Remarks on measurability of operator-valued functions.

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Abstract

The aim of this paper is to consider operator-valued functions that can be approximated in the strong and weak operator topology by countably valued functions. We relate these notions with the classical formulations of measurability and provide conditions for their coincidence. A number of examples and counterexamples are exhibited.

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1 Introduction and preliminaries

The concept of "measurable" function when dealing with functions with values in Banach spaces, or more generally in locally convex topological vector spaces, depends on the possible generalization from the scalar-valued notion that is needed in each case. The notions of strongly measurable, weakly measurable (sometimes called scalarly measurable), weak*-measurable (in the case of dual spaces) and others (see for instance [7, 9, 12]) appear in the literature. However each of them generalize the scalar-valued case in a different way. When dealing with operator-valued functions the situation becomes even more complicated (see [6, 9, 12, 13, 16]) due to the possibilities of considering not only the norm and weak topologies but also the

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strong-operator and weak-operator topologies. While the abstract concept of (Σ_1, Σ_2)-measurability, meaning that $f^{-1}(A) \in \Sigma_2$ for any $A \in \Sigma_1$ does not require the measurable space (Ω, Σ_1) to have an underlying measure μ, in general measurability is not studied in its own right but usually is tied to integrability in one or another sense, that is why negligible sets are considered in most definitions (see [10, 11]). However we would like to develop a theory where only the σ-algebra plays a role. With this in mind we recall that a function $f : \Omega \to \mathbb{K}$ is said to be measurable if $f^{-1}(G) \in \Sigma$ for any open set G. This, can be easily seen to coincide with $f^{-1}(B(\alpha, \varepsilon)) \in \Sigma$ for any $\alpha \in \mathbb{K}$ and $\varepsilon > 0$ where $B(\alpha, \varepsilon) = \{\alpha' \in \mathbb{K}; |\alpha - \alpha'| < \varepsilon\}$. Also this turns out to be equivalent for the function f to be a pointwise limit of simple (meaning finitely valued) functions. For functions with values in spaces of operators we would like to analyze the differences existing when considering any of these three considerations. The main interest is to find weaker assumptions to handle some problems on operator-valued functions that has appeared recently (see [1, 2, 3, 4]) and in particular to establish the basis to consider new problems concerning measurability with respect to the strong operator topology (see [5, 14]).

We would like to call the attention of the differences existing between these possible definitions, and try to find two approaches that cover most of the possible formulations. One point of view (used in the Bochner measurability and integrability) is to consider pointwise limits of countably valued functions and another possible approach (used when dealing with weakly or scalarly measurable functions) is to assume that pre-images of elements in a basis of the topology are measurable sets. According to these two approaches we shall say, for a topological vector space (Y, τ) with a basis $\beta \subseteq \tau$, that $f : \Omega \to Y$ is β-measurable whenever $f^{-1}(A) \in \Sigma$ for any $A \in \beta$ and we shall say that $f : \Omega \to Y$ is τ-approximable whenever f is a pointwise limit of countably valued functions in the τ-topology. Of course, the first examples of τ-measurable functions and τ-approximable functions are the countably valued functions $f = \sum_{k=1}^{\infty} y_k \chi_{A_k}$ where $y_k \in Y$ and $A_k \in \Sigma$ are pairwise disjoint (we keep the name of simple function for those which are finitely valued). Hence for $Y = \mathbb{K}$ we have that $|\cdot|$-approximable is equivalent to β-measurable where $\beta = \{B(\alpha, \epsilon); \alpha \in \mathbb{K}, \varepsilon > 0\}$. However functions taking values in a Banach space E are called (strongly) measurable whenever they are $\|\cdot\|$-approximable, while weakly measurable (or weak* measurable in the case of dual spaces) usually refers to the condition $w \to \langle f(w), x^* \rangle$ is measurable for any $x^* \in E^*$ (respect. $w \to \langle f(w), x \rangle$ is measurable for any $x \in X$), which corresponds to $\mathcal{N}_{weak}$-measurable (respect. $\mathcal{N}_{weak^*}$-measurable), for the standard basis in the weak topology (respect. weak*-topology).

For the space of operators $\mathcal{L}(E_1, E_2)$ endowed with different topologies the notation is even more confusing (see [12, Chapter 3]). A function $f : \Omega \to \mathcal{L}(E_1, E_2)$ is said to be **uniformly** measurable if it is $\|\cdot\|$-approximable, while f is said to be **strongly** measurable if f_x are $\|\cdot\|$-approximable in E_2 for any $x \in E_1$ where $f_x(w) = f(w)(x)$, and f is said to be **weakly** measurable if $\langle f_x, y^* \rangle$ are measurable functions for any $x \in E_1$ and $y^* \in E_2^*$, which would correspond to β-measurability for the standard basis in the WOT topology $\mathcal{L}(E_1, E_2)$. In [16, 13] the name of strong operator measurable and weak operator measurable is used for the last two notions and $(\Sigma, \mathcal{B}_0(\mathcal{L}(H, H))$-measurability is used for the $\mathcal{N}_{SOT}$-measurability in the case of a Hilbert space $E_1 = E_2$ and the standard basis of the strong operator topology.

To avoid misunderstandings we shall use the terms $\|\cdot\|$-, *weak*-, *weak**-, SOT- or WOT-approximable to mean that there exists a sequence of countably valued functions such that $s_n(w)$ converges to $f(w)$ for any $w \in \Omega$, in the norm, weak, weak*, SOT or WOT topologies and we use the terms W-measurable, W^*-measurable, SOT-measurable and WOT-measurable for the corresponding β-measurability with respect to the standard basis in each of the corresponding topologies.

The basic results connecting all these concepts are the Pettis's measurability theorem (see [7, Chapter 2, Thm 2]) and its extension for operator-valued functions due to Dunford (see [12, Thm 3.5.5]) which, in our notation, would read as follows:

(a) $f : \Omega \to E$ is $\|\cdot\|$-approximable if and only if f is W-measurable and $f(\Omega)$ is separable.

(b) $f : \Omega \to \mathcal{L}(E_1, E_2)$ satisfies that f_x is $\|\cdot\|$-approximable for any $x \in E_1$ if and only if f is WOT-measurable and $f_x(\Omega)$ is separable for any $x \in E_1$.

(c) $f : \Omega \to \mathcal{L}(E_1, E_2)$ is $\|\cdot\|$-approximable if and only if f is WOT-measurable and $f(\Omega)$ is separable in $\mathcal{L}(E_1, E_2)$.

Another basic result compares the strong operator measurability with the $(\Sigma, \mathcal{B}_0(\mathcal{L}(H, H))$-measurability (see [13, Theorem 2]), which in our terminology reads: If (Ω, Σ, μ) is a finite complete measure space, and $E_1 = E_2 = H$ is a separable Hilbert space over $\mathbb{C}$ then $f : \Omega \to \mathcal{L}(H, H)$ satisfies that f_x is $\|\cdot\|$-approximable for any $x \in H$ if and only if f is SOT-measurable.

In this paper we differentiate the measurability defined by means of preimages of elements belonging to the basis in the SOT and WOT topologies and the notion defined via approximation by countably valued functions in the mentioned topologies. We shall prove that separability and d-approximability coincide for metric spaces and observe that $\|\cdot\|$-approximable coincide with *weak*-approximable. We shall present a version of Pettis' measurability theorem for the class of SOT -approximable functions. We show that some well known operator-valued functions actually get into this scope and give several procedures to get examples satisfying our definitions.

Throughout the paper (Ω, Σ) stands for a measurable space (i.e. Σ is a σ-algebra over Ω) and E, E_1 and E_2 are always Banach spaces over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). We write B_E for the unit ball, E^* for the dual of E and $\mathcal{L}(E_1, E_2)$ for the space of bounded linear operators between them. As usual, given a sequence of operators $(T_n) \in \mathcal{L}(E_1, E_2)$ we write $T_n \to T$ (or $T_n \xrightarrow{\|\cdot\|} T$) for the convergence in norm, $T_n \xrightarrow{SOT} T$ for the convergence in the strong operator topology, i.e. $\|T_n x - Tx\| \to 0, \quad \forall x \in E_1$, and $T_n \xrightarrow{WOT} T$ for the convergence in weak operator topology, i.e. $\langle T_n x - Tx, y^* \rangle \to 0, \quad \forall x \in E_1$ and $y^* \in E_2^*$.

2 Measurability of operator-valued functions

Let us introduce some notation to be used in the sequel. Given a metric space (Y, d) we denote by $\mathcal{N}_d$ the basis of the topology given by $\mathcal{N}_d = \{B(x, \varepsilon) : x \in Y, \varepsilon > 0\}$ where $B(x, \varepsilon) = \{y \in Y ; d(x, y) < \varepsilon\}$. In the case of normed spaces we write $\mathcal{N}_{\|\cdot\|}$. For a Banach space E, besides the norm topology, we shall consider the weak topology and write $\mathcal{N}_{weak}$ the standard basis consisting in neighborhoods

$$N(x; \mathbf{x}^*, \varepsilon) = \{y \in E : \max_{1 \leq j \leq n} |\langle y - x, x_j^* \rangle| < \varepsilon\}$$

where $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_n^*) \in (E^*)^n$ and $n \in \mathbb{N}$. For a dual Banach space $E = F^*$ we shall also consider the *weak**-topology and write $\mathcal{N}_{weak^*}$ the standard basis consisting of neighborhoods

$$N(x; \mathbf{z}, \varepsilon) = \{y \in E : \max_{1 \leq j \leq n} |\langle y - x, z_j \rangle| < \varepsilon\}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_n) \in F^n$ and $n \in \mathbb{N}$.

In this paper we deal with Banach spaces of bounded linear operators $X = \mathcal{L}(E_1, E_2,)$. Here we have two more topologies to be considered, namely the strong operator topology (SOT) and the weak operator topology (WOT). We denote by $\mathcal{N}_{SOT}$ and $\mathcal{N}_{WOT}$ the corresponding basis given by

$$N(T;\mathbf{x},\varepsilon)=\{S\in X:\max_{1\leq j\leq n}\|(S-T)(x_j)\|<\varepsilon\}$$

and

$$N(T;\mathbf{x},\mathbf{y}^*,\varepsilon)=\{S\in X:\max_{1\leq j\leq n}|\langle (T-S)x_j,y_j^*\rangle|<\varepsilon\}$$

where $\mathbf{x}=(x_1,x_2,\cdots,x_n)\in(E_1)^n$, $\mathbf{y}^*=(y_1^*,y_2^*,\cdots,y_n^*)\in(E_2^*)^n$ and $n\in\mathbb{N}$. Finally in the case that E_2 is a dual space, taking into account the identification $(E\hat{\otimes}_\pi F)^*=\mathcal{L}(E,F^*)$ (see [7, Page 230]), we can also consider $\mathcal{N}_{WOT^*}$ the corresponding basis with respect to the $weak^*$-topology which is given by

$$N(T;\mathbf{x},\mathbf{y},\varepsilon)=\{S\in X:\max_{1\leq j\leq n}|\langle (S-T)x_j,y_j\rangle|<\varepsilon\}.$$

where $\mathbf{x}=(x_1,\cdots,x_n)\in(E_1)^n$, $\mathbf{y}=(y_1,\cdots,y_n)\in(E_2)^n$ and $n\in\mathbb{N}$.

Definition 2.1 *Let (Y,τ) be a topological vector space and $\beta\subseteq\tau$ be a basis of the topology. A function $f:\Omega\to Y$ is said β-measurable whenever $f^{-1}(A)\in\Sigma$ for any $A\in\beta$.*

In particular if X is a Banach space and $f:\Omega\to X$ we say that f is $\|\cdot\|$-measurable or W-measurable whenever $f^{-1}(A)\in\Sigma$ for any $A\in\mathcal{N}_{\|\cdot\|}$ or $A\in\mathcal{N}_{weak}$ respectively, for dual spaces $X=F^*$ we say that f is W^*-measurable whenever $f^{-1}(A)\in\Sigma$ for any $A\in\mathcal{N}_{weak^*}$, for $X=\mathcal{L}(E_1,E_2)$ we say that f is SOT- and WOT-measurable referring to $f^{-1}(A)\in\Sigma$ for any $A\in\mathcal{N}_{SOT}$ and $A\in\mathcal{N}_{WOT}$ respectively. We keep the notations $\tau_{\|\cdot\|}$-measurable, τ_{SOT}-measurable, τ_{WOT}-measurable for τ-measurable when τ is given for the family of open sets with respect to the corresponding topologies.

Let us first establish some equivalent formulations of the above definitions.

Proposition 2.2 *Let $X=\mathcal{L}(E_1,E_2)$, $f:\Omega\to X$ and denote $f_x(w)=f(w)(x)$ and $f_{x,y^*}(w)=\langle f(w)(x),y^*\rangle$ for $x\in E_1$ and $y^*\in E_2^*$. Then*

(i) f is $\|\cdot\|$-measurable $\Longleftrightarrow$ $\|f(\cdot)-T\|$ is measurable for any $T\in X$.

(ii) f is W-measurable $\Longleftrightarrow$ $\langle f(\cdot),T^*\rangle$ is measurable for any $T^*\in X^*$.

(iii) f is SOT-measurable $\Longleftrightarrow$ f_x is $\|\cdot\|$-measurable for any $x\in E_1$.

(iv) f is WOT-measurable $\Longleftrightarrow$ f_{x,y^*} is measurable for any $x\in E_1$ and $y^*\in E_2^*$.

Proof. (i) It follows from the definitions.

(ii) Assume that f is W-measurable and let $T^* \in X^*$. We have to check that $\{\omega \in \Omega : |\langle f(\omega), T^*\rangle - \alpha| < \varepsilon\} \in \Sigma$ for any $\varepsilon > 0$ and $\alpha \in \mathbb{R}$. We may assume that $T^* \neq 0$ and take any $T_0 \in X$ such that $\langle T_0, T^*\rangle = \lambda_0 \neq 0$. Define now $T := \frac{T_0}{\lambda_0}\alpha$ and observe that

$$\{\omega \in \Omega : |\langle f(\omega), T^*\rangle - \alpha| < \varepsilon\} = \{\omega \in \Omega : f(\omega) \in N(T; T^*, \varepsilon)\} \in \Sigma.$$

The converse implication follows using that $N(T; \mathbf{T}^*, \varepsilon) = \cap_{i=1}^n N(T; T_i^*, \varepsilon)$ where $\mathbf{T}^* = (T_1^*, \cdots, T_n^*)$ and $\langle f(\cdot), T_i^*\rangle$ is measurable for $i = 1, \cdots, n$.

(iii) Assume that f is SOT-measurable and let $x \in E_1$. We have to see that $\{\omega \in \Omega : \|f_x(\omega) - y\| < \varepsilon\} \in \Sigma$ for any $\varepsilon > 0$, $y \in E_2$. Select x^* such that $\langle x, x^*\rangle = \|x\|$ and define $T := \frac{x^*}{\|x\|} \otimes y$. With this choice

$$\{\omega \in \Omega : \|f_x(\omega) - y\| < \varepsilon\} = \{\omega \in \Omega : f(\omega) \in N(T; x, \varepsilon)\} \in \Sigma.$$

As above, the converse is immediate.

(iv) Assume that f is WOT-measurable and let $x \in E_1$ and $y^* \in E_2^*$. We need to check that $\{\omega \in \Omega : |f_{x,y*}(\omega)(x) - \alpha| < \varepsilon\} \in \Sigma$ for any $\varepsilon > 0$ and $\alpha \in \mathbb{R}$. We may assume that $x \neq 0$ and $y^* \neq 0$. Then take x^* such that $\langle x, x^*\rangle = \|x\|$, and y_0 with $\langle y_0, y^*\rangle = \lambda_0 \neq 0$. Define $y := \frac{y_0}{\lambda_0}\alpha$ and $T := \frac{x^*}{\|x\|} \otimes \frac{y_0}{\lambda_0}\alpha$. Again we use that

$$\{\omega \in \Omega : |\langle f(\omega)(x), y^*\rangle - \alpha| < \varepsilon\} = \{\omega \in \Omega : f(\omega) \in N(T; x, y^*, \varepsilon)\} \in \Sigma.$$

The converse as above is immediate. ■

There are some trivial implications among these concepts. Of course the topologies $\tau_{WOT} \subset \tau_{SOT} \subset \tau_{\|\cdot\|}$ and $\tau_{weak^*} \subset \tau_{weak} \subset \tau_{\|\cdot\|}$. Also, taking into account that for each $x \in E_1$ and $y^* \in E_2^*$ the map $T \to \langle Tx, y^*\rangle$ belongs to X^* one gets $\mathcal{N}_{WOT} \subset \mathcal{N}_{weak}$. These inclusions give the following remark.

Remark 2.1 *Any $\tau_{\|\cdot\|}$-measurable function is also W-measurable and SOT-measurable and also any W-measurable $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ is also WOT-measurable.*

The converse is false in general as the following example shows.

Example 2.1 *Let $\Omega = [0,1]$, $\Sigma = \mathcal{B}$ the Borel σ-algebra y let A be a non-Borel set. Take $E_1 = \ell^2([0,1])$, $E_2 = \mathbb{K}$ and $f : [0,1] \to \mathcal{L}(\ell^2([0,1]), \mathbb{K}) = \ell^2([0,1])$ given by*

$$t \to e_t \chi_A(t)$$

where $(e_t)_t$ stands for the canonical basis. Then f is W-measurable (and SOT-measurable) but not $\|\cdot\|$-measurable (in particular is not $\tau_{\|\cdot\|}$-measurable).

Proof. Due to Proposition 2.2 we first need to see that $\langle f(\cdot), T^*\rangle$ is measurable Borel for any $T^* \in X^*$. Use that $\ell^2([0,1]) = (\ell^2([0,1]))^*$ and for each $x^* \in \ell^2([0,1])$ there exists $(t_n) \in [0,1]$ such that $x^* = \sum_{n\in\mathbb{N}} \alpha_n e_{t_n}$ with $\sum_n |\alpha_n|^2 < \infty$. Therefore $t \to \langle f(t), x^*\rangle = \sum_{n:t_n\in A} \alpha_n \langle e_{t_n}, e_t\rangle$ is measurable.

However, the set $\{t \in [0,1] : \|f(t)\| < 1/2\} = [0,1] \setminus A$ is not measurable.

■

Recall that in the case $E_2 = \mathbb{K}$ the classical notions of *weak* and *weak**-measurability correspond to W and WOT-measurability respectively. Hence we have the following example at our disposal (see [7, Page 43]).

Example 2.2 *Let $\Omega = [0,1]$, $\Sigma = \mathcal{B}$ the Borel σ-algebra, $E_1 = \ell^1$, $E_2 = \mathbb{K}$ and $f : [0,1] \to \ell^\infty = \mathcal{L}(\ell^1, \mathbb{K})$ given by $f(t) = (r_n(t))_{n\in\mathbb{N}}$ where r_n stand for the Rademacher functions. Then f is WOT-measurable but not W-measurable.*

Let us now see the role of separability in the coincidence of different notions. We start by mentioning that for separable metric spaces we can use the basis of the topology to define measurability.

Proposition 2.3 *Let (Y, d) be a separable metric space. Then $f : \Omega \to Y$ is $\mathcal{N}_d$-measurable if and only if f is τ_d-measurable.*

In particular if $f : \Omega \to E$ is $\mathcal{N}_{\|\cdot\|}$-measurable and E is a separable Banach space then f is $\tau_{\|\cdot\|}$-measurable.

Furthermore, if $X = \mathcal{L}(E_1, E_2)$ where E_1 is separable then if f is SOT-measurable then it is also $\|\cdot\|$-measurable.

Proof. Of course if f is τ_d-measurable then f is $\mathcal{N}_d$-measurable. Assume that f is $\mathcal{N}_d$-measurable. Using that f is β-measurable if and only if f is $\sigma(\beta)$-measurable where $\sigma(\beta)$ stands for the smallest σ-algebra containing β we shall see that $\sigma(\mathcal{N}_d) = \sigma(\tau_d)$ in the case that Y is d-separable. It suffices to see that any open set G belongs to $\sigma(\mathcal{N}_d)$.

Let $\mathcal{A} = (y_n)_{n=1}^\infty$ be a dense set in Y and $G \in \tau_d$. For each $y \in G$, there exist $\varepsilon > 0$ and y_k such that $y \in B(y_k, \varepsilon)$. Selecting ε_k such that $y \in B(y_k, \varepsilon_k) \subseteq G$, we conclude that $G = \cup_k B(y_k, \varepsilon_k)$. This shows that $f^{-1}(G) = \cup_k f^{-1}(B(y_k, \varepsilon_k)) \in \Sigma$.

Assume now that $f : \Omega \to \mathcal{L}(E_1, E_2)$ and E_1 is separable and let (x_n) be a dense set in the unit ball of E_1. Due to Proposition 2.2 each map $\|f_{x_n}(\cdot) - Tx_n\|$ is measurable for each $n \in \mathbb{N}$ and $T \in X$. Since

$$\|f(w) - T\| = \sup_n \|f_{x_n}(w) - Tx_n\|$$

we obtain that f is $\|\cdot\|$-measurable. ■

We know introduce another related notion based on approximation by countably valued functions.

Definition 2.4 *Let (X, τ) be a Haussdorff topological space. A function $f : \Omega \to X$ is said to be countably valued if there exist $(x_n)_n \subset X$ and $(A_n)_n \subset \Sigma$ such that $f = \sum_{n=1}^{\infty} x_n \chi_{A_n}$. In the case that $x_k = 0$ for $k \geq n_0$ it will be called a simple function.*

A function $f : \Omega \to X$ is said to be τ-approximable if there exists a sequence of X-valued countably valued functions $s_n : \Omega \to X$ such that

$$\lim_n s_n(\omega) = f(\omega) \quad \forall \omega \in \Omega.$$

In the case that (X, τ) be a Haussdorff topological vector space and $f : \Omega \to X$, we denote $X_f = \overline{span(f(\Omega))}$ and f is called τ-approximable if $f : \Omega \to X_f$ is τ-approximable, that is to say that we allow the countably valued functions to take values in X_f and not only in $f(\Omega)$.

In particular, a function $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ is said to be *weak*-, *SOT*-, *WOT*- and *weak**-approximable (in the case $E_2 = E^*$) if there exists a sequence of operator-valued countably valued functions $s_n : \Omega \to X_f$ such that

$$\langle s_n(\omega), T^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega), T^* \rangle, \quad \forall T^* \in X^*, \forall \omega \in \Omega,$$

$$\lim_n \|s_n(\omega)(x) - f(\omega)(x)\| = 0 \quad \forall x \in E_1, \quad \forall \omega \in \Omega,$$

$$\langle s_n(\omega)(x), y^* \rangle \xrightarrow[n\to\infty]{} \langle f(\omega)(x), y^* \rangle, \quad \forall x \in E_1, \forall y^* \in E_2^*, \forall \omega \in \Omega$$

and

$$\langle s_n(\omega)(x), y \rangle \xrightarrow[n\to\infty]{} \langle f(\omega)(x), y \rangle, \quad \forall x \in E_1, y \in E \; \forall \omega \in \Omega$$

respectively.

We shall see that for $E_1 = \mathbb{K}$ the notions of $\|\cdot\|$-, *SOT*- and *WOT*-approximable coincide. However, for $E_2 = \mathbb{K}$ the notions *SOT*- and *WOT*-approximable (which correspond to *weak**-approximable) differ from $\|\cdot\|$-approximable as the following example shows.

Example 2.3 *Let* $\Omega = [0,1]$ *with the Borel* σ*-algebra,* $E_1 = \ell^1$ *and* $E_2 = \mathbb{K}$*. Let* $f(t) = (e^{int})_{n\in\mathbb{N}}$*. Then* f *is weak***-approximable but not* $\|\cdot\|$*-approximable.*

Proposition 2.5 *Let* (X,τ) *be a Haussdorff topological space. If* $f:\Omega\to X$ *is* τ*-approximable then* f *is* τ*-measurable and* $f(\Omega)$ *is* τ*-separable.*

Proof. Assume $f = \lim_n s_n$ pointwise and $s_n = \sum_{k=1}^{\infty} x_{n,k}\chi_{A_n,k}$ for some $x_{n,k}\in X$ and $A_{n,k}\in\tau$. If $G\in\tau$ then $s_n^{-1}(G)\in\Sigma$ and

$$\{w : f(w)\in G\} = \limsup\{w : s_n(w)\in G\}\in\Sigma.$$

Finally observe that $f(\Omega)\subseteq\overline{\mathcal{A}}$ where $\mathcal{A} = \{s_n(w) : n\in\mathbb{N}, w\in\Omega\}$ is a countable set in X. ■

Let us show that actually the converse holds true for metric spaces.

Lemma 2.6 *Let* (Y,d) *be a metric space and* $\Sigma = \mathcal{B}(Y)$ *the Borel* σ*-algebra. The following are equivalent:*

(i) Y *is* d*-separable.*

(ii) There exists a sequence $\phi_n : Y\to Y$ *of simple functions such that* $\lim_{n\to\infty} d(\phi_n(w), w) = 0$ *for any* $w\in Y$*.*

(iii) $id:\Omega\to Y$ *is* d*-approximable.*

Proof. (i) $\Longrightarrow$ (ii) Choose $\{y_n : n\in\mathbb{N}\}$ a countable dense subset of Y. For each $n\in\mathbb{N}$, and $1\le k\le n$, we define

$$B_{1,n} = \{y\in Y :\ d(y,y_1)\le \min_{1\le m\le n} d(y,y_m)\},$$

and for each $1<k\le n$, define $B_{k,n}$ as the set $y\in Y$ satisfying

$$d(y,y_k) < d(y,y_m)\ \ 1\le m<k, \quad d(y,y_k)\le d(y,y_m)\ \ k\le m\le n.$$

Clearly $B_{k,n}$ are pairwise disjoint Borel sets.

Now, let us define $\phi_n : Y\to Y$ by $\phi_n(y) = y_k$ for the first y_k attaining $\min_{1\le l\le n}(y,y_l)$, that is to say

$$\phi_n = \sum_{k=1}^{n} y_k\ \chi_{B_{k,n}}. \tag{1}$$

Hence the functions ϕ_n are $\mathcal{B}(Y)$-simple, and we have $\lim_{n\to\infty}\phi_n(y) = y$ $\forall y\in Y$.

(ii) $\Longrightarrow$ (iii) Obvious

(iii) $\Longrightarrow$ (i) It follows from Proposition 2.5. ■

Theorem 2.7 *Let (Y, d) be a metric space. Then $f : \Omega \to Y$ is d-approximable if and only if f is $\mathcal{N}_d$-measurable and $f(\Omega)$ is d-separable.*

Proof. The direct implication follows from Proposition 2.5. Assume now that f is $\mathcal{N}_d$-measurable and $f(\Omega)$ is d-separable. We now use Lemma 2.6 to construct $\phi_n = \sum_{k=1}^n y_k \ \chi_{B_{k,n}}$ with $\phi_n(y) \to y$ for any $y \in f(\Omega)$. Since $B_{k,n} \in \sigma(\mathcal{N}_d)$ and f is $\mathcal{N}_d$-measurable we obtain that $A_{n,k} = f^{-1}(B_{n,k}) \in \Sigma$ and we have that $s_n(w) \to f(w)$ for each $w \in \Omega$ where $s_n(w) = \phi_n(f(w)) = \sum_{k=1}^n y_k \chi_{A_{n,k}}$. Hence f is d-approximable. ■

For Banach spaces $(X, \|\cdot\|)$ we can apply Theorem 2.7 for $d(x, y) = \|x - y\|$, and since $\|\cdot\|$-approximable corresponds to the so-called (strongly)-measurable (see [7, 9]) or uniformly measurable in the case $X = \mathcal{L}(E_1, E_2)$ (see[12]) we recover Pettis (and Dunford generalization) measurability theorem. From Theorem 2.7 and Proposition 2.2 we obtain the following corollary.

Corollary 2.8 *A function $f : \Omega \to \mathcal{L}(E_1, E_2)$ satisfies that f_x is $\|\cdot\|$-approximable for any $x \in E_1$ if and only if f is SOT-measurable and $f_x(\Omega)$ is $\|\cdot\|$-separable for any $x \in E_1$.*

Proposition 2.9 *Let X be a Banach space and $f : \Omega \to X$. Then f is $\|\cdot\|$-approximable if and only if f is weak-approximable.*

Proof. It suffices to use that if $s_n(w) \to f(w)$ weakly then a convex combination of s_n, say $\tilde{s}_n(w)$ converges to $f(w)$ in norm (see [12, Page 36]). Of course if s_n are countably valued, then also $\tilde{s}_n$ are, and the result follows. ■

Proposition 2.10 *Let E be a separable Banach space, $X = E^*$ and let $f : \Omega \to X$ be a bounded function. Then f is weak*-approximable if and only if f is W^*-measurable.*

Proof. We may assume that $f(\Omega) \subseteq B_X$. Since E is separable then the unit ball of $(B_X, weak^*)$ is metrizable (see [9, Page 426]) and $weak^*$-compact (see [12, Page 37]). Hence B_X is a metrizable separable space with the $weak^*$-topology. Applying now Theorem 2.7 and Proposition 2.3 we have the desired result. ■

Let us mention that Proposition 2.10 does not hold without the assumption of separability.

Example 2.4 *Let* $\Omega = [0,1]$, $E = \ell^2[0,1]$ *and* $f : [0,1] \to \ell^2[0,1] = E^*$ *given by* $f(t) = e_t$ *the corresponding element in the canonical basis. It is* W^**-measurable, but not weak***-approximable.*

Proof. For each $x = (\alpha_t)_t \in \ell^2([0,1])$ then $t \to \alpha_t$ is measurable. Let us see that $f([0,1])$ is not *weak**-separable. Assume that $\mathcal{A}$ is *weak**-dense in $f([0,1])$. For any two points from $t, t' \in [0,1]$ and $\varepsilon > 0$ there exists $g \in \mathcal{A}$ such that $g \in N(e_t; e_t, e_{t'}, \varepsilon)$. In other words

$$|\langle g, e_t\rangle - \langle e_t, e_t\rangle| < \varepsilon \text{ and } |\langle g, e_{t'}\rangle - \langle e_t, e_{t'}\rangle| < \varepsilon.$$

so, we have

$$|g(t) - 1| < \varepsilon \text{ and } |g(t')| < \varepsilon.$$

This means that for each pair $(t, t') \in [0,1] \times [0,1]$ with $t \neq t'$, we obtain a function $g_{t,t'} \in \mathcal{A}$ which verifies $g_{t,t'}(t) \sim 1$ and $g_{t,t'}(t') \sim 0$. As there is a non-countable amount of pairs $(t, t') \in [0,1] \times [0,1]$ with $t \neq t'$, we get that $\mathcal{A}$ would contain a non-countable amount of different functions. Therefore $f([0,1])$ is not *weak**-separable and then f cannot be *weak**-approximable. ∎

Of course $\|\cdot\|$-approximable implies SOT-approximable and for dual spaces *weak*-approximable implies *weak**-approximable. However the converse is false as the following example shows.

Example 2.5 *Let* X *be a separable Banach space such that* X^* *is not separable. Let* $\Omega = B_{X^*}$ *with the weak***-topology and* Σ *the Borel* σ*-algebra. Then* $id : \Omega \to \mathcal{L}(X, \mathbb{K})$ *is weak***-approximable (*SOT*-approximable) but it is not weak-approximable (*$\|\cdot\|$*-approximable).*

Proof. Since Ω is *weak**-separable but it is not $\|\cdot\|$-separable then from Lemma 2.6 we obtain $id : \Omega \to X^*$ is *weak**-approximable. However it is not $\|\cdot\|$-approximable, which by Proposition 2.9 gives that it is not *weak*-approximable. ∎

Let us now investigate some conditions on E_1 and E_2 for the WOT and SOT measurability to coincide.

Proposition 2.11 *Let* E_2 *is separable. Then* $f : \Omega \to \mathcal{L}(E_1, E_2)$ *is* WOT*-measurable if and only if* f *is* SOT*-measurable.*

Proof. Assume that f is WOT-measurable, Select $(y_n) \subset E_2$ and $y_n^* \subset E_2^*$ such that (y_n) is dense in E_2 and $\|y_n\| = \langle y_n, y_n^* \rangle$ for each $n \in \mathbb{N}$. Therefore for each $x \in E_1$ and $T \in \mathcal{L}(E_1, E_2)$ we have

$$\|f(w)x - Tx\| = \sup_n |\langle f(w)x, y_n^* \rangle - \langle Tx, y_n^* \rangle|.$$

Therefore f is measurable due to the measurability of f_{x,y_n^*} for each n.

Assume that f is SOT-measurable. Applying Corollary 2.8 we conclude that f_x is $\|\cdot\|$-approximable and in particular f_x is *weak*-approximable. Hence f is WOT-measurable. ∎

Theorem 2.12 *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ and X_f is $\|\cdot\|$-separable. Then f is $\|\cdot\|$-approximable $\Longleftrightarrow$ f is $\tau_{\|\cdot\|}$-measurable $\Longleftrightarrow$ f is $\|\cdot\|$-measurable $\Longleftrightarrow$ f is SOT-measurable $\Longleftrightarrow$ f is WOT-measurable $\Longleftrightarrow$ f is W-measurable.*

Proof. Making use of Theorem 2.7, Proposition 2.3 and Remark 2.1, the only implications that remain to be shown are that if f is either SOT-measurable or WOT-measurable then it is $\|\cdot\|$-measurable. For such a purpose, let (T_n) be a dense set in X_f and select $x_n \in E_1$ and $y_n^* \in E_2^*$ such that $\|T_n\| < \|T_n(x_n)\| + 1/n$ and $\|T_n(x_n)\| = |\langle T_n x_n, y_n^* \rangle|$. It is elementary to see that for each $w \in \Omega$ and $T \in \mathcal{L}(E_1, E_2)$,

$$\|f(w) - T\| = \sup_n \|f(w)x_n - Tx_n\| = \sup_n |\langle (f(w) - T)x_n, y_n^* \rangle|.$$

Hence either SOT- or WOT-measurability implies $\|\cdot\|$-measurability. ∎

As a consequence of Proposition 2.5 and Theorem 2.12 we can recover the Pettis and Dunford measurability theorems at once.

Theorem 2.13 *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$. The following statements are equivalent:*

(i) f is $\|\cdot\|$-approximable.
(ii) f is $\tau_{\|\cdot\|}$-measurable and X_f is separable in X.
(iii) f is WOT-measurable and X_f is separable in X.

Let us now analyze the version of Theorem 2.13 for the SOT topology.

Theorem 2.14 *Let $f : \Omega \to X = \mathcal{L}(E_1, E_2)$ where E_1 is separable. The following statements are equivalent:*

(i) f is SOT-approximable.
(ii) f is τ_{SOT}-measurable and X_f is SOT-separable.
(iii) f is WOT-measurable and X_f is SOT-separable.

Proof. The implication (i) $\Longrightarrow$ (ii) was shown in Proposition 2.5 and the implication (ii) $\Longrightarrow$ (iii) is immediate.

(iii) $\Longrightarrow$ (i) Note that $f = \lim_m f\chi_{mB_X}$ where $X = \mathcal{L}(E_1, E_2)$, and due to the fact that pointwise limit of τ-approximable functions is also τ-approximable we can assume that $K_0 = \sup_{w\in\Omega} \|f(w)\| < \infty$. Assume f is WOT-measurable and X_f is SOT-separable. Since $X_f \cap K_0 B_X$ is SOT-separable we select first $\{T_n\}$ a countable set in $X_f \cap K_0 B_X$ such that $f(\Omega) \subset \overline{\{T_n\}}$ and (x_m) dense set in B_{E_1}. Consider the separable subspace of E_2 given by $\tilde{E}_2 = \overline{span\{T_n(x_k) : n, k \in \mathbb{N}\}}$. Due to Proposition 2.11 we have that f is SOT-measurable.

Denote $N_k = N(0; x_1, \cdots, x_k, \frac{1}{k})$ and define for each $n, k \in \mathbb{N}$ the set

$$A_{k,n} = \{w \in \Omega : f(w) - T_n \in N_k\}$$

which belongs to Σ since f is SOT-measurable. Now consider $B_{k,1} = A_{k,1}$ and

$$B_{k,n} = A_{k,n} \setminus (\cup_{1\leq j<n} B_{k,j})$$

which are pairwise disjoint sets in Σ and $\Omega = \cup_n A_{k,n} = \cup_n B_{k,n}$ for any $k \in \mathbb{N}$. We now define the countably valued function

$$f_k = \sum_n T_n \chi_{B_{k,n}}.$$

It remains to show that $f(w) = SOT - \lim_k f_k(w)$, that is to say that for each $w \in \Omega$, $x \in E_1$ and $\varepsilon > 0$ there exists $k_0 = k(x, \varepsilon)$ such that $f(w) - f_k(w) \in N(0; x, \varepsilon)$ for any $k \geq k_0$. Select $j \in \mathbb{N}$ such that $\|x - x_j\| < \frac{\varepsilon}{4K_0}$, $k \geq \max\{j, \frac{2}{\varepsilon}\}$ and $n \in \mathbb{N}$ such that $w \in B_{k,n}$. Therefore

$$\begin{aligned}
\|f(w)(x) - f_k(w)(x)\| &\leq \|f(w)(x_j) - f_k(w)(x_j)\| + \frac{\varepsilon(\|f(w)\| + \|f_k(w)\|)}{4K_0} \\
&\leq \|f(w)(x_j) - T_n(x_j)\| + \varepsilon/2 \\
&\leq \frac{1}{k} + \varepsilon/2 = \varepsilon.
\end{aligned}$$

This completes the proof. ■

Corollary 2.15 *Let E_1 be separable and $f : \Omega \to \mathcal{L}(E_1, E_2)$. Then f is SOT-approximable if and only if f is WOT-approximable.*

Proof. Assume that f is WOT-approximable. Since $\overline{co(f(\Omega))}^{WOT} = \overline{co(f(\Omega))}^{SOT}$ we can use Theorem 2.14 to obtain that f is SOT-approximable. The converse is obvious. ■

Let us finish this section with a couple of examples showing the importance of separability in the assumptions above.

Example 2.6 *Let $\Omega = [0,1]$ and $\Sigma = \mathcal{B}$ the Borel σ-algebra. Let $E_1 = E_2 = \ell^\infty$. Define $f(t) \in \mathcal{L}(E_1, E_2)$ given by*

$$f(t)((\alpha_n)) = (e^{int}\alpha_n), \quad (\alpha_n) \in \ell^\infty.$$

Then f is not WOT-approximable.

Proof. It suffices to see that $\{f(t) : t \in [0,1]\}$ is not WOT-separable. Assume that this is the case, then selecting $\mathbf{1} = (\alpha_\mathbf{n})$ with $\alpha_n = 1$ for all n we would have that $\{f(t)(\mathbf{1}) : t \in [0,1]\}$ is weakly separable, which is equivalent to norm-separability. However it is clear that $\{(e^{int})_n : t \in [0,1]\}$ is not norm separable. ■

Example 2.7 *Let us consider $f : [0,1] \to \mathcal{L}(\ell^2([0,1]), \ell^2([0,1]))$ given by*

$$f(t)(\sum_s a_s e_s) = \sum_{s \leq t} a_s e_s. \tag{2}$$

Then f is SOT-measurable but not SOT-approximable.

Proof. Let $\sum_s a_s e_s \in \ell^2([0,1])$. Hence $A = \{s : a_s \neq 0\}$ is countable. Hence

$$t \to f(t)(\sum_s a_s e_s) = \sum_{s \in A} a_s e_s \chi_{[s,1]}(t)$$

is countably valued, and then $\|\cdot\|$-approximable.

On the other hand $f([0,1])$ is not SOT-separable. Assume that there exists a set $\mathcal{A}$ of elements in $\overline{span(f([0,1])}$ such that $f([0,1]) \subseteq \overline{\mathcal{A}}^{SOT}$. We shall see that $\mathcal{A}$ is non countable. For each $t \in [0,1]$ there exists $g \in \mathcal{A}$ such that $f(t) - g \in N(0; e_t, 1/2)$. We can choose $g = \sum_j \alpha_j f(t_j)$ for a given sequence of scalars (α_j) and $t_j \in [0,1]$. Hence

$$\|f(t)(e_t) - g(e_t)\| = \|e_t - \sum_{j: t \leq t_j} \alpha_j e_{t_j}\| < 1/2.$$

This implies that $t = t_j$ and $\alpha_j = 1$ for some j. This shows that $\{f(t) : t \in [0,1]\}$ belongs to $\mathcal{A}$ and therefore $\mathcal{A}$ is non countable.

■

3 Examples of SOT-approximable functions

In this section we show that standard functions which appear in different areas are actually SOT-approximable functions but not $\|\cdot\|$-approximable. Also some procedures to construct them are provided.

Proposition 3.1 *Let $\Omega = [0,1]$, $\Sigma = \mathcal{B}$ the Borel σ-algebra, $E_1 = L^1([0,1])$ and $E_2 = C([0,1])$. For each $K : [0,1] \times [0,1] \to \mathbb{R}^+$ measurable and bounded we define $f_K : [0,1] \to \mathcal{L}(E_1, E_2)$ be given by*

$$f_K(t)(\phi)(s) = \int_0^s K(t,u)\phi(u)du, \quad \phi \in L^1([0,1]).$$

Then f_K is SOT-approximable.

Moreover, if $\tilde{K} : [0,1] \to L^\infty([0,1])$ where $\tilde{K}(t)(u) = K(t,u)$ is assumed to be $\|\cdot\|$-approximable then f_K is $\|\cdot\|$-approximable.

Proof. Note first that $f_K(t)$ is well defined.. Indeed, for each t and $\phi \in L^1([0,1])$ we have that $K(t,\cdot)\phi(\cdot) \in L^1([0,1])$ and then $\int_0^s K(t,u)\phi(u)du$ is continuous. Invoking Corollary 2.15 we shall show that f_K is WOT-approximable.

We know from the scalar-valued measurability of K that $K(s,t) = \lim_n K_n(s,t)$ where

$$K_n = \sum_{k=1}^\infty \alpha_{k,n}\chi_{A_{n,k}\times B_{n,k}}, \quad \alpha_{k,n} \in \mathbb{R}^+, A_{n,k}, B_{n,k} \in \Sigma$$

Define

$$f_n(t) = \sum_{k=1}^\infty \alpha_{k,n}\Phi_{n,k}\chi_{A_{n,k}}(t)$$

where $\Phi_{n,k}(\phi)(s) = \int_{[0,s]\cap B_{n,k}} \phi(u)du \in \mathcal{L}(E_1, E_2)$. Therefore for each $\phi \in L^1([0,1])$ and $\mu \in M([0,1]) = E_2^*$ we have

$$t \to \langle f(t)(\phi), \mu\rangle = \lim_n \langle f_n(t)(\phi), \mu\rangle$$

and f_n are countably-valued functions.

Assume that $\tilde{K} = \lim_n \tilde{K}_n$ in the $\|\cdot\|$ topology where $\tilde{K}_n = \sum_{k=1}^\infty \psi_{k,n}\chi_{A_{n,k}}$ for some $\psi_{k,n} \in L^\infty$ and $A_{n,k} \in \Sigma$. Denote

$$f_n(t)(\phi)(s) = \sum_{k=1}^\infty \left(\int_0^s \psi_{k,n}(u)\phi(u)du\right)\chi_{A_{n,k}}(t), \quad \phi \in L^1([0,1]).$$

The result follows using that

$$\|f_K(t)-f_n(t)\| \le \sup_{\|\phi\|_1=1} \int_0^1 |K(t,u)\phi(u)-K_n(t,u)\phi(u)|du \le \|\tilde{K}(t)-\tilde{K}_n(t)\|.$$

■

Corollary 3.2 *Let $t \to f(t) \in \mathcal{L}(L^1([0,1]), C([0,1]))$ given by*

$$f(t)(\phi)(s) = \int_0^{\min\{t,s\}} \phi(u)du$$

is SOT-approximable but not $\|\cdot\|$-approximable.

Proof. Apply Proposition 3.1 for $\tilde{K}(t) = \chi_{[0,t]}(u)$. Note that $f([0,1])$ is not $\|\cdot\|$-separable because for $t < t'$

$$\|f(t) - f(t')\| = \sup_{\phi\ge 0, \|\phi\|_1=1} \int_t^{t'} \phi(u)du \ge 1.$$

This gives that f is not $\|\cdot\|$-approximable. ■

Given $x^* \in E_1^*$ and $y \in E_2$ we denote $x^* \otimes y$ the operator in $\mathcal{L}(E_1, E_2)$ given by $x \to \langle x^*, x\rangle y$. Let us now construct functions with values in the space of operators between general Banach spaces using special sequences of elementary operators.

Proposition 3.3 *Let E_1 and E_2^* be Banach spaces. Let $\phi_n : [0,1] \to \mathbb{C}$ be a sequence of measurable functions such that*

$$M = \sup_n |\phi_n(t)| < \infty.$$

Let $(y_n^*) \in B_{E_2^*}$ and $(x_n^*) \in B_{E_1^*}$ be such that

$$\sum_n |\langle x_n^*, x\rangle\langle y_n^*, y\rangle| < \infty, \quad x \in E_1, \quad y \in E_2.$$

If $f : [0,1] \to \mathcal{L}(E_1, E_2^*)$ given by

$$f(t) = \sum_n \phi_n(t) x_n^* \otimes y_n^*. \tag{3}$$

then f is weak*-approximable.

Furthermore, (i) f is SOT-approximable whenever $\sum_n |\langle x_n^*, x\rangle| < \infty$ for all $x \in E_1$.

(ii) f is $\|\cdot\|$-approximable whenever $\sum_n \|x_n^*\|\|y_n^*\| < \infty$.

Proof. Using the assumption we have that $f(t)$ is well defined, because

$$\langle f(t)(x), y\rangle = \sum_n \phi_n(t)\langle x_n^*, x\rangle\langle y_n^*, y\rangle$$

defines an absolutely convergent series for each $t \in [0,1]$.

Consider $f_N(t) = \sum_{n=1}^N \phi_n(t) x_n^* \otimes y_n^*$ which is a $\|\cdot\|$-approximable function with values in $\mathcal{L}(E_1, E_2^*)$. Hence can be approximated by simple functions in the norm topology. Of course $f_N(t) \to f(t)$ in the WOT^* topology because for each $\varepsilon > 0$ and $x \in E_1$ and $y \in E_2$ there exists N such that $\sum_{n=N}^\infty |\langle x_n^*, x\rangle||\langle y_n^*, y\rangle| < \varepsilon/M$. Hence

$$|\langle f_N(t)(x) - f(t)(x), y\rangle| \leq M \sum_{n=N+1}^\infty |\langle x_n^*, x\rangle||\langle y_n^*, y\rangle| < \varepsilon.$$

Under the assumptions in (i) and (ii) the convergence $f_N(t) \to f(t)$ is in the SOT and $\|\cdot\|$ topologies due to the estimates

$$\|f_N(t)(x) - f(t)(x)\| \leq M \sum_{n=N+1}^\infty |\langle x_n^*, x\rangle| \|y_n^*\|$$

and

$$\|f_N(t) - f(t)\| \leq M \sum_{n=N+1}^\infty \|x_n^*\| \|y_n^*\|.$$

■

Corollary 3.4 *Let $1 < p < \infty$ and (e_n) the canonical basis of ℓ^p. Then $f(t) = \sum_n (e_n \otimes e_n) e^{int}$ is SOT-approximable but not $\|\cdot\|$-approximable.*

Proof. Using Proposition 3.3, together with the fact that $weak^*$-approximable becomes WOT-approximable since ℓ^p is reflexive we obtain that f is WOT-approximable, and using that ℓ^p is separable we also get, invoking Corollary 2.15, that f is SOT-approximable.

However the range is not separable, what follows from the estimate $\|f(t) - f(s)\| = \sup_n |e^{int} - e^{ins}| \geq 1$ whenever $t \neq s$. This gives that is not $\|\cdot\|$-approximable. ■

Other natural examples of SOT-approximable functions which are not $\|\cdot\|$-approximable appear often in different areas.

Example 3.1 *Let $1 \leq p \leq \infty$. Denote the translation operator on $\mathbb{R}$ by $\tau_t \in \mathcal{L}(L^p(\mathbb{R}), L^p(\mathbb{R}))$, that is*

$$\tau_t(\phi)(s) = \phi(s+t)$$

and the dilation operator on $\mathbb{R}^+$ by $D_\delta \in \mathcal{L}(L^p(\mathbb{R}), L^p(\mathbb{R}))$, that is

$$D_\delta(\phi)(s) = \delta^{1/p}\phi(\delta s).$$

Then the functions $f(t) = \tau_t$ and $g(\delta) = D_\delta$ are SOT-approximable but not $\|\cdot\|$-approximable for $1 \leq p < \infty$.

Proof. Using Theorem 2.14 and Proposition 2.5 it suffices to see that f and g are WOT-measurable, $f(\mathbb{R})$ and $g(\mathbb{R}^+)$ are SOT-separable but not $\|\cdot\|$-separable. Note that for $\phi \in L^p(\mathbb{R})$ and $\psi \in L^{p'}(\mathbb{R})$ we have

$$t \to \langle \tau_t(\phi), \psi \rangle = \int_{\mathbb{R}} \phi(t+s)\psi(s)ds$$

and

$$\delta \to \langle D_\delta(\phi), \psi \rangle = \delta^{-1/p} \int_{\mathbb{R}} \phi(\delta s)\psi(s)ds$$

are continuous and hence measurable. Indeed the first case follows using that $\tau_t(\phi) - \tau_{t_0}(\phi) = \tau_{t-t_0}(\tau_{t_0}(\phi)) - \tau_{t_0}(\phi)$ and $\|\tau_\varepsilon(\phi) - \phi\|_p \to 0$ as $\varepsilon \to 0$. For the second one, use that $D_\delta(\phi) - D_{\delta_0}(\phi) = D_{\delta/\delta_0}(D_{\delta_0}(\phi)) - D_{\delta_0}(\phi)$ and the fact $\|D_\delta(\phi) - \phi\|_p \to 0$ as $\delta \to 1$.

Consider $\{f(q) : q \in \mathbb{Q}\}$ and $\{g(q) : q \in \mathbb{Q}, q > 0\}$. The above arguments show that are SOT-dense in $f(\mathbb{R})$ and $g(\mathbb{R}^+)$.

Of course, for each $t > t'$, select $\phi(s) = \frac{1}{(b-a)^{1/p}}\chi_{[a,b]}$ for which $b-a < t-t'$. Hence

$$\|\tau_t - \tau_{t'}\| \geq \left(\int_R |\phi(s+t) - \phi(s+t')|^p ds \right)^{1/p} = 2.$$

Similarly for $\delta' < \delta < 2\delta'$, select $\phi = \chi_{[0,1]}$. Hence, since $(1-a)^{1/p} \geq 1 - a^{1/p}$ for $0 < a < 1$, we have

$$\begin{aligned} \|D_\delta - D_{\delta'}\| &\geq (\delta^{1/p} - \delta'^{1/p})\delta^{-1/p} + \delta'^{1/p}(\delta'^{-1} - \delta^{-1})^{1/p} \\ &\geq 2(1 - (\delta'/\delta)^{1/p}) \geq 2(1 - 2^{-1/p}). \end{aligned}$$

■

Example 3.2 *Let $\Omega = [0,1]$ and $\Sigma = \mathcal{B}$ the Borel σ-algebra. Let $E_1 = C([0,1] \times [0,1])$ and $E_2 = C([0,1])$ the spaces of continuous functions. Let $f : [0,1] \to \mathcal{L}(E_1, E_2)$ be given by*

$$f(t)(\phi) = \phi_t, \quad \phi_t(s) = \phi(t,s), \quad \phi \in C([0,1] \times [0,1]).$$

Then f is SOT-approximable but not $\|\cdot\|$-approximable.

Proof. As above it suffices to see that f is WOT-measurable, $f([0,1])$ is SOT-separable but not $\|\cdot\|$-separable.

Let $\phi \in C([0,1] \times [0,1])$ and $\mu \in M([0,1]) = E_2^*$ we have

$$t \to \langle f(t)(\phi), \mu \rangle = \int_0^1 \phi(t,s) d\mu(s)$$

which is continuous (and hence Borel measurable).

On the other hand $\{f(q) : q \in \mathbb{Q} \cap [0,1]\}$ is SOT-dense in $f([0,1])$. Indeed for any $\phi \in C([0,1] \times [0,1]) = C([0,1], C[0,1])$ and $0 \leq t \leq 1$ we have $\|\phi_t - \phi_{q_n}\| \to 0$ for any sequence $(q_n) \subset \mathbb{Q} \cap [0,1]$ converging to t.

Finally note that for $t \neq t'$ select $\psi \in C([0,1])$ such that $\|\psi\|_\infty = 1$, $\psi(0) = 0$ and $\psi(t - t') = 1$. For $\phi(t,s) = \psi(t - s)$ we have

$$\|f(t) - f(t')\| \geq \|\phi_t - \phi_{t'}\| \geq 1$$

what shows that $f([0,1])$ is not separable. ■

References

[1] N. Ahmed, *A note on Radon-Nikodym theorem for operator valued measures and its applications.* Commun. Korean Math. Soc. 28 (2013), no. 2, 285295.

[2] E. Berkson, *Marcinkiewicz multipliers of higher variation and summability of operator-valued Fourier series.* Studia Math. 222 (2014), no. 2, 123155.

[3] F. Bahrami; A. Bayati Eshkaftaki; S.M: Manjegani, *Operator-valued Bochner integrable functions and Jensen's inequality.* Georgian Math. J. 20 (2013), no. 4, 625640.

[4] R. Bruzual; M. Domnguez, *Representation of operator valued measurable indefinite functions.* Acta Cient. Venezolana 52 (2001), no. 3, 223225.

[5] O. Blasco and J. van Neerven, *Spaces of operator-valued functions measurable with respect to the strong operator topology*, Vector Measures, Integration and related topics. Oper. Theory Adv. Appl., 201, Birkhuser, 2010, 65–78.

[6] A. Badrikian; G.W. Johnson; I. Yoo, *The composition of operator-valued measurable functions is measurable.* Proc. Amer. Math. Soc. 123 (1995), no. 6, 18151820.

[7] J. Diestel and J.J.. Uhl, *Vector measures*, American Mathematical Society, Providence, (1977).

[8] N. Dunford, *On one parameter groups of linear transformations.* Annals of Math., (2) 39 (1938) 569–573.

[9] N. Dunford, J. Schwartz, *Linear Operators, part I: General Theory.* Interscience, New York, (1958).

[10] R.E. Edwards, *Functional Analysis. Theory and Applications.* Dover Pub. Inc, New York (1995).

[11] P. R. Halmos, *Measure Theory.* New York: Springer-Verlag, 1974.

[12] E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc, Providence, R. I., 1957.

[13] G W. Johnson, *The product of strong operator measurable functions is strong operator measurable,* Proc. Amer. Math. Soc. 117 (1993), 1097-1104.

[14] J. H.Fourie,*On operator-valued measurable functions.* Vector measures, integration and related topics, Oper. Theory Adv. Appl., 201, Birkhuser Verlag, Basel, 2010, 205214.

[15] B. J. Pettis, *On integration in vector spaces.* Amer. Math. Soc., 44 (1938) 277–304.

[16] M. Reed and B. Simon, *Methods of modern mathematical physics,* Vol. I, Functional analysis, revised enlarged ed., Academic Press, New York, 1980.

[17] W. Rudin, *Functional analysis.* McGraw-Hill, New York, 1973.

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