largest. Then wait for a time $T - \tau = \dfrac{1}{2R_{max}(\tau)}$. Along that time

$$D^+ R_{max}(t) \leq \sup_{\{x\in M;\; R(x,t)\leq R_{max}(t)\}} \frac{\partial}{\partial t} R(x,t) = R_{max}(t)(R_{max}(t) - r) \leq R_{max}(t)^2,$$

then $\quad D^+\left(-\dfrac{1}{R_{max}}\right) \leq \dfrac{D^+ R_{max}(t)}{R_{max}(t)^2} \leq 1, \quad$ so, $\quad$ for every $t \in [\tau, T]$,

$-\dfrac{1}{R_{max}(t)} + \dfrac{1}{R_{max}(\tau)} \leq t - \tau \leq T - \tau = \dfrac{1}{2R_{max}(\tau)}$, and so

$$R_{max}(t) \leq 2R_{max}(\tau) \text{ for every } t \in [\tau, T]. \tag{10.30}$$

On the other hand, since

$$\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij},$$

integrating this evolution equation we obtain, for every $t \in [\tau, T]$,

$$g_t = g_\tau \exp\left(\int_\tau^t (r - R) dt \right), \quad \text{then}$$

$$g_\tau = g_t \exp\left(\int_\tau^t (R-r)dt\right) \le \exp\left((2R_{max}(\tau)-r)(t-\tau)\right) g_t \le e\ g_t,$$

Hence, if $d(\xi, X)$ is the distance at time T, we will have (cf. Remark 10.1.1.b))

$$\Psi(\xi,\tau,X,T) \le e\,\frac{d(\xi,X)^2}{T-\tau}.$$

Then the Harnack inequality (10.13) gives

$$R(\xi,\tau) \le \frac{e^{rT}-1}{e^{r\tau}-1}\exp\left(e\,\frac{d(\xi,X)^2}{4(T-\tau)}\right) R(X,T).$$

But, if g_τ has not constant curvature, $R_{max}(\tau) > r$, and $1/(2(T-\tau)) = R_{max}(\tau)$, then, if we consider X in a ball around ξ of radius $\rho = \dfrac{\pi}{\sqrt{R_{max}(\tau)}}$ at time T, we have

$$R(\xi,\tau) \le \frac{e^{rT}-1}{e^{r\tau}-1}\exp\left(e\,\frac{\pi^2}{2}\right) R(X,T). \tag{10.31}$$

On the other hand, using again that $T-\tau = 1/(2R_{max}(\tau)) \le 1/(2r)$,

$$\frac{e^{rT}-1}{e^{r\tau}-1} = \frac{e^{r\tau}e^{r(T-\tau)}-1}{e^{r\tau}-1} \le \frac{e^{r\tau}e^{1/2}-1}{e^{r\tau}-1},$$

but the function $\dfrac{e^{r\tau}e^{1/2}-1}{e^{r\tau}-1}$ is decreasing in τ, then, taking $\tau_0 < \tau$ we obtain

$$\frac{e^{rT}-1}{e^{r\tau}-1} \le \frac{e^{r\tau_0}e^{1/2}-1}{e^{r\tau_0}-1} =: C_1. \tag{10.32}$$

If we denote $C_2 := C_1\ \exp(e\pi^2/2)$, by substitution of (10.32) in (10.31), we obtain

$$R(\xi,\tau) \le C_2\ R(X,T) \tag{10.33}$$

Next we shall use this inequality to bound $\ln R_{max}(T)$ by the entropy $N(g_T)$ from above. We shall star with the entropy and we shall bound it. For it we need the following remarks:

Since $R(X,T) \le R_{max}(T) \le 2R_{max}(\tau)$, the Pogorelov-Klingenberg's estimate of the iniectivity radius[40] gives $inj(M, g_T) \ge \pi/R_{max}(\tau) = \rho$.

[40] *Klingerberg's Theorem* If all the sectional curvatures K of a Riemannian manifold M satisfy $0 < K \le \delta$, then the injectivity radius $inj(M)$ of M satisfies $inj(M) \ge \pi/\sqrt{\delta}$ (cf. [76] page 198). For the 2-dimensional case, this theorem had been proved by A. V. Pogorelov, cf. [70].

Then we can apply the Bishop comparison theorem for the area of the geodesic disk $B(\xi,\rho)$ of center ξ and radius ρ in the metric g_T to obtain, for the area, $A(B(\xi,\rho)) \geq A(B_\rho)$, where B_ρ is the geodesic ball of radius ρ in the 2-sphere of sectional curvature $R_{max}(\tau)$, which is all the 2-sphere, then

$$A(B(\xi,\rho)) \geq \frac{4\,\pi}{R_{max}(\tau)}. \tag{10.34}$$

Let us choose ξ satisfying $R_{max}(\tau) = R(\xi,\tau)$. Using (10.30), (10.33) and (10.34), we can estimate the entropy

$$N(g_T) = \int_M R\;\ln R\; d\mu_T \geq \int_{B(\xi,\rho)} R\;\ln R\; d\mu_T \geq \frac{4\pi}{C_2}\ln\left(\frac{1}{2C_2}R_{max}(T)\right).$$

This inequality, combined with Theorem 10.3 shows that $R_{max}(T)$ is bounded, and hence $R_{max}(\tau)$ is bounded. Then we have shown that, gien τ_0 in the interval $[0,\varepsilon[$ where we know that the solution g_t of the normalized Ricci flow exists (by the short time existence theorem), there is a constant C_0 depending on τ_0 such that

$$R(x,t) < C_0 \text{ for every } t \geq \tau_0 \text{ and } x \in M. \tag{10.35}$$

Taking

$$C = \max\{C_0, C_3\}, \text{ where } C_3 = \max_{M\times[0,\tau_0]} R(x,t), \tag{10.36}$$

we obtain $R(x,t) < C$ for every x and t.

Applying again Pogorelov's injectivity estimate, we have $inj(M,g_t) \geq \pi/\sqrt{R_{max}(t)/2} > \pi\sqrt{2/C}$. If we have $m(t)$ points $p_1,...,.p_{m(t)} \in M$ such that $d_{g_t}(p_i,p_j) \geq \pi/\sqrt{R_{max}(t)/2}$, then the geodesic disks $B_{g(t)}(p_i,\pi/\sqrt{R_{max}(t)/2})$ are disjoint and, as above, the Bishop's comparison theorem gives $A(B_{g(t)}(p_i,\pi/\sqrt{R_{max}(t)/2}) \geq 2\pi/R_{max}(t) > 2\pi/C$. Then $A(M,g_0) = A(M,g_t)) > m(t)\;2\pi/C$. Then $m(t) < C\;A(M,g_0)/(2\pi)$, and the diameter of (M,g_t) is bounded from above by some constant $\sqrt{C_4}$. Then, we have an universal bound for $d(\xi,X)$, and the same arguments used in the proof of (10.33) give, choosing $T = \tau+\eta$,

$$R(\xi,\tau) \leq \frac{e^{r(\tau_0+\eta)}-1}{e^{r\tau_0}-1}e^{eC_4/(4\eta)}R(X.\tau+\eta).$$

Then, if we choose ξ satisfying $R(\xi,\tau) > r$, we obtain a constant c, depending only on g_0, τ_0 and η, satisfying

$$R(X,\tau+\eta) \geq c \quad \text{ for every } \tau, \tag{10.37}$$

which is an universal lower bound of R if we take τ_0 and η inside the interval where we know g_t is well defined by the short time existence theorem. □

Using all the estimates obtained above and the Sobolev inequality, one obtains (after non obvious computations)

Corollary 10.5 *If we have a solution for the normalized Ricci flow on a compact surface with $R > 0$ at the start, then there are constants all the derivatives of the curvature remain bounded for all time also.*

10.5. Asymptotic approach to a soliton

Let f be a potential function of the curvature (that is, f satisfies the equation (9.20) $-\Delta f_t = R(t) - r$), and let $\mathbb{M}(t) = \nabla^2 f_t + \frac{1}{2}\Delta_{g_t} f_t \ g_t$ be the tensor defined in (9.26). We have the following evolution equations for $\mathbb{M}(t)$.[41]

Lemma 10.6 *Under the normalized Ricci flow, the tensor $\mathbb{M}$ evolves according to the following formula*

$$\frac{\partial}{\partial t}\mathbb{M}(t) = -\Delta_{g_t}\mathbb{M}(t) + (r - R(t))\mathbb{M}(t) \tag{10.38}$$

Proposition 10.7 *Under the normalized Ricci flow, $|\mathbb{M}|$ satisfies the equation*

$$\frac{\partial}{\partial t}|\mathbb{M}(t)|^2 = -\Delta_{g_t}|\mathbb{M}(t)|^2 - 2|\nabla\mathbb{M}(t)|^2 - 2R|\mathbb{M}|^2 \tag{10.39}$$

And, from this equation, by the maximum principle, we obtain

Corollary 10.8 *If $R \geq c > 0$ for some constant c, then there is a constant C such that*

$$|\mathbb{M}(t)| \leq Ce^{-ct}.$$

Hence $\mathbb{M} \to 0$ exponentially.

10.6. Modified Ricci flow and the final Theorem

Now, we consider the following modification of the Ricci flow:

$$\frac{\partial}{\partial t}g_t = 2\mathbb{M}(t) = (r - R(t))g_t + 2\nabla^2 f_t.$$

[41]see [22] pages 129-130 for a detailed proof of the results in this section.

It is not hard to show that the solutions of this equation differ from those of the Ricci flow only by the action of the 1-parameter family of diffeomorphisms generated by the family of vector fields $\mathrm{grad}_{g_t} f_t$. Since $\mathbb{M}(t)$ converges to zero exponentially, the modified metrics will converge as $t \to \infty$. It is possible to show that their derivatives also converge to zero and the limiting metric is smooth.

Since the solutions of the modified Ricci flow and the normalized Ricci flow correspond by diffeomorphisms, the solutions g_t of the modified Ricci flow will converge also to a C^∞ metric which, by Corollary 10.8, will have $\mathbb{M} = 0$.

Recall that $\mathbb{M} = 0$ for Ricci solitons; so when $R > 0$ at $t = 0$, the metric evolving under the normalized Ricci flow approaches a Ricci soliton as $t \to \infty$.

Then, as a consequence of theorem 9.5 we have the following

Theorem 10.9 *On a compact surface with $R > 0$, the solution of the normalized Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij}$$

converges exponentially to a constant curvature metric.

10.7. Case r = 0

We already know the solution exists for all time and the curvature remains bounded above and below. To reach Hamilton's theorem in this case, it remains to show that the solution converges to a flat metric.

The steps of the proof would be the following:

1. The metrics are uniformly equivalent for all t.
2. $\int |dR|^2 \, d\mu \leq Ce^{-ct}$.
3. $\int (\Delta R)^2 \, d\mu \leq Ce^{-ct}$.
4. R_{max} converges to zero exponentially
5. The metric converges exponentially to the flat metric.

10.8. Chow's Theorem

Without the restrictions on the sign of the scalar curvature R, the following theorem has been proved by B. Chow.

Theorem 10.10 *For any smooth initial metric on S^2, the solution to*

$$\frac{\partial g}{\partial t} = (r - R) g \tag{10.40}$$

exists for all time and converges to a constant curvature metric as $t \to \infty$.

11. MATRIX HARNACK INEQUALITY

In this lecture we shall give some ideas on the general Harnack inequality for higher dimensions. From now on, unless otherwise stated, we shall work only with the Ricci flow (not the normalized Ricci flow).

11.1. Matrix Harnack inequality

Theorem 11.1 *Let $(M, g(t))$ be a solution for the Ricci flow which is either compact or complete with bounded curvature, and suppose the curvature operator is non-negative*[42]*, then for any vector field W and 2-vector U, we have for all $t > 0$*

$$Z := \mathbb{M}_{ab}W^aW^b + 2P_{abc}U^{ab}W^c + R_{abcd}U^{ab}U^{cd} \geq 0 \tag{11.1}$$

where

$$P_{abc} = \nabla_a R_{bc} - \nabla_b R_{ac}$$

and

$$\mathbb{M}_{ab} = \Delta R_{ab} - \frac{1}{2}\nabla^2_{a\ b}R + 2g^{ce}g^{df}R_{acbd}R_{ef} - g^{cd}R_{ac}R_{bd} + \frac{1}{2t}R_{ab}$$

Idea of the proof. . Assume for simplicity the manifold is compact. If Z becomes negative, there will be a first time t_0 when Z is zero; this happens at a point x_0 and in the direction of some U and W. We can extend these any way we like in space and time and still we have $Z \geq 0$ up to t_0, and we can profit by extending them with

$$\nabla_a U_{bc} = \frac{1}{2}(R_{ab}W_c - R_{ac}W_b) + \frac{1}{4t}(g_{ab}W_c - g_{ac}W_b)$$

and

$$\nabla_a W_b = 0$$

[42]We define the curvature operator on 2-vectors as $R : \Lambda^2TM \to \Lambda^2TM$ such that $\langle R(X \wedge Y), (Z \wedge W)\rangle = Rm(X, Y, Z, W)$, where the scalar product over Λ^2TM is defined over basic vector fields as

$$\langle X \wedge Y, Z \wedge W\rangle := \det \begin{pmatrix} \langle X, Z\rangle & \langle X, W\rangle \\ \langle Y, Z\rangle & \langle Y, W\rangle \end{pmatrix}$$

When we say that the curvature operator is non-negative, we mean $\forall U \in \Lambda^2TM$ $\langle RU, U\rangle \geq 0$.

at the critical point where $Z = 0$. We also take

$$(\frac{\partial}{\partial t} + \Delta)W_a = \frac{1}{t}W_a \text{ and } (\frac{\partial}{\partial t} + \Delta)U_{ab} = 0$$

at the critical point. We then compute

$$\begin{aligned}(\frac{\partial}{\partial t} + \Delta)Z = &(P_{abc}W_c + R_{abcd}U_{cd})(P_{abe}W_e + R_{abef}U_{ef}) + 2R_{abcd}M_{cd}W_aW_b \\ &- 2P_{acd}P_{bdc}W_aW_b + 8R_{adce}P_{dbe}U_{ab}W_c + 4R_{aecf}R_{bedf}U_{ab}U_{cd}\end{aligned}$$

Finally, with these elections, we can check that if $Z \geq 0$ then $(\frac{\partial}{\partial t} - \Delta)Z \geq 0$ and apply the maximum principle. Because of the factor $\frac{1}{t}$ in Z we have Z positive for small t and then it must stay positive. □

Next Corollary is a key piece in the study of singularities (see Lecture 14 by Hamilton and Perelman.

Corollary 11.2 *If (M^n, g_t) is a solution of the Ricci flow on a compact manifold with initially positive curvature operator, then for any vector field V on M and all times $t > 0$ such that the solution exists, one has*

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2dR(V) + 2Ric(V, V) \geq 0 \tag{11.2}$$

This corollary follows immediately by taking

$$U_{ij} = \frac{1}{2}(V_iW_j - V_jW_i)$$

in formula (11.1) and tracing over W_i.

Remark 11.2.1 *Letting $V = 0$, we have $\frac{\partial R}{\partial t} + \frac{R}{t} \geq 0$ and multiplying by t this yields $\frac{d}{dt}(tR) \geq 0$ which implies that tR is increasing at each point along the Ricci flow.*

Corollary 11.3 *Let x_1, $x_2 \in M$ and let t_1 and t_2 be two different times with $0 < t_1 < t_2$. Then*

$$R(x_2, t_2) \geq R(x_1, t_1)\frac{t_1}{t_2}e^{-d(x_1,x_2,t_1)^2/2(t_2-t_1)} \tag{11.3}$$

where $d(x_1, x_2, t_1)$ is the distance between x_1 and x_2 at time t_1.

Proof. At time t_1, take a geodesic path $\gamma(t)$ joining x_1 and x_2, namely, $\gamma : [t_1, t_2] \to M$ such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. At time t_1 the constant velocity is $\dfrac{d(x_1, x_2, t_1)}{t_2 - t_1}$, where $d(x_1, x_2, t_1)$ denotes the distance from x_1 to x_2 at time t_1

Now consider a path in space-time $\eta : [t_1, t_2] \to M \times \mathbb{R}$ defined by $\eta(t) = (\gamma(t), t)$; in other words, η is a path joining (x_1, t_1) and (x_2, t_2).

By hypothesis, $Ric \geq 0$; so, using the Ricci flow equation, we have for an arbitrary vector field V

$$\frac{\partial}{\partial t} g_t(V, V) = -2Ric(V, V) \leq 0$$

As a consequence of this, $g_t(V, V)$ is a decreasing function of t.

Thus, if we denote $V \equiv \gamma'(t)$, we get

$$g_t(V, V) \leq g_{t_1}(V, V) = |V|^2_{t_1} = \frac{d(x_1, x_2, t_1)^2}{(t_2 - t_1)^2} \tag{11.4}$$

for every time $t \geq t_1$.

Using the chain rule, we compute

$$\begin{aligned} \frac{dR}{dt}(\eta(t)) = \frac{dR}{dt}(t, \gamma(t)) &= \frac{\partial R}{\partial t}(t, \gamma(t)) + \frac{\partial R}{\partial x^i}\frac{d\gamma^i(t)}{dt} \\ &= \frac{\partial R}{\partial t}(t, \gamma(t)) + dR(\gamma'(t)) \\ &= \frac{\partial R}{\partial t}(t, \gamma(t)) + dR(V) \end{aligned} \tag{11.5}$$

Applying formula (11.2) to $\frac{V}{2}$ instead of V, we have the estimate

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2dR\left(\frac{V}{2}\right) + Ric\left(\frac{V}{2}, \frac{V}{2}\right) \geq 0$$

or, equivalently,

$$\frac{\partial R}{\partial t} + dR(V) \geq -\frac{R}{t} - \frac{1}{2}Ric(V, V) \tag{11.6}$$

Combining (11.5) and (11.6), we reach

$$\frac{dR}{dt}(t, \gamma(t)) \geq -\frac{R}{t} - \frac{1}{2}Ric(V, V)$$

Since Ric is non-negative, it satisfies $R_{ij} - (\sum_i R_{ii})\, g_{ij} \leq 0$, and therefore $Ric(V,V) \leq Rg_t(V,V)$. So, dividing by R last inequality,

$$\begin{aligned}\frac{d\ln R}{dt}(t,\gamma(t)) &\geq -\frac{1}{t} - \frac{1}{2}g_t(V,V) \\ &\geq -\frac{1}{t} - \frac{1}{2}\frac{d(x_1,x_2,t_1)^2}{(t_2-t_1)^2} \qquad \text{(from (11.4))}\end{aligned}$$

Then, by the fundamental theorem of calculus,

$$\begin{aligned}\ln\frac{R(x_2,t_2)}{R(x_1,t_1)} &= \int_{t_1}^{t_2}\frac{d}{dt}\ln R(t,\gamma(t))\,dt \geq -\int_{t_1}^{t_2}\frac{1}{t}\,dt - \frac{1}{2}\int_{t_1}^{t_2}\frac{d(x_1,x_2,t_1)^2}{(t_2-t_1)^2}\,dt \\ &= \ln\frac{t_1}{t_2} - \frac{1}{2}\frac{d(x_1,x_2,t_1)^2}{t_2-t_1}\end{aligned}$$

Taking exponentials, we arrive to

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \geq \frac{t_1}{t_2}e^{-d(x_1,x_2,t_1)^2/2(t_2-t_1)}$$

So, multiplying both sides by $R(x_1,t_1)$ gives the result. □

11.2. How Hamilton derived the Harnack expression

It seems interesting to analyze what was in Hamilton's mind when he deduced the formula (11.2) which, as we have seen above, is fundamental to generalize Harnack's inequality for higher dimensions. Here we give first the reasons Hamilton itself gave. Although we recognize that these reasons are not very helpful, the short introduction to solitons in these pages is interesting.

We need some previous knowledge about Ricci solitons in arbitrary dimensions.

11.2.1. Ricci solitons

Recall that a Ricci soliton is a solution to the Ricci flow which moves by a one parameter group of diffeomorphisms ϕ_t, namely,

$$g(t) = \phi_t^* g(0) \tag{11.7}$$

Derivating respect of t and using the Ricci flow equation, we have[43]

$$\mathcal{L}_V g = -2Ric \tag{11.8}$$

where V is the vector field generated by ϕ_t.

And thus, by the same computations as in section 8.2,

$$\nabla_a V_b + \nabla_b V_a = -2R_{ab} \tag{11.9}$$

Remark 3 *In the original articles of Hamilton the Ricci soliton equations* (11.8) *and* (11.9) *appear without the minus sign before Ric. The reason is possibly a different definition of the Lie derivative.*

If we take $V = -\mathrm{grad} f$,

$$R_{ab} = \nabla^2_{ab} f \quad \text{or } Ric = \nabla^2 f \tag{11.10}$$

In order to check this formula, see computations of section 8.2.

Definition 11.1 *We say g is a gradient Ricci soliton when the Ricci tensor is the Hessian of a function. So* (11.10) *is the gradient Ricci soliton equation.*

Definition 11.2 *A solution to the Ricci flow which moves by a diffeomorphism and at the same time shrinks or expands by a factor is called a homothetic Ricci soliton. It satisfies*

$$-2Ric = \rho g + \mathcal{L}_V g \tag{11.11}$$

where ρ is the homothetic constant. When $\rho > 0$, $\rho = 0$ or $\rho < 0$, the solitons are shrinking, steady or expanding, respectively. The case $V = 0$ is an Einstein metric.

Remark 11.2.1 *The definition of Ricci soliton becomes more natural when we use the normalized Ricci flow. Derivating equation* (11.7) *respect of t and substituting the derivative of the metric by the expression of the normalized Ricci flow, we get*

$$-2Ric + 2\frac{r}{n} g = \mathcal{L}_V g$$

[43] As we remarked in the lecture on surfaces, this is not so obvious, because ϕ_t defines a family of vector fields V_t, and not only one. For more details on solitons see [**?**] pp. 22-23 and [23] pp. 5-8

or, in local coordinates

$$-2R_{ij} + 2\frac{r}{n}g_{ij} = \nabla_i V_j + \nabla_j V_i$$

On the other hand, if we take $V = -\mathrm{grad} f$ *the gradient Ricci soliton equation becomes*

$$-R_{ij} + \frac{r}{n}g_{ij} = \nabla^2_{ij} f$$

Then, if $r > 0$, $r = 0$ *or* $r < 0$, *we say that the soliton is shrinking, steady or expanding, respectively.*

Examples 11.2.1 *The cigar soliton satisfies the steady gradient soliton equation with* $f(s) = -\ln(\cosh s)$.

Definition 11.3 *An eternal solution of the Ricci flow is one that exists for all time.*

An ancient solution of the Ricci flow is one that exists on a maximal time interval $-\infty < t < T$, *where* $T < \infty$.

An immortal solution to the Ricci flow is one which exists on a maximal time interval $\tau < t < \infty$, *where* $\tau > -\infty$.

Shrinking, steady and expanding solitons give examples of ancient, eternal, and immortal solutions, respectively.

We already know the importance of Ricci solitons in dimension 2. The following results illustrate that they are also fundamental in higher dimensions.

Theorem 11.4 (Ivey, [52]) . *There are no three-dimensional solitons on a compact connected manifold* M^3 *other than constant curvature metrics.*

We are interested in solutions which are complete (that is, defined for every point of the manifold), eternal and with Riemannian curvature uniformly bounded for all space and time (because this condition assures *long time existence* of solution of the Ricci flow). Next theorem tells us why this issue is related to Ricci solitons.

Theorem 11.5 (Hamilton, [48]) . *Any complete simply connected eternal solution of the Ricci flow with uniformly bounded curvature and strictly positive curvature operator where the scalar curvature* R *assumes its maximum is necessarily a gradient soliton.*

11.2.2. Motivation of Harnack expression

On an expanding gradient soliton

$$\nabla_a V_b = R_{ab} + \frac{1}{2t} g_{ab} \tag{11.12}$$

since $V_a = \nabla_a f$ implies $\nabla_a V_b = \nabla_b V_a$.

Differentiating (11.12) and commuting give the first order relations

$$\nabla_a R_{bc} - \nabla_b R_{ac} = R_{abcd} V_d$$

and differentiating again gives

$$\nabla_a \nabla_b R_{cd} - \nabla_a \nabla_c R_{bd} = \nabla_a R_{bcde} V_e + R_{ae} R_{bcde} + \frac{1}{2t} R_{bcda}$$

We take the trace of this on a and b to conclude

$$\mathbb{M}_{ab} + P_{cab} V_c = 0$$

where $\mathbb{M}_{ab}$ and P_{abc} are defined as before. The first relation was then

$$P_{cba} V_c + R_{acbd} V_c V_d = 0$$

and in order to get a good expression we add the two equations to make

$$\mathbb{M}_{ab} + (P_{cab} + P_{cba}) V_c + R_{acbd} V_c V_d = 0$$

We apply this to an arbitrary vector W_a and get

$$\mathbb{M}_{ab} W_a W_b + (P_{cab} + P_{cba}) W_a W_b V_c + R_{acbd} W_a V_c W_b V_d = 0$$

If we write $U_{ab} = \frac{1}{2}(V_a W_b - V_b W_a)$ for the wedge product of V and W, the above can be rearranged as

$$Z = \mathbb{M}_{ab} W_a W_b + 2 P_{abc} U_{ab} W_c + R_{abcd} U_{ab} U_{cd}$$

which shows that the Harnack inequality becomes an equality on an expanding gradient soliton.

Since there are other expressions which vanish, one may ask how Hamilton come to select it.

One important criterion is that if $Z \geq 0$ for all choices of W and U then when $Z = 0$ on the soliton must also have $\dfrac{\partial Z}{\partial W} = \dfrac{\partial Z}{\partial U} = 0$. This dictates that we need to take the trace of the second derivative expression, since otherwise we cannot mix it with first derivative expression, and it also shows we must take an equal amount of each.

12. SURVEY ON LENGTH SPACES (I)

The only notion of distance between metrics that we explicitly use in these notes is that of C^m-distance as defined for the Hamilton compactness theorem. However, along the complete and detailed claimed proof of geometrization conjecture, Gromov-Hausdorff distance and Alexandrov spaces appear. For this reason, we give in this and next lectures a survey about Alexandrov spaces and convergence of metric spaces. We shall state the principal results about these items without proving them. Nevertheless, if the reader is interested in the proofs, most of them can be found in [6] and [80].

12.1. Length spaces

Let (X, ρ) be a metric space.

Definition 12.1 *A path $c(t)$ joining $p, q \in X$ is a continuous map $c : [a, b] \to X$ such that $c(a) = p$, $c(b) = q$.*

Definition 12.2 *Given P a partition of the interval $[a, b]$ ($P \in \mathcal{P}([a, b])$), that is, $P = \{t_0, \ldots, t_k\}$ with $a = t_0 \leq t_1 \leq \ldots \leq t_k = b$, we define the sum*

$$L(c, P) := \sum_{i=0}^{k-1} \rho(c(t_i), c(t_{i+1}))$$

A path c is said to be rectifiable if the following set is bounded

$$\{L(c, P) : P \in \mathcal{P}([a, b])\}$$

Definition 12.3 *If c is a rectifiable path, its length is defined as*

$$L(c) := \sup\{L(c, P) : P \in \mathcal{P}([a, b])\}$$

Definition 12.4 *A length space or space with intrinsic metric is a metric space (X, ρ) such that*

$$\rho(x, y) = d(x, y) := \inf\{L(c);\, c : [a, b] \to X,\, c(a) = x,\, c(b) = y\}$$

Remark 12.4.1 *In general, a metric space is not a length space. For example, S^1 with the euclidean distance. Indeed, the euclidean distance between two opposite points is equal to 2, while the length distance between the same points is π.*

Definition 12.5 *Let x, y, z be three distinct points in a metric space (X, d). We define the comparison angle xyz as*

$$\angle_d(x, y, z) := \arccos \frac{d(x, y)^2 + d(y, z)^2 - d(x, z)^2}{2d(x, y)d(y, z)}.$$

Definition 12.6 *Let $\alpha : [0, \varepsilon) \to X$ and $\beta : [0, \varepsilon) \to X$ be two paths in a length space X emanating from the same point $p = \alpha(0) = \beta(0)$. The angle $\angle(\alpha, \beta)$ between α and β is defined by*

$$\angle(\alpha, \beta) = \lim_{s,t\to 0} \angle_d(\alpha(s), p, \beta(t))$$

if such limit exists.

When the angle is not well-defined, we consider *upper angles*, which always exist.

Definition 12.7 *The upper angle $\angle_u(\alpha, \beta)$ is defined as*

$$\angle_u(\alpha, \beta) = \limsup_{s,t\to 0} \angle_d(\alpha(s), p, \beta(t))$$

From now on we'll only consider length spaces (X, ρ) with a complete metric, i.e. such that any two points can be joined by a shortest path[44].

12.2. Alexandrov spaces

12.2.1. Spaces of nonpositive or nonnegative curvature

Definition 12.8 *A triangle in X is a collection of three points a, b, c (vertices) connected by three shortest paths (sides).*

Remark 12.8.1 *Given any two vertices, there may be different shortest paths between them, so the vertices alone may not define a triangle uniquely.*

[44]A theorem states that, in a length space, the properties of being complete and of existing shortest geodesics between two points are equivalent (cfr.[6])

Notation 12.9 *Further, we are going to use the following notation:*
- $\triangle abc \equiv$ *triangle with vertices* a, b, c.
- $\gamma_{ab} \equiv$ *any shortest path joining* a *and* b.
- $\widehat{a} \equiv \angle(\gamma_{ab}, \gamma_{ac})$, *where* a, b, c *are the vertices of a triangle.*

Definition 12.10 *We define the comparison triangle for* $\triangle abc \subset X$ *as a triangle* $\triangle a_* b_* c_*$ *in the euclidean plane with the same lengths of sides, i.e.*

$$\rho(a,b) = |\overrightarrow{a_* b_*}|, \quad \rho(b,c) = |\overrightarrow{b_* c_*}|, \quad \rho(a,c) = |\overrightarrow{a_* c_*}|$$

It is uniquely defined up to a rigid motion.

Definition 12.11 (X, ρ) *is a space of nonpositive (resp. nonnegative) curvature if every point in* X *has a neighborhood* U *such that for every* $\triangle abc \subset U$ *and every point* $d \in \gamma_{ac}$,

$$\rho(d,b) \leq |\overrightarrow{d_* b_*}| \quad (\textit{resp. } \rho(d,b) \geq |\overrightarrow{d_* b_*}|)$$

where d_* *is the point of the side* $\overrightarrow{a_* c_*}$ *of a comparison triangle* $\triangle a_* b_* c_*$ *such that* $\rho(a,d) = |\overrightarrow{a_* d_*}|$.

Intuitively, it is natural to think that we can reformulate definition 12.11 via comparison of angles. Moreover, this point of view appears in Alexandrov's original definition.

Definition 12.12 (X, ρ) *is a space of nonpositive (resp. nonnegative) curvature if every point of* X *has a neighborhood* U *such that, for every triangle* $\triangle abc \subset U$ *the angles* $\widehat{a}$, $\widehat{b}$ *and* $\widehat{c}$ *are well defined and satisfy*

$$\widehat{a} \leq \widehat{a_*}, \quad \widehat{b} \leq \widehat{b_*}, \quad \widehat{c} \leq \widehat{c_*} \qquad (\textit{resp. } \widehat{a} \geq \widehat{a_*}, \quad \widehat{b} \geq \widehat{b_*}, \quad \widehat{c} \geq \widehat{c_*}),$$

where $\triangle a_* b_* c_*$ *is a comparison triangle. Moreover, for nonnegative curvature an additional condition is needed*[45] *: for two shortest path* γ_{pq} *and* γ_{rs}, *where* r *is a inner point of* γ_{pq}, *one has* $\angle(\gamma_{rp}, \gamma_{rs}) + \angle(\gamma_{rs}, \gamma_{rq}) = \pi$.

Obviously, definitions 12.11 and 12.12 are equivalent.

12.2.2. Spaces of curvature $\leq \boldsymbol{k}$ *or* $\geq \boldsymbol{k}$

The definition 12.11 can be generalized by comparing the triangle $\triangle abc$ in X with a comparison triangle $\triangle a_* b_* c_*$ in a two-dimensional model space of constant curvature k ($\equiv k$-plane).

[45]Some authors are not sure if this condition is really necessary or not.

Recall that these model spaces are the sphere of radius $\frac{1}{\sqrt{k}}$ (with its intrinsic metric), if $k > 0$; the hyperbolic plane of curvature k, if $k < 0$, and the euclidean plane, if $k = 0$. Let us denote by D_k the diameter of the model space of constant curvature k.

If we suppose that the perimeter of every triangle in a length space is less than $2D_k$, we can assure that a comparison triangle in a k-plane exists and is unique up to an isometry. Notice that this restriction can be omitted if $k \leq 0$, since in this case $D_k = \infty$.

Notation 12.13 *We will denote:*
- $\overrightarrow{a_*c_*} \equiv$ *any shortest path in a k-plane joining a_* and c_*.*
- $|\overrightarrow{a_*c_*}| \equiv$ *distance between a_* and c_* measured with the intrinsic metric of the k-plane.*

Definition 12.14 *A space of curvature $\leq k$ (resp. $\geq k$) is a length space (X, ρ) which can be covered by a family of open sets $\{U_i\}_{i\in I}$ so that every U_i satisfies:*

1) Every two points in U_i can be connected by a shortest path in U_i.

2) For any $\triangle abc \subset U_i$ of perimeter less than $2D_k$ and a point $d \in \gamma_{ac}$,

$$\rho(d, b) \leq |\overrightarrow{d_*b_*}| \qquad (resp.\ \rho(d, b) \geq |\overrightarrow{d_*b_*}|),$$

where $\triangle a_*b_*c_*$ is a comparison triangle for $\triangle abc$ in the k-plane and d_* is the point in $\overrightarrow{a_*c_*}$ such that $\rho(a, d) = |\overrightarrow{a_*d_*}|$.

Remark 12.14.1 *In fact, it is sufficient to consider only the cases $k = -1, 0, 1$ because all other cases can be reduced to these by rescaling. Moreover, it is enough to check condition 2) when d is a midpoint between a and c.*

Definition 12.15 *We say that a length space X is a space of curvature bounded above (resp. below) if every point $x \in X$ has a neighborhood which is a space of curvature $\leq k$ (resp. $\geq k$) for some $k \in \mathbb{R}$ (possibly varying from one point to another, which makes the difference with Definition 12.14).*

Definition 12.16 *A space with bounded curvature is called Alexandrov space.*

12.2.3. Examples

• Euclidean spaces are clearly Alexandrov spaces (both of nonpositive and nonnegative curvature at the same time).

• The space constructed by gluing at one point several segments has nonpositive curvature. So a locally-finite connected graph is a space of nonpositive curvature.

• A cone over a metric space X is the quotient $X \times [0, \infty[/X \times \{0\}$ with its natural metric. A cone over a circle of length $2\pi r$ is a space of nonnegative curvature if $r \leq 1$ and of nonpositive curvature if $r \geq 1$.

• A two-dimensional polyedral space has nonnegative curvature if in any vertex the sum of the angles is $\leq 2\pi$.

• $\mathbb{R}^2 \setminus D^2$, where D^2 is an open disk, has nonpositive curvature.

• $\mathbb{R}^3 \setminus B^3$, being B^3 an open ball, is a space of curvature ≤ 1.

• A convex hypersurface in $\mathbb{R}^n$ with its length metric is a space of nonnegative curvature for all $n \geq 3$.

• A Riemannian manifold is a space of curvature $\leq k$ (resp. $\geq k$) if and only if its sectional curvature is $\leq k$ (resp. $\geq k$) everywhere.

• Next theorem provide us a way to construct many nontrivial examples of spaces of curvature bounded above.

Theorem 12.1 *(Reshetnyak's Gluing Theorem). Let $\{(X_i, d_i)\}_{i=1,2}$ be two (complete locally compact) spaces of curvature $\leq k$. Suppose that there exists an isometry $f : S_1 \to S_2$ between two convex sets $S_i \subset X_i$ and consider $X = X_1 \cup_f X_2$. Then (X, d)[46] is a space of curvature $\leq k$.*

• We'll give now the statement of another gluing theorem.

Theorem 12.2 *(Alexandrov). Let $X_1, X_2 \subset \mathbb{R}^2$ be two convex sets. Suppose that their boundaries ∂X_1 and ∂X_2 have curvatures $\gamma_i \geq 0$ for $i = 1, 2$ (where the curvatures γ_i are defined in an integral way). Consider an isometry $f : \partial X_1 \to \partial X_2$. Then $X = X_1 \cup_f X_2$ is a space of nonnegative curvature.*

12.2.4. Global geometry of Alexandrov spaces

There is still another definition of Alexandrov space with a more global flavour.

Definition 12.17 *A length space X has curvature $\leq k$ (resp. $\geq k$) globally if it satisfies condition 2 of definition 12.14 for every triangle $\triangle abc$ in the length space for which comparison triangle in a k-plane is well defined (exists and is unique up to a rigid motion), no matter how big it is.*

[46] d is the length metric. See definition 12.4.

- **Globalization theorem for nonpositive curvature**

Definition 12.18 *A Hadamard space is a complete simply connected space of nonpositive curvature.*

Theorem 12.3 *Let X be a Hadamard space of curvature $\leq k$. Then X is a space of curvature $\leq k$ globally.*

Remark 12.3.1 *From the definition of Hadamard space, it is clear that $k \leq 0$ in the statement of the theorem.*

- **Toponogov's globalization theorem**

Theorem 12.4 *Let $k \in \mathbb{R}$ and let X be a complete length space of curvature $\geq k$. Then X has curvature $\geq k$ globally.*

Remark 12.4.1 *This theorem was proved by: Alexandrov, for two dimensions; Topogonov, for Riemannian manifolds of any dimension and Perelman generalized it for Alexandrov spaces.*

12.2.5. Splitting theorem

Definition 12.19 *A geodesic $\gamma : (-\infty, \infty) \to X$ is called a (straight) line if every one of its segments is a shortest path between its endpoints.*

Theorem 12.5 *Let X be a locally compact space of nonnegative curvature. If X contains a line, then it is isometric to a direct product $\mathbb{R} \times Y$, where Y is some space of nonnegative curvature.*

Remark 12.5.1 *Toponogov proved the theorem for Riemannian manifolds and Milka generalized it for Alexandrov spaces. Moreover, for Riemannian manifolds, the theorem also holds assuming nonnegativity of Ricci curvatures instead of sectional curvatures (cf. [15]).*

12.3. Space of metric spaces

In the same way as we work with a real number as a member of the real line, for now on we shall consider every metric space as an element of a class of similar objects called *space of metric spaces.* This allows us to talk about convergence of metric spaces.

Next we are going to motivate this section with some examples.

12.3.1. Examples

We begin with an example of a set that can be constructed as a limit of other sets.

• **Tangent cone of a convex set.** Let $X \subset \mathbb{R}^n$ be a convex set and $p \in X$. We identify p with the origin of $\mathbb{R}^n$ and construct the set $\lambda X = \{\lambda x : x \in X\}$, for any $\lambda > 0$. The tangent cone is obtained as the limit of the family of sets $\{\lambda X\}$ as $\lambda \to \infty$. But, what is the meaning of the word limit here?

From convexity, it follows that $\lambda_1 X \subset \lambda_2 X$ if $\lambda_1 < \lambda_2$; therefore, these sets are contained in their union $\bigcup_{\lambda > 0} \lambda X$. Since the tangent cone is a closed set, the limit is the closure of the aforementioned union.

• **Converging surfaces**

A task previous to the definition of limit is to specify when two spaces have a small distance between them. Intuitively, it seems reasonable to impose that close spaces are homeomorphic and, moreover, that they have similar geometric characteristics, such as distance between their points. Nevertheless, this two conditions lead us to an unsatisfying notion of convergence, as we illustrate with the following examples.

1.- *Spheres with vanishing handles.* Consider the sequence $\{X_n\}$, where each X_n is a sphere S^2 with a handle of diameter less than $1/n$. As n grows, it seems that the handles vanish to a point and X_n converge to S^2. But X_n is not homeomorphic to S^2.

2.- *Homeomorphic surfaces with non-preserved distances.* Consider two spheres of very different radii connected by a long thin tube. Let X be the space obtained by attaching a little handle to the larger sphere and let Y be the same but with the handle in the smaller sphere. Then X and Y are homeomorphic, but any continuous map from X to Y changes distances between some points, no matter how small the handles are. But X and Y are close in the sense that they converge to the same surface when the diameter of the handles tends to 0.

In short, we have found examples of spaces intuitively close but not homeomorphic (1.) and also of homeomorphic spaces with some different geometric properties (2.). So we have to introduce a different concept of convergence. In order to do this, the first step is to define a distance between abstract metric spaces. As we have seen above, this isn't an easy task, in fact, in this section we'll give several definitions of distance suitable for different purposes.

12.3.2. Uniform convergence

Definition 12.20 *A sequence $\{f_n\}$ of real-valued functions on a set X is said to uniformly converge to a function f if*

$$\sup_{x\in X} |f_n(x) - f(x)| \to 0 \qquad \text{as } n \to \infty.$$

We can apply last definition to metrics because every metric on X is a real-valued function defined on $X \times X$. So

Definition 12.21 *We say that a sequence $\{d_n\}$ of metrics on X uniformly converges to a metric d if*

$$\sup_{x,x'\in X} |d_n(x,x') - d(x,x')| \to 0 \qquad \text{as } n \to \infty.$$

This type of convergence has the following nice properties:
- If the metrics d_n are intrinsic, the limit metric is intrinsic too.
- If the curvatures of the converging metrics are uniformly bounded from below, this curvature bound is inherited by the limit metric.

Nevertheless, uniform convergence is a relatively weak type of convergence. For example, a uniform limit of Riemannian metrics can be non Riemannian.

Definition 12.22 *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ an arbitrary map. We define the distortion of f as*

$$dis f = \sup_{x_1,x_2\in X} |d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)|$$

Definition 12.23 *A sequence $\{X_n\}$ of metric spaces uniformly converges to a metric space X if there exist homeomorphisms $f_n : X_n \to X$ such that $dis(f_n) \to 0$ as $n \to \infty$.*

12.3.3. Lipschitz distance

Now we are going to introduce a new distance in order to measure relative difference between metrics. Thus we'll say that two metric spaces X and Y are close to each other in the Lipschitz sense if there is a homeomorphism $f : X \to Y$ such that $d_Y(f(x), f(x'))/d_X(x, x')$ is about 1.

Definition 12.24 *Let X and Y be two metric spaces and $f : X \to Y$ a Lipschitz map. The dilatation of f is defined by*

$$dil f = \sup_{x,x'\in X} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

Definition 12.25 *The Lipschitz distance d_L between two metric spaces X and Y is defined by*

$$d_L(X,Y) = \inf_{f:X\to Y} \ln\left(\max\{dil(f), dil(f^{-1})\}\right)$$

where the infimum is taken over all bi-Lipschitz homeomorphisms $f : X \to Y$.

Remark 12.25.1 *A homeomorphism f is called bi-Lipschitz if both f and f^{-1} are Lipschitz maps.*

Definition 12.26 *A sequence $\{X_n\}_{n=1}^{\infty}$ of metric spaces is said to converge in the Lipschitz sense to a metric space X if $d_L(X_n, X) \to 0$ as $n \to \infty$. If there are no bi-Lipschitz homeomorphisms from X to Y, then $d_L(X,Y) = \infty$.*

In general, d_L is nonnegative, symmetric and satisfies the triangle inequality. Moreover, for compact spaces X and Y, $d_L(X,Y) = 0$ if and only if X and Y are isometric. This tells us that Lipschitz distance is a metric on the "space" of isometry classes of compact metric spaces.

On the other hand, Lipschitz convergence of compact metric spaces implies uniform convergence and they are equivalent in the class of finite metric spaces.

Next we state an important result about Lipschitz convergence of Riemannian manifolds.

Let $\mathcal{C} = \mathcal{C}(n, \Lambda, \delta_0, V_0)$ be the set of all connected compact C^∞ n-dimensional Riemannian manifolds with |sectional curvature| $\leq \Lambda^2$, diameter$<$ δ_0, and $Volume > V_0$.

Theorem 12.6 *Given a sequence $\{M_l\}$ in $\mathcal{C}$ and $\alpha \in (0,1)$, there exists a subsequence $\{M_{l_k}\}$ together with a C^∞ manifold M on which is defined a $C^{1,\alpha}$-Riemannian metric such that $\{M_{l_k}\}$ converges to M with respect to the Lipschitz distance.*

13. SURVEY ON LENGTH SPACES (II)

13.1. Gromov-Hausdorff distance

First we talk about Hausdorff distance, which is a distance between subsets of a metric space, not between abstract metric spaces. Let S be a subset of a metric space (X, ρ) and let us denote $\mathcal{U}_r(S) := \{x : \rho(x,S) < r\}$.

Definition 13.1 *Let A and B subsets of a metric space. The Hausdorff distance between A and B is defined by*

$$d_H(A, B) = \inf\{r > 0 : A \subset \mathcal{U}_r(B) \text{ and } B \subset \mathcal{U}_r(A)\}$$

Proposition 13.1 $(\mathfrak{M}(X), d_H)$ *is a metric space, where* $\mathfrak{M}(X)$ *is the set of closed subsets of* X.

Remark 13.1.1 *There are another important results about the metric space introduced in last proposition.*

1.- If X is complete, then $\mathfrak{M}(X)$ is also complete.

2.- If X is compact, then $\mathfrak{M}(X)$ is also compact.

A corollary of 2 is:

(Blaschke theorem). The set of all compact convex subsets contained in any fixed closed ball in $\mathbb{R}^n$ is compact respect to the Hausdorff distance.

Now we are in position to define the Gromov-Hausdorff distance d_{GH} following two fundamental ideas: distance between isometric spaces is zero and, given two subspaces X, Y of the same metric space, $d_{GH}(X, Y) \leq d_H(X, Y)$.

Definition 13.2 *Let (X, d_X), (Y, d_Y) be two metric spaces and $r > 0$. We say $d_{GH}(X, Y) < r$ if there exists some metric space (Z, d) and subspaces $X', Y' \subset Z$, with the induced metric, satisfying*

(a) $(X', d|_{X'})$ is isometric to (X, d_X).

(b) $(Y', d|_{Y'})$ is isometric to (Y, d_Y).

(c) $d_H(X', Y') < r$.

In other words, $d_{GH}(X, Y) = \inf\{r > 0 : \exists\, Z, X', Y' \text{ for which } (a), (b), (c) \text{ hold}\}$

In practice, this is a very complicated definition. Indeed, if we want to find out the Gromov-Hausdorff distance between X and Y, we have to work with all metric spaces Z containing subspaces isometric to X and Y. Next we shall give another definition, which is equivalent to 13.2.

Definition 13.3 *Given (X, d_X) and (Y, d_Y) two metric spaces. Consider $(X \sqcup Y, d)$, being $X \sqcup Y$ the disjoint union between X and Y, and d a metric such that $d|_X = d_X$, $d|_Y = d_Y$. The Gromov-Hausdorff distance between X and Y is defined by*

$$d_{GH}(X, Y) = \inf\{d_H(X, Y) \text{ in } (X \sqcup Y, d)\},$$

where the infimum is taken over all (semi)-metrics d on $X \sqcup Y$ such that $d|_X = d_X$ and $d|_Y = d_Y$.

Examples 13.3.1 *1. Given (X, d_X) a metric space and $S \subset X$ an ε-net*[47]*, $d_{GH}(X,S) \leq \varepsilon$. In fact, by the definition of ε-net, we have $X \subset \mathcal{U}_\varepsilon(S) = \bigcup_{x \in S} B(x,\varepsilon)$ and, obviously, $S \subset \mathcal{U}_\epsilon(X) = X$. Then $d_H(X,S) \leq \varepsilon$ and so $d_{GH}(X,S) \leq d_H(X,S) \leq \varepsilon$.*

2. Let (X, d_X), (Y, d_Y) be compact metric spaces with diameter $\leq \mathcal{D}$, then $d_{GH}(X,Y) \leq \mathcal{D}/2$. Indeed, we define a distance d on $X \sqcup Y$ such that $d|_X = d_X$, $d|_Y = d_Y$ and $d(x,y) = \mathcal{D}/2$ if $(x,y) \in X \times Y$. It is easy to check that d is a metric on $X \sqcup Y$. Moreover, from the definition of d, it is obvious that $X \subset \mathcal{U}_{\mathcal{D}/2}(Y)$ and $Y \subset \mathcal{U}_{\mathcal{D}/2}(X)$. Therefore, $d_H(X,Y) \leq \mathcal{D}/2$ and then $d_{GH}(X,Y) \leq \mathcal{D}/2$.

3. A generalization of the argument used in example 2 yields the next property: if X and Y are bounded metric spaces, then $d_{GH}(X,Y) < \infty$.

We have already defined the Gromov-Hausdorff distance and now we wonder when d_{GH} is actually a metric.

Theorem 13.2 *$(\mathcal{IC}, d_{GH})$ is a metric space, where $\mathcal{IC} \equiv$ space of isometry classes of compact metric spaces. Moreover, d_{GH} is finite on $\mathcal{IC}$.*

Remark 13.2.1 *The proof consists on checking that d_{GH} is nonnegative, symmetric, satisfies the triangle inequality and, moreover, $d_{GH}(X,Y) = 0$ if and only if X and Y are isometric.*

On the other hand, recall that $(\mathcal{IC}, d_L)$, where d_L is the Lipschitz distance, is also a metric space. However, in this case, d_L could be infinity on $\mathcal{IC}$.

13.1.1. $\boldsymbol{\varepsilon}$-isometries

Definition 13.4 *Let X, Y be metric spaces and $\varepsilon > 0$. A map $f : X \to Y$ (not necessarily continuous) is said to be an ε-isometry if $dis f \leq \varepsilon$ and $f(X)$ is an ε-net in Y.*

Remark 13.4.1 *Recall that $dis f \leq \varepsilon$ means $|d_X(x_1,x_2) - d_Y(f(x_1), f(x_2))| \leq \varepsilon$ for all $x_1, x_2 \in X$ (cf. Definition 12.22).*

The notion of ε-isometry provides us a different way to deal with Gromov-Hausdorff distances.

[47] Let X be a metric space and $\varepsilon > 0$. A set $S \subset X$ is called an ε-net if $dist(x,S) \leq \varepsilon$ for every $x \in X$.

Theorem 13.3 *Consider two metric spaces X, Y and $\varepsilon > 0$. We have the following properties*

1. *If $d_{GH}(X, Y) < \varepsilon$, then there exists a 2ε-isometry $f : X \to Y$.*
2. *If there exists an ε-isometry from X to Y, then $d_{GH}(X, Y) < 2\varepsilon$.*

Next example illustrates this theorem.

Examples 13.4.1 *Let (X, d_X), (Y, d_Y) be compact metric spaces and $\varepsilon > 0$. Consider $\{x_1, \dots, x_n\} \subset X$ and $\{y_1, \dots, y_n\} \subset Y$ such that*
- $|d_X(x_i, x_j) - d(y_i, y_j)| < \varepsilon$ *for all* $i, j = 1, \dots, n$
- $\mathcal{U}_\varepsilon(\{x_1, \dots, x_n\}) = X$
- $\mathcal{U}_\varepsilon(\{y_1, \dots, y_n\}) = Y$.

In other words, $\{x_i\}$, $\{y_j\}$ are ε-nets in X and Y, respectively, and $f(x_i) = y_i$ is an ε-isometry.

We define a metric on $X \sqcup Y$ as $d(x, y) = \min\{d(x, x_i) + \varepsilon + d(y_i, y) : i = 1 \dots, n\}$ for $x \in X$ and $y \in Y$. One can check that with this metric $X \subset \mathcal{U}_{2\varepsilon}(Y)$ and $Y \subset \mathcal{U}_{2\varepsilon}(X)$. In particular, this implies that $d_{GH}(X, Y) \leq 2\varepsilon$.

13.2. Gromov-Hausdorff convergence

Definition 13.5 *A sequence $\{X_n\}_{n=1}^{\infty}$ of compact metric spaces converges to a compact metric space in the Gromov-Hausdorff sense if $d_{GH}(X_n, X) \to 0$ as $n \to \infty$.*

This is the more general concept of convergence we have defined until now. Indeed, uniform convergence implies Gromov-Hausdorff convergence and, as we have seen at the end of Lecture 12, Lipschitz convergence is a particular case of uniform convergence.

Notation 13.6 *For the type of convergence introduced in the last definition, we will use the notation $X_n \underset{GH}{\longrightarrow} X$.*

Examples 13.6.1 *1. Every compact metric space X is a limit of finite spaces. In fact, consider a sequence $\varepsilon_n \to 0$ of positive numbers and take a finite ε_n-net $S_n \subset X$ for every n. Since $d_{GH}(X, S_n) \leq \varepsilon_n$ (see example 1 of 13.3.1), $S_n \underset{GH}{\longrightarrow} X$.*

2. Consider the unit sphere S^3 as a subset of $\mathbb{C}^2$ and the following action of S^1 on S^3

$$(e^{it}, (z, w)) \mapsto (e^{it}z, e^{it}w)$$

This action defines a foliation $\mathcal{F}$ on S^3 whose leafs are the orbits of the action, that is, great circles in S^3. We can consider $S^3/\mathcal{F}$ and the quotient *projection (Hopf fibration)*

$$\pi : S^3 \longrightarrow S^3/\mathcal{F} \equiv M$$

It is possible to prove [48] *that there exists a unique Riemannian metric on M such that π is a Riemannian submersion. Using this, one can show that $S^3/\mathcal{F}$ is isomorphic to S^2 with sectional curvature* 4.

Since π is a Riemannian submersion, we have the decomposition

$$T_xS^3 = V_x \oplus H_x, \quad \text{where } V_x = \pi_{x*}^{-1}(0) = T_x\pi^{-1}(\pi(x)) \text{ and } H_x = (V_x)^{\perp} \tag{13.1}$$

Then $\pi_{x*}| : H_x \longrightarrow T_{\pi(x)}S^2$ is an isomorphism.

We introduce a 1*-parametric family of metrics g_λ on S^3 in the following natural way:*

Using the decomposition 13.1, if $U, W \in T_xS^3$ we can write $U = U^V + U^H$, $W = W^V + W^H$. Then we define

$$g_\lambda(U, W) = \lambda^2 g(U^V, W^V) + g(U^H, W^H), \quad \lambda \in \mathbb{R}$$

We shall denote $g_\lambda = \lambda^2 g_V \oplus g_H$.

For every λ, the Riemannian manifold $(S^3,\ g_\lambda)$ is called a Berger's sphere.

Now, let us consider the family of Berger's spheres (S^3, g_ε) with $0 < \varepsilon < 1$. If we let $\varepsilon \to 0$, intuitively $g_0 = \lim_{\varepsilon\to 0} g_\varepsilon = g_H$ and, as we have seen above, this is the metric of the 2*-sphere $S^2(4)$ with sectional curvature* 4*. So we can conclude $(S^3, g_\varepsilon) \xrightarrow[GH]{} S^2(4)$.*

A more rigorous argument for this, but changing a little bit the family of Berger's spheres (which now will fiber over different S^2) runs as follows. Let $S^3_{\mathbb{C}}(r) := \{x \in \mathbb{C}P^2 : d(p, x) = r\} = \exp_p S^3(r)$ a geodesic sphere of radius r in $\mathbb{C}P^2(1)$. It is a well known fact that, for r before the first cut distance $t_0 = \dfrac{\pi}{2}$ from p, $S^3_{\mathbb{C}}(r)$ is isometric to the Berger's sphere corresponding to the fibration $\pi : S^3(1/\sin r) \longrightarrow S^2(4/\sin r)$ with contraction factor $\varepsilon = \cos r$ on the fiber. Now, it is easy to show that $S^3_{\mathbb{C}}(r) \xrightarrow[GH]{} S^2(4) \equiv \mathbb{C}P^1$, because both $\mathbb{C}P^1$ and $S^3_{\mathbb{C}}(r)$ are in $\mathbb{C}P^2$, and $\mathbb{C}P^1 \subset \mathcal{U}_{\frac{\pi}{2}-r}(S^3_{\mathbb{C}}(r))$ and $S^3_{\mathbb{C}}(r) \subset \mathcal{U}_{\frac{\pi}{2}-r}(\mathbb{C}P^1)$. And, by definition, this implies $d_{GH}(S^3_{\mathbb{C}}(r), \mathbb{C}P^1(1)) \leq \frac{\pi}{2} - r$ and so $S^3_{\mathbb{C}}(r) \xrightarrow[GH]{} \mathbb{C}P^1$.

This phenomenon is called a collapse because the dimension of the limit is smaller than the dimension of the elements in the sequence.

[48] See [66] for more details.

3. Consider the unit cube $[0,1]^3 \subset \mathbb{R}^3$ *and take the grid*

$$X_n := \{(x,y,z) \in [0,1]^3 : \text{at least 2 coordinates are rational numbers with denominator } n\}.$$

Let $Y_n = \partial\mathcal{U}_{r-n}(X_n) \subset \mathbb{R}^3$. *As* $n \to \infty$ *both* X_n *and* Y_n *fill up* $[0,1]^3$ *so* $X_n \to [0,1]^3$ *and* $Y_n \to [0,1]^3$. *This phenomenon is called an explosion, because the limit space has larger dimension than the elements in the sequence.*

4. Fix a point $p \in S^1$ *and consider* $S^1_n = S^1 \backslash \{\text{interval around } p \text{ of length } 1/n\}$. *It is possible to show that* S^1_n *do not converge to* S^1 *as* $n \to \infty$.

On the other hand, let $p \in S^2$ *and take* $S^2_n = S^2 \setminus B_p(\frac{1}{n})$. *In this case,* $S^2_n \to S^2$ *when* $n \to \infty$.

Using suitable ε-nets, one can reduce convergence of arbitrary compact metric spaces to convergence of their finite subsets. With this aim, we shall introduce the concept of (ε,δ)-approximation.

Definition 13.7 *Let* X, Y *be compact metric spaces and* $\varepsilon, \delta > 0$. *We say that* X *and* Y *are* (ε,δ)*-approximations of each other if there exist* $\{x_1,\dots,x_n\}$ *and* $\{y_1,\dots,y_n\}$ ε*-nets in* X *and* Y*, respectively, such that*

$$|d_X(x_i,x_j) - d_Y(x_i,x_j)| < \delta \qquad \forall i,j = 1,\dots,n$$

We use the word ε*-approximation when* $\varepsilon = \delta$.

Proposition 13.4 *Let* X *and* Y *be compact metric spaces.*

1. If Y *is an* (ε,δ)*-approximation of* X*, then* $d_{GH}(X,Y) < 2\varepsilon + \delta$.

2. If $d_{GH}(X,Y) < \varepsilon$*, then* Y *is an* 5ε*-approximation of* X.

Last proposition yields a criterium for convergence of compact metric spaces.

Proposition 13.5 $X_n \xrightarrow[GH]{} X$ *if and only if* $\forall \varepsilon > 0$ *there exists a finite* ε*-net* S_n *in each* X_n *such that* $S_n \xrightarrow[GH]{} S$*, being* S *a finite* ε*-net in* X.

From 13.3, we arrive to another criterium for Gromov-Hausdorff convergence.

Theorem 13.6 $X_n \xrightarrow[GH]{} X$ *if and only if there exists a sequence of maps* $f_n : X_n \to X$ *(or* $f_n : X \to X_n$*) such that every* f_n *is an* ε_n*-isometry and* $\varepsilon_n \to 0$.

13.2.1. (Pre-)compactness theorems

Definition 13.8 *A class $\mathfrak{X}$ of compact metric spaces is uniformly totally bounded if*

1. There is a constant $\mathcal{D}$ such that $diam(X) \leq \mathcal{D}$ for every $X \in \mathfrak{X}$.

2. For every $\varepsilon > 0$ there is a $N(\varepsilon) \in \mathbb{N}$ such that, for all $X \in \mathfrak{X}$ there is an ε-net in X of no more than $N(\varepsilon)$ points.

Theorem 13.7 *$\mathfrak{X}$ satisfying definition 13.8 is pre-compact. This means that any sequence $\{X_n\} \subset \mathfrak{X}$ contains some subsequence convergent (in the Gromov-Hausdorff sense) to some metric space $X \in \mathfrak{X}$.*

There are some classes of Riemannian manifolds which are pre-compact in the Gromov-Hausdorff topology.

$\mathcal{R}(n, V, r) := \{M^n$ Riemannian manifold ; $Volume(M) \leq V$ and injectivity radius $\geq r\}$ is pre-compact.

$\mathcal{R}(n, k, D) := \{M^n$ Riemannian manifold ; $diam(M) \leq D$ and sectional curvature $\geq k\}$ is pre-compact.

The result remains true if we substitute the sectional curvature by the Ricci curvature.

This is a particular case of a more general statement about Alexandrov spaces.

Theorem 13.8 *The class $\mathcal{M}(n, k, D) := \{X : X$ is an Alexandrov space of curvature $\geq k$, $diam(X) \leq D$ and $dim_H(X) \leq n\}$[49], regarded with the Gromov-Hausdorff metric, is compact. Moreover, if $k > 0$, the condition $diam(X) \leq D$ can be removed and the result is still correct.*

13.3. Tangent cone and Asymptotic cone of a metric space

Let (X, p) be a metric space with some fixed point $p \in X$. We call (X, p) pointed metric space.

---

[49] $dim_H(X)$ is the Hausdorff dimension of X defined by

$$dim_H(X) = \inf\{d \geq 0 : \mu_d(X) \neq \infty\},$$

being $\mu_d(X)$ the d-dimensional Hausdorff measure of X, which is given by

$$\mu_d(X) = C(d) \lim_{\varepsilon \to 0} \mu_{d,\varepsilon}(X),$$

where $C(d)$ is a positive constant and, for $\varepsilon > 0$,

$$\mu_{d,\varepsilon} = \inf\{\omega_d(\{S_i\}) : diam(S_i) < \varepsilon \ \forall i\},$$

being $\{S_i\}_{i \in I}$ a finite or countable covering of X and $\omega_d(\{S_i\}) = \sum_i (diam(S_i))^d$ (if $d = 0$ we substitute each -if any- 0^0 term in the formula by 1). By convention, $\mu_d(\emptyset) = 0 \ \forall d \geq 0$.

Definition 13.9 *We say* $(X_n, p_n) \underset{GH}{\longrightarrow} (X, p)$ *if* $\forall r, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ /\forall n > n_0\ \exists f : B_r(p_n) \to X$ *(maybe not continuous) such that:*

1. $f(p_n) = p$.
2. $disf < \varepsilon$.
3. $\mathcal{U}_\varepsilon(f(B_r(p_n))) \supset B_{r-\varepsilon}(p)$.

Notation 13.10 *For a metric space* X *and* $\lambda > 0$*, we represent by* λX *the set of points* X*, but with the original metric multiplied by* λ.

Definition 13.11 *A pointed metric space is called a cone if* $(\lambda X, p)$ *is isometric to* (X, p) *for any* $\lambda > 0$.

Definition 13.12 *Given* (X, p) *a pointed metric space, where* X *is a (boundedly compact*[50]*) metric space. The (Gromov-Hausdorff) tangent cone of* X *at* p *is* $\lim_{\lambda\to\infty}(\lambda X, p)$*, if the limit exists.*

Remark 13.12.1 *The tangent cone is actually a cone in the sense of definition 13.11.*

Definition 13.13 *Let* X *be a (boundedly compact) metric space and* $p \in X$*. The asymptotic cone of* X *is* $\lim_{\lambda\to 0}(\lambda X, p)$*, if the limit exists.*

Remark 13.13.1 *If the asymptotic cone exists, it does not depend on the choice of the fixed point* p.

Examples 13.13.1 *1. As we have seen in previous lectures, the tangent cone at a point* p *to a convex surface* S *in* $\mathbb{R}^3$ *is the smallest convex cone with vertex at* p *which contains* S.

2. Consider a convex surface S *in* $\mathbb{R}^3$ *and a sequence of values for* λ *tending to* 0*. If we choose, in particular,* $0 < \lambda < 1$ *and consider the distance obtained multiplying by* λ *the euclidean distance, we obtain the set* λS*, which lies inside* S*. As* $\lambda \to 0$ *it seems that* λS *tends to a convex cone; but if* $\int_M K\,dM > \pi$*, the sequence* $\{\lambda S\}$ *degenerates to a line. In short, unlike the tangent cone, the asymptotic cone lies inside the surface.*

It is known the following result that assures the existence of asymptotic cone in a Riemannian manifold under certain conditions.

[50]A metric space is said to be boundedly compact if all closed bounded sets in it are compact.

Theorem 13.9 *(Kasue). Let M be a complete non compact Riemannian manifold such that for some $c < \infty$ and $\delta > 0$ satisfies $Sec_x \geq \dfrac{-c}{|px|^{2+\delta}}$, where $|px| = d(p,x)$. Then M has an asymptotic cone.*

Remark 13.13.2 *In particular, a complete noncompact Riemannian manifold with nonnegative curvature always has an asymptotic cone. On the contrary, Lobachebski ($\equiv$ hyperbolic) space has no asymptotic cone.*

14. SINGULARITIES AND DILATIONS ABOUT THEM

14.1. Introduction

Let (M^n, g_t) be a solution of the Ricci flow defined on $[0,T)$. Recall that a maximal solution is said to be non-singular if $T = \infty$ and $\sup_{M\times[0,T)} |Rm| < \infty$ and is called singular if $\sup_{M\times[0,T)} |Rm| = \infty$; in this case, we say that the solution obeys a singularity at the maximal time $T \in (0,\infty]$, which is called singular time.

However, for our purposes, it is convenient to adopt another point of view. Namely, we consider a maximal solution to be non-singular before T and singular at T, because it cannot be extended further in time.

Given a solution $(M^n, g(t))$ of the Ricci flow with a singularity at time $T < \infty$, it is expected that, after a finite number of surgeries on the singularities, the solution becomes non-singular. Therefore, it is important that we understand previously the main properties of non-singular solutions in order to use them in the future.

Before doing the aforementioned surgery and in order to improve the method for it, we need to study in depth two items:

1. The different type of singularities that can arise.
2. The possible limits of dilations of the metric about these singularities.

The method of dilations consists in constructing a suitable sequence of solutions $(M, g_i(t))$ and study its limit $(M_\infty, g_\infty(t))$ (if it exists). If we select carefully such sequence, it is possible to assure that the limit exists and is a complete nontrivial solution of the Ricci flow. One expects that this limit solution (called singularity model) can yield information about the original manifold, at least for points near the singularity and times just before its formation

14.2. Properties of non-singular solutions

First we recall as an important property of these solutions the Harnack inequality (trace form), Corollary 11.2, given in lecture 11. Other important properties are:

Proposition 14.1 *Let $R_{\min}(t) = \inf_{x\in M} R(x,t)$, where M is a compact n-dimensional manifold. Under the normalized Ricci flow, whenever $R_{\min} \leq 0$ it is increasing; whereas if $R_{\min} \geq 0$, it remains so forever.*

Proof. The evolution equation for the scalar curvature under the Ricci flow is (cf. [38])

$$\frac{\partial R}{\partial t} = -\Delta R + 2\left(|Ric|^2 - \frac{1}{n} rR\right) \tag{14.1}$$

We compute the norm of the Ricci tensor considering its decomposition

$$Ric = \overset{\circ}{Ric} + \frac{1}{n} Rg, \tag{14.2}$$

where tr $\overset{\circ}{Ric} = 0$ and $R = \mathrm{tr} Ric$ is the scalar curvature.

By substitution in (14.1), we obtain

$$\frac{\partial R}{\partial t} = -\Delta R + 2\left(|\overset{\circ}{Ric}|^2 + \frac{1}{n} R(R-r)\right) \tag{14.3}$$

Notice that at the points x where $R(x,t) = R_{\min}(t)$, $-\Delta_g R(x,t) \geq 0$, then we have

$$\begin{aligned} D_+ R_{\min} &\geq \inf_{x\in M(t)} \frac{\partial R}{\partial t}(x,t) \geq \inf_{x\in M(t)} 2\left(|\overset{\circ}{Ric}|^2 + \frac{1}{n} R(x,t)(R(x,t)-r)\right) \\ &\geq \frac{2}{n} R_{\min}(R_{\min} - r) \end{aligned}$$

If there exists $t_0 \in \mathbb{R}$ such that $R_{\min}(t_0) \leq 0$ then, since always $R_{\min} \leq r$, we have $D_+ R_{\min} \geq 0$. So $R_{\min}$ is increasing.

On the other hand, suppose that there exists some $t_0 \in \mathbb{R}$ such that $R_{\min}(t_0) \geq 0$. For simplicity in the computations, assume $t_0 = 0$. Consider the ODE associated to (14.3), that is,

$$\frac{dR}{dt} = 2\left(|\overset{\circ}{Ric}|^2 + \frac{1}{n} R(R-r)\right)$$

So we have the inequality $\dfrac{dR}{dt} \geq \dfrac{1}{n} R(R-r)$.

Solving this (with the change of variable $z(t) = \frac{1}{R(t)}$ and applying the Lagrange formula to $z' = rz - \frac{2}{n}$), we reach

$$R(t) \geq \frac{nr}{2\left(e^{rt}\left(\frac{nr}{2R_{\min}(0)} - 1\right) + 1\right)}.$$

Since $0 \leq R_{\min}(0) \leq r$, then $\frac{r}{R_{\min}(0)} \geq 1$ and so $R(t) \geq 0$ if $n \geq 2$. In particular, $R_{\min}(t) \geq 0$ for all t such that the solution exists. □

Proposition 14.2 *If $R_{\min}(t) > 0$ for some t, then the maximal time interval $[0, T[$ of the corresponding solution of the Ricci flow is finite.*

Proof. We have the following evolution equation for the scalar curvature under the Ricci flow

$$\frac{\partial R}{\partial t} = -\Delta R + 2|Ric|^2 \tag{14.4}$$

The decomposition of the Ricci tensor given by (14.2) yields

$$\frac{\partial R}{\partial t} = -\Delta R + 2\left(|\overset{\circ}{Ric}|^2 + \frac{1}{n}R^2\right)$$

And so, considering the associated ODE, we arrive to

$$\frac{dR}{dt} = 2\left(|\overset{\circ}{Ric}|^2 + \frac{1}{n}R^2\right) \geq \frac{2}{n}R^2 \tag{14.5}$$

Suppose that there exists some $t_0 \in \mathbb{R}^+ \cup \{0\}$ such that $R_{\min}(t_0) > 0$. Solving (14.5) and applying the maximum principle, we have

$$R(x,t) \geq R_{\min} \geq \frac{n}{n(R_{\min}(t_0))^{-1} - 2(t - t_0)}.$$

Hence $R_{\min}$ blows up at a finite time $T = \frac{n}{2}(R_{\min}(t_0))^{-1} + t_0$. Notice that $t > 0$ since, by hypothesis, $R_{\min}(t_0) > 0$. □

Proposition 14.3 *Let (M^3, g_t) be a complete solution to the Ricci flow on a 3-manifold which is complete with bounded curvature for $t \geq 0$. Consider the eigenvalues $\lambda \geq \mu \geq \nu$ of the curvature operator and suppose $\inf_{x\in M} \nu(x,0) \geq -1$. Then, at all points and all times where $\nu(x,t) < 0$, the scalar curvature is estimated by*

$$R \geq |\nu|(\ln|\nu| + \ln(1+t) - 3) \tag{14.6}$$

Remark 14.3.1 *1. A detailed proof of this fact can be found in [41] and consists of studying the reaction equations associated to the system of evolution equations for the eigenvalues:*

$$\begin{cases} \partial_t \lambda = \Delta\lambda + \lambda^2 + \mu\nu \\ \partial_t \mu = \Delta\mu + \mu^2 + \nu\lambda \\ \partial_t \nu = \Delta\nu + \nu^2 + \lambda\mu \end{cases}$$

2. Later we shall show that the condition $\inf_{x\in M} \nu(x,0) \geq -1$ *can always be achieved by scaling* $g(0)$ *by a suitable constant.*

3. When we have $|\nu| \geq e^3/(1+t)$ *(which is a condition easy to reach for* t *sufficiently large), the inequality* (14.6) *says that the scalar curvature is nonnegative. Since it is the sum of sectional curvatures, this means that, if there exists any point and time* (x,t) *where a sectional curvature is negative, then it is possible to find a larger positive curvature at the same* (x,t)*.*

The last property gives us a pinching estimate, that is, an inequality which is preserved by the flow. There are other results which yield similar pinching estimates, for instance, here is one referred to the Ricci tensor proved by T. Ivey (see [52] for more details).

Proposition 14.4 *If* (M^3, g_t) *is a solution to the normalized Ricci flow on a compact 3-manifold and has scalar curvature bounded below by a positive constant for all* $t \geq 0$*, then there exists a positive function* $\phi(t)$ *satisfying*

$$Ric(g_t) \geq -\phi(R(t))\ r(t) \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = 0.$$

14.3. Classification of maximal solutions

A first approach to the classification of the maximal solutions to the Ricci flow by their singularity type is given by the maximal time T for which the solution exists. According to this, we distinguish between finite time ($T < \infty$) and infinite time ($T = \infty$) singularities.

On the other hand, from long time existence, we know that if $T < \infty$, then the curvature is unbounded in the sense that $\sup_{M\times[0,T)} |Rm(x,t)| = \infty$. So we cannot use the norm of the curvature tensor to distinguish between different types of finite-time singularities. A way to do so is to make distinctions between slowly and rapidly forming singularities, in other words, we shall do the classification according to the rate at which the curvature tends to infinity as time goes to T.

In short, if $(M^n, g(t))$ is a solution to the Ricci flow which exists up to a maximal time $T \leq \infty$, the following table summarizes its possible singularities.

Singularities		
Finite Time	Type I	$\sup\limits_{M\times[0,T)} \lvert Rm(\cdot,t)\rvert(T-t) < \infty$
$T < \infty$	Type II$_a$	$\sup\limits_{M\times[0,T)} \lvert Rm(\cdot,t)\rvert(T-t) = \infty$
Infinite Time	Type II$_b$	$\sup\limits_{M\times[0,\infty)} \lvert Rm(\cdot,t)\rvert t = \infty$
$T = \infty$	Type III	$\sup\limits_{M\times[0,\infty)} \lvert Rm(\cdot,t)\rvert t < \infty$

Table: Classification of maximal solutions

We are interested in finite-time singularities (and, in particular, in Type I singularities) because they play an important role in Perelman's work towards the proof of the geometrization conjecture for closed 3-manifolds.

14.4. Singularity models

Definition 14.1 *A solution to the Ricci flow is called a singularity model if it is a complete nonflat solution obtained as limit of dilations of a solution* [51] *near a singularity.*

There exist different types of singularity models $(M^n_\infty, g_\infty(t))$ (classified according to their existence time interval and some curvature bound) corresponding to the possible singularities explained in the last section. The classification appears in the following table:

Singularity model	time interval	curvature bound
Ancient Type I	$(-\infty,\omega)/\omega > 0$	$\sup\limits_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert\lvert t\rvert < \infty$
Ancient Type II	$(-\infty,\omega)/\omega > 0$	$\sup\limits_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert\lvert t\rvert = \infty$
Eternal Type II	$(-\infty,\infty)$	$\sup\limits_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert < \infty$
Immortal Type III	$(-\alpha,\infty)/\alpha > 0$	$\sup\limits_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert \cdot t < \infty$

Table: Singularity models

[51] there are many possibilities of taking these dilations, the more usual is that we shall define in (14.8)

14.5. Properties of the ancient solutions

Because we are interested in finite time singularities and the corresponding singularity models are ancient solutions, we shall study its main properties in this section.

Proposition 14.5 *Let $(M^n, g(t))$ be a complete solution of the Ricci flow. Suppose that $R_{\min}(t)$ is finite for all $t \leq 0$ and that there exists a continuous function $\phi(t)$ such that $|K_{sec}(g_t)| \leq \phi(t)$. Then g_t has nonnegative scalar curvature for as long as it exists.*

Proof. Since, for each t, the sectional curvature is bounded, $R_{min}(t)$ exists. As in the proof of Proposition 14.2, we have, for the scalar curvature under the Ricci flow, the evolution equation $\dfrac{\partial R}{\partial t} = -\Delta R + 2|Ric|^2 \geq -\Delta R + \dfrac{2}{n}R^2$, and, applying the maximum priciple we obtain $D_+ R_{min}(t) \geq \frac{2}{n}R^2_{min}$, then $R_{min}(t)$ is increasing with t. Hence, if R becomes nonnegative, it remains so for bigger times and, if there is t_1 where $R_{min}(t_1) < 0$, then $R_{min}(t_0) < 0$ for every $t_0 \leq t_1$ and

$$R_{\min}(t) \geq \frac{n}{n(R_{\min}(t_0))^{-1} - 2(t - t_0)} \geq -\frac{n}{2(t - t_0)} \text{ for all } t > t_0. \quad (14.7)$$

Since the solution is ancient, we may let $t \to -\infty$, obtaining $R_{min}(t) \geq -\lim\limits_{t_0 \to -\infty} \dfrac{n}{2(t - t_0)} = 0$. □

Proposition 14.6 *Let $(M^3, g(t))$ be a complete ancient solution of the Ricci flow. Assume that there exists a continuous function $\phi(t)$ such that $|K_{sec}(g_t)| \leq \phi(t)$. Then g_t has nonnegative sectional curvature for as long as it exists.*

Proof. For the proof we want to use the pinching estimate (14.6). However, to apply this result, we need the condition $\inf_{x \in M} \nu(x, 0) \geq -1$.

If $(M, g(t))$ is a solution of the Ricci flow that exists at least for $0 \leq t < T$ and

$$\nu_0 := \inf_{x \in M} \nu(x, 0) < 0,$$

then, taking $\tilde{t} = |\nu_0| t$, we have $\tilde{g}(t) = |\nu_0| g(\frac{t}{|\nu_0|})$ and $\tilde{g}(t)$ is a solution of the Ricci flow. From $\tilde{\nu} + \tilde{\mu} + \tilde{\lambda} = \widetilde{R} = \frac{1}{|\nu_0|} R = \frac{1}{|\nu_0|}(\nu + \mu + \lambda)$, we get $\tilde{\nu} = \frac{1}{|\nu_0|}\nu$ and, in particular, $\tilde{\nu}(x, 0) = \frac{\nu(x,0)}{|\nu_0|} \leq \frac{\nu_0}{|\nu_0|} = -1$.

Now, if we suppose that $\tilde{\nu}\left(x, \tilde{t}\right) < 0$, we are in position to apply (14.6) to conclude

$$\widetilde{R}(x, \tilde{t}) \geq |\tilde{\nu}(x, \tilde{t})| [\ln |\tilde{\nu}(x, \tilde{t})| + \ln(1 + \tilde{t}) - 3] \quad \forall (x, \tilde{t}) \in M \times [0, T)$$

We can rewrite the last inequality in terms of the metric g to get

$$|\nu_0|R(x,t) \geq \frac{1}{|\nu_0|}|\nu(x,t)| \left(\ln\left(\frac{|\nu(x,t)|}{|\nu_0|}\right) + \ln(1+|\nu_0|t) - 3\right),$$

wherever $\nu(x,t) < 0$.

Simplifying the above expression, we obtain

$$R(x,t) \geq |\nu(x,t)| \left(\ln|\nu(x,t)| + \ln(|\nu_0|^{-1} + t) - 3\right)$$

In particular, if $\nu(x,t) < 0$ at some point $x \in M$ and time $t > 0$, then

$$R(x,t) > |\nu(x,t)| \left(\ln|\nu(x,t)| + \ln t - 3\right),$$

because ln is a strictly increasing function.

Now assume that $g(t)$ exists for $-\alpha \leq t \leq \omega$. Taking $\tilde{t} = t + \alpha$, we have $\tilde{R} = R$ and $\tilde{\nu} = \nu$. Substituting this in the last inequality, we reach

$$R(x,t) > |\nu(x,t)| \left(\ln|\nu(x,t)| + \ln(t+\alpha) - 3\right) \quad \text{wherever } \nu(x,t) < 0$$

Since, by hypothesis, $g(t)$ is an ancient solution, we can take limits when $\alpha \to -\infty$. But, in this case, we get $\lim_{\alpha \to -\infty}(t+\alpha) = \infty$. So R is unbounded; however, this is a contradiction with $|K_{sec}| \leq \phi(t)$, since the scalar curvature R is the sum of the sectional curvatures. Hence $\nu(x,t) \geq 0$ for all x and t. □

Next we state a corollary of the last property. If the reader is interested in the proof, it can be found in [22] p.261.

Proposition 14.7 *In dimension $n = 3$ every Type I limit of Type I singularity has nonnegative sectional curvature for as long as it exists.*

Proposition 14.8 *Let $(M^n, g(t))$ be a solution of the Ricci flow on a compact manifold with initially positive curvature operator. Then the function tR is pointwise nondecreasing for all $t \geq 0$ such that the solution exists. Moreover, if $(M^n, g(t))$ is also ancient, then R itself is pointwise nondecreasing.*

Proof. Taking $X = 0$ in the trace form of the Harnack inequality (11.2), we get $\frac{\partial}{\partial t}(Rt) = t\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) \geq 0$ for all $t \geq 0$ such that the solution exists.

On the other hand, suppose that the solution exists for $-\alpha \leq t < \omega$ and consider $\tilde{t} = t + \alpha$. Then $\widetilde{R} = R$ and so

$$\frac{\partial R}{\partial t} + \frac{R}{t+\alpha} \geq 0$$

or, equivalently,

$$\frac{\partial R}{\partial t} \geq -\frac{R}{t+\alpha}.$$

Since $g(t)$ is an ancient solution, taking limits when $\alpha \to -\infty$, we check the second assertion of the property. □

14.6. Dilations about singularities

Here we shall study how singularities may be removed by dilations of the metric. The first step consists in taking a sequence of points and times where the norm of the curvature tends to infinity and is comparable (in the sense of definition 14.2 below) to its maximum in sufficiently large spatial and temporal neighborhoods of the chosen points and times.

Definition 14.2 *A sequence $\{(x_i, t_i)\}$ is said to be globally curvature essential if it satisfies*

1. $t_i \nearrow T \leq \infty$.

2. There exists a constant $C \geq 1$ and a sequence $r_i \in (0, \sqrt{t_i})$ such that

$$\sup\{|Rm(x,t)| : x \in \overline{B}_{g(t)}(x_i, r_i),\, t \in [t_i - r_i^2, t_i]\} \leq C|Rm(x_i, t_i)|$$

for all $i \in \mathbb{N}$ where $\lim_{i\to\infty} r_i^2 |Rm(x_i, t_i)| = \infty$.

Given a globally curvature essential sequence $\{(x_i, t_i)\}$, we construct a sequence of solutions of the Ricci flow by dilating the metric by a factor $|Rm(x_i, t_i)|$ and the time by a factor $1/|Rm(x_i, t_i)|$, and traslating the origin of time by t_i. That is, we define the family of metrics $\{g_i(t)\}$ given by

$$g_i(t) = |Rm(x_i, t_i)| g\left(t_i + \frac{t}{|Rm(x_i, t_i)|}\right) \tag{14.8}$$

for $0 \leq t_i + \frac{t}{|Rm(x_i,t_i)|} \leq T$.

$(M^n, g_i(t))$ is called the *sequence of parabolic dilations* and exists for

$$-t_i|Rm(x_i, t_i)| \leq t < (T - t_i)|Rm(x_i, t_i)|. \tag{14.9}$$

Derivating respect to t, we check that $g_i(t)$ is a solution to the Ricci flow. In addition, notice that $g_i(0)$ is a homothetic multiple of $g(t_i)$.

By the relation between the curvatures of homothetic metrics, we have constructed a sequence of pointed metric spaces $(M, g_i(t), x_i)$ satisfying

$$|Rm_i(x_i, 0)| = 1$$

This condition guarantees that, if the limit exists, it will not be flat.

In particular, for Type I singularities, it holds from (14.9) that

$$-\infty \leq t < C < \infty$$

Then, we can state the following result.

Proposition 14.9 *If the limit of* $\{(M^n, g_i(t))\}_{i\in\mathbb{N}}$ *(given by the parabolic dilations* (14.8)*) exists, it is an ancient solution.*

14.7. Existence of convergent subsequences

Now our principal aim is to find out conditions which ensure the existence of subsequences of $(M^n, g_i(t))$ convergent to nontrivial limits (that is, nonflat metrics) which are complete solutions to the Ricci flow.

The Compactness Theorem for the Ricci flow, which we shall state below and whose proof can be found in [46], is the main result in this direction. If we check that our sequence of complete solutions of the Ricci flow $(M^n, g_i(t))$ satisfies:

1. Sectional curvatures uniformly bounded from above,
2. Injectivity radius uniformly bounded from below,

the Compactness Theorem assures that it contains a convergent subsequence.

Theorem 14.10 *(Compactness Theorem). Let* $\{M_i^n, g_i(t), O_i, F_i\}_{i\in\mathbb{N}}$*, where, for each* $i \in \mathbb{N}$,

- $g_i(t)$ *is a complete solution to the Ricci flow for* $t \in (\alpha, \omega)$*, being* $-\infty \leq \alpha < 0 < \omega \leq \infty$.

- O_i *is a point in the* n*-dimensional manifold* M_i.

- F_i *is an orthonormal (respect to* $g_i(0)$*) frame at* O_i.

Suppose that the following conditions hold:

1. There is a constant $C < \infty$ *such that* $\sup_{M\times(\alpha,\omega)} |K_{sec}(g_i)| \leq C$ *for all* $i \in \mathbb{N}$.

2. There is a contsnat $\delta > 0$ such that $inj_{g_i(0)}(O_i) \geq \delta$ for all $i \in \mathbb{N}$.

Then there exists a subsequence convergent (in the pointed sense) to a complete solution $(M^n_\infty, g_\infty(t), O_\infty, F_\infty)$ of the Ricci flow existing for $t \in (\alpha, \omega)$ and with the same properties for K_{sec} and inj.

The convergence in the above theorem is the one corresponding to the following definition of distance:

Definition 14.3 *We say that two pointed Ricci flows $(M, g(t), x, \tau)$ and $(M', g'(t), x', \tau')$ are at C^m-distance lower than ε if there are open sets U and U' satisfying $B(x, \tau, 1/\varepsilon) \subset U \subset M$ and $B(x', \tau', 1/\varepsilon) \subset U' \subset M'$ and a diffeomorphism $\phi : U \longrightarrow U'$ such that, for every $t \in]-\frac{1}{\varepsilon^2}, 0]$ one has $\left| \dfrac{\partial^j(\phi^* g')}{\partial t^j}(\tau' + t) - \dfrac{\partial^j(g)}{\partial t^j}(\tau + t) \right|_{C^m} < \varepsilon$ for every $0 \leq j \leq m$.*

14.7.1. A comment about the convergence

It is a remarkable fact that, requiring only bounds on curvature and not on its covariant derivatives, theorem 14.10 provides convergence in the C^∞ topology on compact sets. The reason for this is that a local bound on the curvature of a solution to the Ricci flow implies bounds on all derivatives of the curvature. This property was originally proved by W.X. Shi in [79][52] using N.S. Bernstein ideas; here we just give the statement for the estimation of the first derivative (the results for higher derivatives are similar).

Theorem 14.11 *Let $g(x, t)$ be a solution to the Ricci flow on $\mathcal{U} \times [0, T]$, where $\mathcal{U} \subset M$ is an open neighborhood of $x \in M$. Assume the following conditions are satisfied:*

1. There exists some constant K such that $|Rm(x, t)| \leq K$ for every $(x, t) \in \mathcal{U} \times [0, T]$.

2. There is $r > 0$ such that, at time $t = 0$, the closed ball $\overline{B}_r(x)$ of center x and radius r is a compact set contained in $\mathcal{U}$.

Then we can find a constant $C < \infty$ for which we have the following estimation for the covariant derivatives of the curvature

$$|\nabla Rm(x, T)|^2 \leq CK^2 \left(\frac{1}{r^2} + \frac{1}{T} + K \right)$$

[52] See also [41] section 13 and these notes, Theorem 8.3 for global estimates.

14.7.2. A comment about the injectivity radius bound

In the statement of the theorem 14.10, we have restriction on injectivity radii only in an specific point. We can wonder what happens to the injectivity radius at a point P as the distance of P from the origin O grows to infinity. In [17], it is shown that the injectivity radius $inj(P)$ falls off at worst exponentially; in particular

$$inj(P) \geq \frac{c}{\sqrt{k}} \left(\delta\sqrt{k}\right)^n e^{-C\sqrt{k}\, d(O,P)},$$

where k is an upper bound on $|K_{sec}|$, δ a lower bound on the $inj(O)$ with $\delta \leq \frac{c}{\sqrt{k}}$ and $c > 0$, $C < \infty$ are constants depending only on the dimension n.

The form given here follows from knowing that the volume of the ball of radius r in hyperbolic space grows exponentially in r.

14.8. Limit of dilated solutions

Let us return to our particular problem of finding convergent subsequences of $(M^n, g_i(t))$, where $g_i(t)$ are dilated solutions of the Ricci flow defined as in (14.8). For now on, we assume the global bound

$$\sup\{|Rm(x,t)| : x \in M, t \in [0,t_i]\} \leq C|Rm(x_i,t_i)| \tag{14.10}$$

14.8.1. Bounds on the curvature

Here we wonder how changes the curvature after dilation of the metric. First we need the following result which shows that the blow-up rate of finite time singularities is bounded below.

Lemma 14.12 *Let $(M^n, g(t))$ a solution of the Ricci flow which exists on a maximal time interval $[0,T)$, where $T < \infty$ and M is a compact manifold. Then there exists a constant $c_0 > 0$ (depending only on n) such that*

$$\sup_{x\in M} |Rm(x,t)| \geq \frac{c_0}{T-t}$$

On the other hand, given a Type I singular solution $(M^n, g(t))$ on $[0,T)$ define S by

$$S := \sup_{M\times[0,T)} |Rm(x,t)|(T-t) \tag{14.11}$$

Notice that the definition of Type I singularity assures $S < \infty$.

Now we are in position to give an estimate for the curvature tensor associated to the metric $g_i(t)$. From the definition of $g_i(t)$, it holds

$$|Rm_i(x,t)| = \frac{1}{|Rm(x_i,t_i)|}\left|Rm\left(t_i + \frac{t}{|Rm(x_i,t_i)|}\right)\right| \tag{14.12}$$

Using (14.10), we have

$$\begin{aligned}\frac{1}{|Rm(x_i,t_i)|} &\le \frac{C}{\sup\{|Rm(x,t)| : x\in M, t\in[0,t_i]\}} \le \frac{C}{\sup\{|Rm(x,t_i)| : x\in M\}}\\ &\le \frac{C}{c_0}(T-t_i)\end{aligned} \tag{14.13}$$

Notice that the last inequality follows from lemma 14.12.

Denoting $c := \frac{C}{c_0}$, substituting (14.13) in (14.12) and using the definition of S, we get

$$\begin{aligned}|Rm_i(x,t)| &\le c\,(T-t_i)\frac{S}{T-\left(t_i+\frac{t}{|Rm(x_i,t_i)|}\right)}\\ &= c\,S\frac{T-t_i}{T-t_i-\frac{t}{|Rm(x_i,t_i)|}} = c\,S\frac{1}{1-\frac{t}{(T-t_i)|Rm(x_i,t_i)|}}\\ &\le c\,S\left(1+\frac{|t|}{S}\right)^{-1} \qquad \forall x\in M \text{ and } t_i\in[-\alpha_i,0]\end{aligned} \tag{14.14}$$

Then

$$|t||Rm_i(x,t)| \le c\,S|t|\left(\frac{S}{S+|t|}\right) = c\,S^2\left(\frac{|t|}{S+|t|}\right) \le c\,S^2$$

So we have the estimate

$$\sup_{M\times[-\alpha_i,0]} |t||Rm_i(x,t)| \le c\,S^2 < \infty$$

And this guarantees the first condition of the Compactness Theorem.

14.8.2. Injectivity radius bound

In order to apply the Compactness Theorem to our sequence of pointed dilated solutions $(M_i^n, g_i(t), x_i)$ to the Ricci flow, we still need to find out if it satisfies a suitable estimate on the injectivity radius.

The following theorem, which can be found in [67], give us the condition we were looking for.

Theorem 14.13 *(Perelman's No Local Collapsing Theorem). Let* $(M^n, g(t))$ *be a solution to the Ricci flow that becomes singular in a finite time* T. *Then there exists a constant* $C > 0$ *independent of* t *and a subsequence* (x_i, t_i) *such that*

$$inj(x_i, t_i) \geq \frac{C}{\sqrt{\max_M |Rm(\cdot, t)|}}.$$

This is the most general result that provides the injectivity radius bound which guarantees (by means of theorem 14.10) the existence of the desired limit. But Perelman's Theorem is relatively recent; before it, there are some results (due to Hamilton) which provide a global injectivity radius estimate for particular cases.

14.8.3. Further results

After assuring the existence of limit, we are in position to prove the following result (cf. [22] p. 242).

Proposition 14.14 *Let* $(M^n, g(t))$ *a solution to the Ricci flow with a singularity at time* $T \in (0, \infty)$ *and let* (x_i, t_i) *be a globally curvature essential sequence. Then any limit* $(M^n_\infty, g_\infty(t))$ *is an ancient Type I singularity model with a Type I singularity at some time* $\omega < \infty$.

Some properties of the ancient solutions stated in section 14.5 are used in the proofs of some important results towards the classification of singularities. For instance, property 14.5 is needed to show the next proposition, which plays an important role in the classification of 3-dimensional model singularities (see theorem 6.8).

Proposition 14.15 *(cf.[22] p.275 or [41] p.129). A complete ancient Type I solution* $(N^2, h(t))$ *of the Ricci flow on a surface is a quotient of either a shrinking round* S^2 *or else a flat* $\mathbb{R}^2$.

On the other hand, the case of Type II ancient solutions is proved with the aid of property 14.8.

Finally, it is possible to show (using 14.7) that an ancient solution of positive sectional curvature is either isometric to a spherical space form or else contains points at arbitrarily ancient times where the geometry is arbitrarily close to the product of a surface and a line.

15. SURVEY ON PERELMAN'S WORK (I): NEW FUNCTIONALS AND κ-COLLAPSE

15.1. Introduction

In this and the next lectures we summarize the main results and techniques developed by G. Perelman in his preprint [67], with some hints on [68] and [69].

From the preceding chapters, we know that we have a singularity at time T in a solution to the Ricci flow, we can take a sequence $\{(x_i, t_i)\}$ and dilate the corresponding metrics $g(x_i, t_i)$. If the sequence of dilated metrics $g_i(t)$ converges, it converges to an ancient solution with nonnegative curvature operator and defined on time $-\infty \leq t \leq 0$.

The question is: does this sequence $g_i(t)$ converges?. The first result of Perelman is to answer "yes". Then he studies more properties of ancient solutions which are k-noncollapsing (a concept introduced by him) he researches the asymptotic behaviour of these k-solutions (another Perelman's concept), that is, he takes the limit of these solutions when t goes to $-\infty$ and he shows that it is a gradient shrinking soliton. Then he shows that in a 3-dimensional compact manifols the only possible gradient shrinking solitons are: S^3/Γ, $S^2 \times S^1$ and $S^2 \times \mathbb{R}/\mathbb{Z}_2$. This implies that on our k-solution, except on a compact set, every point has a neighborhood ε-close to a part of a neck $S^2 \times \mathbb{R}$. Then going back at points $g_i(t)$, near T, the singularity is of neck or cap type (which precise definitions we shall see later).

15.2. Ricci flow as a gradient flow

We know that the functional $\int_M R\,dV$ has critical points satisfying $Ric = 0$ or, subject to the conditions of constant volume for M, $Ric - \frac{2}{n}g = 0$. This is the natural functional when we look for Einstein manifolds (as Hamilton did in his program for solving geometrization conjecture). But the flow corresponding to the gradient of this functional has no solution, even for short time. For this reason R. Hamilton introduced his Ricci flow.

The first thing which Perelman did in [67] was to introduce a functional whose gradient could give rise to the Ricci flow. He did so in the following way: let M be a closed manifold, $\mathcal{M}$ the space of metrics on M and $\mathcal{C} := C^\infty(M, \mathbb{R})$, then he defines the functional

$$\mathcal{F} : \mathcal{M} \times \mathcal{C} \to \mathbb{R}; \quad \mathcal{F}(g, f) = \int_M (R_g + |df|_g^2) e^{-f}\ dV_g \tag{15.1}$$

Notice that the expression $R + |\nabla f|^2$ had appeared in Hamilton's work.

A computation (see [56] and [37]) gives the first variation formula for this functional

$$\delta\mathcal{F}(v,h) = \int_M e^{-f}\left(-\langle v,(Ric_g+\nabla^2 f)\rangle\right.$$
$$\left.+\left(\frac{1}{2}\mathrm{tr}_g v - h\right)(R_g - |df|^2 - 2\Delta_g f)\right) dV_g, \quad (15.2)$$

where $v = \delta g$ is a 2-covariant tensor field on M and $h = \delta f$ is a C^∞ function on M.

If we consider only the variations with $dm = e^{-f}\, dV_g$ constant, then $\frac{1}{2}\mathrm{tr}_g v - h = 0$. But to consider only those variations for the functional $\mathcal{F}$ is equivalent to consider, given a fixed volume form dm, any variation of the functional

$$\mathcal{F}^m : \mathcal{M} \to \mathbb{R}; \mathcal{F}^m = \int_M (R + |\nabla f|^2)\, dm \quad (15.3)$$

where $f = \ln\left(\frac{dV}{dm}\right)$ (then f depends on g in this approach).

From $\frac{1}{2}\mathrm{tr}_g v - h = 0$ for dm fixed and (15.2), it follows that the L^2-gradient of $\mathcal{F}^m$ at $g \in \mathcal{M}$ is the symmetric 2-tensor field, on M, $\mathrm{grad}\mathcal{F}^m(g) = -2(Ric_g + \nabla^2 f)$. To this gradient (vector field on $\mathcal{M}$) corresponds the flow (equation giving the integral curves of $\mathrm{grad}\mathcal{F}^m$ in $\mathcal{M}$)

$$\frac{\partial g}{\partial t} = -2(Ric_g + \nabla^2 f) \quad (15.4)$$

and, from $f = \ln\left(\frac{dV}{dm}\right)$, it follows that $\frac{\partial f}{\partial t} = \frac{1}{2}\mathrm{tr}_g \frac{\partial g}{\partial t}$ and, using (15.4),

$$\frac{\partial f}{\partial t} = -R_g + \Delta_g f \quad (15.5)$$

The solution of system (15.4) and (15.5) exists if and only if it is the Ricci flow modified by a diffeomorphism. Now, the fixed points of this flow are not only Einstein manifolds, but also solitons.

Notice that (15.4) had already appeared in Hamilton's paper about surfaces.

Example 6 *In order to have some intuition, we consider the so-called basic example in [56] notes, where $M = \mathbb{R}^n$, a non closed manifold on which we can still define the flow given by* (15.4) *and* (15.5). *Let us consider* $\mathbb{R}^n$

with its standard metric, constant in time. Let $\tau = t_0 - t$, with t_0 fixed. We take the following function

$$f(t,x) = \frac{|x|^2}{4\tau} + \frac{n}{2}\ln(4\pi\tau) \tag{15.6}$$

This implies $e^{-f} = (4\pi\tau)^{-n/2}\, e^{-|x|^2/4\tau}$, that is, e^{-f} is the standard heat kernel for τ going from 0 to t_0, that is, for t going from T_0 to 0 (see (4.3)). Then f satisfies (15.5), because we are considering g_t to be the euclidean metric for every t. Moreover, the well-known property that

$$\int_{\mathbb{R}^n} e^{-|x|^2/4\tau}\, dV = (4\pi\tau)^{-n/2}, \tag{15.7}$$

assures that f is properly normalized. From (15.6),

$$\nabla f = x/2\tau \text{ and } |\nabla f|^2 = |x|^2/4\tau \tag{15.8}$$

Differentiating (15.7) with respect of τ, we get

$$\int_{\mathbb{R}^n} \frac{|x|^2}{4\tau^2}\, e^{-|x|^2/4\tau}\, dV = (4\pi\tau)^{-n/2}\cdot\frac{n}{2\tau} \tag{15.9}$$

So, using (15.8), we have $\displaystyle\int_{\mathbb{R}^n} |\nabla f|^2 e^{-f}\, dV = \frac{n}{2\tau}$. Then $\mathcal{F}(t) = \frac{n}{2\tau}$. In particular, this is a nondecreasing function of $t \in [0, t_0)$.

15.3. Shrinking solitons

A main question in Hamilton's program is to classify the solitons, which are a particular type of solutions to the Ricci flow (cf. definitions 9.1 and 11.2). In section 2 of the aforementioned preprint, Perelman proved that there are no nontrivial (that is, with nonconstant Ricci curvature) steady or expanding solitons on closed M. This result was proved before by Hamilton and Ivey (cf. [52] and [**?**]). However, the same question for shrinking solitons was open and Perelman deals with this in [67]§3. Having this purpose in mind, he introduces a new functional $\mathcal{W}(g, f, \tau)$ (using ideas connected to statistical physics).

First he introduces the concept of breather

Definition 15.1 *A metric g solution of the Ricci flow is called a breather if, for some $t_1 < t_2$ and $\alpha > 0$, the metrics $\alpha g(t_1)$ and $g(t_2)$ differ only by a diffeomorphism. We say that a breather is steady if $\alpha = 1$, shrinking if $\alpha < 1$ and expanding if $\alpha > 1$*

Definition 15.2 *A breather is called trivial if it is a breather for every t_1, t_2 in the interval of existence of the solution. Then, a Ricci soliton is a trivial breather and reciprocally.*

Now we consider the functional $\mathcal{W} : \mathcal{M} \times \mathcal{C} \times (0, \infty) \to \mathbb{R}$ defined by

$$\mathcal{W}(g, f, \tau) = \int_M \big(\tau(|df|_g^2 + R_g) + f - n\big)\,(4\pi\tau)^{-n/2}\,e^{-f}\,dV_g \qquad (15.10)$$

and restricted to the f satisfying the following additional condition (to keep the dm-volume constant)

$$\int_M (4\pi\tau)^{-n/2}\,e^{-f}\,dV = 1, \qquad \tau > 0 \qquad (15.11)$$

Notice that $\mathcal{W}$ is a generalization of the functional $\mathcal{F}$ defined by (15.1). On the other hand, $\mathcal{W}$ is invariant under simultaneous scaling of τ and g. The evolution equations (which generalize (15.4) and (15.5)) are

$$\begin{aligned} \frac{\partial g}{\partial t} &= -2Ric \\ \frac{\partial f}{\partial t} &= \Delta f + |df|^2 - R + \frac{n}{2\tau} \\ \frac{\partial \tau}{\partial t} &= -1 \end{aligned} \qquad (15.12)$$

The evolution equation for f can also be written as $\Box^* u = 0$, being $\Box^* = \Delta + R - \dfrac{\partial}{\partial t}$ the conjugate heat operator and $u = (4\pi\tau)^{-n/2}\,e^{-f}$.

We can compute a formula for the first variation of $\mathcal{W}$ in the direction of the curve $(g(\cdot, t), f(\cdot, t), \tau(t))$:

$$\frac{d\mathcal{W}}{dt} = \int_M 2\tau \Big|\underbrace{R + \nabla^2 f - \frac{1}{2\tau} g_{ij}}_{(\star)}\Big|^2 (4\pi\tau)^{-n/2}\,e^{-f}\,dV \qquad (15.13)$$

Notice that $(\star) = 0$ is the equation of a gradient shrinking soliton.

Perelman call (15.13) a monotonicity formula.

Now, we introduce the quantities

$$\mu(g, \tau) = \inf\{\mathcal{W}(g, f, \tau) : f \in \mathcal{C} \text{ and satisfying (15.11)}\}$$

$$\nu(g) = \inf_{\tau > 0} \mu(g, \tau)$$

From (15.13), it follows that, for every $\tau \in \mathbb{R}^+$, $t \mapsto \mu(g_t, \tau - t)$ is nondecreasing along the Ricci flow, for t in $]-\infty, \tau[$.

Here Perelman states the first result related to the introduced functionals.

Proposition 15.1 *For an arbitrary $g \in \mathcal{M}$, $\mu(g,\tau) < 0$ for small $\tau > 0$ and $\lim_{\tau\to 0} \mu(g,\tau) = 0$.*

The proof consists in arriving to a contradiction with the Gaussian logarithmic Sobolev inequality [53] in euclidean space:

$$\int_{\mathbb{R}^n} (f^2 \ln f)\,\rho\,dx - \left(\int_{\mathbb{R}^n} f^2\,\rho\,dx\right) \ln \left(\int_{\mathbb{R}^n} f^2\,\rho\,dx\right)^{1/2} \leq \int_{\mathbb{R}^n} |\nabla f|^2\,\rho\,dx \tag{15.14}$$

where $\rho(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$ and f is any nonnegative function which is square integrable and with $|\nabla f|^2$ integrable on $\mathbb{R}^n$ respect to the measure $\rho\,dx$.

An important example of a gradient shrinking soliton is the Gaussian soliton, which corresponds to the example 6. In fact, as we know, a gradient Ricci soliton has to satisfy $Ric + cg + \frac{1}{2}(\nabla_i\nabla_j a + \nabla_j\nabla_i a) = 0$, $a \in \mathcal{C}$, which works for the basic example with $c = 1$ and $a = -|x|^2/2$.

As a corollary of this first result, Perelman obtains that "*there are no shrinking breathers other than gradient solitons*".

In dimension 3, Ivey (cf.[52]) have proved even more: "*there are no other shrinking breathers other than trivial (i.e. Einstein) Ricci solitons*".

15.4. No Local Collapsing Theorem

Definition 15.3 *A smooth solution $g(t)$ of the Ricci flow defined on a time interval $[0,T)$ is said to be locally collapsing at T if there exists a sequence of times $t_k \to T$ and metric balls $B_k = B_{t_k}(p_k, r_k)$ satisfying*

1. *r_k^2/t_k is bounded (when $T < \infty$, this is equivalent to r_k^2 bounded).*
2. *$|Rm_{g(t_k)}| \leq 1/r_k^2$ in B_k.*
3. *$r_k^{-n} Volume(B_k) \xrightarrow[k\to\infty]{} 0$ (i.e. $Volume(B_k) \ll Volume(\text{euclidean ball of radius } r_k)$).*

[53] Recall that the usual Sobolev inequality (which works for M Riemannian manifold either compact or non compact) is given by

$$\int_M C \left(\int_M f^{\frac{n}{n-1}}\,dV\right)^{\frac{n-1}{n}} dV \leq \int_M |\nabla f|\,dV,$$

for some constant C.

The following result is an application of formula (15.13) to the analysis of singularities of the Ricci flow.

Theorem 15.2 (No local collapsing theorem) *If M is closed and $T < \infty$ then $g(t)$ is not locally collapsing at time T.*

Idea of the proof. [54]. In the case of flat $\mathbb{R}^n$, put $\tau = t_0 - t$ for some $t_0 > 0$ fixed. Taking $f(x,t) = |x|^2/4\tau$, it is possible to check that (g_t, f_t, τ_t) satisfies the conditions (15.11) and (15.12) of the previous section.

From the definition of f, we obtain

$$\tau(|\nabla f|^2 + R) + f - n = \tau\frac{|x|^2}{4\tau^2} + \frac{|x|^2}{4\tau} - n = \frac{|x|^2}{2\tau} - n$$

So, using (15.7) and (15.9), this yields $\mathcal{W}(t) = 0\, \forall t \in [0, t_0)$, so $\mathcal{W}(g, f, \tau) = 0$.

Therefore, if we put $\tau = r_k^2$ and $e^{-f_k}(x) = e^{-|x|^2/4r_k^2}$, we have $\mathcal{W}(g, f_k, r_k^2) = 0$.

But euclidean space is non collapsing. In the collapsing case, the idea is to use a test function f_k so that

$$e^{-f_k}(x) \sim e^{-c_k}\, e^{-dist_{t_k}(x,p_k)^2/4r_k^2}, \tag{15.15}$$

where c_k is defined by the normalization condition

$$\int_M (4\pi r_k^2)^{-n/2} e^{-f_k}\, dV = 1 \tag{15.16}$$

The main difference between computing (15.16) in M and in $\mathbb{R}^n$ comes from the difference in volumes, which give us $e^{-c_k} \sim \dfrac{1}{r_k^{-n} Volume(B_k)}$. In particular, $c_k \to -\infty$ as $k \to \infty$.

Now that f_k is normalized correctly, the main difference between computing $\mathcal{W}(g(t_k), f_k, r_k^2)$ in M and the analogous computation for the Gaussian case comes from the f term in the integrand of $\mathcal{W}$. Since $f_k \sim c_k$, we get

$$\mathcal{W}(g(t_k), f_k, r_k) \xrightarrow[k\to\infty]{} -\infty$$

and so $\mu(g(t_k), r_k^2) \to -\infty$.

From (15.13), it is true that, for any t_0, $\mu(g(t), t_0 - t)$ is nondecreasing in t. Then

$$\mu(g(0), t_k + r_k^2) \leq \mu(g(t_k), r_k^2) \to -\infty$$

[54] See [56] for a complete proof.

So $\mu(g(0), t_k + r_k^2) \to -\infty$. But this is a contradiction, because T is finite and thus t_k, r_k^2 are bounded. Hence the infimum is also finite. In conclusion, $g(t)$ is not locally collapsing at T. □

Very related with Definition 15.3 is the following

Definition 15.4 *The metric g is said to be κ-noncollapsed on the scale ρ if every metric ball B_r of radius $r < \rho$, with curvature $|Rm(x)| \leq \frac{1}{r^2}$ for every $x \in B$, satisfies $Volume(B_r) \geq \kappa r^n$.*

Examples 15.4.1 *1. Consider the cigar solitons, that is, the following metric on $\mathbb{R}^2 \times \mathbb{R}$*

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2} + dz^2$$

This metric has positive sectional curvature which decays exponentially fast to 0 in the geodesic distance to some basic point, then $|Rm| \leq c/r^2$. As the volume growth satisfies $Volume_g(B_s) \sim s$, then g is κ-collapsed for some values of ρ and not for others.

2. $(\mathbb{R}^n, g_{can})$ is ω_n-noncollapsed[55]. Nevertheless, the flat product $\mathbb{R}^{n-1} \times S^1$ is ω_n-noncollapsed for small balls, but highly collapsed for a large ball.

An important corollary of Theorem 15.2 is

Corollary 15.3 *Let $g(t)$, $t \in [0, T)$ be a solution to the Ricci flow on a closed manifold M, $T < \infty$. Suppose that, for some sequence $\{(p_k, t_k)\}$, where $p_k \in M$ and $t_k \to T$, and some constant c, we have $Q_k := |Rm|(p_k, t_k) \to \infty$ and $|Rm|(x, t) \leq cQ_k$ for $t < t_k$. Then there exists a subsequence of $Q_k g(t_k, p_k)$ convergent to a complete ancient solution to the Ricci flow, which is κ-noncollapsed on all scales for some $\kappa > 0$.*

Remark 15.3.1 *This theorem is an important refinement of previous results of R. Hamilton. It uses Hamilton compactness Theorem 14.10. The difficult problem of checking the hypothesis on the injectivity radius is solved by the k-noncollapsing results, because, given the sectional curvature bounds, the lower bound on the injectivity radii is equivalent to a lower bound on the volumes of balls. This equivalence follows from [16], Theorem 4.7.*

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[55]Recall that $\omega_n = Volume(S^{n-1})$.

16. SURVEY ON PERELMAN'S WORK (II): κ-SOLUTIONS

16.1. The L-distance

Our first aim is to understand singularities. Before we gave a theorem where dilations near singularities converge to an ancient solution. By reversing the time, an ancient solution can be considered as an immortal solution, but with the time reversed. For this reason, in this lecture we shall work with the *backward Ricci flow* on a manifold M, that is,

$$\frac{\partial g}{\partial \tau} = 2Ric \tag{16.1}$$

Let $g(\tau)$ be a Riemannian metric evolving under the backward Ricci flow. Suppose that either M is closed, or the metrics are complete with uniformly bounded curvatures.

In space-time $M^n \times \mathbb{R}$, we introduce the $\mathcal{L}$-length of a curve $\gamma(\tau)$, $0 < \tau_1 \leq \tau \leq \tau_2$ by the following formula

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left(R_{g_\tau}(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2_{g_\tau}\right)\, d\tau \tag{16.2}$$

After this definition, it is possible to derive a formula for the first variation and then deduce the equation of the $\mathcal{L}$-*geodesics*:

$$\nabla_X X - \frac{1}{2}\text{grad}R + \frac{1}{2\tau}X + 2Ric(X, \cdot) = 0 \tag{16.3}$$

where $X(\tau) = \dot{\gamma}(\tau)$.

Fixing a point p, we denote by $L(q, \overline{\tau})$ the $\mathcal{L}$-length of the $\mathcal{L}$-shortest curve $\gamma(\tau)$, $0 \leq \tau \leq \overline{\tau}$, joining p and q. In other words,

$$L(q, \overline{\tau}) = \inf\{\mathcal{L}(\gamma)/\, \gamma : [0, \overline{\tau}] \to M \text{ with } \gamma(0) = p, \gamma(\overline{\tau}) = q\}$$

The following inequality holds

$$-\Delta L \leq -2\sqrt{\tau}R + \frac{n}{\sqrt{\tau}} - \frac{1}{\tau}K \tag{16.4}$$

being $K = K(\gamma, \overline{\tau}) = \displaystyle\int_0^{\overline{\tau}} \tau^{3/2} H(X)\, d\tau$ and $H(X)$ is the Hamilton expression for the Harnack inequality (trace version) with $t = -\tau$, that is,

$$H(X) = \frac{\partial R}{\partial t} + \frac{R}{t} + 2\,\langle \text{grad}R, X\rangle + 2Ric(X, X) \tag{16.5}$$

The above inequality (16.4) and the next ones (16.8), (16.9), are analogs to the following Laplacian Comparison Theorem in Riemannian geometry: *Let M^n be a complete Riemannian manifold with $Ric(M) \geq -(n-1)k^2$, $k \geq 0$. Denote by ρ_M the distance function of M and by ρ_N the distance function of N, which is a space of constant curvature $-k^2$. If $x \in M$ and ρ_M is differentiable at x, then for any $y \in N$ with $\rho_N(y) = \rho_M(x)$*

$$-\Delta_M \rho(x) \leq -\Delta_N \rho(y) = \frac{n-1}{\rho}(1 + k\rho)$$

We call

$$l(q,\tau) = \frac{1}{2\sqrt{\tau}} L(q,\tau) \tag{16.6}$$

the reduced distance and denote

$$\overline{L}(q,\tau) = 2\sqrt{\tau} L(q,\tau) \tag{16.7}$$

After computing the second variation formula for $\mathcal{L}$ and the equation for the $\mathcal{L}$-Jacobi fields, we arrive to the following inequalities

$$l_{\overline{\tau}} - \Delta l + |\nabla l|^2 - R + \frac{n}{2\overline{\tau}} \geq 0 \tag{16.8}$$

$$\overline{L}_{\overline{\tau}} + \Delta \overline{L} \leq 2n \tag{16.9}$$

From this follows that the minimum of $\overline{L}(\cdot,\overline{\tau}) - 2n\overline{\tau}$ is non increasing and, in particular,

$$\min_{\overline{\tau}>0} l(\cdot,\overline{\tau}) \leq n/2. \tag{16.10}$$

An analogy with this ideas can be found in the original proof of the Harnack inequality given by Li and Yau (cf. [58]). They proved that, under the assumption $Ric(M) \geq -k$, a positive solution of the heat equation $\left(\frac{\partial}{\partial t} + \Delta\right) u = 0$ satisfies the gradient estimate: $\frac{|\mathrm{grad} u|^2}{u^2} - \frac{u_t}{t} \leq \frac{n}{2t}$. Along the proof of this fact, they define the function $F(x,t) := t(|\mathrm{grad} f|^2 - f_t)$, where $f = \ln u$, and show that $\max F \leq n/2$.

Example 7 *(Kleiner-Lott) Consider $\mathbb{R}^n$ with the flat metric and fix some point p. For some q, $\gamma(\tau) = \left(\frac{\tau}{\overline{\tau}}\right)^{1/2} \vec{q}$ and then*

$$L(q,\overline{\tau}) = \frac{1}{2}\overline{\tau}^{-1/2}|q|^2, \quad l(q,\overline{\tau}) = \frac{|q|^2}{4\overline{\tau}}, \quad \overline{L}(q,\overline{\tau}) = |q|^2.$$

Now we state a result which compares l with the distance.

Lemma 16.1 *(cf. [87]). Suppose that the curvature operator is nonnegative for each τ. For a fixed point $x \in M$, we have*

$$-l(x,\tau) - 1 + C_1 \frac{d^2(x,q,\tau)}{\tau} \leq l(q,\tau) \leq l(x,\tau) + C_2 \frac{d^2(x,q,\tau)}{\tau} \quad \forall q \in M,$$

where $C_1, C_2 > 0$ are constants depending only on M and $d(x,q,\tau)$ denotes the distance between the points x, q with respect to the metric $g(\tau)$.

16.2. The reduced volume

Definition 16.1 *Given a solution of the backward Ricci flow, the reduced volume function is defined as*

$$\widetilde{V}(\tau) = \int_M \tau^{-n/2} \exp\left(-l(q,\tau)\right) dq, \tag{16.11}$$

being dq the volume form of $g(\tau)$.

Notice that the integrand is the heat kernel in the euclidean space.

In the next Proposition we summarize the main properties of this function.

Proposition 16.2 ([87]) *1). If Ric is bounded from below on $[0,\tau]$ for each τ, then $\widetilde{V}(\tau)$ is a nonincreasing function.*

2). If we assume that at least one of the following conditions holds

(a) Ric is bounded on $[0,\tau]$ for all τ.
(b) The curvature operator is nonnegative for each τ.

Then $\widetilde{V}(\tau) \leq (4\pi)^{n/2}$ for all τ.

3). If we assume that either

(a) K_{sec} is bounded on $[0,\tau]$ for all τ, or
(b) Rm is nonnegative.

Then we have the following equality

$$\widetilde{V}(\tau_2) - \widetilde{V}(\tau_1) = -\int_{\tau_1}^{\tau_2} \int_M \left(l_\tau - R + \frac{n}{2\tau}\right) e^{-l} \tau^{-n/2} \, dq \, d\tau \tag{16.12}$$

Remark 16.2.1 *Notice that $(4\pi)^{n/2} = \int_{\mathbb{R}^n} \tau^{-n/2} e^{-l(q,\tau)} \, dq$, which is the same integral of the reduced function, but in the euclidean space.*

16.3. No local collapsing theorem II

Definition 16.2 *We say that a solution to the Ricci flow is κ-collapsed at (x_0, t_0) on the scale $r > 0$ if*

i). $|Rm|(x,t) \leq 1/r^2$ for all $(x,t) \in B_r^{g_{t_0}}(x_0) \times t \in [t_0 - r^2, t_0]$.

ii). $Volume(B_{t_0}(x_0, r^2)) < \kappa r^n$, where $B_{t_0}(x_0, r^2)$ is the ball of center x_0 and radius r^4 in the metric $g(t_0)$.

Now we state a result which is an application of the reduced length and reduced volume functions.

Theorem 16.3 *For any $A > 0$ there exists a constant $\kappa = \kappa(A) > 0$ such that if $g(t)$ is a solution to the Ricci flow on $[0, r_0^2]$ satisfying*

i. $|Rm(x,t)| \leq \frac{1}{r_0^2}$ for every (x,t) such that $dist_0(x, x_0) < r_0$.

ii. $Volume(B_0(x_0, r_0)) \geq A^{-1} r_0^n$.

Then $g(t)$ cannot be κ-collapsed on the scales less than r_0 at a point (x, r_0^2) with $dist_{r_0^2}(x, x_0) < A^{-1} r_0$.

Idea of the proof. It is possible to scale the metric and suppose $r_0 = 1$; we may also assume $dist_1(x, x_0) = A$. If $g(t)$ is collapsed at x on the scale $r \leq 1$, then the reduced volume $\widetilde{V}(r^2)$ must be very small. On the other hand, $\widetilde{V}(1)$ cannot be small unless $\min l(x, \frac{1}{2})$ over x satisfying $dist_1(x, x_0) \leq \frac{1}{10}$ is large. thus all we need is to estimate l, or equivalently, $\overline{L}$, in that ball. Indeed, $\overline{L}(q, \overline{\tau}) \geq -n/3$ for $\overline{\tau} \in [0, \frac{1}{2}]$ (Kleiner-Lott) $\square$

16.4. κ-solutions

Until now, we have stated some results without dimensional or curvature restrictions. Here we begin the work on details of Hamilton program for geometrization of three manifolds. Hence our first aim is to study ancient solutions, which (as we showed in the last lecture) arise as limits of dilations about singularities.

Definition 16.3 *A smooth ancient solution to the Ricci flow $\partial_t g = -2Ric$, $-\infty \leq t \leq 0$ is called κ-solution if it satisfies the following properties*

1. For each t, $g(t)$ is a complete non-flat metric of bounded curvature and nonnegative curvature operator.

2. $g(t)$ is κ-noncollapsed on all scales, for some $\kappa > 0$ fixed.

Let $\tilde{g}(t)$ be a κ-solution for some $\kappa > 0$ on a manifold M. Pick an arbitrary point $(p, t_0) \in M \times (-\infty, 0]$ and set $\tau := t_0 - t$ for $t \leq t_0$. Then

$g(\tau) = \tilde{g}(t_0 - \tau)$, $\tau \in [0, \infty]$ is a solution of the backward Ricci flow (16.1). Recall that, since $\widetilde{V}(\tau)$ is nonincreasing, there exists $\widetilde{V}_\infty := \lim_{\tau \to \infty} \widetilde{V}(\tau)$.

Now take $x = x(\tau)$ a minimum point for $l(\cdot, \tau)$, by (16.10), $l(x, \tau) \leq n/2$. We shall use x as the center for pointed convergence. The reference point p is not suitable for this purpose, because the estimates around p are not good enough after rescaling.

Given $\overline{\tau} > 0$, we set $g_{\overline{\tau}}(\tau) = \frac{1}{\overline{\tau}} g(\overline{\tau}\tau)$.

Theorem 16.4 *Consider a sequence $\tau_k \to \infty$. Then the pointed flow $(M, g_{\tau_k}(\tau), x(\tau_k))$ with $\tau \in (0, \infty)$ has a subsequence convergent to a non-flat gradient shrinking soliton $(M_\infty, g_\infty, x_\infty)$, that is, a solution satisfying*

$$Ric_{g_\infty} + \nabla^2_{g_\infty} f - \frac{1}{2\tau} g_\infty = 0,$$

where $f = l_\infty$ (limit of the reduced distance l).

These limits are called *asymptotic solitons* of g.

For dimension 3, if the asymptotic soliton has non strictly positive sectional curvature, a Hamilton's maximum principle applies to show that it admits a local metric spplitting, and, in this case, the soliton is either $S^2 \times \mathbb{R}$, with its canonical metric, or its $\mathbb{Z}_2$ quotient (in this quotient the $\mathbb{Z}_2$-action is $(x, t) \mapsto (-x, -t)$) which topologicaly is $S^2 \times \mathbb{R}^+$, with the boundary glued to $\mathbb{R}P^2$ by the canonical projection $S^2 \longrightarrow \mathbb{R}P^2$. If the asymptotic soliton is compact and has strictly positive sectional curvature, then it has to be a quotient metric of the round sphere (by [38]). On the other hand, Perelman proves that *there is no complete oriented 3-dimensional noncompact κ-nonccollapsed gradient shrinking soliton with bounded positive sectional curvature*, which finishes the classification of asymptotic solitons.

Let M be a complete Riemannian manifold of nonnegative Ricci curvature. For a fixed point $p \in M$, the function $Volume(B(p, r))\ r^{-n}$ is non increasing in $r > 0$. Thus we can give the next definition.

Definition 16.4 *We call asymptotic volume ratio to the limit*

$$\mathcal{V} := \lim_{r \to \infty} Volume(B(p, r))\ r^{-n}$$

Proposition 16.5 *For a κ-solution, $\mathcal{V} = 0$ for each t.*

For the proof, Perelman defines the asymptotic scalar curvature ratio

$$\mathcal{R} := \limsup_{d(x) \to \infty} R(x, t_0) d^2(x),$$

where $d(x)$ denotes the distance, at time t_0, from x to some fixed point x_0. If $\mathcal{R} = \infty$, the proof follows by using Topogonov's splitting theorem.

A key theorem that Perelman proves using the invariants used in this lecture is the following compacity theorem:

Theorem 16.6 *In dimension 3, the set of complete κ-solutions is compact modulo scaling; that is, from any sequence of such solutions and points $(x_k, 0)$ with $R(x_k, 0) = 1$, we can find a smoothly converging subsequence, whose limit satisfies the same conditions.*

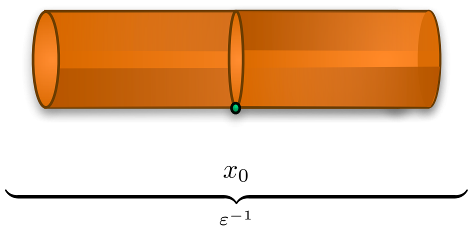
This theorem and his proof an important corollary about the structure of a κ-solution. Before stating it, we give some definitions.

Definition 16.5 *Let us denote by $B(x, t, r)$ the open ball of radius r and center x with respect to the metric $g(t)$. We shall call a parabolic neighborhood $P(x, t, r, \Delta t)$ of (x, t) to the set of all points (x', t') such that $x' \in B(x, t, r)$ and $t' \in [t + \Delta t, t]$ or $t' \in [t, t + \Delta t]$*

This kind of neighborhoods have their motivation in Harnack's inequalities, which gives control on the curvature at past or future times from the value of curvature at present. Perelman will use the preceding definition also to give future and past bounds on the curvature.

Definition 16.6 *A ball $B(x, t, \varepsilon^{-1}r)$ is called an ϵ-neck if, after scaling the metric with factor r^{-2}, it is ε-close to the standard neck $S^2 \times I$ with the product metric, where S^2 has constant scalar curvature one and I has length $2\varepsilon^{-1}$. Here ε-close refers to the C^N-topology with $N > \varepsilon^{-1}$. x is called the center of the ϵ-neck.*

Definition 16.7 *A parabolic neighborhood $P(x, t, \varepsilon^{-1}r, r^2)$ is called a strong ϵ-neck if, after scaling the metric with factor r^{-2}, it is ε-close to the evolving standard neck, which at each time $t' \in [-1, 0]$ has length $2\varepsilon^{-1}$ and scalar curvature $1 - t'$.*



ϵ-neck centered at x_0

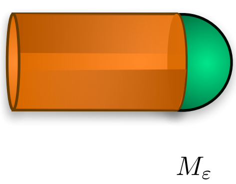


illustration of Corollary 16.7

From Theorem 16.6 and its proof, it follows that

Corollary 16.7 *For any $\varepsilon > 0$, there exists $C = C(\varepsilon, \kappa) > 0$ such that, if $g(t)$ is a κ-solution of a non compact 3-manifold and M_ε denotes the set of points which are not centers of ε-necks, then M_ε is compact, diam$(M_\varepsilon) \leq C\ Q^{-\frac{1}{2}}$, and $C^{-1}\ Q \leq R(x, 0) \leq C\ Q$ whenever $x \in M_\varepsilon$, where $Q = R(x_0, 0)$ for some $x_0 \in \partial M_\varepsilon$*

From Theorem 16.4, the classification of asymptotic solitons, and Theorem 16.6 and its Corollary, one obtains a classification of the κ-solutions in dimension 3:

• If such a solution has a compact asymptotic soliton, then it is itself a metric quotient of the 3-sphere of constant sectional curvature.

• If the asymptotic soliton is $S^2 \times \mathbb{R}/\mathbb{Z}_2$, then the solution has a $\mathbb{Z}_2$ cover, whose asymptotic soliton is $S^2 \times \mathbb{R}$ with its standard metric (up to a scale).

• Finally, if the asymptotic soliton is the cylinder $S^2 \times \mathbb{R}$, then the solution can be either non-compact or compact.

A consequence of the above classification and the same quoted theorems and their proofs is a estimate on the gradient and the temporal derivative of R, and a classification of the local look of the κ-solutions. We shall state this result after giving a lemma, consequence of 16.4, which allows to apply Theorem 16.6. Before, another definition.

Definition 16.8 *A metric on B^3 or $\mathbb{R}P^3$ such that each point outside some compact subset is contained in some ε-neck is called an ε-cap if the scalar curvature stays bounded*

Lemma 16.8 *There exists $\kappa_0 > 0$ such that every ancient κ-solution is either a κ_0-solution or a metric quotient of the round sphere.*

Theorem 16.9 *There is an universal constant η such that, at each point of every ancient κ-solution we have the estimates*

$$|\nabla R| < \eta R^{\frac{3}{2}}, \quad |R_t| < \eta R^2 \tag{16.13}$$

For every sufficiently small $\varepsilon > 0$ one can find constants C_1, C_2 such that for every point (x, t) in every ancient κ-solution there is a radius $r < C_1 R(x, t)^{-1/2}$ and a neighborhood B, $B(x, t, r) \subset B \subset B(x, t, 2r)$ which falls into one of the collowing categories:

(a) B is the slice of a strong ε-neck at its maximal time,

(b) B is a ε-cap,

(c) B is a closed manifold, diffeomorphic to S^3 or $\mathbb{R}P^3$,

(d) B is a closed manifold of constant positive sectional curvature.

Moreover, the scalar curvature in B at time t is between $C_2^{-1}R(x,t)$ and $C_2R(x,t)$, its volume in cases (a), (b), (c) is greater than $C_2^{-1}R(x,t)^{-3/2}$, and, in case (c), the sectional curvature in B at time t is greater than $C_2^{-1}R(x,t)$.

From the above theorems, Perelman obtains the next principal result, valid for almost nonnegatively curved manifolds. First we give this concept on the curvature:

Definition 16.9 *Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a decreasing function satisfying that $\lim_{R\to\infty} \phi(R) = 0$. A solution to the Ricci flow is said to have ϕ-almost nonnegative curvature if it satisfies $Rm(x,t) \geq -\phi(R(x,t))\ R(x,t)$.*

Theorem 16.10 (of the canonical neighborhood) *Given $\varepsilon > 0$, $\kappa > 0$ and a function ϕ as above, there is $r_0 > 0$ with the property that, if $g(t)$, $0 \leq t \leq T$, is a solution to Ricci flow on a closed 3-manifold M, which has ϕ-almost nonnegative curvature and is κ- noncollapsed on scales $< r_0$, then, for any point (x_0,t_0) with $t_0 \geq 1$ and $Q = R(x_0,t_0) \geq r_0^{-2}$, the solution in $\{(x,t);\ \mathrm{dist}^2_{\mathrm{t_0}}(\mathrm{x},\mathrm{x_0}) < (\varepsilon\mathrm{Q})^{-1},\ \mathrm{t_0} - (\varepsilon\ \mathrm{Q})^{-1} \leq \mathrm{t} \leq \mathrm{t_0}\}$ is, after scaling by the factor Q, ϵ-close to the corresponding subset of some κ-solution.*

17. SURVEY ON PERELMAN'S WORK (III): THE WAY TO THE END

Before to continue we simplify technically the problem by *considering only Ricci flow with normalized initial conditions*:

Definition 17.1 *We shall call a Riemannian manifold (M,g) normalized if*

1. M is a closed oriented 3-manifold and the sectional curvature of g does not excede 1 in absolute value, and

2. the volume of every metric ball of radius 1 is at least half the volume of the Euclidean limit.

For smooth Ricci flow with normalized initial data, it is a consequence of a pinching estimates by Hamilton and Ivey (cf. [43] and [52]) that *the* *solution metric is ϕ-almost nonnegatively curved* on finite intervals of time, with ϕ satisfying the conditions of Definition 16.9.

Let us consider a smooth solution $g(t)$ of the Ricci flow on $M \times [0, T[$, where M is a closed oriented 3-manifold with normalized data, $T < \infty$, and the solution becomes singular at $t = T$ (i.e. the curvature goes to ∞ as t goes to T).

Then, by Theorems 16.9 and 16.10, we can find $r = r(\epsilon)$ such that each point (x, t) with $R(x, t) \geq r^{-2}$ satisfies the estimates (16.13) and has a neighborhood which is either an ε-neck or an ε-cap or a closed positively curved manifold.

In the later case the solution becomes extint at time T, so we do not need to consider it any more. If it is not the case, let Ω be the set of all points in M where the curvature stays bounded as $t \to T$. The estimates (16.13) imply that Ω is open and that $R(x, t) \to \infty$ as $t \to \infty$ for each $x \in M - \Omega$. If Ω is empty, then the soliton becomes extinct at time T and it is entirely covered by an ε-neck and caps strictly before that time, and M is diffeomorphic toeither S^3 or $\mathbb{R}P^3$ or $S^2 \times S^1$ or $\mathbb{R}P^3 \sharp \mathbb{R}P^3$.

If Ω is not empty, we left the metric to flow until time T, and consider on Ω the metric $\overline{g} = \lim_{t \to T} g(t)$. For $\rho < r$, let Ω_ρ denote the set of points $x \in \Omega$ where the scalar curvature $\overline{R}$ of $\overline{g}$ satisfies $\overline{R}(x) \leq \rho^{-2}$. Ω_ρ is compact. In order to describe $\Omega - \Omega_\rho$, it is convenient to give the following definitions:

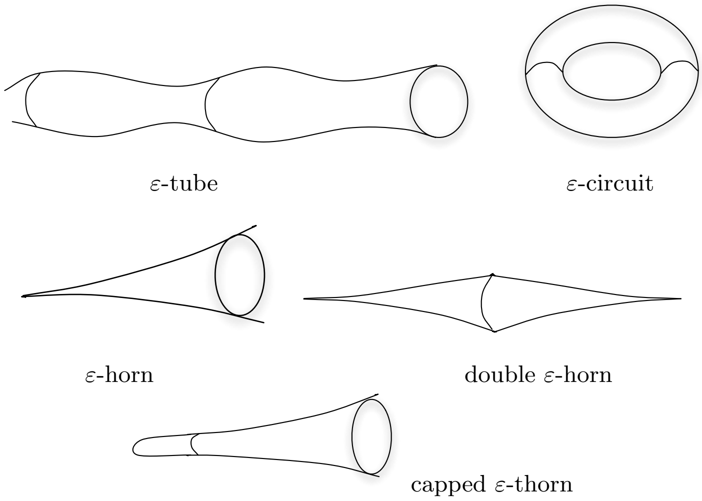
Definition 17.2 *An ε-tube in Ω is a submanifold diffeomorphic to $S^2 \times [0, 1]$ such that each point is the center of an ϵ-neck in Ω (then its scalar curvature stays bounded on both ends).*

An ε-circuit in Ω is a component of Ω which is a closed manifold and each one of its points is the center of an ϵ-neck. It is diffeomorphic to $S^2 \times S^1$.

An ε-horn H is a closed subset of Ω diffeomorphic to $S^2 \times [0, 1[$ with boundary contained in Ω_ρ such that every point in H is the center of an ϵ-neck in Ω. The scalar curvature is bounded at the end in Ω_ρ and goes to infinity at the other.

A double ε-horn is a closed subset of Ω diffeomorphic to $S^2 \times]0, 1[$ such that every point is the center of an ε-neck in Ω. The scalar curvature goes to infinity at each end of it.

A capped ε-horn is a closed subset of Ω diffeomorphic to the open ball D^3 such that each point is either the center of an ε-neck or is contained in an ε-cap. The scalar curvature goes to infinity near the end of a capped ε-horn.



It follows from the classification given when $t < T$ that,

Corollary 17.1 *At time T every $x \in \Omega - \Omega_\rho$, is contained in one of the following:*

(1) a component of Ω containing x is diffeomorphic to a quotient of a sphere,
(2) an ε-circuit diffeomorphic to $S^2 \times S^1$,
(3) an ε-tube with boundary components in Ω_ρ,
(4) an ε-cap with boundary in Ω_ρ,
(5) an ε-horn with boundary in Ω_ρ,
(6) a capped ε-horn,
(7) a double ε-horn.

By looking at our solutions for times t just before T, the topology of M can be reconstructed as follows. Take the components Ω_j of Ω which contains points of Ω_ρ, truncate their ϵ-horns and glue to the boundary components of truncated Ω_j a collection of tubes $S^2 \times I$ and caps B^3 or $\mathbb{R}P^3 - B^3$. That M is diffeomorphic to a connected sum of $\overline{\Omega}_j$ with a finite number of $S^2 \times S^1$ and a finite number of $\mathbb{R}P^3$.

But we still need to know about the topology of Ω_j, and this is the point where Perelman introduces surgery. The idea is the following. When we arrive to singular time T, we remove all the subsets of Ω of types (1), (2), (6) and (7) in Corollary 17.1. On the other pieces, we cut along well chosen S^2 satisfying appropriate bounds of curvature and radius and paste, with appropriate metric, the "standard solution", defined by Perelman, and which has the topology of $\mathbb{R}^3$ with a metric close to the standard metric on the upper hemisphere of S^3 union the product metric on $S^2 \times [0, \infty[$. After the gluing is done, one checks that: a) the number of surgeries at time t is finite, and b) the new manifold obtained still satisfies the essential properties of the original one (ϕ-almost nonnegative curvature and satisfies the theorem of the canonical neighborhood). Part a) is a consequence on the bounds on the volume lost in each surgery process. Part b) has a very delicated proof. Then we continue the flow until next singularity appears, and do surgery as before. The numbers of surgeries in each finite interval of time is bounded (then finite), again using the argument of the bound of the volume lost in each surgery process. We continue this flow with surgery until the time when the next theorems allows us to say something about the topology of the resulting Ω_j.

Theorem 17.2 *For any $\omega > 0$ there exist $K = K(\omega) < \infty$, $\rho = \rho(\omega) > 0$ such that for sufficiently large t the manifold M admits a thick-thin decomposition $M = M_{thick} \cup M_{thin}$ with the following properties*

(a) For every $x \in M_{thick}$ we have an estimate $|\overline{Rm}| \leq K$ in the ball $B(x, \rho(\omega), t)$ and the volume of this ball is at least $\frac{1}{10}\omega(\rho(\omega)\sqrt{t})^n$.

(b) For every $y \in M_{thin}$ there exists $r = r(y)$, $0 < r < \rho(\omega)\sqrt{t}$, such that for all points in the ball $B(y, r)$ we have $Rm \geq -r^{-2}$, and the volume of this ball is $< \omega r^n$.

From the arguments of Hamilton in [43] it follows that either M_{thick} is empty for large t, or, for an appropriate sequence of $t \to \infty$ and $\omega \to 0$, it converges to a (possibly disconnected) complete hyperbolic manifold of finite volume which boundary (if there is any) is an incompressible torus. On the other hand, for M_{thin}, a result of Shioya and Yamaguchi ([81]) says

Theorem 17.3 *There are positive numbers ε and δ such that if an oriented 3-dimensional manifold has a complete metric with sectional curvature $K \geq 1$ and $Volume(M) < \varepsilon$, the one of the following conditions holds:*

i) M is homeomorphic to a graph manifold [56]

ii) $diam(M) \leq \delta$ and M has finite fundamental group

The proof of the geometric conjecture is then finished for case i) of the Shioya-Yamaguchi theorem, the case ii) does not conclude, but it is covered by the third preprint of Perelman [69].

[56] Result connected with the Cheeger-Gromov theorem: "If M^3 has a complete metric g, with $|K| \leq 1$ and volume sufficiently small, then it is a graph manifold"

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