AVERAGING OPERATORS, BEREZIN TRANSFORMS AND ATOMIC DECOMPOSITION ON BERGMAN-HERZ SPACES

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ABSTRACT. We study the class of weight functions W in the unit disk for which the averaging operators $\mathcal{A}_r\phi(z) = \frac{1}{|D(z,r)|}\int_{D(z,r)} \phi(w)dA(w)$ are bounded on $L^p(W)$, where $D(z,r)$ is the disk centered at z and radius r in the hyperbolic metric. We also show the atomic decompositions on weighted Bergman-Herz spaces $A^p_q(W)$ for weights in the above class for which the Bergman projection is continuous on the Herz spaces $\mathcal{K}^p_q(W)$.

1. Introduction and preliminaries

The purpose of this paper is to study weights W in the unit disk $\mathbb{D}$ for which the averaging operators

$$\mathcal{A}_r\phi(z) = \frac{1}{|D(z,r)|}\int_{D(z,r)} \phi(w)dA(w) \tag{1}$$

are continuous in $L^p(W)$ of the disk, where dA denotes the normalized Lebesgue measure in $\mathbb{D}$ and $D(z,r)$ is the disk centered at z and radius r with respect to the hyperbolic metric in $\mathbb{D}$

$$D(z,r) = \{w \in \mathbb{D} : |\varphi_z(w)| \leq \tanh(r)\}, \quad 0 < r < \infty,$$

where, as usual, we write $\varphi_z(u) = \frac{z-u}{1-\bar{z}u}$ for the Möbius transformation. It is well known and easy to see that $\mathcal{A}_r$ is bounded on $L^p(dA_\alpha)$ for any $\alpha > -1$ and $1 \leq p \leq \infty$, where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. For further results about its boundedness for radial weights and on more general spaces the reader is referred to [1] and references therein.

We will also study Berezin-type operators of the form $b_{(\varepsilon_1,\varepsilon_2)}$

$$b_{(\varepsilon_1,\varepsilon_2)}(\phi)(z) = (1-|z|^2)^{\varepsilon_1}\int_{\mathbb{D}} \frac{(1-|w|^2)^{\varepsilon_2}}{|1-z\bar{w}|^{\varepsilon_1+\varepsilon_2+2}}\phi(w)dA(w).$$

The operators $\mathcal{A}_r$ and $b_{(\varepsilon_1,\varepsilon_2)}$ are comparable, in fact for $\phi \geq 0$

$$\mathcal{A}_r(\phi) \leq C_r b_{(\varepsilon_1,\varepsilon_2)}(\phi)$$

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and for certain weights they share some continuity properties. It is of special interest the case $b_{(0,\alpha)} = P_\alpha^*$ associated to the Forelli-Rudin Bergman projections P_α and $b_{(\alpha,\alpha)}$ that are the α-Berezin transforms used to study Toeplitz operators on the Bergman space.

The weighted inequalities for the averaging operators $\mathcal{A}_r$ will be based on two properties of the weights: on one hand a weak doubling property denote by D_r given by $W(D(z,2r)) \leq C_r W(D(z,r))$ where $D(z,r)$ is the disk centered at z in the hyperbolic geometry in $\mathbb{D}$, and on the other hand the property that we denote by b_p^r which is the Muckenhoupt class A_p restricted to hiperbolic disks of the same radius r. Weights in these classes (even the Lebesgue measure) are not in general doubling in the hyperbolic geometry making impossible the treatment of the averaging operators via the Hardy-Littlewood maximal function acting in a space of homogeneous.

In this paper we shall also study Bergman-Herz spaces and in particular we shall prove that atomic decompositions are possible in these spaces. The Bergman-Herz spaces, that we denote by $A_q^p(W)$ consist of all the holomorphic functions belonging to the Herz space on $\mathbb{D}$ defined by the norm

$$\|f\|_{\mathcal{K}_q^p(W)} = \left(\sum_{n=1}^{\infty} \|f\|_{L^p(A_n,W)}^q\right)^{1/q} < \infty,$$

with $A_n = \{z \in \mathbb{D}, 1 - 2^{-(n-1)} \leq |z| < 1 - 2^{-n}\}$. Atomic decomposition on weighted Bergman spaces have been extensively studed and constructed for Békollé weights by Békollé-Bonami [4], Luecking [10] and Constantin [5]. In this work we use the classes b_p^r to propose "weighted Kellog spaces" as the natural sequence space to base atomic decompositions for weighted Bergman-Herz spaces.

In Section 2 we introduce classes of weakly doubling weights and study the continuity of the Berezin-type transforms $b_{(\epsilon_1,\epsilon_2)}$ in $L^1(W)$. In Section 3 we obtain a full characterization of weights W for which there exists $r > 0$ such that $\mathcal{A}_r$ is continuous in $L^p(W)$. Weights in D_r where the doubling constant C_r grows like e^{Mr} will be called M-doubling. We will prove that for these weights $\mathcal{A}_r$ and $b_{(\varepsilon_1,\varepsilon_2)}$ have common continuity properties in $L^1(W)$. Then in Section 4 we study Bergman-Herz spaces with weights satisfying the property b_p^r including the sequence space where the sample sequences $(f(z_k))_k$ taken from an r-lattice $(z_k)_k$ lie for $f \in A_q^p(W)$. In Section 5 we prove that that atomic decompositions are posible for the elements the Herz space $\mathcal{A}_q^p(W)$, $1 \leq p, q < \infty$, provided the operator P^* is continuous in the Herz space $\mathcal{A}_q^p(W)$.

By a weight we will always mean a function $W : \mathbb{D} \to (0, \infty)$ which is locally integrable with respect to dA. We write $dW(z) = W(z)dA(z)$, $dW_\varepsilon(z) = (1 - |z|^2)^\varepsilon dW(z)$. For $1 \leq p \leq \infty$ we denote $\|f\|_{L^p(W)} = (\int_{\mathbb{D}} |f(z)|^p W(z) dA(z))^{1/p}$ and $W(E) = \int_E W dA$.

Throughout the paper $hol(\mathbb{D})$ is the space of all holomorphic functions in $\mathbb{D}$ and $A^p = L^p(\mathbb{D}) \cap hol(\mathbb{D}\ )$ the Bergman space for $1 \leq p \leq \infty$.

We write $A^p(W)$ for the space of all holomorphic functions in $L^p(W)$.

Since we want that the polynomials (in particular constant functions) belong to $A^p(W)$ we assume that $W \in L^1(dA)$ when dealing with spaces of holomorphic functions. We have the chain of inclusions $A^\infty(\mathbb{D}) \subset A^{p_2}(W) \subset A^{p_1}(W)$ for $p_1 \leq p_2 \leq \infty$.

Denote the Bergman-type projections, for $\alpha > -1$,

$$P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{\alpha+2}} dA_\alpha(w).$$

P_α is the orthogonal projection of $L^2(dA_\alpha)$ onto $L^2(dA_\alpha) \cap hol(\mathbb{D})$.

The case $\alpha = 0$ is the standard Bergman projection. We also denote

$$P_\alpha^* f(z) = \int_{\mathbb{D}} \frac{f(w)}{|1 - z\overline{w}|^{\alpha+2}} dA_\alpha(w).$$

We will write $P = P_0$ and $P^* = P_0^*$. It is well known that P_α and P_α^* are continuous on $L^p(\mathbb{D}, (1 - |z|^2)^\varepsilon dA(z))$ for $0 < \varepsilon + 1 < p(\alpha + 1)$, (see [8, Theorem 1.9]).

In fact, for $1 < p < \infty$, the complete characterization of weights for which P_α and P_α^* are bounded on $L^p(W_\alpha)$ was given by D. Bekollé (see [3]). By using the pseudo-distance

$$d(z, w) = \Big||z| - |w|\Big| + \left|\frac{z}{|z|} - \frac{w}{|w|}\right|$$

and writing $B(z, R) = \{w : d(w, z) < R\}$, it was shown that a P_α^* is bounded on $L^p(W_\alpha)$ is equivalent to the existence of a constant $C_p^\alpha(W) > 0$

$$\Big(\frac{1}{A_\alpha(B)} \int_B W dA_\alpha\Big)\Big(\frac{1}{A_\alpha(B)} \int_B W^{-1/(p-1)} dA_\alpha\Big)^{p-1} \leq C_p^\alpha(W) \tag{2}$$

for any $B = B(z, R)$ such that $\overline{B} \cap \partial\mathbb{D} \neq \emptyset$.

Let us finally recall the notion of *r-lattice* (see [12]) : for every $0 < r < \infty$ there exists a set that we will call an *r-lattice* $\mathcal{D}_r = \{z_i\}$ of points in $\mathbb{D}$ and an integer N (independent of r) such that

P1) $\{D(z_i, r/4)\}_i$ are disjoint,
P2) $\mathbb{D} = \cup_i D(z_i, r)$,
P3) Every point of $\mathbb{D}$ belongs to at most N elements of $\{D(z_i, 2r)\}_i$.

For this set $\mathcal{D}_r$ we can find subsets D_n such that

$$D(z_n, r/4) \subset D_n \subset D(z_n, r) \tag{3}$$

for all $n \geq 1$, and $\{D_n\}_{n\in\mathbb{N}}$ is a disjoint covering of $\mathbb{D}$.

We will write $A \sim B$ if there exists $C > 1$ such that $C^{-1}A \leq B \leq CA$.

2. Doubling weights and Berezin-type transforms

We will consider two doubling conditions for the measures defined by weights. To start off we mention a basic estimate for the area measure. We first recall that $D(z,r)=\Delta(C(z,r),R(z,r))$ with

$$C(z,r)=\frac{1-s^2}{1-s^2|z|^2}z,\quad R(z,r)=\frac{1-|z|^2}{1-s^2|z|^2}s$$

where we use the notation $s=\tanh r\in(0,1)$ and $\Delta(w,r')$ for the euclidean ball of center w and radius r'.

In particular,

$$|D(z,r)|=\frac{(1-|z|^2)^2s^2}{(1-|z|^2s^2)^2},\quad s=\tanh(r). \tag{4}$$

From (4) we can obtain uniform estimates in z for $\frac{|D(z,2r)|}{|D(z,r)|}$, in fact

$$\frac{|D(z,2r)|}{|D(z,r)|}=\left(\frac{\tanh(2r)}{\tanh(r)}\right)^2\left(\frac{1-|z|^2\tanh^2(r)}{1-|z|^2\tanh^2(2r)}\right)^2\le C\left(\frac{1-|z|\tanh(r)}{1-|z|\tanh(2r)}\right)^2.$$

Since $\frac{1-|z|\tanh(r)}{1-|z|\tanh(2r)}$ is increasing in $|z|$ we find that

$$\frac{|D(z,2r)|}{|D(z,r)|}\le C\left(\frac{1-\tanh(r)}{1-\tanh(2r)}\right)^2\le Ce^{4r}. \tag{5}$$

Definition 1. *Let $0<r<\infty$. We say that a weight $W\in D_r$ if there exists $C_r>0$ such that*

$$W(D(z,2r))\le C_rW(D(z,r)) \tag{6}$$

for all $z\in\mathbb{D}$.

Using (5) we have that condition $W\in D_r$ is equivalent to

$$\mathcal{A}_{2r}(W)(z)\sim\mathcal{A}_r(W)(z).$$

Observe that $W=1\in\cap_{r>0}D_r$ and that if $W\in\cap_{r>0}D_r$ then for each $0<r_1<r_2<\infty$ one has

$$W(D(z,r_1))\sim W(D(z,r_2)),z\in\mathbb{D}. \tag{7}$$

A special subclass of weights in $\cap_{r>0}D_r$ is given by those where $C_r=Ce^{Mr}$ for certain $M\ge 0$.

Definition 2. *Let $0<W(z)<\infty$ be locally integrable and $M\ge 0$. We say that W is M-doubling if there exists $C>0$ such that*

$$\frac{W(D(z,2r))}{W(D(z,r))}\le Ce^{Mr}, \tag{8}$$

for all $z\in\mathbb{D}$ and $r>0$.

Remark 3. *If W satisfies the M-doubling condition then there exists $\beta>0$ such that*

$$W(D((z,kr))\le k^\beta e^{Mkr}W(z,r)$$

for $k\ge 2$.

Indeed, assume $W(D(z,2r)) \leq Ce^{Mr}W(D(z,r))$ *and set* $N = [log_2 k]$. *Then for each* $r > 0$

$$\frac{W(D(z,kr))}{W(D(z,r))} \leq C^N \Pi_{j=1}^N (e^{\frac{Mkr}{2^j}}) \leq e^{Mkr}k^\beta,$$

with $\beta = log_2(C)$.

Proposition 4. *Let* $\alpha > -1$. *Then* dA_a *satisfies an* $(4+6|\alpha|)$*-doubling condition.*

Proof. Let $|z| < 1$ and $r > 0$ and set $s = \tanh r$. Due to the fact that we deal with radial weights we have that $A_\alpha(D(z,r)) = A_\alpha(D(|z|,r))$. Since

$$D(|z|,r) \cap \mathbb{R} = (\frac{|z|-s}{1-s|z|}, \frac{|z|+s}{1+s|z|})$$

then

$$D(|z|,r) \subset \{w : \max\{\frac{|z|-s}{1-s|z|}, 0\} \leq |w| < \frac{|z|+s}{1+s|z|}\}.$$

In particular

$$\frac{(1-|z|)(1-s)}{2} \leq 1-|w| \leq \min\{1, 2\frac{1-|z|}{1-s}\}, \quad w \in D(|z|,r). \tag{9}$$

By (9) we have for any $\alpha > -1$,

$$\left(\frac{2}{1-s}\right)^{-|\alpha|}(1-|z|)^\alpha \leq (1-|w|)^\alpha \leq \left(\frac{2}{1-s}\right)^{|\alpha|}(1-|z|)^\alpha, \tag{10}$$

Now observe that if $s' = \tanh(2r)$, we have that $s' = \frac{2s}{1+s^2}$. Hence, using that $\frac{(1-s)^2}{2} \leq 1-s' = \frac{(1-s)^2}{1+s^2} \leq (1-s)^2$ and $1-s = \frac{2}{e^{2r}+1}$ we conclude that

$$\begin{aligned}
A_\alpha(D(z,2r)) &\leq C\frac{(1-|z|)^\alpha}{(1-s')^{|\alpha|}}A(D(z,2r)) \leq Ce^{4r}\frac{(1-|z|)^\alpha}{(1-s')^{|\alpha|}}A(D(z,r)) \\
&\leq C\frac{e^{4r}}{(1-s)^{|\alpha|}(1-s')^{|\alpha|}}A_\alpha(D(z,r)) \leq C\frac{e^{4r}}{(1-s)^{3|\alpha|}}A_\alpha(D(z,r)) \\
&\leq Ce^{(4+6|\alpha|)r}A_\alpha(D(z,r)).
\end{aligned}$$

□

Recall that for $\alpha > -1$ one defines the α-Berezin transform of $\phi \in L^1(dA_\alpha)$ by the formula

$$B_\alpha(\phi)(z) = (1-|z|^2)^{2+\alpha}\int_{\mathbb{D}} \frac{\phi(w)}{|1-z\bar{w}|^{4+2\alpha}}dA_\alpha.$$

Let us consider the following definition (see [8]) which allows to consider P_α^* and B_α as special cases.

Definition 5. *Let $\varepsilon_i > -1$ for $i = 1, 2$. We shall define*

$$b_{(\varepsilon_1,\varepsilon_2)}(\phi)(z) = (1-|z|^2)^{\varepsilon_1} \int_{\mathbb{D}} \frac{(1-|w|^2)^{\varepsilon_2}}{|1-z\bar{w}|^{\varepsilon_1+\varepsilon_2+2}} \phi(w) dA(w)$$

for $\phi \in L^\infty(dA)$.

Remark 6. *Let $1 \leq p < \infty$ and $\delta \in \mathbb{R}$. Then $b_{(\varepsilon_1,\varepsilon_2)}$ is bounded on $L^p(dA_\delta)$ iff $-p\varepsilon_1 < \delta + 1 < p(\varepsilon_2 + 1)$ (see* [8, Thm 1.9]*).*

Lemma 7. *For each $R > 0$ there exist $C_R > 0$, such that for every $\phi \geq 0$ measurable,*

$$\mathcal{A}_r(\phi) \leq \frac{C_R}{r^2} b_{(\varepsilon_1,\varepsilon_2)}(\phi), \quad 0 < r \leq R. \tag{11}$$

Proof. Denote $s = \tanh r$. Clearly we have

$$(1-|z|^2)^2 s^2 \leq |D(z,r)| \leq \frac{(1-|z|^2)^2 s^2}{(1-(\tanh R)^2)^2}, 0 < r \leq R. \tag{12}$$

and also, using the well-known formulas for $w = \varphi_z(u)$

$$|1 - z\bar{w}| = \frac{1-|z|^2}{|1-z\bar{u}|} \tag{13}$$

and

$$(1-|w|^2) = \frac{(1-|u|^2)(1-|z|^2)}{|1-z\bar{u}|^2}. \tag{14}$$

one concludes that for $w \in D(z,r)$ one gets

$$1 - |z|^2 \leq |1 - \bar{w}z| \leq \frac{(1-|z|^2)}{1-s},$$

and one gets

$$(1-s^2)(1-|z|^2) \leq (1-|w|^2) \leq \frac{2(1-|z|^2)}{1-s}.$$

Hence, since $r \leq \frac{e^{2r}-1}{2} \leq s \leq e^{2r} - 1$ and $0 < s < \tanh R$ we obtain

$$\begin{aligned}
\mathcal{A}_r(\phi)(z) &\leq \frac{1}{r^2(1-|z|^2)^2} \int_{D(z,r)} \phi(w) dA(w) \\
&\leq C_R \frac{(1-|z|^2)^{\varepsilon_1}}{r^2} \int_{D(z,r)} \frac{(1-|w|^2)^{\varepsilon_2}}{|1-z\bar{w}|^{\varepsilon_1+\varepsilon_2+2}} \phi(w) dA(w) \\
&\leq \frac{C_R}{r^2} b_{(\varepsilon_1,\varepsilon_2)}(\phi)(z).
\end{aligned}$$

□

Theorem 8. *Let W be a weight satisfying the M-doubling condition. If $min\{\varepsilon_2, 2+\varepsilon_1\} > M/2$ then for each $r > 0$ there exists $K_r > 0$ such that*

$$b_{(\varepsilon_1,\varepsilon_2)}(W) \leq K_r \mathcal{A}_r(W).$$

Proof. Using (13) and (14) one easily concludes that for $w \in D(z,r), u = \varphi_z(w)$ and $r > 0$

$$\frac{(1-|z|^2)^{\varepsilon_1}(1-|w|^2)^{\varepsilon_2}}{|1-z\bar{w}|^{\varepsilon_1+\varepsilon_2+2}} = \frac{|1-z\bar{u}|^{\varepsilon_1-\varepsilon_2+2}(1-|u|^2)^{\varepsilon_2}}{(1-|z|^2)^2}. \tag{15}$$

In particular for $\varepsilon_1 - \varepsilon_2 + 2 \geq 0$ one has

$$\frac{(1-|z|^2)^{\varepsilon_1}(1-|w|^2)^{\varepsilon_2}}{|1-z\bar{w}|^{\varepsilon_1+\varepsilon_2+2}} \leq C\frac{(1-|u|)^{\varepsilon_2}}{(1-|z|^2)^2} \tag{16}$$

and for $\varepsilon_1 - \varepsilon_2 + 2 < 0$

$$\frac{(1-|z|^2)^{\varepsilon_1}(1-|w|^2)^{\varepsilon_2}}{|1-z\bar{w}|^{\varepsilon_1+\varepsilon_2+2}} \leq C\frac{(1-|u|)^{2+\varepsilon_1}}{(1-|z|^2)^2}, \tag{17}$$

where we have used the estimate $|1-z\bar{u}| \geq 1-|u|$. Take $\delta = \min\{\varepsilon_2, 2+\varepsilon_1\} > 0$ and decompose

$$\mathbb{D} = D(z,r) \cup \left(\cup_{k=1}^{\infty} D(z,(k+1)r) \setminus D(z,kr)\right).$$

Note that for $w \notin D(z,kr)$ one has that $|u| > \tanh(kr)$ and therefore $1-|u| < \frac{2}{e^{2kr}-1} \leq 2e^{-2kr}$. Hence from (16) and (17)

$$\frac{(1-|z|^2)^{\varepsilon_1}(1-|w|^2)^{\varepsilon_2}}{|1-z\bar{w}|^{\varepsilon_1+\varepsilon_2+2}} \leq \frac{Ce^{-2kr\delta}}{(1-|z|^2)^2}, w \in D(z,(k+1)r) \setminus D(z,kr).$$

This shows, using Remark 3, that

$$\begin{aligned}(1-|z|^2)^2 b_{(\varepsilon_1,\varepsilon_2)}(W)(z) &\leq CW(D(z,r)) + C\sum_{k=1}^{\infty} e^{-2kr\delta} W(D(z,(k+1)r)) \\ &\leq C\Big(\sum_{k=0}^{\infty} e^{r(M-2\delta)k}k^{\beta}\Big) W(D(z,r)).\end{aligned}$$

Denoting $B_r = \sum_{k=0}^{\infty} e^{r(M-2\delta)k}k^{\beta}$, one gets that $B_r < \infty$ since $2\delta > M$ and for a constant C_r that $b_{(\varepsilon_1,\varepsilon_2)}(W) \leq C_r B_r \mathcal{A}_r(W)$. □

Now we study the weights for which $b_{(\varepsilon_1,\varepsilon_2)}$ is bounded on $L^p(W)$ for $1 \leq p < \infty$.

Proposition 9. *Let $\epsilon_1 + \epsilon_2 > -1$, $1 < p < \infty$ and W be a weight such that $W^{-1/(p-1)}$ is also locally integrable. The following statements are equivalent.*

- i) $b_{(\varepsilon_1,\varepsilon_2)}$ *extends to a bounded operator on* $L^p(W)$.
- ii) $b_{(\varepsilon_2,\varepsilon_1)}$ *extends to a bounded operator on* $L^{p'}(W^{-1/(p-1)})$.
- iii) $P^*_{\varepsilon_1+\varepsilon_2}$ *extends to a bounded operator on* $L^p(W_{\varepsilon_1 p})$.
- iv) $P^*_{\varepsilon_1+\varepsilon_2}$ *extends to a bounded operator on* $L^{p'}((W^{-1/(p-1)})_{\varepsilon_1 p'})$.

Moreover the norms coincide.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) follows from the fact that $b_{(\varepsilon_1,\varepsilon_2)}$ is the transpose of $b_{(\varepsilon_2,\varepsilon_1)}$ with respect to the duality of $L^p(W)$ and $L^{p'}(W^{-1/(p-1)})$ given by $\int_{\mathbb{D}} f\bar{g}dA$.

The equivalence (iii) and (iv) is the symmetry of Bekolle's condition (2).

Finally for (iii) $\iff$ (i) use that $b_{(0,\varepsilon)} = P_\varepsilon^*$, and

$$b_{(\varepsilon_1,\varepsilon_2)}(\phi) = (1-|z|^2)^\delta b_{(\varepsilon_1-\delta,\varepsilon_2+\delta)}((1-|w|^2)^{-\delta}\phi).$$

for $\delta \in \mathbb{R}$. Hence $b_{(\varepsilon_1,\varepsilon_2)}$ is bounded on $L^p(W)$ if and only if $b_{(\varepsilon_1-\delta/p,\varepsilon_2+\delta/p)}$ is bounded on $L^p(W_\delta)$.

□

Corollary 10. *Let $\alpha > -1$ and $1 < p < \infty$. Then P_α^* is bounded on $L^p(W_\alpha)$ if and only if $b_{(\alpha/p,\alpha/p')}$ is bounded on $L^p(W)$.*

Proposition 11. *Let W be a locally integrable weight. Then $b_{(\varepsilon_1,\varepsilon_2)}$ extends to a bounded operator on $L^1(W)$ if and only if $b_{(\varepsilon_2,\varepsilon_1)}(W) \le CW \quad a.e.$*

In particular for $\alpha > -1$, P_α^* is bounded on $L^1(W)$ if and only if $P_\alpha^*(W) \le CW \quad a.e.$

Proof. For each non negative $f \in L^1(\mathbb{D})$ one has $W^{-1}f \in L^1(W)$. Therefore

$$\int_{\mathbb{D}} fW^{-1}b_{(\varepsilon_2,\varepsilon_1)}(W)dA(w) = \int_{\mathbb{D}} b_{(\varepsilon_1,\varepsilon_2)}(fW^{-1})WdA. \tag{18}$$

giving directly the continuity of $b_{(\varepsilon_1,\varepsilon_2)}$ if $b_{(\varepsilon_2,\varepsilon_1)}(W) \le CW$ a.e. Conversely if $b_{(\varepsilon_1,\varepsilon_2)}$ is continuous then using that the dual of $L^1(\mathbb{D})$ is $L^\infty(\mathbb{D})$ we conclude by (18) that $b_{(\varepsilon_2,\varepsilon_1)}(W) \le CW$ a.e. □

3. Averaging operators

To study the $\mathcal{A}_r$ it will be convenient to introduce the following related averaging operator.

Definition 12. *Let $0 < W(z) < \infty$ be locally integrable and $0 < r < \infty$. We define*

$$\mathcal{A}_r^W(\phi)(z) = \frac{1}{W(D(z,r))}\int_{D(z,r)} \phi(w)W(w)dA(w).$$

Proposition 13. *For each $0 < W(z) < \infty$ locally integrable and $0 < r < \infty$ the operator $\mathcal{A}_r^W$ is bounded on $L^p(W)$ for $1 < p \le \infty$ and of weak type (1,1) on $L^1(W)$.*

Proof. Since $\|\mathcal{A}_r^W(f)\|_\infty \le \|f\|_\infty$ then using interpolation we shall simply see that $\mathcal{A}_r^W$ is weak type (1,1). Let $\Omega = \{z : \mathcal{A}_r^W(\phi)(z) > \lambda\}$.

Consider an *r/2-lattice* $\mathcal{D}_{r/2} = \{z_n\}$. For each $z \in \Omega$ there exists $n = n(z)$ such that $z \in D_n \subset D(z_n, r/2)$. Hence

$$\begin{aligned} W(D_{n(z)}) &\le W(D(z_n,r/2)) \le W(D(z,r)) \\ &\le \frac{1}{\lambda}\int_{D(z,r)} \phi(w)W(w)dW \\ &\le \frac{C}{\lambda}\int_{D(z_n,3r/2)} \phi(w)W(w)dW. \end{aligned}$$

Hence writing $\Omega = \cup_n(\Omega \cap D_n)$ we have

$$\begin{aligned} W(\Omega) &\leq \frac{C}{\lambda}\sum_{n\in\mathbb{N}}\int_{D(z_n,3r/2)}\phi(w)W(w)dW \\ &\leq \frac{C'}{\lambda}\int_{\mathbb{D}}\phi(w)W(w)dW \end{aligned}$$

where we use that there is a finite number of overlappings of $D(z_n, 3r/2)$ in the last estimate. $\square$

Proposition 14. *Let $0 < r < \infty$ and $W \in D_r$. Then the operator $\mathcal{A}_r^W$ is bounded on $L^1(W)$.*

Proof. Assume that $W \in D_r$. Since $D(w,r) \subset D(z, 2r)$ for any $w \in D(z,r)$

$$C^{-1} \leq \frac{W(D(w,r))}{W(D(z,r))} \leq C, w \in D(z,r).$$

This allows to write

$$\begin{aligned} \int_{\mathbb{D}}\mathcal{A}_r^W(\phi)(z)W(z)dA(z) &= \int_{\mathbb{D}}\int_{\mathbb{D}}\frac{\chi_{D(z,r)(w)}}{W(D(z,r))}\phi(w)W(w)W(z)dA(w)dA(z) \\ &\leq C\int_{\mathbb{D}}\Big(\int_{D(w,r)}\frac{W(z)}{W(D(w,r))}dA(z)\Big)\phi(w)W(w)dA(w) \\ &\leq C\int_{\mathbb{D}}\phi(w)W(w)dA(w). \end{aligned}$$

$\square$

Let us now consider the Muckenhuupt A_p condition restricted to hyperbolic disks with fixed radius r.

Definition 15. *Let $0 < r < \infty$ and $1 \leq p < \infty$. We say that a weight is a b_p^r weight, for short $W \in b_p^r$, if*

$$\|W\|_{b_p^r} = \sup_{z\in\mathbb{D}}\Big(\mathcal{A}_r(W)(z)\Big)^{1/p}\Big(\mathcal{A}_r\Big(W^{-1/(p-1)}\Big)(z)\Big)^{1/p'} < \infty, \textit{ for } 1 < p < \infty,$$

and

$$\|W\|_{b_1^r} = \sup_{z\in\mathbb{D}}\mathcal{A}_r(W)(z)\sup_{\xi\in D(z,r)}W^{-1}(\xi) < \infty.$$

Proposition 16. *Let $1 \leq p < \infty$ and $0 < r < \infty$. If $W \in b_p^r$ then $W \in D_{r/2}$.*

Proof. Let $p > 1$ and $W \in b_p^r$. We shall show that $\mathcal{A}_rW(z) \leq C_r\mathcal{A}_{r/2}W(z)$. Since

$$|D(z,r)| \leq (W(D(z,r))^{1/p}\left(\int_{D(z,r)}W^{-1/(p-1)}dA\right)^{1/p'}, r > 0, z \in \mathbb{D}.$$

the b_p^r condition implies

$$\begin{aligned}\mathcal{A}_r(W)(z) &\leq C|D(z,r)|^{p-1}\left(\int_{D(z,r)} W^{-1/(p-1)}dA\right)^{1-p}\\ &\leq C|D(z,r/2)|^{p-1}\left(\int_{D(z,r/2)} W^{-1/(p-1)}dA\right)^{1-p}\\ &\leq C_r\mathcal{A}_{r/2}(W)(z).\end{aligned}$$

In the case $p=1$ we have $\mathcal{A}_r(W)(z)\leq W(\xi)$ for all $\xi\in D(z,r)$. Then the result follows integrating both sides of this inequality on $D(z,r/2)$. □

Lemma 17. *Let $0<r<\infty$, $1<p<\infty$ and W a locally integrable weight. Then $W\in b_p^r$ if and only if there exists a constant $C>0$ such that*

$$\mathcal{A}_r(\phi)(z)\leq C(\mathcal{A}_r^W(\phi^p)(z))^{1/p}$$

for any measurable $\phi\geq 0$.

Proof. First assume $W\in b_p^r$. Hence for $\phi\geq 0$ we have the following estimate

$$\begin{aligned}\mathcal{A}_r(\phi)(z) &\leq \frac{C}{|D(z,r)|}\left(\int_{D(z,r)}\phi^p W dA\right)^{1/p}\left(\int_{D(z,r)} W^{-p'/p}dA\right)^{1/p'}\\ &\leq C(\frac{1}{W(D(z,r))}\int_{D(z,r)}\phi^p W dA)^{1/p}.\end{aligned}$$

Hence

$$\mathcal{A}_r(\phi)(z)\leq C(\mathcal{A}_r^W(\phi^p)(z))^{1/p}. \tag{19}$$

Assume now that $\mathcal{A}_r(\phi)\leq C(\mathcal{A}_r^W(\phi^p))^{1/p}$. Selecting $\phi=W^{-\frac{1}{p-1}}$ we have $\phi^p W=\phi$ and therefore for any disc $D(z,r)$,

$$\mathcal{A}_r(\phi)(z)=\frac{1}{|D(z,r)|}\int_{D(z,r)} W^{-1/(p-1)}dA$$

and

$$(\mathcal{A}_r^W(\phi^p)(z))^{1/p}=(\frac{1}{W(D(z,r))}\int_{D(z,r)} W^{-1/(p-1)}dA)^{1/p}.$$

This gives $W\in b_p^r$.

□

Theorem 18. *Let $0<r<\infty$, $1<p<\infty$ and W a locally integrable weight. The following are equivalent*

i) *$W\in b_p^r$.*
ii) *$W\in D_{r/2}$, $W^{-p'/p}\in D_{r/2}$ and the averaging operator $\mathcal{A}_r$ is of weak-type (p,p) on $L^p(W)$.*

Proof. (i) $\Longrightarrow$ (ii) Taking into account that $W \in b_p^r$ is equivalent to $W^{-p'/p} \in b_{p'}^r$, Proposition 16 gives $W \in D_{r/2}$ and $W^{-p'/p} \in D_{r/2}$.

Therefore, using Lemma 17 and Proposition 13, we have $\mathcal{A}_r$ is weak-type (p,p) on $L^p(W)$.

(ii) $\Longrightarrow$ (i) Consider $\phi(w) = W^{-1/p}(w)g(w)\chi_{D(z,r/2)}(w)$ for some $g \in L^p(D(z,r/2))$ non negative and with norm 1. Hence for $\xi \in D(z,r/2)$ one has that $D(z,r/2) \subset D(\xi,r) \subset D(z,3r/2)$ and therefore

$$\begin{aligned}\mathcal{A}_r(\phi)(\xi) &= \frac{1}{|D(\xi,r)|}\int_{D(\xi,r)\cap D(z,r/2)} gW^{-1/p}dA \\ &\geq \frac{C}{|D(z,r/2)|}\int_{D(z,r/2)} gW^{-1/p}dA.\end{aligned}$$

Therefore

$$W(D(z,r/2)) \leq W\left(\{\xi : \mathcal{A}_r(\phi)(\xi) > \frac{C}{|D(z,r/2)|}\int_{D(z,r/2)} gW^{-1/p}dA\}\right).$$

Hence

$$\left(\frac{1}{|D(z,r/2)|}\int_{D(z,r/2)} gW^{-1/p}dA\right)(W(D(z,r/2)))^{1/p} \leq \|\mathcal{A}_r\|_{L^p\to L^p_{weak}}.$$

and taking the supremum over functions g in the unit ball of $L^p(D(z,r/2))$ one gets

$$\frac{1}{|D(z,r/2)|}(\int_{D(z,r/2)} W^{-p'/p}dA)^{1/p'}(W(D(z,r/2)))^{1/p} \leq \|\mathcal{A}_r\|_{L^p\to L^p_{weak}}$$

and, taking into account that $W \in D_{r/2}$ and $W^{-p'/p} \in D_{r/2}$ we obtain that $W \in b_p^r$. □

Corollary 19. *Let $1 \leq p < \infty$, $r > 0$ and W a weight. Consider the following statements:*

i) $W \in b_p^r$.
ii) $\mathcal{A}_{r/2}$ *is bounded on* $L^p(W)$.
iii) $\mathcal{A}_{r/2}$ *is of weak-type* (p,p) *on* $L^p(W)$.
iv) $W \in b_p^{r/4}$.

Then (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv).

Proof. (i) $\Longrightarrow$ (ii) Assume that $W \in b_p^r$ for some $r > 0$ then in particular $W \in b_p^{r/2}$ and $W \in D_{r/2}$ by Proposition 16. From Lemma 17 one obtains

$$\mathcal{A}_{r/2}(\varphi)(z) \leq C(\mathcal{A}_{r/2}^W(\varphi^p)(z))^{1/p}$$

(the case $p = 1$ is similar and left to the reader). Hence $\mathcal{A}_{r/2}$ is bounded on $L^p(W)$ using Proposition 14.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (iv) It is shown in the proof of Theorem 18 that if $\mathcal{A}_{r/2}$ is of weak-type (p,p) on $L^p(W)$ then $W \in b_p^{r/4}$ (same argument works for $p=1$). $\square$

Same arguments show the following situation for weights $W \in \cap_{r>0} D_r$.

Corollary 20. *Let $0<r<\infty$, $1 \le p < \infty$ and $W \in \cap_{s>0} D_s$. The following are equivalent*

i) *$W \in b_p^r$.*
ii) *$\mathcal{A}_r$ is bounded on $L^p(W)$.*
iii) *$\mathcal{A}_r$ is of weak-type (p,p) on $L^p(W)$.*

As an application of the continuity of $\mathcal{A}_r$ we prove that for $W \in b_p^r$, the well known inequality $\|(1-|z|^2)f'\|_{L^p} \le C\|f\|_{A^p}$ has an extension in $A^p(W)$.

Proposition 21. *Let $1 \le p < \infty$ and $W \in \cup_{r>0} b_p^r$. Then there exist $r_0 > 0$ and $C > 0$ such that*

$$\|(1-|z|^2)f'\|_{L^p(W)} \le \frac{C}{s^3}\|f\|_{A^p(W)}, \quad f \in A^p(W), 0 < s \le \tanh(r_0).$$

Proof. Using Corollary 19 there exists r_0 such that $\mathcal{A}_{r_0}$ is bounded on $L^p(W)$. For each $0<\rho<1$,

$$\rho f'(0) = 2\int_0^{2\pi} f(\rho e^{it})e^{-it}\frac{dt}{\pi}$$

and integrating over $(0,s)$ with respect to $\rho d\rho$ we have

$$s^3 f'(0) = 6\int_{|w|\le r} f(w)\frac{\bar{w}}{|w|}dA(w).$$

We shall show the pointwise estimate

$$(1-|z|^2)|f'(z)| \le \frac{C}{s^3}\mathcal{A}_{r_0}(|f|)(z), 0 < s \le \tanh(r_0). \tag{20}$$

For $0 < s \le \tanh(r_0)$, applying (20) to $f \circ \varphi_z$ and using (12) we obtain

$$\begin{aligned}
(1-|z|^2)|f'(z)| &\le \frac{6}{s^3}\int_{D(z,r)} |f(u)|\frac{(1-|z|^2)^2}{|1-z\bar{u}|^4}dA(u) \\
&\le \frac{96}{s^3(1-|z|^2)^2}\int_{D(z,r)} |f(u)|dA(u) \\
&\le \frac{96 s_0^2}{s^3(1-s_0^2)^2}\frac{1}{|D(z,r_0)|}\int_{D(z,r)} |f(u)|dA(u) \\
&\le \frac{C}{s^3}\mathcal{A}_{r_0}(|f|)(z).
\end{aligned}$$

We conclude the proof using that $\mathcal{A}_{r_0}$ is bounded on $L^p(W)$. $\square$

Definition 22. *We write $\mathcal{W}$ for the set of weights W such that there exist $r_0 > 0$ and $C > 0$ such that*

$$\frac{W(D(z,r_0))}{|D(z,r_0)|} \le CW(z), \quad z \in \mathbb{D}.$$

Remark 23. *For every $r > 0$, $b_1^r \subseteq \mathcal{W}$, since*

$$\frac{W(D(z,r))}{|D(z,r)|} \leq C \inf_{\xi \in D(z,r)} W(\xi), \quad z \in \mathbb{D},$$

for every $W \in b_1^r$.

Remark 24. *For weights $W \in \cap_{r>0} D_r$, it follows from (7) that $W \in \mathcal{W}$ if and only if for any $r > 0$ there exists $C_r > 0$ so that*

$$\mathcal{A}_r(W)(z) \sim \frac{W(D(z,r))}{(1-|z|^2)^2} \leq C_r W(z), \quad z \in \mathbb{D}.$$

We end this section by showing that for M-doubling weights the membership of W to b_1^r is also related to the continuity of $b_{(\varepsilon_1,\varepsilon_2)}$ to $L^1(W)$.

Proposition 25. *Let W be an M-doubling weight for some $M \geq 0$. The following are equivalent.*

i) $W \in \mathcal{W}$.
ii) $W \in b_1^r$ *for some* $r > 0$.
iii) $W \in b_1^r$ *for all* $r > 0$.

Proof. Of course (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

We only need to show that (i) $\Longrightarrow$ (iii). Let $W \in \mathcal{W}$. Since every M-doubling weight belongs to $\cap_{r>0} D_r$ we have by Remark 23 for any $r > 0$ that $\mathcal{A}_r(W) \leq CW$. Using Theorem 8 if we select $(\varepsilon_1, \varepsilon_2)$ such that $\varepsilon_2 > M/2, \varepsilon_1 + 2 > M/2$ one has $b_{(\varepsilon_1,\varepsilon_2)}(W) \sim \mathcal{A}_r(W) \leq CW$. Hence from Proposition 11, $b_{(\varepsilon_2,\varepsilon_1)}$ is continuous in $L^1(W)$, which using (11) gives that $\mathcal{A}_r$ is bounded on $L^1(W)$ for any $r > 0$ and therefore by Corollary 19, $W \in b_1^{r/2}$ for any $r > 0$. □

Proposition 26. *Let W be an M-doubling weight for some $M \geq 0$. The following are equivalent.*

i) $W \in \mathcal{W}$.
ii) $b_{(\varepsilon_2,\varepsilon_1)}$ *is continuous in* $L^1(W)$ *for all* $(\varepsilon_1, \varepsilon_2)$ *such that* $\varepsilon_2 > M/2, \varepsilon_1 + 2 > M/2$.
iii) There exists $(\varepsilon_1, \varepsilon_2)$ *such that* $b_{(\varepsilon_1,\varepsilon_2)}(W) \leq CW$.

Proof. (i) $\Rightarrow$ (ii) is part of the proof of (i) $\Rightarrow$ (iii) in Proposition 25.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious.

□

Corollary 27. *Let $M < 4$, $M/2 < \delta < 2 + \alpha - M/2$, $1 \leq p < \infty$ and let $W \in \mathcal{W}$ be M-doubling such that $W \in L^1(dA)$. Then P_α^* is bounded on $L^1(W_\delta)$. If W is radial, then P_α^* is bounded on $L^p(W_\delta)$, for $1 \leq p < \infty$.*

Proof. We notice that

$$\|P_\alpha^*(\phi)\|_{L^1(W_\delta)} = \|b_{(\delta,\alpha-\delta)}((1-|w|^2)^\delta \phi)\|_{L^1(W dA)}.$$

Since by Proposition 26 we have that $b_{(\delta,\alpha-\delta)}$ is bounded on $L^1(W)$, the result follows for $p=1$. The extension to $p>1$ for radial weights follows from Bekolle's condition (see [2, Remark 2.2]). $\square$

4. Weighted Bergman-Herz spaces

Let us now study some properties of the weighted Bergman-Herz spaces.

Definition 28. *Let W be a weight in $\mathbb{D}$ and $1 \le p,q \le \infty$. We define $\mathcal{K}^p_q(W)$ as the space consisting of all complex measurable functions on $\mathbb{D}$ such that*

$$\|f\|_{\mathcal{K}^p_q(W)} = \left(\sum_{n=1}^{\infty} \|f\|^q_{L^p(A_n,W)}\right)^{1/q} < \infty,$$

where for $n \ge 1$, $A_n = \{z \in \mathbb{D}, r_{n-1} \le |z| < r_n\}$, and $r_n = 1-2^{-n}$. We write $A^p_q(W) = \mathcal{K}^p_q(W) \cap hol(\mathbb{D})$.

We have that $\mathcal{K}^p_p(W) = L^p(W)$ and $\mathcal{A}^p_p(W) = A^p(W)$.

Remark 29. *Note that $f \in \mathcal{K}^p_q(W)$ if and only if $W^{1/p}f \in \mathcal{K}^p_q(dA)$. Hence, using*

$$< f,g >= \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z) = \int_{\mathbb{D}} f(z)W(z)^{1/p}\overline{g(z)}W(z)^{-1/p}dA(z)$$

we have the duality $(\mathcal{K}^p_q(W))^* = \mathcal{K}^{p'}_{q'}(W^{-1/(p-1)})$ for $1 \le p,q < \infty$.

Definition 30. *We denote $\ell^W_{\mathcal{D}_r}(p,q)$, the Kellog space adapted to the set $\mathcal{D}_r$ consisting of all sequences $(a_n)_{n\ge 0}$ for which*

$$\|(a_n)\|_{\ell^W_{\mathcal{D}_r}(p,q)} = \Big(\sum_{n=1}^{\infty}(\sum_{\{k\in\mathbb{N}: z_k\in A_n\}} W(D(z_k,r))|a_k|^p)^{q/p}\Big)^{1/q} < \infty.$$

Lemma 31. *Let $R>0$. There exists $M>0$ such that for all $0<r\le R$*

$$D(z,r) \subset \cup_{|k-n(z)|\le M} A_k$$

where $z \in A_{n(z)}$.

Proof. Using (14) there exist $0<C_1<1$ and $C_2>1$ such that

$$C_1(1-|z|^2) \le 1-|w|^2 \le C_2(1-|z|^2),$$

$w \in D(z,r), 0<r\le R$. If $z \in A_n$ then

$$C_1 2^{-n} \le 1-|w| \le 4C_2 2^{-n}, w \in D(z,r), 0<r\le R.$$

We then have for some $k_1,k_2 \in \mathbb{Z}$, that $2^{-n-k_1} \le 1-|w| \le 2^{-n+k_2}$ for any $w \in D(z,r)$, $0<r\le R$ and $z \in A_n$. This gives the result. $\square$

Lemma 32. *Let $R > 0$. Then there exists $C > 0$ such that*
$$\|h\|_{\mathcal{K}^p_q(W)} \leq C\big(\sum_{n=0}^{\infty}(\sum_{z_k\in A_n}\int_{D_k}|h(z)|^p\,W(z)dA(z))^{q/p}\big)^{1/q},$$
for any h measurable function and $0 < r \leq R$.

Proof. Due to Lemma 31 one has for each $n \in \mathbb{N}$,
$$A_n = \cup_{|n-l|\leq M}\cup_{z_k\in A_l}(D_k\cap A_n).$$
Hence
$$\begin{aligned}
\|h\|^q_{\mathcal{K}^p_q(W)} &= \sum_{n=0}^{\infty}\left(\int_{A_n}|h(z)|^p\,W(z)dA(z)\right)^{q/p}\\
&\leq \sum_{n=0}^{\infty}\left(\sum_{|n-l|\leq M}\sum_{z_k\in A_l}\int_{D_k}|h(z)|^p\,W(z)dA(z)\right)^{q/p}\\
&\leq C\sum_{n=0}^{\infty}\sum_{|n-l|\leq M}\left(\sum_{z_k\in A_l}\int_{D_k}|h(z)|^p\,W(z)dA(z)\right)^{q/p}\\
&\leq C(2M+1)\sum_{l=0}^{\infty}\left(\sum_{z_k\in A_l}\int_{D_k}|h(z)|^p\,W(z)dA(z)\right)^{q/p}.
\end{aligned}$$
□

For an r-lattice $\mathcal{D}_r = (z_k)_k$ we consider the sampling operator defined in $A^p_q(W)$
$$\mathcal{T}_r(f) = (f(z_k))_k$$
and the operator
$$\mathcal{R}_r(f) = \sum_{k=1}^{\infty} f(z_k)\chi_{D_k},$$
where $\{D_k\}$ are the regions associated to the r-lattice $\mathcal{D}_r = (z_k)_k$.

Lemma 33. *Let $1 \leq p, q < \infty$. If $W \in b^{r_0}_p$ for some $r_0 > 0$ then $\mathcal{T}_r$ is bounded from $A^p_q(W)$ into $\ell^W_{\mathcal{D}_r}(p,q)$ for $0 < r \leq r_0$.*

Moreover, there exist $C > 0$ such that for $0 < r \leq r_0$
$$\|\Big(f(z_k)\Big)_k\|_{\ell^W_{\mathcal{D}_r}(p,q)} \leq C\|W\|_{b^r_p}\,\|f\|_{A^p_q(W)}\,.$$

Proof. First notice that since $|D(z,r_0)|/|D(z,r)| \leq C(r,r_0)$, then $W \in b^{r_0}_p$ implies that $W \in b^r_p$ for every $r \leq r_0$.

There exists $C_1 > 0$ (see [8, p.69]) such that for any holomorphic function on $\mathbb{D}$ we have
$$|f(z_k)| \leq \frac{C_1}{|D(z_k,r)|}\int_{D(z_k,r)}|f(w)|\,dA(w) \leq$$

$$\frac{C_1}{|D(z_k,r)|}\left(\int_{D(z_k,r)}|f|^p\,W dA\right)^{1/p}\left(\int_{D(z_k,r)}W^{-1/(p-1)}dA\right)^{1/p'}.$$

Hence

$$W(D(z_k,r))\,|f(z_k)|^p\le C_1^p\|W\|_{b_p^r}^p\int_{D(z_k,r)}|f(w)|^p\,W(w)dA(w).$$

Hence for $r\le r_0$, since $\cup_{z_k\in A_n}D(z_k,r)\subset\cup_{|l-n|\le M}A_l$ by Lemma 31 we obtain

$$\begin{aligned}&\left(\sum_{z_k\in A_n}W(D(z_k,r))\,|f(z_k)|^p\right)^{1/p}\\ \le\ & C_1\|W\|_{b_p^r}\left(\int_{\cup_{z_k\in A_n}D(z_k,r)}|f(w)|^p\,W(w)dA(w)\right)^{1/p}\\ \le\ & C_1\|W\|_{b_p^r}N^{1/p}\left\{\sum_{|l-n|\le M}\left(\int_{A_l}|f(w)|^p\,W(w)dA(w)\right)^{1/p}\right\}\end{aligned}$$

Now the lemma follows by Minkowski's inequality. □

Lemma 34. *Let $1<p<\infty$, $1\le q<\infty$ and $W\in b_p^{r_0}$ for some $r_0>0$. Then $\mathcal{R}_r$ is bounded from $A_q^p(W)$ into $\mathcal{K}_q^p(W)$ and there exists $C>0$ such that for $0<r<r_0/2$, and $s=\tanh r$,*

$$\|(Id-\mathcal{R}_r)(f)\|_{\mathcal{K}_q^p(W)}\le Cs\|f\|_{A_q^p(W)}.$$

Proof. A combination of Lemmas 32 and 33 show that $\mathcal{R}_r$ is bounded from $A_q^p(W)$ into $\mathcal{K}_q^p(W)$.

As in the proof of [12, Lemma 4.4.3], there exists a constant $C_1>0$ such that for $z\in D(z_k,r)$ and $0<r<r_0/2$

$$|f(z)-f(z_k)|\le\frac{C_1s(1-|z_k|^2)}{(1-\tanh r_0/2)^2}\sup\{|f'(w)|:w\in D(z_k,r_0/2)\},$$

for every $f\in hol(\mathbb{D})$.

Hence, using that $(1-|z_k|^2)\sim(1-|w|^2)$ for $w\in D(z_k,r)$ and $0<r<r_0/2$ together with (20)

$$\begin{aligned}|f(z)-f(z_k)|&\le Cs\sup\{\mathcal{A}_{r_0/2}(|f|)(w):w\in D(z_k,r_0/2)\}\\ &\le Cs\frac{1}{|D(z_k,r_0)|}\int_{D(z_k,r_0)}|f(u)|dA(u)\\ &\le\frac{Cs}{|D(z_k,r_0)|}\left(\int_{D(z_k,r_0)}|f|^pW dA\right)^{1/p}\left(\int_{D(z_k,r_0)}W^{\frac{-1}{p-1}}dA\right)^{1/p'}.\end{aligned}$$

This gives

$$\int_{D_k}|f(z)-f(z_k)|^p\,W(z)dA(z)\le C^ps^p\|W\|_{b_p^{r_0}}^p\int_{D(z_k,r_0)}|f|^p\,W dA. \tag{21}$$

Therefore

$$\sum_{z_k\in A_n}\int_{D_k}|f(\cdot)-f(z_k)|^p\,WdA\le C_1s^p\|W\|_{b_p^{r_0}}^p N\int_{\cup_{z_k\in A_n}D(z_k,r_0)}|f|^p\,WdA.$$

Hence we have

$$\left(\sum_{z_k\in A_n}\int_{D_k}|f(\cdot)-f(z_k)|^p\,WdA\right)^{1/p}\le C_2s\|W\|_{b_p^{r_0}}\left(\sum_{|l-n|\le M}\int_{A_l}|f|^pWdA\right)^{1/p}.$$

Thus, using Minkowski's inequality,

$$\left(\sum_{n=0}^{\infty}\left(\sum_{z_k\in A_n}\int_{D_k}|f(z)-f(z_k)|^p\,W(z)dA(z)\right)^{q/p}\right)^{1/q}\le Cs\,\|f\|_{A^p_q(W)}\,.$$

Finally applying Lemma 32 to $h=(Id-\mathcal{R}_r)(f)$ we obtain

$$\begin{aligned}\|(Id-\mathcal{R}_r)(f)\|_{\mathcal{K}^p_q(W)} &\le \left(\sum_{n=0}^{\infty}\left(\sum_{z_k\in A_n}\int_{D_k}|f(z)-f(z_k)|^p\,W(z)dA(z)\right)^{q/p}\right)^{1/q}\\ &\le Cs\,\|f\|_{A^p_q(W)}\,.\end{aligned}$$

□

5. Atomic decomposition for $A^p_q(W)$

Definition 35. *Let $0<r<\infty$. We define*

$$\mathcal{S}_rf(z)=\sum_{k=1}^{\infty}|D_k|\,f(z_k)K(z_k,z).$$

where $K(w,z)=\frac{1}{(1-\bar{w}z)^2}$ denotes the Bergman kernel and $\mathbb{D}=\cup_kD_k$ where D_k are corresponding disjoint sets associated to the r-lattice $\mathcal{D}_r=\{z_k\}$.

Lemma 36. *Let $1<p<\infty$, $1\le q<\infty$. If P^* is bounded in $\mathcal{K}^q_p(W)$ then $\mathcal{S}_r$ is bounded on $A^p_q(W)$.*

Proof. First notice that P^* is bounded in $K^q_p(W)$ implies that $W\in\cap_{r>0}b^r_p$. Indeed, from Lemma 31 we conclude that for each $\psi\in\mathcal{K}^p_q$ supported in $D(z,r)$,

$$\|W^{-1/p}\psi\|_{\mathcal{K}^p_q(W)}\sim\|\psi\chi_{D(z,r)}\|_{L^p}.$$

On the other hand, if $w\in D(z,r)$,

$$P^*(W^{-1/p}\psi\chi_{D(z,r)})(w)\ge C(1-|z|^2)^{-2}\left(\int_{D(z,r)}W^{-1/p}\psi\right)\chi_{D(z,r)}(w).$$

Therefore we have

$$\begin{aligned}\|\psi\chi_{D(z,r)}\|_{L^p} &\geq C\|P^*(W^{-1/p}\psi\chi_{D(z,r)})\chi_{D(z,r)}\|_{L^p(W)}\\ &\geq C(1-|z|^2)^{-2}\left(\int_{D(z,r)} W^{-1/p}\psi\right) W^{1/p}(D(z,r)).\end{aligned}$$

Which by duality gives

$$(\int_{D(z,r)} W^{-p'/p})^{1/p'} W^{1/p}(D(z,r)) \leq C_r|D(z,r)|,$$

proving that $W \in b_p^r$. Now since P^* is also bounded on $\mathcal{K}_{q'}^{p'}(W^{-1/(p-1)})$, we have that the Bergman projection

$$P(f)(z) = \int_{\mathbb{D}} K(w,z)f(w)dA(w)$$

is bounded from $\mathcal{K}_{q'}^{p'}((W^{-1/(p-1)})$ into $A_{q'}^{p'}((W^{-1/(p-1)})$. Let $f \in A_q^p(W)$ and $h \in \mathcal{K}_{q'}^{p'}(W^{-1/p-1})$ and denote $g = P(h)$.

First write

$$\langle \mathcal{S}_r(f)), h\rangle = \sum_{k=1}^{\infty} f(z_k)|D_k|\langle K(z_k,.), h\rangle = \sum_{k=1}^{\infty} f(z_k)|D_k|\overline{g(z_k)}.$$

Since both $W \in b_p^r$ and $W^{-1/(p-1)} \in b_{p'}^r$, we can use Lemma 33 twice to obtain the following estimates

$$\begin{aligned}
&|\langle \mathcal{S}_r(f), h\rangle| \leq C\sum_{k=1}^{\infty} |f(z_k)|\Big(\int_{D(z_k,r)} dA(w)\Big)|g(z_k)|\\
&\leq C\sum_{n=1}^{\infty}\sum_{z_k\in A_n} |f(z_k)|\Big(\int_{D(z_k,r)} W^{-1/(p-1)}dA\Big)^{1/p'} W(D(z_k,r))^{1/p}|g(z_k)|\\
&\leq C\left(\sum_{n=1}^{\infty}\Big(\sum_{z_k\in A_n}(\int_{D(z_k,r)} W^{-1/(p-1)}dA)\,|g(z_k)|^{p'}\Big)^{q'/p'}\right)^{1/p'} \|(f(z_n))\|_{\ell^W_{\mathcal{D}_r}(p,q)}\\
&\leq C\|W^{-1/(p-1)}\|_{b_{p'}^r}\,\|g\|_{A_{q'}^{p'}(W^{-1/(p-1)})}\,\|(f(z_n))\|_{\ell^W_{\mathcal{D}_r}(p,q)}\\
&\leq C\|W\|_{b_p^r}\|W^{-1/(p-1)}\|_{b_{p'}^r}\,\|h\|_{\mathcal{K}_{q'}^{p'}(W^{-1/(p-1)})}\,\|f\|_{A_q^p(W)}\\
&\leq C\|W\|_{b_p^r}\|W^{-1/(p-1)}\|_{b_{p'}^r}\,\|h\|_{\mathcal{K}_{q'}^{p'}(W^{-1/(p-1)})}\,\|f\|_{A_q^p(W)}\,.
\end{aligned}$$

□

Lemma 37. *Let $p > 1$, $1 \leq q < \infty$ and W a weight such that P^* is bounded on $\mathcal{K}_q^p(W)$. If $r > 0$ is small enough, then $\mathcal{S}_r$ is invertible in $A_q^p(W)$.*

Proof. It suffices to prove that $I-\mathcal{S}_r$ is a contraction for $0<r$ small enough. The assumption gives that the Bergman projection is bounded on $\mathcal{K}^p_q(W)$. Hence we have $(A^p_q(W))^* = A^{p'}_{q'}(W^{-1/(p-1)})$, $W \in b^r_p$ and $W^{-1/(p-1)} \in b^r_{p'}$.

Let $f \in A^p_q(W)$ and $g \in A^{p'}_{q'}(W^{-1/(p-1)})$. We can write

$$
\begin{aligned}
\langle (I-\mathcal{S}_r)f, g\rangle &= \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z) - \sum_{k=1}^{\infty}\int_{\mathbb{D}} \frac{|D_k|\, f(z_k)}{(1-\overline{z_k}z)^2}\overline{g(z)}dA(z)\\
&= \sum_{k=1}^{\infty}\int_{D_k}\left(f(z)\overline{g(z)} - f(z_k)\overline{g(z_k)}\right)dA(z)\\
&= \sum_{k=1}^{\infty}\int_{D_k} f(z)\left(\overline{g(z)}-\overline{g(z_k)}\right)dA(z)\\
&+ \sum_{k=1}^{\infty}\int_{D_k}\left(f(z)-f(z_k)\right)\overline{g(z_k)}dA(z)\\
&= \int_{\mathbb{D}} f(z)(g-\mathcal{R}_r g)(z)dA(z) + \int_{\mathbb{D}}(f-\mathcal{R}_r)(z)\mathcal{R}_r g(z)dA(z).
\end{aligned}
$$

Then the proof follows from Lemma 34. $\square$

Next, the main theorem of this section.

Theorem 38. *Let $1<p,q<\infty$ and let W be such that P^* is bounded on $\mathcal{K}^p_q(W)$. Let $\mathcal{D}_r=\{z_n\}$ for $r>0$ small enough so that S_r is invertible on $A^p_q(W)$.*

(i) If $(a_n)\in \ell^W_{\mathcal{D}_r}(p,q)$ then

$$f(z)=\sum_n \frac{a_n|D_n|}{(1-\overline{z_n}z)^2}\in A^p_q(W)$$

and $\|f\|_{A^p_q(W)} \le C\,\|(a_n)\|_{\ell^W_{\mathcal{D}_r}(p,q)}$.

(ii) If $f\in A^p_q(W)$, there exists a sequence $(a_n)\in \ell^W_{\mathcal{D}_r}(p,q)$ such that

$$f(z)=\sum_n \frac{a_n|D_n|}{(1-\overline{z_n}z)^2}$$

and $\|(a_n)\|_{\ell^W_{\mathcal{D}_r}(p,q)} \le C\,\|f\|_{A^p_q(W)}$.

Proof. (i) It follows using duality and Lemma 33.

(ii) Given $f\in A^p_q(W)$, take $g=\mathcal{S}_r^{-1}f\in A^p_q(W)$. Define $a_n=g(z_n)$. Then $f(z)=\mathcal{S}_r(g)(z)=\sum_n \frac{a_n|D_n|}{(1-\overline{z_n}z)^2}$. The estimate follows using the boundedness of S_r. $\square$

Let us finish by showing some sufficient conditions to get that P^* is continuous on $\mathcal{K}^p_q(W)$.

Lemma 39. *Let $1 \leq p < \infty$ and W a locally integrable weight. If S is a linear operator bounded in $L^p(W_{\pm\varepsilon})$ for some $\varepsilon > 0$ then S is also bounded on $\mathcal{K}_q^p(W)$ for $1 \leq q < \infty$.*

Proof. Since

$$\int_{\mathbb{D}} (1-|z|^2)^{\pm\varepsilon} |Sf(z)|^p W(z) dA(z) \leq C \int_{\mathbb{D}} (1-|z|^2)^{\pm\varepsilon} |f(z)|^p W(z) dA(z),$$

it follows that if *supp* $f \subset A_n$,

$$\int_{A_m} |Sf(z)|^p W(z) dA(z) \leq C 2^{\pm\varepsilon(m-n)} \int_{\mathbb{D}} |f(z)|^p W(z) dA(z).$$

Splitting $f = \sum f_n$, with $f_n = f\chi_{A_n}$ we have

$$\begin{aligned} \|Sf\|_{L^p(A_m,W)} &\leq C \sum_n 2^{\pm\varepsilon(m-n)/p} \|f\|_{L^p(A_n,W)} \\ &\leq C X * Y(m), \end{aligned}$$

where $X = (x_n)$ and $Y = (y_n)$ with $x_n = 2^{-\varepsilon|n|/p}$ and

$$y_n = \begin{cases} \|f\|_{L^p(A_n,W)}, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

The lemma follows from Young's inequality. $\square$

A nice consequence of this is that weighted Bergman-Herz spaces can be defined using the derivative for weights where the averaging operator is bounded.

Theorem 40. *Let $1 \leq p, q < \infty$ and W a weight.*
If $W \in b_p^r$ for some $r > 0$ then

$$\|(1-|z|^2)f'\|_{\mathcal{K}_q^p(W)} \leq C_r \|f\|_{A_q^p(W)}, \quad f \in A_q^p(W).$$

Proof. Note that $W_{\pm\varepsilon} \in b_p^r$ for any $\varepsilon > 0$. Hence the result follows from Proposition 21 and Lemma 39. $\square$

Remark 41. *We mention examples of weights for which the operator P^* is bounded in Herz spaces and the atomic decomposition of Theorem 38 holds. (see* [2] *for (a) and (b)).*

a) *P^* is bounded on $\mathcal{K}_q^p(dA_\delta)$ for $-1 < \delta < p-1 < \infty$.*
b) *Let W be a radial weight, $1 < p < \infty$ and $1 \leq q < \infty$. If for some $\gamma > 1$,*

$$\int_0^1 \frac{W(r)^\gamma}{1-rt} r dr \leq C W(t)^\gamma,$$

then P^* is continuous on $\mathcal{K}_q^p(W)$.

c) *Let* $M < 4$, $M/2 < \delta < 2+\alpha - M/2$, $1 \leq p < \infty$ *and let* $W \in \mathcal{W}$ *be M-doubling such that* $W \in L^1(dA)$. *Then* P^* *is continuous on* $\mathcal{K}^1_q(WdA_\delta)$ *for any* $q > 1$. *And if* W *is also radial then* P^* *is continuous on* $\mathcal{K}^p_q(WdA_\delta)$ *for any* $1 < p, q < \infty$. *In fact, for such* δ *we let* ϵ *such that* $M/2 < \delta \pm \epsilon < 2 + \alpha - M/2$. *Then by Corollary 27,* P^* *is continuous on* $L^1(W_{\delta\pm\epsilon})$ *and on* $L^p(W_{\delta\pm\epsilon})$ *for radial weights. Then the claimed continuity of* P^* *in Herz spaces follows from Lemma 39.*

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