BLOCH FUNCTIONS ON THE UNIT BALL OF AN INFINITE DIMENSIONAL HILBERT SPACE

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Abstract. The Bloch space has been studied on the open unit disk of $\mathbb{C}$ and some homogeneous domains of $\mathbb{C}^n$. We define Bloch functions on the open unit ball of a Hilbert space E and prove that the corresponding space $\mathcal{B}(B_E)$ is invariant under composition with the automorphisms of the ball, leading to a norm that- modulo the constant functions - is automorphism invariant as well. All bounded analytic functions on B_E are also Bloch functions.

Introduction

The classical Bloch space $\mathcal{B}$ of analytic functions on the open unit disk $\mathbf{D}$ of $\mathbb{C}$ plays an important role in geometric function theory and it has been studied by many authors.

R. M. Timoney ([6] and [7]) extended the notion of Bloch function by considering bounded homogeneous domains in $\mathbb{C}^n$, such as the unit ball B_n and the polydisk $\mathbf{D}^n$.

In this article, Bloch functions on the unit ball B_E of an infinite-dimensional Hilbert space E are introduced. We prove that a number results about Bloch functions on $\mathbf{D}$ and B_n can be extended to this infinite dimensional setting. Among them, several characterizations of Bloch functions on B_E known to hold in the finite dimensional case.

First, we will recall some background about the classical Bloch space and the space of Bloch functions on B_n. In Section 2, we will introduce the definition of $\mathcal{B}(B_E)$, the space of Bloch functions defined on B_E. A function $f : B_E \to \mathbb{C}$ is said to be a Bloch function if

$$\sup_{x\in B_E} (1 - \|x\|^2)\|\nabla f(x)\| < \infty.$$

Section 2 is devoted to the connection between functions in $\mathcal{B}(B_E)$ and their restrictions to one-dimensional subspaces seen as functions defined on $\mathbf{D}$ or either to their restrictions to finite-dimensional ones, resulting the fact that if for a given n, the restrictions of the function to the n-dimensional subspaces have their Bloch norms uniformly bounded, then the function is a Bloch one and conversely. We also introduce an equivalent norm for $\mathcal{B}(B_E)$ obtained by replacing the gradient by the radial derivative. We exhibit in Section 3 another equivalent norm for $\mathcal{B}(B_E)$ which is invariant- modulo the constant functions - under the action of the automorphisms of the ball. This is achieved without appealing to the invariant Laplacian and relying only on properties of automorphisms of B_E. Further, we are able to show that the space $H^\infty(B_E)$ of bounded analytic functions is contractively embedded in $\mathcal{B}(B_E)$, as it occurs in the finite dimensional case. Examples of unbounded Bloch functions are also shown.

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1. Background

1.1. **The classical Bloch space $\mathcal{B}$.** The classical *Bloch space* $\mathcal{B}$ (see [4]) is the space of analytic functions $f : \mathbf{D} \longrightarrow \mathbb{C}$ satisfying

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)| < \infty$$

endowed with the norm

$$\|f\|_{Bloch} = |f(0)| + \|f\|_{\mathcal{B}} < \infty$$

so that $(\mathcal{B}, \|\cdot\|_{Bloch})$ becomes a Banach space.

It is well-known that the seminorm $\|\cdot\|_{\mathcal{B}}$ is invariant by automorphisms, that is, $\|f\circ\varphi\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ for any $f \in \mathcal{B}$ and $\varphi \in Aut(\mathbf{D})$. The following basic result can be proved applying Schwarz's lemma (see for instance [9]).

Proposition 1.1. *H^∞ is properly contained in $\mathcal{B}$ and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$ for any $f \in H^\infty$.*

For further information and references about the Bloch space $\mathcal{B}$, the reader is referred to [1, 9].

1.2. **The Bloch space on the unit ball of $\mathbb{C}^n$.** R. M. Timoney extended Bloch functions to bounded homogeneous domains $\mathcal{D}$ of $\mathbb{C}^n$ (see [6] and [7]). K. Zhu studied further these functions on the unit ball B_n of $\mathbb{C}^n$ and many of his results are compiled in [8]. The Bloch space of functions on B_n can be defined in several ways. It is natural to consider the space $\mathcal{B}(B_n)$ of analytic functions $f : B_n \longrightarrow \mathbb{C}$ satisfying

$$\sup_{z\in B_n}(1-\|z\|^2)\|\nabla f(z)\| < \infty,$$

where $\nabla f(z) := (\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z))$ and whose norm is the Euclidean one or to consider analytic functions satisfying

$$\sup_{z\in B_n}(1-\|z\|^2)|Rf(z)| < \infty,$$

where $Rf(z) := \langle \nabla f(z), \bar{z}\rangle$ is the so-called radial derivative of f in z, here $\bar{z} = (\bar{z_1}, \ldots, \bar{z_n})$ or even to define the space reducing to functions defined on $\mathbf{D}$ by the condition

$$\sup\{\|f_x\|_{\mathcal{B}} : x \in \mathbb{C}^n \ , \|x\| = 1\} < \infty,$$

where $f_x(z) = f(zx),\ z \in \mathbf{D}$.

Adding up $|f(0)|$ to each of the quantities above to avoid constant functions to have norm 0 we get Banach spaces. It was shown by Timoney that they are equivalent norms in the space $\mathcal{B}(B_n)$.

However, these definitions using the gradient or the radial derivative, do not allow us to define a semi-norm invariant by automorphisms. It seems that this is why Timoney used the Bergman metric on B_n to define the norm $Q_f(z)$ of df as a cotangent vector, so f is said to belong to the Bloch functions space on B_n, $\mathcal{B}(B_n)$, if $\|f\|_{\mathcal{B}(B_n)} := \sup_{z\in B_n} Q_f(z) < \infty$. This expression satisfies $\|f\circ\varphi\|_{\mathcal{B}(B_n)} = \|f\|_{\mathcal{B}(B_n)}$ for any $\varphi \in Aut(B_n)$ since the Bergman metric is also invariant by automorphisms. Timoney also proved that this is equivalent to the previous formulations and got the bonus of the invariance under automorphisms.

Recall that the invariant gradient of a holomorphic function $f : B_n \to \mathbb{C}$ at $z \in B_n$ is $\widetilde{\nabla} f(z) := \nabla(f\circ\varphi_z)(0)$. Zhu proved that a holomorphic function $f : B_n \longrightarrow \mathbb{C}$ belongs to the Bloch space $\mathcal{B}(B_n)$ if and only if $\sup_{z\in B_n}\|\widetilde{\nabla} f(z)\| < \infty$ (see [8]).

2. The space of Bloch functions on the unit ball of a Hilbert space E

2.1. **Holomorphic functions on E.** A function $f : B_E \to \mathbb{C}$ is said to be holomorphic if it is Fréchet differentiable at every $x \in B_E$ or, equivalently, if $f(x) = \sum_{n=1}^{\infty} P_n(x)$ for all $x \in B_E$, where P_n is an $n-$homogeneous polynomial, that is, the restriction to the diagonal of a continuous $n-$linear form on the n-fold space $E \times \cdots \times E$. The space $H^\infty(B_E)$ is given by $\{f : B_E \to \mathbb{C} : f \textit{ is holomorphic and bounded } \}$ and it becomes a uniform Banach algebra when endowed with the sup-norm $\|f\|_\infty = \sup\{|f(x)| : x \in B_E\}$. It is, obviously, the analogue of the space H^∞ for E.

Let $(e_k)_{k\in\Gamma}$ be an orthonormal basis of E that we fix at once. Then every $z \in E$ can be written as $z = \sum_{k\in\Gamma} z_k e_k$ and we write $\overline{z} = \sum_{k\in\Gamma} \overline{z_k} e_k$.

Given a holomorphic function $f : B_E \to \mathbb{C}$ and $x \in B_E$, we will denote, as usual, by $\nabla f(x)$ the gradient of f at x, that is, the unique element in E representing the linear operator $f'(x) \in E^*$. It may be written $\nabla f(x) = \left(\frac{\partial f}{\partial x_k}(x)\right)_{k\in\Gamma}$, and so

$$f'(x)(z) = \sum_{k\in\Gamma} \frac{\partial f}{\partial x_k}(x) z_k = \langle z, \overline{\nabla f(x)}\rangle.$$

Bearing in mind the classical Bloch spaces defined on the unit ball of $\mathbb{C}$ and $\mathbb{C}^n$, and their possible definitions, we set the following possible norms on the space for the unit ball of E.

Definition 2.1. We define $\mathcal{B}(B_E)$ as the space of holomorphic functions $f : B_E \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{B}(B_E)} := \sup_{x\in B_E} (1 - \|x\|^2)\|\nabla f(x)\| < \infty.$$

As usual $\|f\|_{Bloch(B_E)} := |f(0)| + \|f\|_{\mathcal{B}(B_E)}$ is a complete norm on $\mathcal{B}(B_E)$.

We first observe that the study of Bloch functions defined on the unit ball of E can be reduced to studying functions defined on finite dimensional subspaces. For each $z \in E$ and each finite subset ν of Γ write $z_\nu = \sum_{k\in\nu} z_k e_k$.

Let us use the following notations: Let $n \in \mathbb{N}$, write $\mathbf{z_n} = (z_1, \cdots, z_n) \in B_n$ and denote

$$SO_n = \{\mathbf{y} = (y_1, \cdots, y_n) : y_k \in E, \langle y_k, y_j\rangle = \delta_{k,j}\},$$

that is to say the family of orthonormal systems of order n for $n \geq 2$ and SO_1 the unit sphere of E. Now for each $\mathbf{y} \in SO_n$ and $f : B_E \to \mathbb{C}$ holomorphic we define

$$f_{\mathbf{y}}(\mathbf{z_n}) = f\Big(\sum_{k=1}^{n} z_k y_k\Big). \tag{2.1}$$

We have for each $1 \leq k \leq n$

$$\frac{\partial f_{\mathbf{y}}}{\partial z_k}(\mathbf{z_n}) = \Big\langle y_k, \overline{\nabla f}\Big(\sum_{j=1}^{n} z_j y_j\Big)\Big\rangle.$$

This gives

$$\big\|\nabla f_{\mathbf{y}}(\mathbf{z_n})\big\| = \Big\|\nabla f\Big(\sum_{j=1}^{n} z_j y_j\Big)\Big\|. \tag{2.2}$$

For each finite subset ν of Γ we denote $f_\nu = f_{\mathbf{y}}$ for $\mathbf{y} = \{e_k : k \in \nu\}$.

Proposition 2.2. *Let $f : B_E \to \mathbb{C}$ be holomorphic. Then the following statements are equivalent:*

(i) $f \in \mathcal{B}(B_E)$,

(ii) $\sup\{\|f_\nu\|_{\mathcal{B}(B_{|\nu|})}: \ \nu \subset \Gamma$ *finite* $\} < \infty$,

(iii) $\sup_{\mathbf{y}\in SO_m} \|f_{\mathbf{y}}\|_{\mathcal{B}(B_m)} < \infty$ *for all* $m \geq 2$,

(iv) *There exists* $m \geq 2$ *such that* $\sup_{\mathbf{y}\in SO_m} \|f_{\mathbf{y}}\|_{\mathcal{B}(B_m)} < \infty$.

Moreover, for each $m \geq 2$,

$$\|f\|_{\mathcal{B}(B_E)} = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{B}(B_n)} = \sup_{\mathbf{y}\in SO_m} \|f_{\mathbf{y}}\|_{\mathcal{B}(B_m)}.$$

Proof. (i) $\Longrightarrow$ (ii) Let $\nu \subset \Gamma$ finite, $n = |\nu|$ and $\mathbf{z_n} \in B_n$. According to (2.2),

$$\big\|\nabla f_\nu(\mathbf{z_n})\big\| = \big\|\nabla f\big(\sum_{j\in\nu} z_j e_j\big)\big\|. \tag{2.3}$$

Since $\|\sum_{j\in\nu} z_j y_j\| = \|\mathbf{z_n}\|$ we obtain $\|f_\nu\|_{\mathcal{B}(B_n)} \leq \|f\|_{\mathcal{B}(B_E)}$. In particular

$$\sup\{\|f_\nu\|_{\mathcal{B}(B_{|\nu|})}: \ \nu \subset \Gamma \text{ finite } \} \leq \|f\|_{\mathcal{B}(B_E)}.$$

(ii) $\Longrightarrow$ (i) Let $x = \sum_{k\in\Gamma} z_k e_k \in B_E$, actually a series whose partial sums we denote s_n. Since f is holomorphic one has that

$$\|\nabla f(x)\| = \lim_n \|\nabla f(s_n)\| \leq \sup\{\|\nabla f_\nu(\mathbf{z}_{|\nu|})\| : \nu \subset \Gamma \text{ finite}\}.$$

Hence, since $\|s_n\| = \|\mathbf{z}_{|\nu|}\| \leq \|x\|$ with $n = |\nu|$, we obtain

$$(1 - \|x\|^2)\|\nabla f(x)\| \leq \sup\big\{(1 - \|\mathbf{z}_{|\nu|}\|^2)\|\nabla f_n(\mathbf{z}_{|\nu|})\| : \ \nu \subset \Gamma \text{ finite } \big\}.$$

(i) $\Longrightarrow$ (iii) follows analogously to (i) $\Longrightarrow$ (ii).

(iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (i) Assume now that $\sup_{\mathbf{y}\in SO_m} \|f_{\mathbf{y}}\|_{\mathcal{B}(B_m)} < \infty$ for some $m \geq 2$. Fix $x \in B_E \setminus \{0\}$ and choose $y \in E$ a unit vector such that $\|\nabla f(x)\| = |\langle y, \nabla f(x)\rangle|$. Now write $y = \sum_{j=1}^m \alpha_j y_j$ for some $(\alpha_1, \cdots, \alpha_m) \in \mathbb{C}^m$ where $\mathbf{y} \in SO_m$ such that $y_1 = \frac{x}{\|x\|}$. Using now (2.2) for $\mathbf{z_m} = (\|x\|, 0, \cdots, 0)$ we obtain $\|\mathbf{z_m}\| = \|x\|$ and $\|\nabla f_{\mathbf{y}}(\mathbf{z_m})\| = \|\nabla f(x)\|$. This gives

$$\|f\|_{\mathcal{B}(B_E)} \leq \sup_{\mathbf{y}\in SO_m} \|f_{\mathbf{y}}\|_{\mathcal{B}(B_m)}.$$

□

We now show that descriptions in terms of the radial derivative and the one dimensional case can be obtained as well.

Definition 2.3. For a holomorphic function $f : B_E \to \mathbb{C}$ we set

$$\|f\|_{\mathcal{R}} := \sup_{x\in B_E} (1 - \|x\|^2)|\mathcal{R}f(x)|,$$

where $\mathcal{R}(f)(x) = \langle\nabla f(x), \bar{x}\rangle$, $x \in B_E$ and

$$\|f\|_{weak} := \sup_{\|y\|=1} \|f_y\|_{\mathcal{B}},$$

where $f_y(z) = f(zy)$, $|z| < 1$, for each $y \in E$ with $\|y\| = 1$. Notice that $zf_y'(z) = \mathcal{R}f(zy)$.

We denote $\mathcal{B}_{\mathcal{R}}(B_E)$ and $\mathcal{B}_{weak}(B_E)$ the space of holomorphic functions on B_E for which $\|f\|_{\mathcal{R}} < \infty$ and $\|f\|_{weak} < \infty$ respectively. As usual, $|f(0)| + \|f\|_{\mathcal{R}}$ and $|f(0)| + \|f\|_{weak}$ are complete norms in these spaces.

Our aim is to show that the three spaces, $\mathcal{B}(B_E)$, $\mathcal{B}_{\mathcal{R}}(B_E)$ and $\mathcal{B}_{weak}(B_E)$ coincide and that their norms are equivalent actually. Comparing $\mathcal{B}_{\mathcal{R}}(B_E)$ and $\mathcal{B}_{weak}(B_E)$ is rather elementary.

Proposition 2.4. *The spaces $\mathcal{B}_{\mathcal{R}}(B_E)$ and $\mathcal{B}_{weak}(B_E)$ coincide. Moreover there exists $C > 0$ such that for every f in the spaces*

$$\|f\|_{\mathcal{R}} \leq \|f\|_{weak} \leq C\|f\|_{\mathcal{R}}.$$

Proof. Let $f \in \mathcal{B}_{weak}(B_E)$. Since

$$\mathcal{R}f(x) = \|x\| f'_{\frac{x}{\|x\|}} \|x\|, \ x \in B_E \setminus \{0\}$$

we have that $\|f\|_{\mathcal{R}} \leq \|f\|_{weak}$ for any holomorphic f defined on B_E, so $f \in \mathcal{B}_{\mathcal{R}}(B_E)$.

Assume now that $f \in \mathcal{B}_{\mathcal{R}}(B_E)$. We check that $f \in \mathcal{B}_{weak}(B_E)$. Let $y \in E$ with $\|y\| = 1$.

Since f is holomorphic at 0, its derivative $f' : B_E \to E^*$ is also holomorphic and thus, bounded on some ball $\bar{B}(0, r)$, $0 < r < 1$, hence there is $M > 0$ such that $|f'(x)| \leq M$ for any $x \in \bar{B}(0, r)$. Thus

$$\sup_{|z| \leq r} (1 - |z|^2)|f'_y(z)| \leq M.$$

While for $|z| > r$ and taking into account that $zf'_y(z) = \mathcal{R}f(zy)$ and that the function $\frac{1-t}{t}$ is decreasing, we have

$$\begin{aligned}
(1 - |z|^2)|f'_y(z)| &= (1 - |z|^2)(1 - |z|)|f'_y(z)| + (1 - |z|^2)|z||f'_y(z)| \\
&\leq (1 - |z|^2)|z|\frac{1 - r}{r}|f'_y(z)| + (1 - \|zy\|^2)|\mathcal{R}f(zy)| \\
&\leq \Big((1 - \|zy\|^2)\frac{1 - r}{r} + (1 - \|zy\|^2)\Big)\mathcal{R}f(zy)|
\end{aligned}$$

Hence

$$\sup_{|z| > r} (1 - |z|^2)|f'_y(z)| \leq \frac{1}{r} \sup_{\|x\| < 1} (1 - \|x\|^2)|\mathcal{R}f(x)|.$$

Therefore $f \in \mathcal{B}_{weak}(B_E)$ as wanted, and so the spaces coincide. Now the open mapping theorem yields the equivalence of both complete norms. □

To compare $\mathcal{B}(B_E)$ and $\mathcal{B}_{weak}(B_E)$ we follow the arguments used by Timoney (see Theorem 4.10 in [6]) applying the following lemma for functions in two variables.

Lemma 2.5 (Lemma 4.11 in [6])**.** *Let $F : B_2 \to \mathbb{C}$ be an analytic function. If there exists $M \geq 0$ such that for any $(z_1, z_2) \in B_2$, the function $F_{(z_1,z_2)}(z) := F(zz_1, zz_2)$, $|z| < 1$, satisfies $\|F_{(z_1,z_2)}\|_{\mathcal{B}} \leq M$, then*

$$\|\nabla F(z_1, 0)\|(1 - |z_1|^2) \leq 3M \log 2 \ \ \textit{for all } z_1 \in \mathbb{C} \ , |z_1| < 1.$$

Although the coming proof follows the same pattern as Theorem 4.10 in [6], we include it for the sake of completeness.

Theorem 2.6. *The spaces $\mathcal{B}(B_E)$, $\mathcal{B}_{\mathcal{R}}(B_E)$ and $\mathcal{B}_{weak}(B_E)$ coincide. Moreover*

$$\|f\|_{\mathcal{R}} \leq \|f\|_{\mathcal{B}(B_E)} \leq (3 \log 2)\|f\|_{weak}.$$

Proof. Of course $|\mathcal{R}f(x)| \leq \|\nabla f(x)\|$ and therefore $\|f\|_{\mathcal{R}} \leq \|f\|_{\mathcal{B}(B_E)}$. Let us show that $\|f\|_{\mathcal{B}(B_E)} \leq (3 \log 2)\|f\|_{weak}$. The result follows using Proposition 2.4.

Let $f \in \mathcal{B}_{weak}(B_E)$ with $\sup_{\|y\|=1} \|f_y\|_{\mathcal{B}} = 1$. Fix $x \in B_E$. Choose $y \in E$ a unit vector such that $\|\nabla f(x)\| = |\langle y, \nabla f(x)\rangle|$. Now consider two orthogonal unit vectors $x_1, x_2 \in B_E$ such that $x = \alpha x_1$ and $y = \alpha_1 x_1 + \alpha_2 x_2$ for some $\alpha, \alpha_1, \alpha_2 \in \mathbb{C}$. Define $F : B_2 \to \mathbb{C}$ by $F(z_1, z_2) = f(z_1 x_1 + z_2 x_2)$. If $L : \mathbb{C}^2 \to E$ is the linear mapping $L(z_1, z_2) := z_1 x_1 + z_2 x_2$, $F = f \circ L$, and the chain rule yields $\nabla F(z_1, z_2) = \big(\langle \nabla f(z_1 x_1 + z_2 x_2), \overline{x_1}\rangle, \langle \nabla f(z_1 x_1 + z_2 x_2, \overline{x_2}\rangle\big)$. Then F satisfies the assumptions of Lemma 2.5, so $|\langle(\alpha_1, \alpha_2), \nabla F(\alpha, 0)\rangle| (1 - |\alpha|^2) \leq 3 \log 2$ and we have the aimed inequality

$$\|\nabla f(x)\|(1 - \|x\|^2) = |\langle y, \nabla f(x)\rangle|(1 - \|x\|^2) \leq 3 \log 2.$$

□

Using Theorem 2.6 and classical results on Bloch functions on the unit disk, one can obtain the following result.

Proposition 2.7. *There exists $C > 0$ such that for the Taylor series $f(x) = \sum_{n=0}^{\infty} P_n(x)$ of any $f \in \mathcal{B}(B_E)$ one has*

$$\|P_n\|_\infty \leq C\|f\|_{\mathcal{B}(B_E)} \text{ for every } n = 1, 2, \dots \tag{2.4}$$

Proof. It suffices to notice that for any $z \in \mathbf{D}$,

$$f_y(z) = \sum_{n=0}^{\infty} P_n(zy) = \sum_{n=0}^{\infty} P_n(y) z^n$$

and to use that for any $\varphi \in \mathcal{B}$ one has (see [1, Lemma 2.1])

$$|a_n| \leq \sqrt{e}\|\varphi\|_{\mathcal{B}}, \ n \geq 1.$$

The result follows now considering $a_n = P_n(y)$ and applying Theorem 2.6. □

Proposition 2.8. *Let (P_k) be a sequence of 2^k-homogeneous polynomials on E with*

$$M = \sup_{k \in \mathbb{N}, y \in B_E} |P_k(y)| < \infty.$$

Then $f(x) = \sum_{k=0}^{\infty} P_k(x) \in \mathcal{B}(B_E)$.

Proof. To see that f is holomorphic in the unit ball, simply observe that if $\|x\| < 1$, then

$$\sum_k |P_k(x)| \leq M \sum_k \|x\|^{2^k} \leq \frac{C}{1 - \|x\|}.$$

From that we get the uniform convergence on compact sets and therefore f is holomorphic. To finish the proof use again that

$$f_y(z) = \sum_{n=0}^{\infty} P_k(y) z^{2^k}$$

and now the fact (see [1, Lemma 2.1])

$$\|\varphi\|_{\mathcal{B}} \leq C \sup_k |a_k|$$

whenever $\varphi(z) = \sum_k a_k z^{2^k}$ together with Theorem 2.6. □

Notice that $f \in H^\infty(B_E)$ is equivalent to $\sup_{\|y\|=1} \|f_y\|_\infty = \|f\|_\infty$. The use of Proposition 1.1 and the fact that $\mathcal{B}(B_E) = \mathcal{B}_{weak}(B_E)$ imply

$$H^\infty(B_E) \subseteq \mathcal{B}(B_E).$$

Let us finish this section with a basic example of an unbounded Bloch function on B_{ℓ_2}.

Example 2.9. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} (\sum_{k=1}^{n} z_k^2)^n, \ z = \sum_{k=1}^{\infty} z_k e_k \in B_{\ell_2}.$$

Then $f \in \mathcal{B}(B_{\ell_2}) \setminus H^\infty(B_{\ell_2})$.

Proof. Let $0 < r < 1$ and $\|z\| \le r$. The series $\sum_{n=0}^{\infty} \frac{1}{n+1} (\sum_{k=1}^{n} z_k^2)^n$ converges uniformly in $\|z\| \le r$ because

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \sup_{\|z\| \le r} \left|(\sum_{k=1}^{n} z_k^2)^n\right| \le \sum_{n=0}^{\infty} \frac{1}{n+1} r^{2n} \le \log(\frac{1}{1-r^2}).$$

Hence f is holomorphic on the unit ball.

On the other hand, for $j \in \mathbb{N}$,

$$\frac{\partial f}{\partial z_j}(z) = \sum_{n=j}^{\infty} \frac{2n}{n+1} (\sum_{k=1}^{n} z_k^2)^{n-1} z_j.$$

Hence

$$|\frac{\partial f}{\partial z_j}(z)| \le \sum_{n=1}^{\infty} \frac{2n}{n+1} (\sum_{k=1}^{n} |z_k|^2)^{n-1} |z_j| \le \sum_{n=1}^{\infty} \frac{2n}{n+1} \|z\|^{2n-2} |z_j|$$

and

$$\|\nabla f(z)\| \le \frac{2}{1-\|z\|^2}.$$

Finally we observe that selecting $x = (z, 0, \cdots,)$ we have $f(x) = \log(\frac{1}{1-z^2})$ and therefore $f \notin H^\infty(B_{\ell_2})$. □

3. A Möbius invariant norm for the Bloch space on the unit ball B_E

Now our aim is to prove that the Bloch space $\mathcal{B}(B_E)$ is invariant under the action of the automorphisms of the ball. We begin by collecting the necessary information about such automorphisms.

3.1. **Automorphisms and the pseudohyperbolic distance on B_E.** The analogues of Möbius transformations on E are the mappings $\varphi_a : B_E \longrightarrow B_E$, $a \in B_E$, defined according to

$$\varphi_a(x) = (s_a Q_a + P_a)(m_a(x)) \tag{3.1}$$

where $s_a = \sqrt{1-\|a\|^2}$, $m_a : B_E \longrightarrow B_E$ is the analytic map

$$m_a(x) = \frac{a-x}{1-\langle x, a\rangle}, \tag{3.2}$$

$P_a : E \longrightarrow E$ is the orthogonal projection along the one-dimensional subspace spanned by a, that is,

$$P_a(x) = \frac{\langle x, a\rangle}{\langle a, a\rangle} a$$

and $Q_a : E \longrightarrow E$, is the orthogonal complement, $Q_a = Id - P_a$. Recall that P_a and Q_a are self-adjoint operators since they are projections, so $\langle P_a(x), y\rangle = \langle x, P_a(y)\rangle$ and $\langle Q_a(x), y\rangle = \langle x, Q_a(y)\rangle$ for any $x, y \in E$.

The automorphisms of the unit ball B_E turn to be compositions of such analogous Möbius transformations with unitary transformations U of E, that is, self-maps of E satisfying $\langle U(x), U(y)\rangle = \langle x, y\rangle$ for all $x, y \in E$.

We will also need the following facts about the pseudohyperbolic distance in B_E. It is given by

$$\rho_E(x, y) = \|\varphi_{-y}(x)\| \text{ for any } x, y \in B_E \tag{3.3}$$

and it satisfies that

$$\rho(f(x), f(y)) \le \rho_E(x, y) \quad \text{for } f \in H^\infty(B_E) \text{ with } \|f\|_\infty \le 1, \tag{3.4}$$

where $\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|$ for any $z, w \in \mathbf{D}$ is the pseudohyperbolic distance in $\mathbf{D}$. Actually,

$$\rho_E(x,y)^2 = 1 - \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-\langle x,y\rangle|^2} \tag{3.5}$$

For all these facts and further information on the automorphisms of B_E and the pseudohyperbolic distance, see [3].

3.2. **The invariance under automorphisms.** As it happens in the finite dimensional case $n \geq 2$, it is not true that $\|f \circ \varphi\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ for all $f \in \mathcal{B}(B_E)$. This is also false for the norms $\|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_{weak}$. Thus, in Timoney' spirit, we are led to find a semi-norm on $\mathcal{B}(B_E)$ that is invariant under the automorphisms of the ball. Our goal is to give a direct proof using only properties of automorphisms of B_E and not the invariant Laplacian.

Definition 3.1. Let $f : B_E \longrightarrow \mathbb{C}$ be a holomorphic function. The invariant gradient $\widetilde{\nabla} f$ is defined by $\widetilde{\nabla} f(x) = \nabla(f \circ \varphi_x)(0)$ for any $x \in B_E$.

Let us first relate the invariant gradient with the standard one and the radial derivative. We need first following easy fact.

Lemma 3.2. *Let $x \in B_E$. Then, $\varphi_x'(0) = -s_x^2 P_x - s_x Q_x$.*

Proof. Recall that $\varphi_x(y) = (P_x + s_x Q_x)(m_x(y))$, where $m_x(y) = \frac{x-y}{1-\langle x,y\rangle}$.
The derivative of m_x is given by

$$m_x'(y)(t) = \frac{-(1-\langle x,y\rangle)t + \langle t,x\rangle x - \langle t,x\rangle y}{(1-\langle x,y\rangle)^2},$$

so $m_x'(0) = -Id + \|x\|^2 P_x$ and, by the chain rule,

$$\begin{aligned}\varphi_x'(0) &= (P_x + s_x Q_x)'(m_x(0)) \circ m_x'(0) = (P_x + s_x Q_x) \circ (-Id + \|x\|^2 P_x) \\ &= -P_x - s_x Q_x + \|x\|^2 P_x = -s_x^2 P_x - s_x Q_x.\end{aligned}$$

□

Lemma 3.3 is a generalization of Lemma 2.13 in [8] for the infinite dimensional case.

Lemma 3.3. *Let $f : B_E \to \mathbb{C}$ be a holomorphic function. Then,*

$$\|\widetilde{\nabla} f(x)\|^2 + (1-\|x\|^2)|\mathcal{R}f(x)|^2 = (1-\|x\|^2)\|\nabla f(x)\|^2.$$

Proof. By Lemma 3.2, $\varphi_x'(0) = -s_x^2 P_x - s_x Q_x$, so by the chain rule and bearing in mind that P_x, Q_x are self-adjoint, we get

$$\begin{aligned}\|\widetilde{\nabla} f(x)\|^2 &= \left(\sup_{y\in B_E} |(\nabla f(x) \circ \varphi_x'(0))(y)|\right)^2 \\ &= \left(\sup_{y\in B_E} |\langle \varphi_x'(0)(y), \overline{\nabla f(x)}\rangle|\right)^2 \\ &= \|\varphi_x'(0)(\overline{\nabla f(x)})\|^2 \\ &= s_x^4\|P_x(\overline{\nabla f(x)})\|^2 + s_x^2\|Q_x(\overline{\nabla f(x)})\|^2 \\ &= s_x^4\|P_x(\overline{\nabla f(x)})\|^2 + s_x^2\|\overline{\nabla f(x)}\|^2 - s_x^2\|P_x(\overline{\nabla f(x)})\|^2 \\ &= s_x^2(s_x^2-1)\|P_x(\overline{\nabla f(x)})\|^2 + s_x^2\|\nabla f(x)\|^2 \\ &= s_x^2\|\nabla f(x)\|^2 - s_x^2\|x\|^2\frac{|\langle\overline{\nabla f(x)}, x\rangle|^2}{\|x\|^2} \\ &= s_x^2(\|\nabla f(x)\|^2 - |\mathcal{R}f(x)|^2)\end{aligned}$$

□

Definition 3.4. Denote $\mathcal{B}_{inv}(B_E)$ the set of holomorphic functions $f : B_E \to \mathbb{C}$ such that

$$\|f\|_{inv} := \sup_{x\in B_E} \|\widetilde{\nabla} f(x)\| < \infty.$$

It is clear that $\|f \circ \varphi\|_{inv} = \|f\|_{inv}$ for any $f \in \mathcal{B}(B_E)$ and any automorphism φ of B_E.

Note that $|\mathcal{R}f(x)| \leq \|\nabla f(x)\|\|x\|$ and then

$$\|\nabla f(x)\|^2 - |\mathcal{R}f(x)|^2 \geq (1 - \|x\|^2)\|\nabla f(x)\|^2.$$

Hence, using Lemma 3.3, we have

$$(1 - \|x\|^2)\|\nabla f(x)\| \leq \|\widetilde{\nabla} f(x)\| \leq \sqrt{1 - \|x\|^2}\|\nabla f(x)\|. \tag{3.6}$$

In particular, $\mathcal{B}_{inv}(B_E) \subseteq \mathcal{B}(B_E)$ and $\|f\|_{\mathcal{B}(B_E)} \leq \|f\|_{inv}$.

We shall show that $\mathcal{B}(B_E) = \mathcal{B}_{inv}(B_E)$. We will neither use the facts related to the Bergman metric or $Q_f(z)$ of the finite dimensional case used by Timoney nor the properties of the invariant gradient and its connection to the invariant Laplacian used by Zhu.

We shall use the following lemma, which generalizes Theorem 3.1 in [8] for the infinite dimensional case. It gives a different explicit formula for $\|\widetilde{\nabla} f(x)\|$ which is closely related to the expression of $Q_f(z)$ for the finite dimensional case. This calculation will help us in proving Theorem 3.8.

Lemma 3.5. *Let $f : B_E \to \mathbb{C}$ be a holomorphic function. Then,*

$$\|\widetilde{\nabla} f(x)\| = \sup_{w\neq 0} \frac{|\langle \nabla f(x), w\rangle|(1 - \|x\|^2)}{\sqrt{(1 - \|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}}.$$

Proof. Notice that

$$\|\widetilde{\nabla} f(x)\|^2 = \|\varphi_x'(0)(\nabla f(x))\|^2 = \sup_{w\neq 0} \frac{\langle \varphi_x'(0)(\nabla f(x)), w\rangle}{\|w\|}$$

and bearing in mind that $\varphi_x'(0)$ is a linear invertible and self-adjoint operator, we have that

$$\|\widetilde{\nabla} f(x)\|^2 = \sup_{w\neq 0} \frac{|\langle \varphi_x'(0)(\nabla f(x)), \varphi_x'(0)^{-1}(w)\rangle|}{\|\varphi_x'(0)^{-1}(w)\|} = \sup_{w\neq 0} \frac{|\langle \nabla f(x), w\rangle|}{\|\varphi_x'(0)^{-1}(w)\|}.$$

Since $\varphi_x'(0)^{-1} = -\frac{1}{s_x^2}P_x - \frac{1}{s_x}Q_x$, we have that

$$\begin{aligned}
\|\varphi_x'(0)^{-1}(w)\|^2 &= \frac{1}{s_x^4}\|P_x(w)\|^2 + \frac{1}{s_x^2}\|Q_x(w)\|^2 \\
&= \frac{1}{s_x^4}\|P_x(w)\|^2 + \frac{1}{s_x^2}(\|w\|^2 - \|P_x(w)\|^2) \\
&= \frac{1 - s_x^2}{s_x^4}\frac{|\langle w, x\rangle|^2}{\|x\|^2} + \frac{s_x^2}{s_x^4}\|w\|^2 \\
&= \frac{(1 - \|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}{(1 - \|x\|^2)^2},
\end{aligned}$$

so we conclude that

$$\|\widetilde{\nabla} f(x)\| = \sup_{w\neq 0} \frac{|\langle \nabla f(x), w\rangle|(1 - \|x\|^2)}{\sqrt{(1 - \|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}}.$$

□

We shall use a result about holomorphic functions in two complex variables due to Timoney.

Lemma 3.6. *(see Lemma 4.8 in* [6]*) Let $F : B_2 \to \mathbb{C}$ be a holomorphic function satisfying*

$$\|F\|_{\mathcal{B}(B_2)} := \sup_{(z_1,z_2)\in B_2} (1-|z_1|^2-|z_2|^2)\big\|\nabla F(z_1,z_2)\big\|.$$

Then for each $z \in \mathbf{D}$,

$$\left|\frac{\partial F}{\partial w}(z,0)\right|(1-|z|^2)^{1/2} \leq \frac{\sqrt{31}}{2}\ \|F\|_{\mathcal{B}(B_2)}.$$

The previous lemma was also generalized in [6] to the unit ball of $\mathbb{C}^n$. We give the proof for B_E for the sake of completeness.

Lemma 3.7. *Let $f : B_E \to \mathbb{C}$ be a holomorphic function satisfying $\|f\|_{\mathcal{B}(B_E)} < \infty$. Let $x_0 \in B_E$ and $y \in E$, $\|y\| = 1$, such that $\langle x_0, y\rangle = 0$. Then*

$$|\langle \nabla f(x_0), y\rangle|(1-\|x_0\|^2)^{1/2} \leq \frac{\sqrt{31}}{2}\|f\|_{\mathcal{B}(B_E)}.$$

Proof. Set $x_0' = x_0/\|x_0\|$. We consider the linear mapping $L : \mathbb{C}^2 \to E$ given by $L(z_1,z_2) = x_0' z_1 + y z_2$ and define the holomorphic function $F : B_2 \to \mathbb{C}$ by $F(z_1,z_2) = (f\circ L)(z_1,z_2)$. Applying the Chain Rule, we have

$$\nabla F(z_1,z_2) = \big(\langle \nabla f(z_1x_0'+z_2y), \overline{x_0'}\rangle, \langle \nabla f(z_1x_0'+z_2y), \overline{y}\rangle\big).$$

Using that x_0' and y are orthonormal vectors in E we conclude that $\|F\|_{\mathcal{B}(B_2)} \leq \|f\|_{\mathcal{B}(B_E)}$. Therefore, we can apply Lemma 3.6 to obtain

$$\left|\frac{\partial F}{\partial z_2}(\|x_0\|,0)\right|(1-\|x_0\|^2)^{1/2} \leq \frac{\sqrt{31}}{2}\|f\|_{\mathcal{B}(B_E)}.$$

Since

$$\frac{\partial F}{\partial z_2}(\|x_0\|,0) = \langle \nabla f(x_0), y\rangle$$

we finish the proof. □

Theorem 3.8. *Let $f : B_E \to \mathbb{C}$ be a holomorphic function. Then $f \in \mathcal{B}(B_E)$ if and only if $f \in \mathcal{B}_{inv}(B_E)$. The underlying semi-norms satisfy*

$$\|f\|_{\mathcal{B}(B_E)} \leq \|f\|_{inv} \leq C\|f\|_{\mathcal{B}(B_E)}$$

for some constant $C \leq (1+\frac{\sqrt{31}}{2})$.

Proof. We already mentioned that $\|f\|_{\mathcal{B}(B_E)} \leq \|f\|_{inv}$.

Let us show that $\|f\|_{inv} \leq C\|f\|_{\mathcal{B}(B_E)}$. By Lemma 3.5, we have that

$$\|\widetilde{\nabla} f(x)\| = \sup_{w\neq 0} \frac{|\langle \nabla f(x), w\rangle|(1-\|x\|^2)}{\sqrt{(1-\|x\|^2)\|w\|^2+|\langle w,x\rangle|^2}}.$$

Fix $x \in B_E$. Every $v \in E$ can be decomposed as $v = \lambda x + y$, where y is orthogonal to x. Then,

$$\begin{aligned}
\frac{|\langle \nabla f(x), v\rangle|(1-\|x\|^2)}{\sqrt{(1-\|x\|^2)\|v\|^2+|\langle v,x\rangle|^2}} &= \frac{|\langle \nabla f(x), v\rangle|(1-\|x\|^2)}{\sqrt{|\lambda|^2\|x\|^2+(1-\|x\|^2)\|y\|^2}} \\
&= \frac{|\langle \nabla f(x), \lambda x\rangle|(1-\|x\|^2)}{\sqrt{|\lambda|^2\|x\|^2+(1-\|x\|^2)\|y\|^2}} + \frac{|\langle \nabla f(x), y\rangle|(1-\|x\|^2)}{\sqrt{|\lambda|^2\|x\|^2+(1-\|x\|^2)\|y\|^2}} \\
&\leq \frac{|\langle \nabla f(x), \lambda x\rangle|(1-\|x\|^2)}{\sqrt{|\lambda|^2\|x\|^2}} + \frac{|\langle \nabla f(x), y\rangle|(1-\|x\|^2)}{\sqrt{|(1-\|x\|^2)\|y\|^2}} \\
&= |\langle \nabla f(x), \frac{x}{\|x\|}\rangle|(1-\|x\|^2) + |\langle \nabla f(x), \frac{y}{\|y\|}\rangle|\sqrt{1-\|x\|^2}.
\end{aligned}$$

Hence, applying Lemma 3.7,

$$\|f\|_{inv} = \sup_{x\in B_E}\left(\sup_{v\neq 0}\frac{|\langle\nabla f(x),v\rangle|(1-\|x\|^2)}{\sqrt{(1-\|x\|^2)\|v\|^2+|\langle v,x\rangle|^2}}\right)\leq (1+\frac{\sqrt{31}}{2})\|f\|_{\mathcal{B}(B_E)}.$$

□

4. The embeddeding of $H^\infty(B_E)$ into $\mathcal{B}(B_E)$

As we remarked before, $H^\infty(B_E)\subset\mathcal{B}(B_E)$. However, the use of the equivalence of the norms $\|f\|_{\mathcal{B}(B_E)}$ and $\|f\|_{\mathcal{R}}$ does not give the best constant for the embedding. To improve the result we give firstly an elementary lemma.

Lemma 4.1. *Let $x,y\in B_E$. Then,*

$$\rho_E(x,y)\leq\frac{\|x-y\|}{|1-\langle x,y\rangle|}.$$

Proof. We have

$$\rho_E(x,y)^2 = 1-\frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-\langle x,y\rangle|^2}=\frac{-2\Re\langle x,y\rangle+|\langle x,y\rangle|^2+\|x\|^2+\|y\|^2-\|x\|^2\|y\|^2}{|1-\langle x,y\rangle|^2}.$$

Using now $|\langle x,y\rangle|^2-\|x\|^2\|y\|^2\leq 0$ we have

$$\rho_E(x,y)^2\leq\frac{-2\Re\langle x,y\rangle+\|x\|^2+\|y\|^2}{|1-\langle x,y\rangle|^2}=\frac{\|x-y\|^2}{|1-\langle x,y\rangle|^2}.$$

□

Theorem 4.2. *Let $f\in H^\infty(B_E)$ such that $\|f\|_\infty\leq 1$. For any $x\in B_E$, we have that*

$$(1-\|x\|^2)\|\nabla f(x)\|\leq 1-|f(x)|^2.$$

Proof. Notice that $f'(x)$ is the functional on E given by

$$f'(x)(y)=\lim_{t\to 0}\frac{f(x+ty)-f(x)}{t}\quad\text{for any } y\in E.$$

There is $y\in E$, $\|y\|=1$ such that $\|f'(x)\|=|f'(x)(y)|$. Put $\delta:=1-\|x\|$ and consider only t such that $|t|<\delta$. For those t, $\|x+ty\|\leq\|x\|+\delta<1$, so $x+ty\in B_E$ and we have, applying Lemma 4.1, that

$$\rho_E(x+ty,x)\leq\frac{|t|}{|1-\langle x,x+ty\rangle|}\quad\text{if}\quad |t|<\delta.$$

On the other hand

$$\begin{aligned}\left|\frac{f(x+ty)-f(x)}{t}\right| &= \left|\frac{f(x+ty)-f(x)}{1-\overline{f(x)}f(x+ty)}\right|\left|\frac{1-\overline{f(x)}f(x+ty)}{t}\right|\\ &= \rho\big(f(x+ty),f(x)\big)\left|\frac{1-\overline{f(x)}f(x+ty)}{t}\right|\\ &\leq \rho_E(x+ty,x)\left|\frac{1-\overline{f(x)}f(x+ty)}{|t|}\right|\\ &\leq \frac{|1-\overline{f(x)}f(x+ty)|}{|1-\langle x,x+ty\rangle|},\end{aligned}$$

where we have used that the pseudohyperbolic distance is contractive for f.

Hence

$$\left|f'(x)(y)\right| \leq \limsup_{t\to 0} \frac{|1-\overline{f(x)}f(x+ty)|}{|1-\langle x, x+ty\rangle|} = \frac{1-|f(x)|^2}{1-\|x\|^2}.$$

Therefore,

$$(1-\|x\|^2)\|f'(x)\| \leq 1-|f(x)|^2.$$

□

So we get the following extension of Proposition 1.1.

Corollary 4.3. *The inclusion $i : H^\infty(B_E) \longrightarrow \mathcal{B}(B_E)$ is a linear operator satisfying*

$$\|f\|_{\mathcal{B}(B_E)} \leq \|f\|_\infty.$$

Corollary 4.4. *Let $f \in H^\infty(B_E)$ with $\|f\|_\infty = 1$ and $\varphi \in \mathcal{B}$. Then $g = \varphi \circ f \in \mathcal{B}(B_E)$ and $\|g\|_{\mathcal{B}(B_E)} \leq \|\varphi\|_{\mathcal{B}}$.*

In particular, $f(x) = \log(1-\langle x, e_1\rangle) \in \mathcal{B}(B_E) \setminus H^\infty(B_E)$.

Proof. Using the chain rule for $g(x) = \varphi(f(x))$ we have $\nabla g(x) = \varphi'(f(x))\nabla f(x)$. Therefore, using Theorem 4.2 we conclude

$$\|\nabla g(x)\| = |\varphi'(f(x))|\|\nabla f(x)\| \leq \frac{\|\varphi\|_{\mathcal{B}}\|\nabla f(x)\|}{1-|f(x)|^2} \leq \|\varphi\|_{\mathcal{B}}.$$

□

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