FINITELY DETERMINED SINGULARITIES OF RULED SURFACES IN $\mathbb{R}^3$

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ABSTRACT. We study local singularities of ruled surfaces in $\mathbb{R}^3$. We show that any map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ with a simple singularity is $\mathcal{A}$-equivalent to a ruled surface. Moreover, we give a topological classification of $\mathcal{A}$-finitely determined singularities of ruled surfaces and show that there are just eleven topological classes.

1. INTRODUCTION

Ruled surfaces are surfaces generated by straight lines or rulings and have been studied for centuries by geometers. In fact, many introductory textbooks about differential geometry of curves and surfaces include a section dedicated to this subject. We find simple examples of ruled surfaces like the cylinder, the hyperbolic paraboloid, the hyperboloid of one sheet, the helicoid or the Möbius strip. From the singularity point of view, we also see that the simplest singular surface, the Whitney umbrella or cross cap, is also a ruled surface (see figure 1).

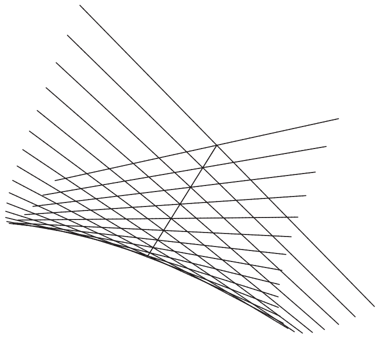

FIGURE 1. The Whitney umbrella or cross cap

In [6], Izumiya and Takeuchi consider the question of how different are the singularities of ruled surfaces comparing to developable surfaces or general surfaces in $\mathbb{R}^3$. They show that generically, ruled surfaces present the same local singularities as general surfaces, namely, Whitney umbrella type singularities.

In this paper, we try to answer this question for $\mathcal{A}$-finitely determined singularities of ruled surfaces in $\mathbb{R}^3$. We consider the C^∞ classification: two map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are $\mathcal{A}$-equivalent if there are C^∞-diffeomorphism germs $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $g = \psi \circ f \circ \phi$. We also consider the topological classification,

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which means that ϕ, ψ are homeomorphisms instead of diffeomorphisms. The main results of this paper are given in the two following theorems.

Theorem 1.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map germ with a simple singularity. Then f is $\mathcal{A}$-equivalent to a germ of ruled surface.*

Theorem 1.2. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface. Then the topological type of f is determined by one of the eleven links of figure 2.*

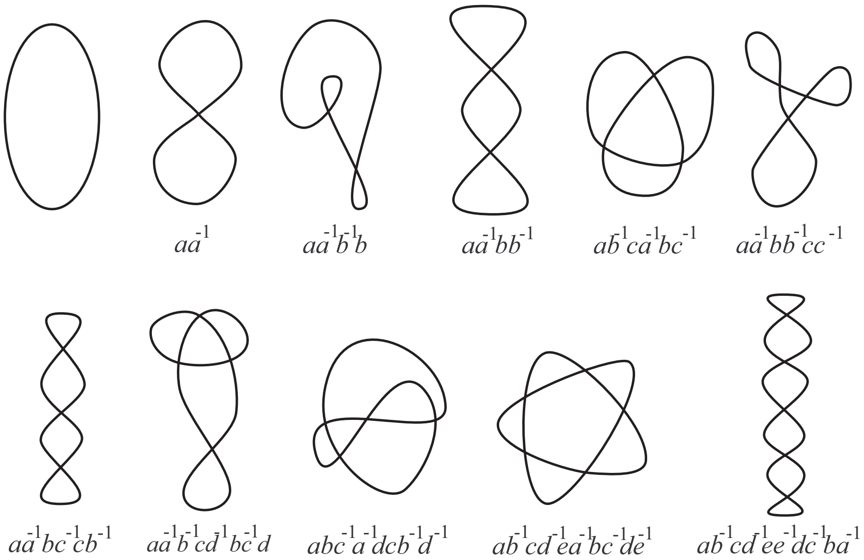

Figure 2. Links of finitely determined ruled surfaces and their Gauss words

In theorem 1.1 we use the classification of simple singularities obtained by Mond [10]: apart from the immersion or the Whitney umbrella, the only simple singularities are the series $S_k^\pm$, $B_k^\pm$, $C_k^\pm$, H_k and the exceptional germ F_4. We consider germs of ruled surfaces of the form $f(x, y) = (x, y^2 + 2xy, h(y) + xg(y))$. We show that it is possible to choose the functions h, g conveniently so that we find all the singularities $S_k^\pm$, $B_k^\pm$, $C_k^\pm$ or F_4. Note that the H_k series has a normal form which is itself a ruled surface $f(x, y) = (x, xy + y^{3k-1}, y^3)$.

For the topological classification, we consider the link of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. This is defined by intersecting f with a small enough sphere S^2_ϵ in $(\mathbb{R}^3, 0)$ so that we obtain a curve immersed in S^2_ϵ with only transverse double points. By using a theorem of Fukuda [3], we know that if f is finitely determined, then it is topologically equivalent to the cone of its link. In order to codify the topology of the link, we also consider the Gauss word, which is a finite sequence $a_1^{\epsilon_1} \dots a_k^{\epsilon_k}$ so that each letter a_i corresponds to a double point and appears twice, one with exponent $\epsilon_i = +1$ and another one with exponent $\epsilon_i = -1$. This Gauss word contains all the topological information of the link and hence, of the map germ f.

Since we are using just rulings to construct our singular surfaces, one should expect that only very simple topological configurations can appear. However, we find in figure 2 several links (specially those having 4 or 5 crossings) which are difficult to believe they correspond to singularities of ruled surfaces.

The use of links and Gauss words has been used previously by the second author and Marar in [9] to classify topologically singularities of finitely determined map germs

$f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$. We should also mention the work of Ishikawa [5], where he obtains the topological classification of singularities of developable surfaces.

2. Singularities of ruled surfaces

A *ruled surface* in $\mathbb{R}^3$ is locally the image of a map $f: I\times\mathbb{R}\to\mathbb{R}^3$ given by

$$f(t,u)=\alpha(t)+u\gamma(t), \tag{1}$$

where $I\subset\mathbb{R}$ is an interval, $\alpha: I\to\mathbb{R}^3$ is a space curve and $\gamma: I\to\mathbb{R}^3$ is a vector field along the curve such that $\gamma(t)\neq 0$, for any $t\in I$. The curve α is called the *base curve* and the vector field γ is called the *director curve.* Moreover, the straight lines $u\mapsto\alpha(t)+u\gamma(t)$ are called *rulings* of the ruled surface. Finally, if $(t_0,u_0)\in I\times\mathbb{R}$ and $f(t_0,u_0)=p$, then the image of the map germ $f:(I\times\mathbb{R},(t_0,u_0))\to(\mathbb{R}^3,p)$ will be called a *germ of ruled surface.*

Given $\alpha: I\to\mathbb{R}^3$ and $\gamma: I\to\mathbb{R}^3$, we can see both curves together as a map $(\alpha,\gamma): I\to\mathbb{R}^3\times\mathbb{R}^3$. We will denote by $f_{(\alpha,\gamma)}$ the ruled surface associated to these curves as in (1). The starting point is the following result of Izumiya and Takeuchi [6], which states that the generic singularities of the ruled surfaces coincide with the generic singularities of general surfaces in $\mathbb{R}^3$.

We recall that for a generic map $f: U\subset\mathbb{R}^2\to\mathbb{R}^3$, then the germ of f at any point of U is either an immersion or a singularity of *Whitney umbrella type*: it is $\mathcal{A}$-equivalent to the map germ $(x,y)\mapsto(x,y^2,xy)$.

Theorem 2.1. *There is an open and dense subset $\mathcal{O}\subset C^\infty_{pr}(I,\mathbb{R}^3\times\mathbb{R}^3)$ with the Whitney C^∞-topology such that if $(\alpha,\gamma)\in\mathcal{O}$, then the germ of the ruled surface $f_{(\alpha,\gamma)}$ at any point $(t_0,u_0)\in I\times\mathbb{R}$ is either an immersion or a Whitney umbrella.*

Proof. See [6]. □

Remark 2.2. In fact, in [6] it is considered $C^\infty_{pr}(I,\mathbb{R}^3\times S^2)$ instead of $C^\infty_{pr}(I,\mathbb{R}^3\times\mathbb{R}^3)$. Since we are assuming that $\gamma(t)\neq 0$, for any $t\in I$, this does not make any difference.

By taking translation in either I, $\mathbb{R}$ or $\mathbb{R}^3$, we can assume without loss of generality that $t_0=0$, $u_0=0$ and $p=0$. We will also assume that all the map germs are C^∞.

Lemma 2.3. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ which parametrizes a ruled surface, with $f(t,u)=\alpha(t)+u\gamma(t)$. Then:*

1. *f is singular if and only if $\alpha'(0)$ and $\gamma(0)$ are collinear.*
2. *If f is singular, it is a Whitney umbrella if and only if $\alpha''(0)$, $\gamma(0)$ and $\gamma'(0)$ are not collinear.*

Proof. The first part follows from the definition, since

$$\frac{\partial f}{\partial t}(0,0)=\alpha'(0),\quad \frac{\partial f}{\partial u}(0,0)=\gamma(0).$$

To see the second part, let us assume for instance that $\gamma_1(0)\neq 0$ and let us consider the Taylor expansion $j^2f(t,u)=$

$$\Big(a_{11}t+\frac{a_{12}}{2}t^2+u(b_{10}+b_{11}t),a_{21}t+\frac{a_{22}}{2}t^2+u(b_{20}+b_{21}t),a_{31}t+\frac{a_{32}}{2}t^2+u(b_{30}+b_{31}t)\Big)$$

where $a_{ij}=\alpha_i^{(j)}(0)$ and $b_{ij}=\gamma_i^{(j)}(0)$. We assume that f is singular, so that $b_{10}a_{21}-b_{20}a_{11}=b_{10}a_{31}-b_{30}a_{11}=0$. Moreover, consider the parameter change

$$x=a_{11}t+\frac{a_{12}}{2}t^2+u(b_{10}+b_{11}t),$$

which gives $j^2 f(x,t) = (x, p(x,t), q(x,t))$, where

$$p(x,t) = \frac{b_{20}}{b_{10}}x + \frac{b_{10}b_{21} - b_{20}b_{11}}{b_{10}^2}xt + \frac{(a_{22}b_{10}^2 - b_{20}a_{12}b_{10} + 2b_{11}b_{20}a_{11} - 2b_{21}a_{11}b_{10})}{2b_{10}^2}t^2,$$
$$q(x,t) = \frac{b_{30}}{b_{10}}x + \frac{b_{10}b_{31} - b_{30}b_{11}}{b_{10}^2}xt + \frac{(a_{32}b_{10}^2 - b_{30}a_{12}b_{10} + 2b_{11}b_{30}a_{11} - 2b_{31}a_{11}b_{10})}{2b_{10}^2}t^2.$$

We recall that with this notation, f has a Whitney umbrella singularity if and only if

$$\begin{vmatrix} p_{xt} & p_{tt} \\ q_{xt} & q_{tt} \end{vmatrix} \neq 0,$$

where subscripts denote partial derivatives. This condition gives

$$\begin{vmatrix} b_{10}b_{21} - b_{20}b_{11} & a_{22}b_{10}^2 - b_{20}a_{12}b_{10} + 2b_{11}b_{20}a_{11} - 2b_{21}a_{11}b_{10} \\ b_{10}b_{31} - b_{30}b_{11} & a_{32}b_{10}^2 - b_{30}a_{12}b_{10} + 2b_{11}b_{30}a_{11} - 2b_{31}a_{11}b_{10} \end{vmatrix} \neq 0$$

which is also equivalent to

$$b_{10}a_{22}b_{31} - b_{10}b_{21}a_{32} + b_{20}b_{11}a_{32} - b_{20}a_{12}b_{31} + b_{21}b_{30}a_{12} - a_{22}b_{11}b_{30} \neq 0.$$

To finish the proof, just note that the left hand side corresponds to the determinant of $\alpha''(0)$, $\gamma(0)$ and $\gamma'(0)$. □

Because of condition $\gamma(t) \neq 0$, it follows that any singularity of ruled surface $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ will have corank 1. Then we have four types of singularity according to the 2-jet of f.

We denote by $J^2(2,3)$ the space of polynomial map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ of degree ≤ 2. The subspace of corank 1 map germs is denoted by $\Sigma^1 J^2(2,3)$. On this space we consider the action of $\mathcal{A}^2$, the group of 2–jets of elements of $\mathcal{A}$.

Proposition 2.4 (Classification of 2-jets). *There exist four orbits in $\Sigma^1 J^2(2,3)$ under the action of $\mathcal{A}^2$, which are*

$$(x, y^2, xy),\ (x, xy, 0),\ (x, y^2, 0),\ (x, 0, 0).$$

Proof. See [10]. □

In the next lemma, we see that it is possible to make a change of coordinates such that any ruled surface germ can be written in the form $f(x,y) = (x, f_2(x,y), f_3(x,y))$, where f_2, f_3 are linear in x.

Lemma 2.5. *Given a germ of ruled surface, we can choose affine coordinates in $\mathbb{R}^3$ such that it is parametrized by a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ of the form*

$$(2) \qquad f(x,y) = \beta(y) + x\delta(y), \quad \beta(y) = (0, \beta_2(y), \beta_3(y)), \quad \delta(y) = (1, \delta_2(y), \delta_3(y)).$$

Proof. Assume that the ruled surface is the image of a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, with $f(t,u) = \alpha(t) + u\gamma(t)$ and so that $\gamma_1(0) \neq 0$.

We reparametrize it by taking the change of coordinates $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given by

$$\phi(x,y) = (y, \frac{x - \alpha_1(y)}{\gamma_1(y)}).$$

Then,

$$f \circ \phi(x,y) = (x, x\delta_2(y) + \beta_2(y), x\delta_3(y) + \beta_3(y)),$$

where

$$\delta_2(y)=\frac{\gamma_2(y)}{\gamma_1(y)},\quad \beta_2(y)=\alpha_2(y)-\frac{\alpha_1(y)\gamma_2(y)}{\gamma_1(y)},$$
$$\delta_3(y)=\frac{\gamma_3(y)}{\gamma_1(y)},\quad \beta_3(y)=\alpha_3(y)-\frac{\alpha_1(y)\gamma_3(y)}{\gamma_1(y)}.$$

□

Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ of the form (2). Then lemma 2.3 has a simpler statement. It is easy to see that f is singular if and only if $\beta'(0)=0$ and it is a Whitney umbrella if and only if $\beta'(0)=0$ and $\beta''(0),\delta'(0)$ are not collinear.

Proposition 2.6. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ of the form (2). We assume that f has a singularity more degenerated than a Whitney umbrella. Then the 2-jet of f belongs to the orbit of*

$$\begin{cases}(x,y^2,0), & \text{if } \beta''(0)\neq 0;\\ (x,xy,0), & \text{if } \beta''(0)=0,\ \delta'(0)\neq 0;\\ (x,0,0), & \text{if } \beta''(0)=\delta'(0)=0.\end{cases}$$

Proof. Since f has a singularity more degenerated than a Whitney umbrella, we must have the following Taylor expansions

$$\beta(t)=(0,a_{22}t^2+\dots,a_{32}t^2+\dots),\quad \delta(t)=(1,b_{21}t+b_{22}t^2+\dots,b_{31}t+b_{32}t^2+\dots),$$

with $a_{22}b_{31}-a_{32}b_{21}=0$, so that the 2-jet of f is given by

$$j^2f(0)=\left(x,a_{22}t^2+b_{21}tx,a_{32}t^2+b_{31}xt\right).$$

The statement follows easily from the classification of $j^2f(0)$ in terms of the four coefficients $a_{22},b_{31},a_{32},b_{21}$. □

We finish this section by showing that the orbit of $(x,0,0)$ does not contain any finitely determined germ of ruled surface. We see this by looking at the invariant $C(f)$ introduced by Mond [10].

Definition 2.7. Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ. We define

$$C(f)=\dim_{\mathbb{R}}\frac{\mathcal{E}_2}{J(f)},$$

where $\mathcal{E}_2$ is the local algebra of function germs from $(\mathbb{R}^2,0)$ to $\mathbb{R}$ and $J(f)$ is the ideal generated by the maximal minors of the jacobian matrix of f.

The main properties of $C(f)$ are:

(1) $C(f)$ is $\mathcal{A}$-invariant;
(2) if f is finitely determined, then $C(f)<\infty$ and it is equal to the number of Whitney umbrellas that appear near the origin in a stable perturbation of a complexification of f.

(See [10, 11] for a proof.)

Proposition 2.8. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be germ of ruled surface whose 2-jet is in the orbit of $(x,0,0)$. Then $C(f)=\infty$ and hence, f is not finitely determined.*

Proof. By lemma 2.5, we can assume that f has the form

$$f(x,y)=\beta(y)+x\delta(y),\quad \beta(y)=(0,\beta_2(y),\beta_3(y)),\quad \delta(y)=(1,\delta_2(y),\delta_3(y)).$$

Then, the jacobian ideal $J(f)$ has just two generators in $\mathcal{E}_2$, namely, $\beta_2'(t)+x\delta_2'(t)$ and $\beta_3'(t)+x\delta_3'(t)$.

By lemma 2.6, if the 2-jet is in the orbit of $(x,0,0)$, we must have $\delta_2'(0)=\delta_3'(0)=0$. In particular, the two generators of $J(f)$ do not present any pure term in x in their Taylor expansions. It is well known that this implies the ideal cannot have finite codimension in $\mathcal{E}_2$. □

3. Simple singularities

We give in this section the proof of theorem 1.1 that any map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ with a simple singularity is $\mathcal{A}$-equivalent to a germ of ruled surface. We recall that a singularity f is said to be *simple* if for any deformation f_t of f, we only find a finite number of $\mathcal{A}$-classes in f_t near the origin. The classification of simple singularities from $(\mathbb{R}^2,0)$ to $(\mathbb{R}^3,0)$ was obtained by Mond.

Theorem 3.1. [10] *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ with a simple singularity. Then f is $\mathcal{A}$-equivalent to one of the germs of table 3.1:*

Normal form	*Name*	
(x,y^2,xy)	*Whitney umbrella*	
$(x,y^2,y^3+\pm x^{k+1}y)$	$S_k^\pm$	$k\geq 1$
$(x,y^2,x^2y\pm y^{2k+1})$	$B_k^\pm$	$k\geq 2$
$(x,y^2,xy^3\pm x^ky)$	$C_k^\pm$	$k\geq 3$
$(x,xy+y^{3k-1},y^3)$	H_k	$k\geq 2$
(x,y^2,x^3y+y^5)	F_4	

Table 1. Simple singularities

By looking at this table, it follows that the Whitney umbrella and the members of the H_k series are already ruled surfaces. Then we only need to look at the remaining cases $S_k^\pm, B_k^\pm, C_k^\pm$ and F_4. We remark that all of these singularities have a 2-jet which belongs to the $(x,y^2,0)$ orbit. For map germs in this orbit, we have a pre-normal form which is very useful in order to recognize the singularity type.

Lemma 3.2. [10] *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ whose 2-jet is in the $(x,y^2,0)$ orbit. Then f is $\mathcal{A}$-equivalent to a germ of the form*

$$(x,y)\mapsto(x,y^2,yp(x,y^2)),$$

for some function germ $p\in\mathcal{E}_2$.

Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ of the form $f(x,y)=(x,y^2,yp(x,y^2))$. We assume that f has a simple singularity. Let us denote by $\sum_{i+j\geq 1}a_{i,2j}x^iy^{2j}$ the Taylor expansion of $p(x,y^2)$. Then we have the following criteria which allow to recognize the singularity type of f just by analysing its 4-jet (see [10]):

1. If $a_{1,0}\neq 0$, then f is a Whitney umbrella.
2. If $a_{1,0}=0$ but $a_{2,0}\neq 0\neq a_{0,2}$, then f has type $S_1^\pm$.
3. If $a_{1,0}=a_{2,0}=0$ but $a_{0,2}\neq 0$, then f has type $S_k^\pm$, for some $k\geq 2$.

(4) If $a_{1,0} = a_{0,2} = 0$ but $a_{2,0} \neq 0$, then f has type $B_k^\pm$, for some $k \geq 2$.
(5) If $a_{1,0} = a_{2,0} = a_{0,2} = 0$ but $a_{1,2} \neq 0$, then f has type $C_k^\pm$, for some $k \geq 3$.
(6) If $a_{1,0} = a_{2,0} = a_{0,2} = a_{1,2} = 0$ but $a_{3,0} \neq 0 \neq a_{0,4}$, then f has type F_4.

Moreover, in case (3), the number k is determined by the first non zero term in $p(x, y^2)$ of the form x^{k+1}. Analogously, in cases (4) or (5), the number k corresponds to the first non zero term of the form y^{2k} or x^k, respectively. We use this criteria to show the following result, which completes the proof of theorem 1.1.

Proposition 3.3. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a germ of ruled surface of the form $f(x, t) = (x, t^2 + 2xt, h(t) + xg(t))$. Then it is possible to choose the function germs h, g in such a way f has any of the types $S_k^\pm, B_k^\pm, C_k^\pm$ or F_4.*

Proof. We consider the coordinate change in the target given by $(X, Y, Z) \mapsto (X, Y - X^2, Z)$ and then the coordinate change in the source defined by $y = x + t$. Then f is equivalent to

$$\tilde{f}(x, y) = (x, y^2, h(y - x) + xg(y - x)),$$

in such a way that

$$p(x, y^2) = \frac{1}{2y} \left(h(x + y) - h(x - y) + x(g(x + y) - g(x - y))\right).$$

Assume that $h(t) = a_2t^2 + a_3t^3 + \dots$ and $g(t) = b_1t + b_2t^2 + \dots$. Then we can compute explicitly the 4-jet of $p(x, y^2)$:

$$\begin{aligned} j^4p(x, y^2) = {} & a_1 + (2a_2 + b_1)x + (3a_3 + 2b_2)x^2 + a_3y^2 + (3b_3 + 4a_4)x^3 + (b_3 + 4a_4)y^2x \\ & + (4b_4 + 5a_5)x^4 + (4b_4 + 10a_5)y^2x^2 + a_5y^4. \end{aligned}$$

The result is now an easy consequence of the above criteria. □

We finish this section with an explicit list of all the simple singularities constructed with ruled surfaces. We left to the reader the work of checking that the proposed germs verify the required conditions.

Type	Ruled surface	
Immersion	$(0, 0, t) + u(1, 0, 0)$	-
Whiney umbrella	$(0, t^2, 0) + u(1, 0, t)$	-
$S_k^\pm$	$(0, t^2, t^3) + u(1, 2t, [\frac{3}{2} \pm \frac{-1}{2}]t^2)$	k=1
	$(0, t^2, t^3) + u(1, 2t, \frac{3}{2}t^2 \pm \frac{t^k+1}{k+1})$	$k \geq 2$
$B_k^\pm$	$(0, t^2, \pm t^{2k+1}) + u(1, 2t, \frac{t^3}{3})$	$k \geq 2$
$C_k^\pm$	$(0, t^2, -t^4) + u(1, 2t, [-1 \pm 1]t^3)$	$k = 3$
	$(0, t^2, \frac{-3}{8}t^4 \mp \frac{t^{k+1}}{k+1}) + u(1, 2t, -\frac{t^3}{2})$	$k \geq 5$ odd
	$(0, t^2, \frac{-3}{8}t^4) + u(1, 2t, \frac{-1}{2}t^3 \mp \frac{t^k}{k})$	$k \geq 4$ even
F_4	$(0, t^2, \frac{t^4}{8} + t^5) + u(1, 2t, \frac{t^3}{2} + \frac{5}{2}t^4)$	-
H_k	$(0, t^{3k-1}, t^3) + u(1, t, 0)$	$k \geq 2$

TABLE 2. Ruled surfaces in the orbits of simple singularities

4. The link and the Gauss word of a finitely determined singularity

The study of the topology of a finitely determined map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ has been done in general (that is, not only for ruled surfaces) by the second author and Marar [9]. We recall here the main definitions and properties. First of all, we use a result of Fukuda [3] to see that any finitely determined map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ has a cone structure over its link. This link is obtained by intersecting f with a small enough sphere S^2_ϵ in $(\mathbb{R}^3,0)$.

Theorem 4.1. [3] *Suppose $n\le p$. Then given a semi-algebraic subset W of $J^r(n,p)$, there exist an integer s, depending only on n, p and r, and a closed semi-algebraic subset Σ_W of $(\pi^s_r)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ mapping $f:\mathbb{R}^n\to\mathbb{R}^p$ with $j^sf(0)$ belonging to $(\pi^s_r)^{-1}(W)\setminus\Sigma_W$, there exists a positive number ϵ_0 such that for any number ϵ with $0<\epsilon\le\epsilon_0$ we have*

1. *$\tilde S^{n-1}_\epsilon=f^{-1}(S^{p-1}_\epsilon)$ is a homotopy $(n-1)$-sphere which, if $n\ne 4,5$ is diffeomorphic to the natural $(n-1)$-sphere S^{n-1},*
2. *the restricted mapping $f|\tilde S^{n-1}_\epsilon:\tilde S^{n-1}_\epsilon\to S^{p-1}_\epsilon$ is topologically stable (C^∞ stable if (n,p) is a nice pair),*
3. *letting $\tilde D^n_\epsilon=f^{-1}(D^p_\epsilon)$, the restricted mapping $f|\tilde D^n_\epsilon\setminus\{0\}:\tilde D^n_\epsilon\setminus\{0\}\to D^p_\epsilon\setminus\{0\}$ is proper, topologically stable (C^∞ stable if (n,p) is nice) and topologically equivalent (C^∞ equivalent if (n,p) is nice) to the product mapping*
$$(f|\tilde S^{n-1}_\epsilon)\times \mathrm{id}_{(0,\epsilon)}:\tilde S^{n-1}_\epsilon\times(0,\epsilon)\to S^{p-1}_\epsilon\times(0,\epsilon)$$
defined by $(x,t)\mapsto(f(x),t)$, and
4. *consequently, $f|\tilde D^n_\epsilon:\tilde D^n_\epsilon\to D^p_\epsilon$ is topologically equivalent to the cone*
$$C(f|\tilde S^{n-1}_\epsilon):\tilde S^{n-1}_\epsilon\times[0,\epsilon)/\tilde S^{n-1}_\epsilon\times\{0\}\to S^{p-1}_\epsilon\times[0,\epsilon)/S^{p-1}_\epsilon\times\{0\}$$
of the stable mapping $f|\tilde S^{n-1}_\epsilon:\tilde S^{n-1}_\epsilon\to S^{p-1}_\epsilon$ defined by $C(f|\tilde S^{n-1}_\epsilon)(x,t)=(f(x),t)$.

Corollary 4.2. *Suppose $n\le p$ and let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a finitely determined map germ. Then there is a representative $f:U\subset\mathbb{R}^n\to\mathbb{R}^p$ of the map germ and there exists a positive number ϵ_0 such that any number ϵ with $0<\epsilon\le\epsilon_0$ verifies (1), (2), (3) and (4) of the above theorem.*

Proof. Assume that f is r-determined for some r and let $W=\{j^rf(0)\}$. By the above theorem there is s, and a closed semi-algebraic subset Σ_W of $(\pi^s_r)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ mapping $g:\mathbb{R}^n\to\mathbb{R}^p$ with $j^sg(0)$ belonging to $(\pi^s_r)^{-1}(W)\setminus\Sigma_W$, there exists a positive number ϵ_0 such that any number ϵ with $0<\epsilon\le\epsilon_0$ verifies (1), (2), (3) and (4).

Since $(\pi^s_r)^{-1}(W)\setminus\Sigma_W\ne\emptyset$, we can take a map $g:\mathbb{R}^n\to\mathbb{R}^p$ with $j^sg(0)\in(\pi^s_r)^{-1}(W)\setminus\Sigma_W$. This implies that $j^rg(0)=j^rf(0)$ so that g is $\mathcal{A}$-equivalent to f. Hence, properties (1), (2), (3) and (4) are also true for f. ☐

Definition 4.3. Suppose $n\le p$ and let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a finitely determined map germ. We say that the stable map $f|\tilde S^{n-1}_\epsilon:\tilde S^{n-1}_\epsilon\to S^{p-1}_\epsilon$ is the *link* of f, where f is a representative and $0<\epsilon\le\epsilon_0$ verify (1), (2), (3) and (4) of the above theorem.

Note that when $n=2$ and $p=3$, the link $f|\tilde S^1_\epsilon:\tilde S^1_\epsilon\to S^2_\epsilon$ is nothing but a closed curve in the sphere S^2_ϵ which is immersed and has only double transverse points (see figure 3).

In order to describe the topology of a closed curve immersed with normal crossings in the sphere we use the Gauss word. This concept was introduced by Gauss [4] who

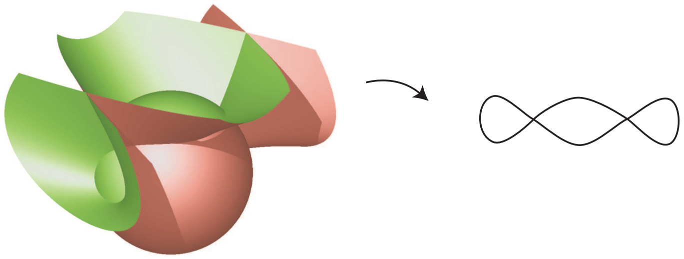

FIGURE 3. The link of the S_1^- singularity

studied the problem of "realizability" of Gauss words, a problem similar in nature to the "planarity" problem for graphs.

Definition 4.4. Let $\gamma : S^1 \to S^2$ be an immersed closed curve in the sphere with r transverse double points. We choose r letters $a_1, \dots, a_r$ which label these double points. We also fix orientations on both S^1 and S^2 and choose a base point $z_0 \in S^1$. We consider a permutation

$$\sigma : \{1, \dots, 2r\} \to \{a_1^{+1}, \dots, a_r^{+1}, a_1^{-1}, \dots, a_r^{-1}\},$$

constructed as follows: We denote by $z_1, \dots, z_{2r} \in S^1$ the source double points ordered such that $z_0 \leq z_1 < \cdots < z_{2r}$. Assume that $\gamma(z_i) = \gamma(z_j) = a_k$ with $i < j$. Then we put $\sigma(i) = a_k^{+1}$ and $\sigma(j) = a_k^{-1}$ if the pair $(\gamma'(z_i), \gamma'(z_j))$ is positively oriented in S^2 or $\sigma(i) = a_k^{-1}$ and $\sigma(j) = a_k^{+1}$ otherwise (see figure 4).

As usual when working with permutations, in order to simplify the notation, we will identify the permutation σ with the sequence $\sigma(1) \dots \sigma(2r)$. This sequence is called the *Gauss word* of γ.

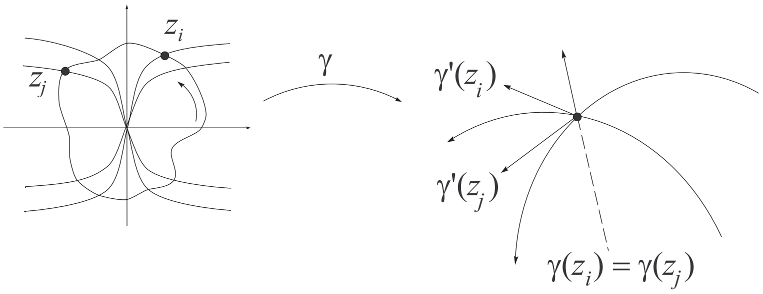

FIGURE 4. A double point with $\sigma(i) = a_k^{+1}$ and $\sigma(j) = a_k^{-1}$.

It is obvious that the Gauss word is not uniquely determined, since it depends on: the labels $a_1, \dots, a_r$, the chosen orientations on both S^1 and S^2 and the base point $z_0 \in S^1$. A change in the choices will produce the following changes in the Gauss word:

1. permuting the alphabet set $a_1, \dots, a_r$;
2. cyclically permuting the sequence which defines the Gauss word;
3. reversing the sequence;
4. changing all the exponents from $+1$ to -1 and vice versa.

Up to this equivalence, the Gauss word is well defined and has the following property: two curves are topologically equivalent if and only if their Gauss words are equivalent.

There are also known classifications of curves with a low number of double points or crossings. For instance, there is just one class with one crossing (this is different from the case of plane curve where we have two classes). If the number of crossings is either 2,3 or 4, then the number of topological classes is either 2, 6 or 19 respectively (we refer to [1] for details and pictures).

5. TOPOLOGICAL CLASSIFICATION OF THE $(x, y^2, 0)$ ORBIT

In this section, we give the topological classification of all the finitely determined germs of ruled surfaces $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ whose 2-jet is in the $(x, y^2, 0)$ orbit. In this case, we have the following result which has been obtained by the second author and W. Marar. It applies not only to germs or ruled surfaces but to general map germs whose 2-jet belongs to the $(x, y^2, 0)$ orbit (in fact, we will see in the next section that the theorem is also valid when the 2-jet belongs to the $(x, xy, 0)$ orbit and has fold type, see theorem 6.3).

Theorem 5.1. [9] *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ whose 2-jet belongs to the $(x, y^2, 0)$. Then the link has Gauss word equivalent to*

$$a_1 a_2^{-1} \dots a_r^{\pm 1} a_r^{\mp 1} \dots a_2 a_1^{-1}.$$

In particular, two map germs of this type are topologically equivalent if and only if their double point curves have the same number of branches (see figure 5).

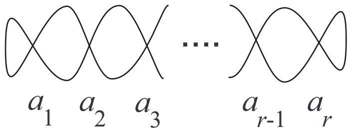

FIGURE 5. The link of a map germ of type $(x, y^2, 0)$

We recall that the double point curve is defined as

$$D(f) = \overline{\{x \in \mathbb{R}^2 : f^{-1}(f(x)) \neq \{x\}\}},$$

where $\overline{A}$ denotes the topological closure of A. It is well known that if f is finitely determined, then we can assume that $D(f)$ is the germ of a 1-dimensional analytic set with isolated singularity at the origin (see [11]). In the particular case that $f(x, y) = (x, y^2, yp(x, y^2))$ we easily see that the double point curve is defined by the equation $p(x, y^2) = 0$.

As a consequence of the theorem, in order to classify the ruled surfaces of this type, we only need to control the possible values for the number of branches of the double point curve of f. We will show that if f is a ruled surface, then the number of branches will be always ≤ 3.

We begin with the following lemma, which is useful in order to compute an upper bound of the number of branches of a plane curve.

Lemma 5.2. *Let $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be an analytic function germ and assume that $g = g_k + g_{k+1} + \dots$, where each g_i is a homogeneous polynomial of degree i and $g_k \neq 0$. Then the number of branches of the plane curve $g^{-1}(0)$ is less than or equal to the number of real linear factors of g_k, counted with multiplicity.*

Proof. Assume that the curve has r branches, $g^{-1}(0) = X_1 \cup \cdots \cup X_r$, where each branch X_i has a reduced equation $f_i(x, y) = 0$, with $f_i \in \mathcal{E}_2$, $i = 1, \dots, r$. It follows that each f_i divides g, so that we can write $g = f_1 \dots f_r h$, for some $h \in \mathcal{E}_2$.

Let us denote by $\text{in}(\alpha)$ the initial part of a function germ $\alpha \in \mathcal{E}_2$, then we have

$$g_k = \text{in}(g) = \text{in}(f_1) \dots \text{in}(f_r)\, \text{in}(h).$$

Moreover, each homogeneous polynomial $\text{in}(f_i)$ has at least one real factor, which concludes the proof. ☐

Lemma 5.3. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface whose 2-jet belongs to the $(x, y^2, 0)$ orbit. Then f is $\mathcal{A}$-equivalent to a map germ of the form*

$$(x, y) \mapsto (x, y^2, a(y + cx)^k + bx(y + cx)^{k-1} + g(x, y)),$$

where $k \geq 3$, $a^2 + b^2 \neq 0$, $c \in \mathbb{R}$ and $g \in m_2^{k+1}$.

Proof. The assumption on f implies that we can assume it has the form

$$f(x, y) = (x, y^2 - 2cxy + h(x, y), ay^k + bxy^{k-1} + g(x, y)),$$

for some $a^2 + b^2 \neq 0$, $c \in \mathbb{R}$, $h \in m_2^3$ and $g \in m_2^{k+1}$. Following the steps of the proof of proposition 2.6, we see that f is $\mathcal{A}$-equivalent to

$$(x, y^2 + h(x, y + cx), a(y + cx)^k + bx(y + cx)^{k-1} + g(x, y + cx)).$$

Now we use the arguments of [10]: since (x, y^2) is 2-determinated, it is possible to find diffeomorphisms ϕ and φ whose 2-jet is the identity, such that f is $\mathcal{A}$-equivalent to

$$(x, y^2, a(y + cx)^k + bx(y + cx)^{k-1} + \tilde{g}(x, y)),$$

with $\tilde{g} \in m_2^{k+1}$. ☐

Theorem 5.4. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface whose 2-jet belongs to the $(x, y^2, 0)$ orbit. Then the double point curve $D(f)$ has at most three branches.*

Proof. By the above lemma, f is $\mathcal{A}$-equivalent to

$$(x, y^2, a(y + cx)^k + bx(y + cx)^{k-1} + g(x, y)),$$

where $k \geq 3$, $a^2 + b^2 \neq 0$, $c \in \mathbb{R}$ and $g \in m_2^{k+1}$. If $c = 0$, then we have that it is $\mathcal{A}$-equivalent to either $(x, y^2, y^k + \dots)$, if k is odd, or $(x, y^2, xy^{k-1} + \dots)$, if k is even. It is not difficult to see that the double point curve has either 0 or 1 branches respectively. Thus, we can assume that $c \neq 0$ and by taking the obvious linear coordinate changes, we reduce it to $c = 1$.

Following the arguments of Mond in [10], f is $\mathcal{A}$-equivalent to $(x, y^2, yp(x, y^2))$ and the equation of $D(f)$ is $p(x, y^2) = 0$. By lemma 5.2, it is enough to analyze the initial part of $p(x, y^2)$, which is equal to

$$h(x, y^2) = \frac{1}{2y}\{a(y + x)^k + bx(y + x)^{k-1} - a(-y + x)^k - bx(-y + x)^{k-1}\}.$$

We will separate our analysis in two cases. First we assume k odd, then

$$\begin{aligned} h(x, y^2) =& ay^{k-1} + \left[a\binom{k}{2} + b\binom{k-1}{1}\right] y^{k-3}x^2 + \left[a\binom{k}{4} + b\binom{k-1}{3}\right] y^{k-5}x^4 \\ & \dots + \left[a\binom{k}{k-1} + b\binom{k-1}{k-2}\right] x^{k-1}. \end{aligned}$$

If $a=0$, we have that

$$h(x,y^2)=bx^2\left\{\binom{k-1}{1}(y^2)^{r-1}+\binom{k-1}{3}(y^2)^{r-2}x^2+\cdots+\binom{k-1}{k-2}(x^2)^{r-1}\right\},$$

where $k=2r+1$. It is obvious that $h(x,y^2)$ has only 2 real linear factors, counted with multiplicity. Otherwise, we have $a\neq 0$ and we can assume $a=1$ (since $h(x,y^2)$ is linear in a,b). Moreover, the number of real linear factors of $h(x,y^2)$ is equal to twice the number of positive real roots of the polynomial $\tilde{h}(z)=h(1,z)$, that is,

$$\begin{aligned}\tilde{h}(z)=&\left[z^r+\binom{k}{2}z^{r-1}+\binom{k}{4}z^{r-2}+\ldots+\binom{k}{k-1}\right]\\&+b\left[\binom{k-1}{1}z^{r-1}+\binom{k-1}{3}z^{r-2}+\ldots+\binom{k-1}{k-2}\right].\end{aligned}$$

If $b\geq 0$, then all the coefficients of $\tilde{h}(z)$ are also positive and thus, it does not have any positive real root. Otherwise, if $b<0$, the number of positive real roots is equal to the number of positive preimages of b through the function

$$g(z)=-\frac{\left[z^r+\binom{k}{2}z^{r-1}+\binom{k}{4}z^{r-2}+\ldots+\binom{k}{k-1}\right]}{\left[\binom{k-1}{1}z^{r-1}+\binom{k-1}{3}z^{r-2}+\ldots+\binom{k-1}{k-2}\right]}.$$

It is not difficult to arrive to the following formula for the derivative of $g(z)$:

$$g'(z)=-\frac{\binom{k}{1}z^{2r-2}+2\binom{k}{3}z^{2r-3}+\ldots+2\binom{k}{3}}{\left[\binom{k-1}{1}z^{r-1}+\binom{k-1}{3}z^{r-2}+\ldots+\binom{k-1}{k-2}\right]^2}.$$

Hence, if $z\geq 0$ we have $g'(z)<0$, so that g is strictly decreasing on $[0,+\infty)$. This implies that b has at most one preimage through g and thus, $\tilde{h}(z)$ has at most one positive real root and $D(f)$ has at most 2 branches by lemma 5.2.

In the case k is even we obtain:

$$\begin{aligned}h(x,y^2)=&x\Big\{\left[a\binom{k}{1}+b\binom{k-1}{0}\right]y^{k-2}+\left[a\binom{k}{3}+b\binom{k-1}{2}\right]y^{k-4}x^2\\&\ldots+\left[a\binom{k}{k-1}+b\binom{k-1}{k-2}\right]x^{k-2}\Big\}.\end{aligned}$$

By using similar arguments we conclude that in this case $D(f)$ has at most 3 branches. □

Corollary 5.5. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a finitely determined germ of ruled surface whose 2-jet belongs to the $(x,y^2,0)$ orbit. Then the topological type of f is determined by one of the following four links:*

$$\emptyset,\quad aa^{-1},\quad ab^{-1}ba^{-1},\quad ab^{-1}cc^{-1}ba^{-1}.$$

Example 5.6. We find germs of ruled surfaces whose whose 2-jet belongs to the $(x,y^2,0)$ orbit and which present the four topological types. In fact, the four classes are obtained by considering respectively the following ruled surfaces:

Ruled surface	Link
$(x, y^2 - 2xy, y^3 + xy^2)$	$\emptyset$
$(x, y^2 - 2xy, y^4 + xy^3)$	aa^{-1}
$(x, y^2 - 2xy, y^3 - 2xy^2)$	$ab^{-1}ba^{-1}$
$(x, y^2 - 2xy, y^4 - 2xy^3)$	$ab^{-1}cc^{-1}ba^{-1}$

TABLE 3. Ruled surfaces in the $(x, y^2, 0)$ orbit

On the other hand, after theorem 5.4, it is also very easy to find a map germ whose 2-jet belongs to the $(x, y^2, 0)$ orbit and which is not $\mathcal{A}$-equivalent to any ruled surface. Consider the map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$f(x, y) = (x, y^2, y(x^2 - y^2)(x^2 - \frac{1}{4}y^2)).$$

The double point curve is defined by the equation $p(x, y^2) = (x^2 - y^2)(x^2 - \frac{1}{4}y^2) = 0$, which has 4 branches. Therefore, it cannot be topologically equivalent (and hence $\mathcal{A}$-equivalent) to any ruled surface (see figure 5).

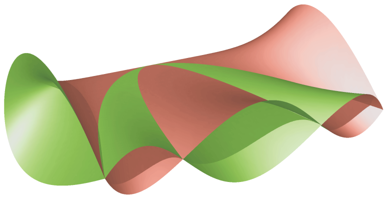

FIGURE 6. A “non ruled” singularity in the $(x, y^2, 0)$ orbit

6. TOPOLOGICAL CLASSIFICATION OF THE $(x, xy, 0)$ ORBIT

As we have seen in the previous section, the double point curve of a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ plays an important role in order to describe its topological behavior. Unfortunately, in general it is not easy to give an analytic description of $D(f)$, unless we are in the $(x, y^2, 0)$ orbit. However, if the map germ has corank 1, we can assume that it is given by $f(x, y) = (x, g(x, y), h(x, y))$ and we can introduce a new double point set which is easier to work with:

$$D^2(f) = \overline{\{(x, y, u) \in \mathbb{R}^3 : f(x, y) = f(x, u),\ y \neq u\}}.$$

According to [8], it follows that $D^2(f)$ is given by equations:

$$\frac{g(x, y) - g(x, u)}{y - u} = \frac{h(x, y) - h(x, u)}{y - u} = 0.$$

Moreover, we have that $D(f) = p_3(D^2(f))$, where p_3 is the projection $p_3(x, y, u) = (x, y)$.

Assume now that f is a map germ whose 2-jet is in the $(x, xy, 0)$ orbit. After some coordinate changes, we can assume that f is given by

$$f(x,y) = (x, xy + p(y), xq(y) + r(y)),$$

for some function germs p, q, r. Then the defining equations of $D^2(f)$ are

$$x + \frac{p(y)-p(u)}{y-u} = x\frac{q(y)-q(u)}{y-u} + \frac{r(y)-r(u)}{y-u} = 0.$$

In particular, we get that $D^2(f)$ is isomorphic to $p_1(D^2(f))$, where $p_1(x,y,u) = (u,y)$ and this is defined by equation:

$$\frac{p(y)-p(u)}{y-u}\frac{q(y)-q(u)}{y-u} - \frac{r(y)-r(u)}{y-u} = 0. \tag{3}$$

We use this to prove the following theorem which gives an upper bound for the number of branches of the double point curve also in this case as we did in theorem 5.4.

In order to simplify the computations we introduce the following notation:

$$F_k(y,u) = y^k + y^{k-1}u + \cdots + u^k, \qquad f_k(y) = F_k(y,1),$$

$$G_{k,m}(y,u) = \frac{F_k(y,u)F_m(y,u)}{F_{k+m}(y,u)}, \qquad g_{k,m}(y) = G_{k,m}(y,1).$$

Theorem 6.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface whose 2-jet belongs to the $(x, xy, 0)$ orbit. Then the double point curve $D(f)$ has at most five branches.*

Proof. Let f be the map germ given by

$$f(x,y) = (x, xy + p(y), xq(y) + r(y)),$$

for some function germs p, q, r. We also write

$$p(t) = at^k + a_1t^{k+1} + \dots, \quad q(t) = bt^m + b_1t^{m+1} + \dots, \quad r(t) = ct^n + c_1t^{n+1} + \dots,$$

where $a, b, c \neq 0$. After taking linear coordinate changes in the source and target, we can assume without loss of generality that $a = b = 1$. We analyse the initial part of the equation of the projection of the double point curve $p_1(D^2(f))$. We will see that the number of real linear factors counted with multiplicity is always ≤ 5 which implies the result by lemma 5.2.

In fact, the initial part of equation (3) is given by $H(y,u) = 0$, where

$$H(y,u) = \begin{cases} F_{k-1}(y,u)F_{m-1}(y,u), & \text{if } n > m+k-1, \\ F_{n-1}(y,u), & \text{if } n < m+k-1, \\ F_{k-1}(y,u)F_{m-1}(y,u) - cF_{k+m-2}(y,u), & \text{if } n = m+k-1. \end{cases}$$

If $n > m+k-1$, then $H(y,u)$ has at most two real linear factors, both equal to $y+u$. Analogously, if $n < m+k-1$, then $H(y,u)$ has at most one real linear factor, equal to $y+u$. Hence, we can assume $n = m+k-1$.

If $c = 1$, it is obvious that the real linear factors of $H(y,u)$ are either yu, if $k+m$ is even, or $yu(y+u)$, if $k+m$ is odd. Otherwise, we have $c \neq 1$. Since y^{k+m-2} and u^{k+m-2}

have coefficient $1-c\neq 0$, we can dehomogenize $H(y,u)$: the number of real linear factors of $H(y,u)$ is exactly equal to the number of real roots of the polynomial

$$h(y)=H(y,1)=f_{k-1}(y)f_{m-1}(y)-cf_{k+m-2}(y).$$

If $k+m$ is even, then $f_{k+m-2}(y)$ never vanishes. Hence, y is a root of $h(y)$ if and only if $g_{k-1,m-1}(y)=c$, where

$$g_{k-1,m-1}(y)=\frac{f_{k-1}(y)f_{m-1}(y)}{f_{k+m-2}(y)}.$$

Thus, the number of roots of $h(y)$ coincides with the number of inverse images $g^{-1}_{k-1,m-1}(c)$ (counting with multiplicity). Otherwise, if $k+m$ is odd, we have one more root of $h(y)$, namely $y=-1$. In this case, the number of real roots of $h(y)$ is equal to the number of inverse images $g^{-1}_{k-1,m-1}(c)$ plus one.

On the other hand, we observe that

$$g_{k-1,m-1}(y)\begin{cases}<1, & \text{if } y<0,\\ =1, & \text{if } y=0,\\ >1, & \text{if } y>0.\end{cases}$$

This implies that if $c>1$ (resp. $c<1$), then all the inverse images in $g^{-1}_{k-1,m-1}(c)$ are positive (resp. negative).

In order to count the number of real roots of $h(y)$ we consider another polynomial:

$$\tilde{h}(y)=(y-1)^2h(y)=(1-c)y^{k+m}+cy^{m+k-1}-y^m-y^k+cy+(1-c).$$

The number of negative real roots of $\tilde{h}(y)$ is equal to that of $h(y)$, but the number of positive real roots of $\tilde{h}(y)$ is equal to that of $h(y)$ plus two. We use the well known Descartes rule of signs: "the number of positive real roots of a polynomial, counted with multiplicity, is less than or equal to the number of sign changes in the coefficients of the polynomial".

Since $\tilde{h}(y)$ has only 6 non-zero coefficients, we have at most 5 sign changes. Therefore, $\tilde{h}(y)$ has at most 5 positive real roots and 5 negative real roots. However, taking into account the above observation, we deduce that $h(y)$ has at most 3 real roots when $c>1$, or it has at most 5 real roots, if $c<1$. This concludes the proof. □

It follows from the proof of theorem 6.1 that the number of branches of the double point curve is controlled by the number of preimages of $c=g_{k-1,m-1}(y)$. We present in figure 7 the graph of this function depending on the parity of p,r, where $p=\min\{k,m\}$ and $r=\max\{k,m\}$. We also remark that the function $g_{k-1,m-1}$ has the following basic properties:

1. $g_{k-1,m-1}(1/y)=g_{k-1,m-1}(y)$ for any $y\neq 0$.
2. $g_{k-1,m-1}(y)>1$, if $y>0$.
3. $0\leq g_{k-1,m-1}(y)<1$, if $y<0$.

In order to determine completely the graph of $c=g_{k-1,m-1}(y)$, we will look at the number of real roots of the polynomial

$$(1-c)y^{k+m}+cy^{m+k-1}-y^m-y^k+cy+(1-c).$$

By using the Descartes rule of signs, the number if real roots is determined by taking into account the number of sign changes of the coefficients of the polynomial. This will depend on the parameter c, as well as the parities of k and m.

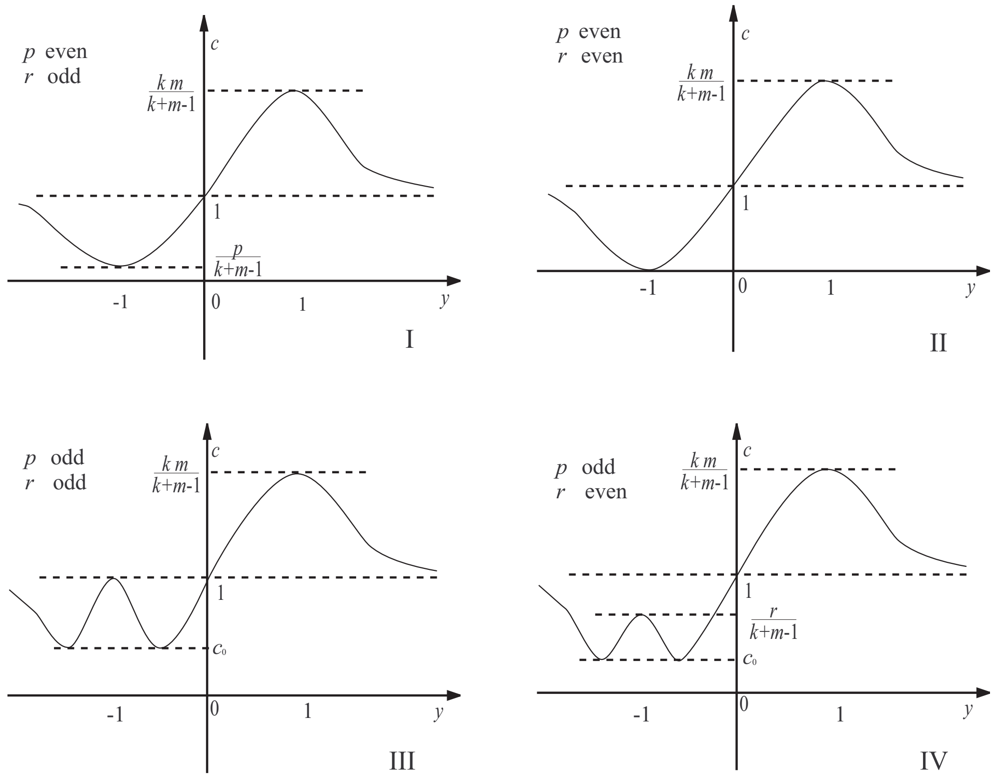

FIGURE 7. Graphs of $c = g_{k-1,m-1}(y)$

From the analysis of the function $c = g_{k-1,m-1}(y)$ we deduce that its maximum value is always equal to $km/(k+m-1)$ and it is attained at $y = 1$. If p is even, the minimum value of c is attained at $y = -1$, being equal to $p/(k+m-1)$ if r is odd, or 0 if r is even. Finally, if p is odd, the minimum value of c is denoted by c_0, with $0 < c_0 < 1$. Because of symmetry, it is attained at two points y_0 and $1/y_0$ with $-1 < y_0 < 0$.

According to [9], the map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ whose 2-jet belongs to the $(x, xy, 0)$ orbit can be divided into two groups: germs of fold and cusp type. Then they show that any germ of fold type is topologically equivalent to a map germ in the $(x, y^2, 0)$ orbit. As a consequence, theorem 5.1 is extended easily to all the germs of fold type.

Definition 6.2. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ whose 2-jet belongs to the $(x, xy, 0)$ orbit. We can assume that f is written in the form

$$f(x, y) = (x, xy + g(x, y), h(x, y)),$$

for some function germs $g, h \in m_2^3$. Since f is finitely determined, it follows that g must have finite order in y, that is, $g(0, y) = a_k y^k + a_{k+1} y^{k+1} + \dots$ with $a_k \neq 0$, $k \geq 3$. We say that f has *fold type* if k is even or *cusp type* if k is odd.

Theorem 6.3. [9] *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ whose 2-jet belongs to the $(x, xy, 0)$ and has fold type. Then the link has Gauss word equivalent to*

$$a_1 a_2^{-1} \dots a_r^{\pm 1} a_r^{\mp 1} \dots a_2 a_1^{-1}.$$

As a consequence of theorems 6.1 and 6.3, if $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a ruled surface germ of fold type, then its link is given by $a_1 a_2^{-1} \dots a_r^{\pm 1} a_r^{\mp 1} \dots a_2 a_1^{-1}$, with $r \leq 5$. Moreover, the case $r = 4$ (i.e., the double point curve has 4 branches) cannot occur. This follows from the analysis of the graph of $g_{k-1,m-1}(y)$ (see figure 7).

If $k \leq m$, then $p = k$ is even. Thus, the graph of $g_{k-1,m-1}(y)$ has type I or II and $r \leq 3$. Otherwise, if $k > m$, then $r = k$ is even and the graph of $g_{k-1,m-1}(y)$ has type II or IV. If it has type II, then $r \leq 2$. If it has type IV, then $r \leq 3$ unless that $c_0 \leq c \leq k/(k+m-1)$. But even if $c_0 \leq c \leq k/(k+m-1)$, the initial part has 5 real branches counted with multiplicity. Since f has no singularities in the initial part, it follows that r must be also odd and hence, $r = 1, 3, 5$.

Corollary 6.4. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface whose 2-jet belongs to the $(x, xy, 0)$ orbit and has fold type. Then the topological type of f is determined by one of the following five links:*

$$\emptyset, \quad aa^{-1}, \quad ab^{-1}ba^{-1}, \quad ab^{-1}cc^{-1}ba^{-1}, \quad ab^{-1}cd^{-1}ee^{-1}dc^{-1}ba^{-1}.$$

6.1. **The weighted homogeneous case.** Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface whose 2-jet belongs to the $(x, xy, 0)$ orbit. Apart from trivial cases and after taking linear coordinate changes, we can assume that it is written in the form $f = f_c + \tilde{f}$, where f_c is a weighted homogeneous map germ of weights $(k-1, 1)$,

$$f_c(x, y) = (x, xy + y^k, xy^m + cy^{k+m-1}),$$

and $\tilde{f}$ has only terms of higher weighted degree. By using a result of Damon [2], if f_c is finitely determined, then f is topologically equivalent to its initial part f_c. Note that f_c is finitely determined, except for a finite number of values of the parameter c.

By using the Mather-Gaffney determinacy criterion (see [13]), we see that if the map germ $f_c : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is finitely determined, then $f_c : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3$ is an immersion with only transverse double points. Conversely, if f_c is an immersion with only transverse double points, then $f_c : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is not necessarily finitely determined. However, in this case we also have a well defined Gauss word and we will see that if we add higher order terms to f_c so that it becomes finitely determined, then the Gauss word is the same.

We will compute first the Gauss words of all the weighted homogeneous polynomial maps f_c in the case that c is not a critical value of the function $g_{k-1,m-1}(y)$. In the last part, we will show how to extend the classification in the case that c is a critical value.

Definition 6.5. We say that f_c is *non-degenerate* if $f_c : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3$ is an immersion with only transverse double points. It is obvious this condition fails in three cases:

1. f_c has a curve of singularities;
2. f_c has a curve of triple points; or
3. f_c has a curve of non transverse self-intersection.

These three phenomena can be detected by looking at the double point curves $D^2(f_c)$ and $D(f_c)$. In fact, $D^2(f_c)$ is given by equations

$$x + F_{k-1}(y, u) = 0, \quad F_{k-1}(y, u)F_{m-1}(y, u) - cF_{k+m-2}(y, u) = 0.$$

In particular, $D^2(f_c)$ is isomorphic to the curve in the (y, u)-plane defined by the second equation. Since the equation is homogeneous, this is equal to a finite union of lines: Let $\lambda_1, \dots, \lambda_r$ be the solutions of $c = g_{k-1,m-1}(\lambda)$ and let $\lambda_0 = -1$. Then $D^2(f_c)$ is defined by

$$\begin{cases} (u - \lambda_1 y) \dots (u - \lambda_r y) = 0, & \text{if } k + m \text{ is even;} \\ (u - \lambda_0 y)(u - \lambda_1 y) \dots (u - \lambda_r y) = 0, & \text{if } k + m \text{ is odd.} \end{cases}$$

We deduce that $\lambda_i \neq \lambda_j$, $\forall i \neq j$ if and only if c is not a critical value of the function $g_{k-1,m-1}(y)$. The critical values of c can be due either to the existence of a line of singularities of f_c or a line of non transverse self-intersection.

The double point curve $D(f_c)$ is equal to the projection of $D^2(f_c)$ into the (x, y)-plane. We obtain its equation by eliminating the u variable in the equations of $D^2(f_c)$. This gives

$$\begin{cases} (x+\mu_1 y^{k-1})\dots(x+\mu_r y^{k-1}) = 0, & \text{if } k+m \text{ is even;} \\ (x+\mu_0 y^{k-1})(x+\mu_1 y^{k-1})\dots(x+\mu_r y^{k-1}) = 0, & \text{if } k+m \text{ is odd,} \end{cases}$$

where $\mu_i = f_{k-1}(\lambda_i)$ for any $i = 0, \dots, r$. Then, f_c is non degenerate if and only if $\mu_i \neq \mu_j$, $\forall i \neq j$. Moreover, f_c has a line of triple points if and only if c is not a critical value of $g_{k-1,m-1}(y)$ but $\mu_i = \mu_j$, for some $i \neq j$.

Lemma 6.6. *Assume that $-1 < s < t < 0$ and k, m are even. Then,*

$$f_k(s) = f_k(t) \Longrightarrow \frac{f_m(s)}{f_{k+m}(s)} > \frac{f_m(t)}{f_{k+m}(t)}.$$

Proof. We show the following identity:

$$\begin{aligned} f_m(s)f_{m+k}(t) - f_m(t)f_{m+k}(s) =& (f_k(t) - f_k(s))s^m f_m(t) \\ &+ (t-s)(f_k(t)-1)\sum_{j=0}^{m-1} t^j s^{m-1-j} f_j(s). \end{aligned}$$

This will imply the result, since if $-1 < s < t < 0$ and k, m are even, then we have $(t-s) > 0$, $(f_k(t) - 1) < 0$, $t^j s^{m-1-j} < 0$ and $f_j(s) > 0$. The above identity can be proved by means of an intensive use of the formula $f_m(t) = \frac{t^{m+1}-1}{t-1}$. Details are left to the reader. ☐

Proposition 6.7. *The ruled surface $f_c(x, y) = (x, xy + y^k, xy^m + cy^{k+m-1})$ is degenerate exactly in each one of the following cases:*

1. *f_c has a curve of singularities if and only if $c = \frac{km}{k+m-1}$.*
2. *f_c has a curve of triple points if and only if k is odd and $c = 1$.*
3. *f_c has a curve of non transverse intersection if and only if:*

$$c = \begin{cases} \frac{p}{k+m-1}, & \text{if } p \text{ even and } r \text{ odd,} \\ 0, & \text{if } p, r \text{ both even,} \\ 1, c_0, & \text{if } p, r \text{ both odd,} \\ \frac{r}{k+m-1}, c_0, & \text{if } p \text{ odd and } r \text{ even,} \end{cases}$$

where $p = \min\{k, m\}$ and $r = \max\{k, m\}$.

Proof. (1) It is easy to compute the singular set of f_c, which is defined by equations

$$x + ky^{k-1} = 0, \quad (c(k+m-1) - km)y^{k+m-2} = 0.$$

Then, f_c has a line of singularities if and only if $c = \frac{km}{k+m-1}$.

(2) According to the equations of the double point curves, f_c will have a line of triple points if and only if there are $\lambda_i \neq \lambda_j$ such that $g_{k-1,m-1}(\lambda_i) = g_{k-1,m-1}(\lambda_j) = c$ and $f_{k-1}(\lambda_i) = f_{k-1}(\lambda_j)$.

First we note that if k is even, then $f_{k-1}(y)$ is a 1-1 function. Hence, in this case we cannot a have a line of triple points. We assume that k is odd and we distinguish three subcases:

(a) If $c > 1$, we have $\lambda_i, \lambda_j > 0$. Since $f_{k-1}(y)$ is also 1-1 in $(0, \infty)$, we have no triple points in this case.

(b) If $c < 1$, we have $\lambda_i, \lambda_j < 0$. By looking at the graph of $f_{k-1}(y)$ in $(-\infty, 0)$, we see that $f_{k-1}(\lambda_i) = f_{k-1}(\lambda_j)$ implies $-1 < \lambda_i, \lambda_j < 0$. By lemma 6.6, this gives $g_{k-1,m-1}(\lambda_i) \neq g_{k-1,m-1}(\lambda_j)$. Hence, we have no triple points.

(c) Finally, if $c = 1$, we have a line of triple points. In fact, we find that $\lambda_i = -1$ and $\lambda_j = 0$ verify the required conditions.

(3). The ruled surface f_c has either a line of singularities or a line of non transverse intersection if and only if c is a critical value of $g_{k-1,m-1}(y)$. Moreover, the list of critical values of $g_{k-1,m-1}(y)$ is given by

$$c = \begin{cases} \frac{km}{k+m-1}, \frac{p}{k+m-1}, & \text{if } p \text{ even and } r \text{ odd,} \\ \frac{km}{k+m-1}, 0, & \text{if } p, r \text{ both even,} \\ \frac{km}{k+m-1}, 1, c_0, & \text{if } p, r \text{ both odd,} \\ \frac{km}{k+m-1}, \frac{r}{k+m-1}, c_0, & \text{if } p \text{ odd and } r \text{ even.} \end{cases}$$

Now the result now follows from (1). □

In order to prove the main theorem of this section, we need the following lemma, which will be useful to compute the sign of the Gauss word in some cases.

Lemma 6.8. *Let λ be such that $g_{k-1,m-1}(\lambda) = c$ and let (x, y, u) be such that $x + f_{k-1}(\lambda)y^{k-1} = 0$ and $u = \lambda y$. Then*

$$\det\left(\frac{\partial f_c}{\partial x}(x,y), \frac{\partial f_c}{\partial x}(x,u), \frac{\partial f_c}{\partial y}(x,y)\right) = -y^{k+m-1}(\lambda - 1)\lambda(\lambda^{k+m-1} - 1)g'_{k-1,m-1}(\lambda).$$

Proof. It is a tedious but straightforward computation. □

Theorem 6.9. *Assume that the ruled surface $f_c(x, y) = (x, xy + y^k, xy^m + cy^{k+m-1})$ is non-degenerate and has cusp type. Then its topological type is given in table 4 (see also figure 2):*

m even	*Link*
$c < c_0$	aa^{-1}
$c_0 < c < \frac{m}{k+m-1}$	$ab^{-1}cd^{-1}ea^{-1}bc^{-1}de^{-1}$
$\frac{m}{k+m-1} < c < 1$	$ab^{-1}ca^{-1}bc^{-1}$
$1 < c < \frac{km}{k+m-1}$	$aa^{-1}bb^{-1}cc^{-1}$
$\frac{km}{k+m-1} < c$	aa^{-1}

m odd	*Link*
$c < c_0$	$\emptyset$
$c_0 < c < 1$	$ab^{-1}c^{-1}dba^{-1}dc^{-1}$
$1 < c < \frac{km}{k+m-1}$	$aa^{-1}b^{-1}b$
$\frac{km}{k+m-1} < c$	$\emptyset$

Table 4

Proof. Because of corollary 6.4 we only need to consider the case that f_c has cusp type, that is, k is odd. We also denote by r the number of branches of the double point curve $D(f_c)$. If $r \leq 2$, there is nothing to show since the list includes all the possible Gauss words. We will consider the three cases $r = 3, 4, 5$.

(1) $r = 5$. Looking at the possible graphs of $g_{k-1,m-1}(y)$ (figure 7), we must have $k < m$, m is even and $c_0 < c < m/(k+m-1)$. Moreover, we denote

$$g_{k-1,m-1}^{-1}(c) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\},$$

where $\lambda_4 < \lambda_3 < -1 < \lambda_2 < \lambda_1 < 0$ and $\lambda_1 = 1/\lambda_4$, $\lambda_2 = 1/\lambda_3$. Then, the double point curve $D^2(f_c)$ is defined by

$$x + (y^{k-1} + \cdots + u^{k-1}) = 0, \quad (u+y)(u - \lambda_1 y)\dots(u - \lambda_4 y) = 0.$$

By eliminating the u variable, we compute the equation of the double point curve $D(f_c)$,

$$(x + y^{k-1})(x + \mu_1 y^{k-1})\dots(x + \mu_4 y^{k-1}) = 0,$$

where $\mu_i = f_{k-1}(\lambda_i)$, for any $i = 1, \dots, 4$. Note that $k-1$ is even, so that $D(f_c)$ is the union of five parabolas passing through the origin. Moreover, we have that either $\mu_4 > \mu_3 > 1 > \mu_2 > \mu_1 > 0$ or $\mu_4 > \mu_3 > 1 > \mu_1 > \mu_2 > 0$. The relative position of the five parabolas in both cases is presented in figure 8.

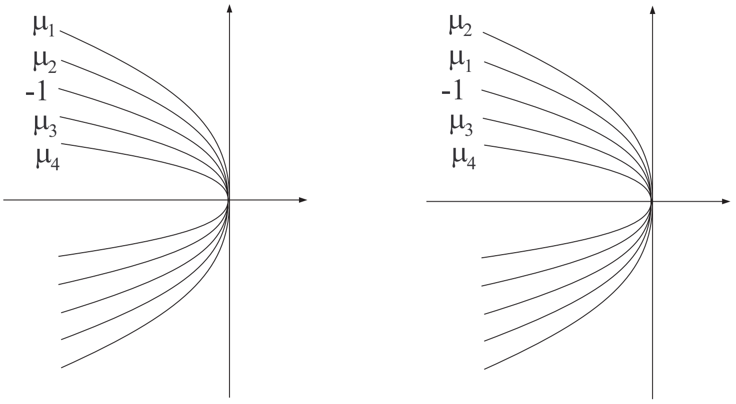

FIGURE 8. Relative position of the five branches of $D(f_c)$.

Now we compute the Gauss word up to the sign. Assume that $\mu_4 > \mu_3 > 1 > \mu_2 > \mu_1 > 0$. Let us consider double point (x, y) in the branch $x + \mu_1 y^{k-1} = 0$ in the positive half plane $y > 0$. We label this double point with the letter a. The symmetric point of (x, y) is given by (x, u) with $u = \lambda_1 y$. Since $\lambda_4 = 1/\lambda_1$, this point belongs to the branch $x + \mu_4 u^{k-1} = 0$ and it is located in the negative half plane $u < 0$. Therefore, the letter a will appear in the first and the sixth positions of the Gauss word. A similar argument with the remaining branches shows that the Gauss word is equal, up to the sign, to *abcdeabcde* (see figure 8 left).

In the other case, we have $\mu_4 > \mu_3 > 1 > \mu_1 > \mu_2 > 0$. We follow the same process and arrive to the Gauss word *abcdebaced* (see figure 8 right). The problem is that this Gauss word is not realizable on the sphere. Thus, this case is not possible.

Finally, we compute the signs in the Gauss word *abcdeabcde*. By analyzing all the possibilities, we see that there is only one which is realizable on the sphere, namely, $ab^{-1}cd^{-1}ea^{-1}bc^{-1}de^{-1}$. This concludes the proof in the case $r = 5$.

(2) $r = 4$. The proof is essentially the same as $r = 5$. We have that m is odd, $c_0 < c < 1$ and

$$g_{k-1,m-1}^{-1}(c) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\},$$

where $\lambda_4 < \lambda_3 < -1 < \lambda_2 < \lambda_1 < 0$ and $\lambda_1 = 1/\lambda_4$, $\lambda_2 = 1/\lambda_3$. Then, the double point curves $D^2(f_c)$ and $D(f_c)$ are defined respectively by

$$x + (y^{k-1} + \cdots + u^{k-1}) = 0, \quad (u - \lambda_1 y) \dots (u - \lambda_4 y) = 0,$$

and

$$(x + \mu_1 y^{k-1}) \dots (x + \mu_4 y^{k-1}) = 0,$$

where $\mu_i = f_{k-1}(\lambda_i)$, for any $i = 1, \dots, 4$. Again we have two possibilities either $\mu_4 > \mu_3 > 1 > \mu_2 > \mu_1 > 0$ or $\mu_4 > \mu_3 > 1 > \mu_1 > \mu_2 > 0$. This will produce two possible Gauss words up to the sign, namely, *abcdabcd* and *abcdbadc* respectively. However, in this case, the first one is not realizable and necessarily we must have the second one *abcdbadc*. Again, this Gauss word has a unique choice of signs so that it is realizable on the sphere, $ab^{-1}c^{-1}dba^{-1}d^{-1}c$, which concludes the proof.

(3) $r = 3$. In this case we have that m is even and either $m/(k+m-1) < c < 1$ or $1 < c < km/(k+m-1)$. Then,

$$g^{-1}_{k-1,m-1}(c) = \{\lambda_1, \lambda_2\},$$

where $\lambda_1 = 1/\lambda_2$ and either $\lambda_2 < -1 < \lambda_1 < 0$ if $c < 1$ or $\lambda_2 > 1 > \lambda_1 > 0$ if $c > 1$. Moreover, the double point curves $D^2(f_c)$ and $D(f_c)$ are defined respectively by

$$x + (y^{k-1} + \cdots + u^{k-1}) = 0, \quad (u + x)(u - \lambda_1 y)(u - \lambda_2 y) = 0,$$

and

$$(x + y^{k-1})(x + \mu_1 y^{k-1})(x + \mu_2 y^{k-1}) = 0,$$

where $\mu_i = f_{k-1}(\lambda_i)$, for any $i = 1, 2$. We have either $\mu_1 < 1 < \mu_2$, if $c < 1$, or $1 < \mu_1 < \mu_2$, if $c > 1$. Therefore, the Gauss word is equal, up to the sign, to either *abcabc* if $c < 1$, or *abbcca* if $c > 1$.

If $c < 1$, there is a unique choice of the signs such that the Gauss word *abcabc* is realizable on the sphere, $ab^{-1}ca^{-1}bc^{-1}$. However, when $c > 1$, there are several choices of the signs for the Gauss word *abbcca* which are realizable on the sphere. We compute the signs explicitly in this case by means of lemma 6.8.

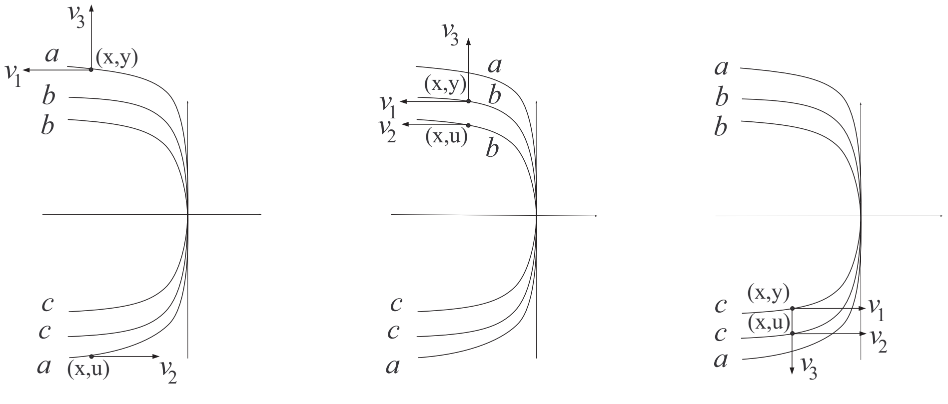

FIGURE 9. Computation of the signs in *abbcca*.

We take the first double point labeled by a. It is given by a pair of points (x, y) and (x, u), with $x + y^{k-1} = 0$, $u = -y$ and $y > 0$. By looking at the relative position of the branches, we deduce that the sign of the Gauss word is equal to the sign of the determinant

$\det(v_1, v_2, v_3)$, where $v_1 = -\frac{\partial f_c}{\partial x}(x, y)$, $v_2 = \frac{\partial f_c}{\partial x}(x, u)$ and $v_3 = \frac{\partial f_c}{\partial y}(x, y)$ (see figure 9). An easy computation shows that

$$\det(v_1, v_2, v_3) = 2y^{k+m-1}(c(k + m - 1) - m) > 0,$$

since $c > m/(k + m - 1)$.

We consider now the second double point labeled by b, which is the image of a pair (x, y) and (x, u), with $x + \mu_1 y^{k-1} = 0$, $u = \lambda_1 y$ and $y > 0$. In this case, we look again at the relative position of the branches. We see that the sign in the Gauss word coincides with the sign of the determinant $\det(v_1, v_2, v_3)$, where now $v_1 = -\frac{\partial f_c}{\partial x}(x, y)$, $v_2 = -\frac{\partial f_c}{\partial x}(x, u)$ and $v_3 = \frac{\partial f_c}{\partial y}(x, y)$ (see figure 9). It follows from lemma 6.8 that

$$\det(v_1, v_2, v_3) = -y^{k+m-1}(\lambda_1 - 1)\lambda_1(\lambda_1^{k+m-1} - 1)g'_{k-1,m-1}(\lambda_1) < 0,$$

since $0 < \lambda_1 < 1$ and $g'_{k-1,m-1}(\lambda_1) > 0$.

Finally, the third double point labeled by c is the image of a pair (x, y) and (x, u), with $x + \mu_2 y^{k-1} = 0$, $u = \lambda_2 y$ and $y < 0$. Again, the sign in the Gauss word is determined by the relative position of the branches. In this case, it is equal to the sign of the determinant $\det(v_1, v_2, v_3)$, where now $v_1 = \frac{\partial f_c}{\partial x}(x, y)$, $v_2 = \frac{\partial f_c}{\partial x}(x, u)$ and $v_3 = -\frac{\partial f_c}{\partial y}(x, y)$ (see figure 9). In this case, lemma 6.8 gives that

$$\det(v_1, v_2, v_3) = y^{k+m-1}(\lambda_2 - 1)\lambda_2(\lambda_2^{k+m-1} - 1)g'_{k-1,m-1}(\lambda_2) < 0,$$

since $\lambda_2 > 1$, $g'_{k-1,m-1}(\lambda_2) < 0$. Therefore, the complete Gauss word with signs is equal to $ab^{-1}bc^{-1}ca^{-1}$, which is equivalent to $aa^{-1}bb^{-1}cc^{-1}$ and this concludes the proof. □

6.2. **The general case.** We finish by showing that the topological classification of non degenerate ruled surfaces f_c can be extended to all the finitely determined germs or ruled surfaces in the $(x, xy, 0)$ orbit.

We will assume that f is a finitely determined germ of ruled surface in the $(x, xy, 0)$ orbit written in the form $f = f_c + \tilde{f}$, where $f_c(x, y) = (x, xy + y^k, xy^m + cy^{k+m-1})$ and $\tilde{f}$ has only terms of higher order with respect to the weighted grading determined by f_c.

If f_c is finitely determined itself, it follows that f is topologically equivalent to f_c by a result of Damon [2]. Even if f_c is not finitely determined but it is non degenerate, then the Gauss word of f is determined by the initial part f_c and then they are also topologically equivalent.

Proposition 6.10. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a finitely determined germ of ruled surface defined by* $f = f_c + \tilde{f}$ *such that* f_c *is non-degenerate. Then the Gauss word of* f *is equal to the Gauss word of* f_c.

Proof. We assume that $f_c(x, y) = (x, xy + y^k, xy^m + cy^{k+m-1})$ and $\tilde{f}$ has only higher order terms, that is, $\tilde{f}(x, y) = (0, g_1(y), xg_3(y) + g_2(y))$, where $g_1 \in m_1^{k+1}$, $g_2 \in m_1^{m+1}$ and $g_3 \in m_1^{k+m}$. The equation for the double point curve $D^2(f)$ is given by

$$\begin{cases} x + F_{k-1}(y, u) + \frac{g_1(y) - g_1(u)}{y - u} = 0, \\ F_{k-1}(y, u)F_{m-1}(y, u) - cF_{k+m-2}(y, u) + Q(y, u) = 0, \end{cases}$$

where $Q \in m_2^{k+m-1}$. We denote $P_0(y, u) = F_{k-1}(y, u)F_{m-1}(y, u) - cF_{k+m-2}(y, u)$ and $P(y, u) = P_0(y, u) + Q(y, u)$. Since f_c is non-degenerate, the initial part P_0 admits the following factorization:

$$P_0(y, u) = (u - \lambda_1 y) \dots (u - \lambda_r y)H_0(y, u),$$

for some $\lambda_1 < \cdots < \lambda_r$ and $H_0^{-1}(0) = \{(0,0)\}$. By using a result of [7], we deduce that the plane curve $P(y,u) = 0$ has the same number of real branches as its initial part $P_0(y,u) = 0$. In particular, P will also factorize as

$$P(y,u) = (u - \lambda_1 y + K_1(y,u)) \dots (u - \lambda_r y + K_r(y,u))H(y,u),$$

where $K_i \in m_2^2$ and $H^{-1}(0) = \{(0,0)\}$.

We use now the implicit function theorem, so that each branch $u - \lambda_i y + K_i(y,u) = 0$ can be parameterized as $u = \lambda_i y + k_i(y)$ for some $k_i \in m_1^2$. By substituting in the first equation of $D^2(f)$ we get the equation of $D(f)$:

$$(x + \mu_1 y^{k-1} + R_1(y)) \dots (x + \mu_r y^{k-1} + R_r(y)) = 0,$$

for some functions $R_i \in m_1^k$. Again, the hypothesis that f_c is non-degenerate implies that $\mu_i \neq \mu_j$ if $i \neq j$. If $\mu_i < \mu_j$, for instance, we will have that $\mu_i y^{k-1} + R_i(y) < \mu_j y^{k-1} + R_j(y)$ for any $y \in (0, \epsilon)$. Thus, we have shown that the relative position of the branches of $D(f)$ is determined by $\mu_1, \dots, \mu_r$. In particular, the Gauss word of f is equal, up to the signs, to the Gauss word of f_c.

Finally, we compute the signs in the Gauss word. Consider a double point given by the image of a pair (x,y) and (x,u), with $(x,y,u) \in D^2(f)$. The sign in the Gauss word is defined by the sign of the determinant of three vectors $\det(v_1, v_2, v_3)$, where $v_1 = \delta_1 \frac{\partial f}{\partial x}(x,y)$, $v_2 = \delta_2 \frac{\partial f}{\partial x}(x,u)$ and $v_3 = \delta_3 \frac{\partial f}{\partial y}(x,y)$, $\delta_i = \pm 1$. Moreover, the three signs δ_i depend only on the relative position of the branches (in fact, they only depend on the signs of y and u).

We assume that $u = \lambda_i y + k_i(y)$, for some $i = 1, \dots, r$. Then, lemma 6.8 gives

$$\det(v_1, v_2, v_3) = -\delta_1 \delta_2 \delta_3 y^{k+m-1} (\lambda_i - 1) \lambda_i (\lambda_i^{k+m-1} - 1) g'_{k-1,m-1}(\lambda_i) + \dots$$

Since f_c is non-degenerate, $\lambda_i \neq 0, 1, -1$ and $g'_{k-1,m-1}(\lambda_i) \neq 0$. Hence the initial part is not zero and the sign is equal to the sign in the Gauss word of f_c. □

Theorem 6.11. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface whose 2-jet belongs to the $(x, xy, 0)$ orbit. Then its topological type is determined by one of the eleven links of figure 2.*

Assume that f is defined by $f = f_c + \tilde{f}$. If f_c is non-degenerate, then the result follows from proposition 6.10 and theorem 6.9. Thus, we only need to solve the cases where c is one of the critical values. We also know from corollary 6.4 that the problem is solved if f has fold type. Hence we can assume that f_c has cusp type, i.e., k is odd. Finally, if we denote by r the number of branches of $D(f)$ we can also assume that $3 \leq r \leq 5$. We start with $r = 5$ which can occur only when m is even and either $c = c_0$ or $c = m/(k+m-1)$.

Proposition 6.12. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface of the form $f = f_c + \tilde{f}$. Assume that k is odd, m is even, $c = c_0$ and $r = 5$. Then the Gauss word of the link is $ab^{-1}cd^{-1}ea^{-1}bc^{-1}de^{-1}$.*

Proof. The five branches of the double point curve $D^2(f)$ are defined in the (y,u) plane by the following equations:

$$\begin{cases} u + y + p_1(y,u) = 0, \\ u - \lambda_1 y + p_2(y,u) = 0, \\ u - \lambda_1 y + p_3(y,u) = 0, \\ u - \lambda_2 y + p_4(y,u) = 0, \\ u - \lambda_2 y + p_5(y,u)) = 0, \end{cases}$$

with $\lambda_2 < -1 < \lambda_1 < 0$, $\lambda_2 = 1/\lambda_1$, and $p_i \in m_2^2$, $i = 1, \dots, 5$. By the implicit function theorem, the five branches can be parameterized by

$$\begin{cases} u = -y + q_1(y), \\ u = \lambda_1 y + q_2(y), \\ u = \lambda_1 y + q_3(y), \\ u = \lambda_2 y + q_4(y), \\ u = \lambda_2 y + q_5(y), \end{cases}$$

for some $q_i \in m_1^2$, $i = 1, \dots, 5$. Now we get the equations of the branches of $D(f)$ in the (x, y)-plane:

$$\begin{cases} x = -y^{k-1} + r_1(y), \\ x = -\mu_1 y^{k-1} + r_2(y), \\ x = -\mu_1 y^{k-1} + r_3(y), \\ x = -\mu_2 y^{k-1} + r_4(y), \\ x = -\mu_2 y^{k-1} + r_5(y), \end{cases}$$

being $\mu_1 = f_{k-1}(\lambda_1)$, $\mu_2 = f_{k-1}(\lambda_2)$ and $r_i \in m_1^k$, $i = 1, \dots, 5$.

Since k is odd and $0 < \mu_1 < 1 < \mu_2$, we have five parabolas on the halfplane $x \leq 0$ like in figure 8. Moreover, we also have that if a double point is the image of a pair (x, y) and (x, u) then y and u have opposite signs. This implies that the Gauss word must be, up to the signs, of the form $abcde - - - - -$.

By looking at the relative position of the branches we see that the Gauss word, up to the signs, has the form $abcde**c\#\#$, where $* = a, b$ and $\# = d, e$. There is a unique solution of the problem which is realizable on the sphere, namely, $ab^{-1}cd^{-1}ea^{-1}bc^{-1}de^{-1}$. □

Proposition 6.13. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface of the form $f = f_c + \tilde{f}$. Assume that k is odd, m is even, $c = m/(k + m - 1)$ and $r = 5$. Then the Gauss word of the link is $ab^{-1}cd^{-1}ea^{-1}bc^{-1}de^{-1}$.*

Proof. We follow an argument similar to the case $c = c_0$. to In this case, the branches of $D^2(f)$ in the (y, u) plane are

$$\begin{cases} u + y + p_1(y, u) = 0, \\ u + y + p_2(y, u) = 0, \\ u + y + p_3(y, u) = 0, \\ u - \lambda_1 y + p_4(y, u) = 0, \\ u - \lambda_2 y + p_5(y, u)) = 0, \end{cases}$$

with $\lambda_2 < -1 < \lambda_1 < 0$, $\lambda_2 = 1/\lambda_1$, and $p_i \in m_2^2$, $i = 1, \dots, 5$. The equations of the branches of $D(f)$ in the (x, y)-plane are:

$$\begin{cases} x = -y^{k-1} + r_1(y), \\ x = -y^{k-1} + r_2(y), \\ x = -y^{k-1} + r_3(y), \\ x = -\mu_1 y^{k-1} + r_4(y), \\ x = -\mu_2 y^{k-1} + r_5(y), \end{cases}$$

being $\mu_1 = f_{k-1}(\lambda_1)$, $\mu_2 = f_{k-1}(\lambda_2)$ and $r_i \in m_1^k$, $i = 1, \dots, 5$.

Again we have five parabolas on the halfplane $x \leq 0$ giving a Gauss word of the form $abcde - - - - -$, up to the signs. Since $0 < \mu_1 < 1 < \mu_2$, the relative position of the parabolas imply, in this case, that the Gauss word will be of the form $abcdea * * * e$, with $* = b, c, d$. But again the only Gauss word of this form which is realizable on the sphere is $ab^{-1}cd^{-1}ea^{-1}bc^{-1}de^{-1}$. ☐

The next case is when $D(f)$ has four branches, i.e., $r = 4$. This can happen only when m is odd and either $c = c_0$ or $c = 1$.

Proposition 6.14. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a finitely determined germ of ruled surface of the form* $f = f_c + \tilde{f}$. *Assume that* k *and* m *are odd,* $c = c_0$ *and* $r = 4$. *Then the Gauss word of the link is* $ab^{-1}c^{-1}dba^{-1}d^{-1}c$.

Proof. The proof is similar to that of the case $r = 5$ and $c = c_0$. In fact, we arrive to four parabolas in the (x, y)-plane defined by equations

$$\begin{cases} x = -\mu_1 y^{k-1} + r_1(y), \\ x = -\mu_1 y^{k-1} + r_2(y), \\ x = -\mu_2 y^{k-1} + r_3(y), \\ x = -\mu_2 y^{k-1} + r_4(y), \end{cases}$$

where $0 < \mu_1 < 1 < \mu_2$ and $r_i \in m_1^k$, $i = 1, \ldots, 4$. This implies that the Gauss word must be, up to the sign, of the form $abcd - - - -$. Moreover, the analysis of the relative position of the branches gives a Gauss word of the form $abcd * * \# \#$, where $* = a, b$ and $\# = c, d$. The only Gauss word of this type which is realizable on the sphere is equal to $ab^{-1}c^{-1}dba^{-1}d^{-1}c$. ☐

In order to analyze the case $c = 1$ we need the following lemma, which we state without proof.

Lemma 6.15. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a finitely determined germ of ruled surface of the form* $f = f_c + \tilde{f}$, *with* $c = 1$. *Let us consider the branch of the double point curve parametrized by* $x = -y^{k-1} + r(y)$ *and* $u = q(y)$, *with* $r \in m_1^k$ *and* $q \in m_1^2$. *If* $q(y) = a_j y^j + \ldots$ *with* $a_j \neq 0$ *and* $j \geq 2$, *then*

$$\det\left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial x}(x, u), \frac{\partial f}{\partial y}(x, y)\right) = -j a_j y^{k+m+j-2} + \ldots$$

Proposition 6.16. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a finitely determined germ of ruled surface of the form* $f = f_c + \tilde{f}$. *Assume that* k *and* m *are odd,* $c = 1$ *and* $r = 4$. *Then the Gauss word of the link is either* $ab^{-1}c^{-1}dba^{-1}d^{-1}c$ *or* $ab^{-1}c^{-1}cda^{-1}bd^{-1}$.

Proof. The four branches in the (y, u)-plane are defined by

$$\begin{cases} u + y + p_1(y, u) = 0, \\ u + y + p_2(u, y) = 0, \\ u + p_3(y, u) = 0, \\ y + p_3(u, y) = 0, \end{cases}$$

where $p_1, p_2 \in m_2^2$. Now, we parameterize them by

$$\begin{cases} u = -y + q_1(y), \\ u = -y + q_2(y), \\ u = q_3(y), \\ y = q_3(u), \end{cases}$$

for some $q_i \in m_1^2$, $i = 1, 2, 3$. Finally, we arrive to the equations of the four branches in the (x, y)-plane:

$$\begin{cases} P_1: \ x = -y^{k-1} + r_1(y), \\ P_2: \ x = -y^{k-1} + r_2(y), \\ P_3: \ x = -y^{k-1} + r_3(y), \\ Q: \ \begin{cases} x = -u^{k-1} + r_3(u), \\ y = q_3(u), \end{cases} \end{cases}$$

with $r_i \in m_1^k$, $i = 1, 2, 3$. The four branches are three parabolas P_1, P_2, P_3 and one cusp Q, all of them located on the halfplane $x \leq 0$ (see figure 10).

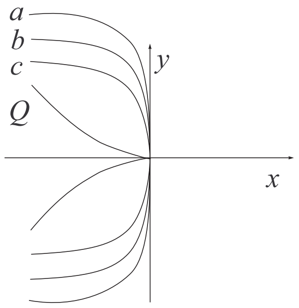

FIGURE 10. Double point curve with $c = 1$ and $r = 4$.

We compute first the Gauss word without signs. With independence of the relative position of P_1, P_2, P_3, we can state that the Gauss word is, up to the sign, of the form $abc - - - - -$. In fact, the three first letters label the positive halves of P_1, P_2, P_3. Since P_1 and P_2 have the corresponding double point in the halfplane $y < 0$, their letters will appear again in positions 6, 7 or 8. Analogously, P_3 has the corresponding double point in Q and its letter will appear again in positions 4 or 5.

Now, the realizability conditions for Gauss words on the sphere implies that there are just three possibilities for the fourth letter: $abca - - - -$, $abcc - - - -$ or $abcd - - - -$. Since the fifth letter labels necessarily the second half of the cusp Q, we get $abcad - - -$, $abccd - - -$ or $abcdb - - -$ respectively. We apply again the realizability conditions, which give only four possibilities: *abcadcbd*, *abccdabd*, *abcdbadc* or *abcdbcda*. Finally, note that *abcadcbd* is equivalent to *abcdbadc* and *abcdbcda* is equivalent to *abccdabd*.

In the second part, we compute the signs in the Gauss word. For *abcdbadc* there is only one choice of the signs which is realizable, namely, $ab^{-1}c^{-1}dba^{-1}d^{-1}c$. Unfortunately, for *abccdabd*, there are two possible choices, $ab^{-1}cc^{-1}da^{-1}bd^{-1}$ and $ab^{-1}c^{-1}cda^{-1}bd^{-1}$. We see that only the second one is possible.

We will compute the sign of the first c and the last d in the Gauss word. Both letters are on the parabola P_3 and we use lemma 6.15. Assume that $q_3(y) = a_j y^j + \dots$ with $a_j \neq 0$ and $j \geq 2$. We distinguish four cases:

(1) j odd and $a_j > 0$. Let us consider the first c in the Gauss word. We have one point $(x, y) \in P_3$ with $y > 0$ and its partner $(x, u) \in Q$, also with $u > 0$. The sign is determined by the sign of the determinant

$$\det\left(-\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = -ja_j y^{k+m+j-2} + \dots < 0$$

Analogously, we consider now the last d. Now we have $(x, y) \in P_3$ and $(x, u) \in Q$ with $y, u < 0$. The sign is determined by

$$\det\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), -\frac{\partial f}{\partial y}(x,y)\right) = ja_j y^{k+m+j-2} + \dots < 0$$

(2) j odd and $a_j < 0$. This case is not possible. On one hand, the cusp Q has the two halves on the two halfplanes $y > 0$ and $y < 0$. On the other hand, the points (x, y) and (x, u) should be located on different halfplanes. But this is not compatible with a Gauss word of the form *abccdabd*.

(3) j even and $a_j > 0$. The cusp Q is contained in the positive halfplane. For the first c we have $(x, y) \in P_3$ with $y > 0$ and $(x, u) \in Q$, also with $u > 0$. The sign is given by

$$\det\left(-\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = -ja_j y^{k+m+j-2} + \dots < 0$$

For the last d we have $(x, y) \in P_3$ with $y < 0$ and $(x, u) \in Q$ with $u > 0$. This gives

$$\det\left(\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), -\frac{\partial f}{\partial y}(x,y)\right) = -ja_j y^{k+m+j-2} + \dots < 0$$

(4) j even and $a_j < 0$. The cusp Q is contained in the negative halfplane. For the first c we have $(x, y) \in P_3$ with $y > 0$ and $(x, u) \in Q$, with $u < 0$. Then,

$$\det\left(-\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = ja_j y^{k+m+j-2} + \dots < 0$$

For the last d we take $(x, y) \in P_3$ and $(x, u) \in Q$ with $y, u < 0$. This implies

$$\det\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), -\frac{\partial f}{\partial y}(x,y)\right) = ja_j y^{k+m+j-2} + \dots < 0$$

We see that in all the cases, both the first c and the last d have sign -1. This concludes the proof. □

To finish our classification, we only need to analyze the cases where the double point curve has three branches, i.e., $r = 3$. These are possible only if m is even and either $c = c_0$, $c = m/(k + m - 1)$, $c = 1$ or $c = km/(k + m - 1)$.

Proposition 6.17. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface of the form $f = f_c + \tilde{f}$. Assume that k is odd, m is even, $c = c_0$ and $r = 3$. Then the Gauss word of the link is $ab^{-1}ca^{-1}bc^{-1}$.*

Proof. The proof is similar to the case $c = c_0$ and $r = 5$. Here the three branches of $D^2(f)$ are:

$$\begin{cases} u + y + p_1(y,u) = 0, \\ u - \lambda_1 y + p_2(y,u) = 0, \\ u - \lambda_2 y + p_3(y,u) = 0, \end{cases}$$

with $\lambda_2 < -1 < \lambda_1 < 0$, $\lambda_2 = 1/\lambda_1$, and $p_i \in m_2^2$, $i = 1,2,3$. The three branches of $D(f)$ in the (x,y)-plane are:

$$\begin{cases} x = -y^{k-1} + r_1(y), \\ x = -\mu_1 y^{k-1} + r_2(y), \\ x = -\mu_2 y^{k-1} + r_3(y), \end{cases}$$

being $\mu_1 = f_{k-1}(\lambda_1)$, $\mu_2 = f_{k-1}(\lambda_2)$ and $r_i \in m_1^k$, $i = 1,2,3$. Since k is odd and $0 < \mu_1 < 1 < \mu_2$, the Gauss word is equal, up to the signs, to $abcabc$. Finally, the signs of the Gauss word are $ab^{-1}ca^{-1}bc^{-1}$. □

Proposition 6.18. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface of the form $f = f_c + \tilde{f}$. Assume that k is odd, m is even, $c = m/(k + m - 1)$ and $r = 3$. Then the Gauss word of the link is $ab^{-1}ca^{-1}bc^{-1}$.*

Proof. The proof is the same as the case $c = c_0$. Note that the three branches of $D^2(f)$ are defined again by

$$\begin{cases} u + y + p_1(y,u) = 0, \\ u - \lambda_1 y + p_2(y,u) = 0, \\ u - \lambda_2 y + p_2(u,y)) = 0, \end{cases}$$

with $\lambda_2 < -1 < \lambda_1 < 0$, $\lambda_2 = 1/\lambda_1$, and $p_1, p_2 \in m_2^2$. □

Proposition 6.19. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined germ of ruled surface of the form $f = f_c + \tilde{f}$. Assume that k is odd, m is even, $c = 1$ and $r = 3$. Then the Gauss word of the link is either $ab^{-1}ca^{-1}bc^{-1}$ or $aa^{-1}bb^{-1}cc^{-1}$.*

Proof. The proof is similar to the case $c = 1$ and $r = 4$. The three branches in the (y,u)-plane are parameterized by

$$\begin{cases} u = -y + q_1(y), \\ u = q_2(y), \\ y = q_2(u), \end{cases}$$

for some $q_1, q_2 \in m_1^2$. The equations of the three branches in the (x,y)-plane are:

$$\begin{cases} P_1 : \ x = -y^{k-1} + r_1(y), \\ P_2 : \ x = -y^{k-1} + r_2(y), \\ Q : \ \begin{cases} x = -u^{k-1} + r_2(u), \\ y = q_2(u), \end{cases} \end{cases}$$

with $r_1, r_2 \in m_1^k$. We have two parabolas P_1, P_2 and one cusp Q on the halfplane $x \le 0$ (see figure 11).

The Gauss word is, up to the sign, of the form $ab----$. In fact, the two first letters label the positive halves of P_1, P_2. Since P_1 has the corresponding double point in the

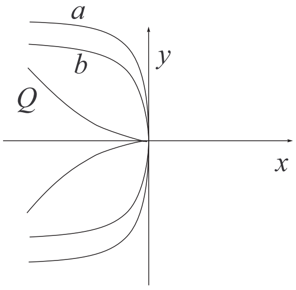

Figure 11. Double point curve with $c = 1$ and $r = 3$.

halfplane $y < 0$, the letter will appear again in positions 5 or 6. Analogously, P_2 has the corresponding double point in Q and its letter will appear again in positions 3 or 4.

By the realizability conditions, we have only two possibilities for the third letter: $abc---$ or $abb---$. The fourth letter labels the second half of the cusp Q, which gives $abca--$ or $abbc--$ respectively. The only realizable Gauss words are $abcabc$ or $abbcca$ respectively.

Note that for $abcabc$ there is only one choice of the signs which is realizable, namely, $ab^{-1}ca^{-1}bc^{-1}$. However, for $abbcca$, there are many possible choices. To compute the signs, we take the vectors v_1, v_2 and v_3 as in the proof of theorem 6.9 (see also figure 9).

In the first a, we have two points $(x, y), (x, u) \in P_1$, with $y > 0$ and $u < 0$. The sign is determined by the sign of the determinant

$$\det\left(-\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = (2+k)y^{k+m-1} + \cdots > 0$$

We compute now the signs of the first b and the last c in the Gauss word. Both letters are on the parabola P_2 and we use lemma 6.15. Assume that $q_2(y) = a_j y^j + \ldots$ with $a_j \neq 0$ and $j \geq 2$. We distinguish four cases:

(1) j odd and $a_j > 0$. For the first b we have $(x, y) \in P_2$ with $y > 0$ and $(x, u) \in Q$, also with $u > 0$. The sign is determined by the sign of the determinant

$$\det\left(-\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = -ja_j y^{k+m+j-2} + \cdots < 0$$

For the last c we have $(x, y) \in P_2$ and $(x, u) \in Q$ with $y, u < 0$. The sign is determined by

$$\det\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), -\frac{\partial f}{\partial y}(x,y)\right) = ja_j y^{k+m+j-2} + \cdots > 0$$

(2) j odd and $a_j < 0$. This case is not possible.

(3) j even and $a_j > 0$. For the first b we have $(x, y) \in P_2$ with $y > 0$ and $(x, u) \in Q$, also with $u > 0$. The sign is given by

$$\det\left(-\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = -ja_j y^{k+m+j-2} + \cdots < 0$$

For the last d we have $(x,y)\in P_2$ with $y<0$ and $(x,u)\in Q$ with $u>0$. This gives

$$\det\left(\frac{\partial f}{\partial x}(x,y),-\frac{\partial f}{\partial x}(x,u),-\frac{\partial f}{\partial y}(x,y)\right)=-ja_jy^{k+m+j-2}+\dots>0$$

(4) j even and $a_j<0$. The cusp Q is contained in the negative halfplane. For the first c we have $(x,y)\in P_2$ with $y>0$ and $(x,u)\in Q$, with $u<0$. Then,

$$\det\left(-\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial x}(x,u),\frac{\partial f}{\partial y}(x,y)\right)=ja_jy^{k+m+j-2}+\dots<0$$

For the last d we take $(x,y)\in P_2$ and $(x,u)\in Q$ with $y,u<0$. This implies

$$\det\left(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial x}(x,u),-\frac{\partial f}{\partial y}(x,y)\right)=ja_jy^{k+m+j-2}+\dots>0$$

We see that in all the cases, the first b has sign -1 but the last c has sign $+1$. Thus, the signed Gauss word is $ab^{-1}bc^{-1}ca^{-1}$. □

For the case $c=km/(k+m-1)$, we need the following lemma, which is similar to lemma 6.15. It will be necessary in order to compute the signs of the Gauss word.

Lemma 6.20. *Let* $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ *be a finitely determined germ of ruled surface of the form* $f=f_c+\tilde{f}$, *with* $c=km/(k+m-1)$. *Let us consider the branch of the double point curve parametrized by* $x=-ky^{k-1}+r(y)$ *and* $u=y+q(y)$, *with* $r\in m_1^k$ *and* $q\in m_1^2$. *If* $q(y)=a_jy^j+\dots$ *with* $a_j\neq 0$ *and* $j\geq 2$, *then*

$$\det\left(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial x}(x,u),\frac{\partial f}{\partial y}(x,y)\right)=\frac{1}{6}k(k-1)m(m-1)a_j^3y^{k+m+3j-4}+\dots$$

Proposition 6.21. *Let* $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ *be a finitely determined germ of ruled surface of the form* $f=f_c+\tilde{f}$. *Assume that* k *is odd,* m *is even,* $c=km/(k+m-1)$ *and* $r=3$. *Then the Gauss word of the link is* $aa^{-1}bb^{-1}cc^{-1}$.

Proof. The proof is similar to the case $c=1$ and $r=4$. The three branches in the (y,u)-plane are parameterized by

$$\begin{cases}u=-y+q_1(y),\\u=y+q_2(y),\\u=y+q_3(u),\end{cases}$$

for some $q_1,q_2,q_3\in m_1^2$. The equations of the three branches in the (x,y)-plane are:

$$\begin{cases}x=-y^{k-1}+r_1(y),\\x=-ky^{k-1}+r_2(y),\\x=-ky^{k-1}+r_3(y),\end{cases}$$

with $r_1,r_2,r_3\in m_1^k$. The relative position of the three parabolas implies that the Gauss word is equal, up to the signs, to *abbcca*. To compute the signs, we take the vectors v_1,v_2 and v_3 as in the proof of theorem 6.9 (see also figure 9).

In the first a, we have two points $(x,y),(x,u)$ on the first parabola, with $y>0$ and $u<0$. The sign is determined by the sign of the determinant

$$\det\left(-\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial x}(x,u),\frac{\partial f}{\partial y}(x,y)\right)=(2+k)my^{k+m-1}+\dots>0$$

We compute now the signs of the first b and the first c in the Gauss word. Both letters are on the parabolas P_2, P_3 and we can use lemma 6.20. Assume that $q_2(y) = a_j y^j + \dots$ with $a_j \neq 0$ and $j \geq 2$. Then, by symmetry, we have that $q_3(y) = -a_j y^j + \dots$ By interchanging P_2 and P_3 if necessary, we can assume that $a_j > 0$.

We distinguish two cases:

(1) j odd. For the first b we have (x, y) and (x, u) with $0 < u < y$. This implies that necessarily $(x, y) \in P_3$ and $(x, u) \in P_2$. The sign is determined by the sign of the determinant

$$\det\left(-\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = \frac{1}{6}k(k-1)m(m-1)(-a_j)^3 y^{k+m+3j-4} + \dots < 0$$

For the first c we have (x, y) and (x, u) with $u < y < 0$. Then, $(x, y) \in P_2$ and $(x, u) \in Q$ with $y, u < 0$. The sign is determined by

$$\det\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), -\frac{\partial f}{\partial y}(x,y)\right) = a_j y^{k+m+j-2} + \dots > 0$$

(2) j odd and $a_j < 0$. This case is not possible.

(3) j even and $a_j > 0$. For the first b we have $(x, y) \in P_2$ with $y > 0$ and $(x, u) \in Q$, also with $u > 0$. The sign is given by

$$\det\left(-\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = -a_j y^{k+m+j-2} + \dots < 0$$

For the last d we have $(x, y) \in P_2$ with $y < 0$ and $(x, u) \in Q$ with $u > 0$. This gives

$$\det\left(\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial x}(x,u), -\frac{\partial f}{\partial y}(x,y)\right) = -a_j y^{k+m+j-2} + \dots > 0$$

(4) j even and $a_j < 0$. The cusp Q is contained in the negative halfplane. For the first c we have $(x, y) \in P_2$ with $y > 0$ and $(x, u) \in Q$, with $u < 0$. Then,

$$\det\left(-\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,y)\right) = a_j y^{k+m+j-2} + \dots < 0$$

For the last d we take $(x, y) \in P_2$ and $(x, u) \in Q$ with $y, u < 0$. This implies

$$\det\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,u), -\frac{\partial f}{\partial y}(x,y)\right) = a_j y^{k+m+j-2} + \dots > 0$$

We see that in all the cases, the first b has sign -1 but the last c has sign $+1$. Thus, the signed Gauss word is $ab^{-1}bc^{-1}ca^{-1}$. □

We finish with the following example, which shows that the link with Gauss word $ab^{-1}c^{-1}cda^{-1}bd^{-1}$ is also realizable as the link of a finitely determined surface. Note that this is the only one from the list of eleven links of figure 2 which cannot appear in the non degenerate case, but which in fact appears in a degenerate ruled surface.

Example 6.22. Let us consider the ruled surface $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ given by

$$f(x,y) = (x, xy + y^3 + \frac{3}{10}y^4, xy^5 + y^7 - \frac{2}{10}xy^6 + \frac{1}{10}y^8).$$

The double point curve $D^2(f)$ can be computed easily in the (y,u)-plane by equation:

$$\begin{aligned}(u+y)(3u^7+3yu^6+6y^2u^5-5yu^5+6y^3u^4-5y^2u^4-50yu^4+6y^4u^3-20y^3u^3\\ -50y^2u^3+6y^5u^2-5y^4u^2-50y^3u^2+3y^6u-5y^5u-50y^4u+3y^7)=0.\end{aligned}$$

This gives only four real branches parametrized by:

$$\begin{cases} u=-y,\\ u=-y-\frac{1}{5}y^2+\dots,\\ u=\frac{3}{50}y^3+\dots,\\ y=\frac{3}{50}y^3+\dots,\end{cases}$$

and the corresponding branches in the (x,y)-plane for $D(f)$ are:

$$\begin{cases} P_1: \ x=-y^2,\\ P_2: \ x=-y^2-\frac{1}{5}y^3+\dots,\\ P_3: \ x=-y^2-\frac{3}{10}y^3+\dots,\\ Q: \ \begin{cases} x=-y^2-\frac{3}{10}y^3+\dots,\\ y=\frac{3}{50}y^3+\dots\end{cases}\end{cases}$$

The relative position of the branches is $P_1P_2P_3QQP_1P_2P_3$ and hence, the Gauss word is $ab^{-1}c^{-1}cda^{-1}bd^{-1}$. We present in figure 12 a picture of the link of f obtained with the computer program `SphereXSurface` by A. Montesinos-Amilibia [12].

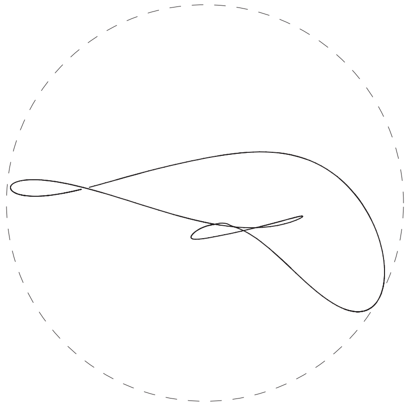

FIGURE 12. The link of f

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