Bilinear multipliers and transference.

Oscar Blasco∗

Abstract
Let \( m(\xi, \eta) \) be a regulated function in \( \mathbb{R} \times \mathbb{R} \), \( 1 \leq p_1, p_2, p_3 \leq \infty \) and \( 1/p_1 + 1/p_2 = 1/p_3 \). It is shown that \( m \) defines a bilinear bounded \( (p_1, p_2) \)-multiplier on \( \mathbb{R} \times \mathbb{R} \) if and only if there exists a constant \( K \) so that
\[
| \sum_{(t,s) \in \mathbb{R} \times \mathbb{R}} m(t, s) \mu\{t\} \nu\{s\} \lambda\{t + s\} | \leq K \| \hat{\mu} \|_{B^{p_1}} \| \hat{\nu} \|_{B^{p_2}} \| \hat{\lambda} \|_{B^{p_3^*}}
\]
for all measures \( \mu, \nu, \lambda \) supported on a finite number of points, where
\[
\| \hat{\mu} \|_{B^{p}} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} | \hat{\mu}(\xi) |^p d\xi \right)^{1/p}.
\]

1 Introduction.

Let \((p_1, p_2, p_3)\) such that \( 1 \leq p_1, p_2, p_3 \leq \infty \), \( 1/p_1 + 1/p_2 = 1/p_3 \) and let \( m(\xi, \eta) \) be a bounded measurable function in \( \mathbb{R}^2 \). It is said to be a bilinear \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) if
\[
C_m(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i x (\xi + \eta)} d\xi d\eta
\]
(defined for functions \( f, g \) in the Schwartz class \( S \)) extends to a bounded bilinear operator from \( L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \) into \( L^{p_3}(\mathbb{R}) \).

The theory of these multipliers has been tremendously developed after the results proved by M. Lacey and R. Thiele ([20, 21, 22]) which establish that \( m(\xi, \eta) = \text{sign}(\xi + \alpha \eta) \) are \((p_1, p_2)\)-multipliers for each triple \((p_1, p_2)\) such that \( 1 < p_1, p_2 \leq \infty \), \( p_3 > 2/3 \) and each \( \alpha \in \mathbb{R} \setminus \{0, 1\} \).

The study of such multipliers was started by R. Coifman and Y. Meyer (see [3, 5, 6]) for smooth symbols and new results for non-smooth symbols, extending the ones given by the bilinear Hilbert transform, have been achieved

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by J.E. Gilbert and A.R. Nahmod (see [10, 11, 12]) and also by J. Muscalu, T. Tao and C. Thiele (see [19]). We refer the reader also to [18, 17, 9, 13] for new results on bilinear multipliers and related topics.

In a recent paper (see [9]) D. Fan and S. Sato have shown certain DeLeeuw type theorems for transferring multilinear operators on Lebesgue and Hardy spaces $L^p(\mathbb{R})$ and get a characterization which allows us to transfer not only to the bilinear multipliers on $\mathbb{T}$ but also on $\mathbb{Z}$. Our approach will follow closely the ideas in the original paper by DeLeeuw (see [8]) and will provide an alternative proof to some results in [9].

Let us start by setting up natural analogue versions of bilinear multipliers in the periodic and discrete cases. Let $(m_{k,k'})$ be a bounded sequence and $\tilde{m}$ be a periodic function defined on $\mathbb{T} \times \mathbb{T}$. Define

$$P_m(f,g)(\theta) = \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \hat{f}(k)\hat{g}(k')m_{k,k'}e^{2\pi i \theta (k+k')}$$

for periodic functions $f, g$ defined on $\mathbb{T} \times \mathbb{T}$. Define

$$D_{\tilde{m}}(a,b)(k) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} P(t)Q(s)\tilde{m}(t,s)e^{2\pi i x(t+s)}dtds$$

for sequences $(a(n))_{n \in \mathbb{Z}}$ and $(b(n))_{n \in \mathbb{Z}}$ where $P(t) = \sum_{n \in \mathbb{Z}} a(n)e^{2\pi int}$ and $Q(t) = \sum_{n \in \mathbb{Z}} b(n)e^{2\pi int}$.

Now we say that $(m_{k,k'})$ (respect. $\tilde{m}$) is a a bilinear $(p_1, p_2)$-multiplier on $\mathbb{Z} \times \mathbb{Z}$ (respect. $\mathbb{T} \times \mathbb{T}$ ) if $P_m$ (respect. $D_{\tilde{m}}$) defines a bounded bilinear operator from $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ (respect. $l^{p_1}(\mathbb{Z}) \times l^{p_2}(\mathbb{Z})$ into $l^{p_3}(\mathbb{Z})$).

Of course we can see these three cases as instances of the general bilinear multiplier acting on different groups. Let $G$ be a locally compact abelian group and $\hat{G}$ its dual. Let $1 \leq p_1, p_2 \leq \infty$ and $m$ be a bounded measurable function defined on $\hat{G} \times \hat{G}$. We say that $m$ is a $(p_1, p_2)$-multiplier on $\hat{G} \times \hat{G}$ if the operator

$$T_m(f,g)(x) = \int_{\hat{G}} \int_{\hat{G}} \mathcal{F}f(\gamma_1)\mathcal{F}g(\gamma_2)m(\gamma_1,\gamma_2)\gamma_1(-x)\gamma_2(-x)dm(\gamma_1)dm(\gamma_2)$$

(defined for simple functions $f$ and $g$) extends to a bounded bilinear operator from $L^{p_1}(G) \times L^{p_2}(G)$ to $L^{p_3}(G)$ where $1/p_1 + 1/p_2 = 1/p_3$. 2
The first transference results on linear multiplier were given by K. Deleuwe (see [8]). He showed, among other things, that if $m$ is regulated (all its points are Lebesgue points) and $m$ is a $p$-multiplier on $\mathbb{R}$ then $(m(\varepsilon k))_k$ are uniformly bounded $p$-multipliers for all $\varepsilon > 0$ on $\mathbb{Z}$. See [25] page 264 for the converse of this result for continuous multipliers.

In [9] the multilinear version this result was shown, namely that for continuous functions $m(\xi, \eta)$ one has that $m$ is a $(p_1, p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if $m(\varepsilon k, \varepsilon k')_{k,k'}$ are uniformly bounded multipliers on $\mathbb{Z} \times \mathbb{Z}$. An extension of the result to Lorentz spaces is achieved in [2].

We shall first characterize the boundedness of bilinear multipliers on $\mathbb{R} \times \mathbb{R}$ by the existence of a constant $K$ such that

$$
| \sum_{(t,s) \in \mathbb{R} \times \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) | \leq K \|\hat{\mu}\|_{B_{p_1}} \|\hat{\nu}\|_{B_{p_2}} \|\hat{\lambda}\|_{B_{p_3}}
$$

for all measures $\mu, \nu, \lambda$ of finite supports.

This allows us to present an alternative proof of the result in [9].

We also obtain the transference from the continuous case $C_m$ to the periodic case $P_m$. Our main result establishes that $m$ is $(p_1, p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if $D_\varepsilon m = m(\varepsilon x, \varepsilon y)$ are uniformly bounded $(p_1, p_2)$-multipliers on $\mathbb{T} \times \mathbb{T}$.

Throughout the paper $1 \leq p_1, p_2, p_3 \leq \infty$ and $1/p_3 = 1/p_1 + 1/p_2$. For a given finite Borel measure on $\mathbb{R}$ we write $\mu(\xi) = \int_\mathbb{R} e^{2\pi i \xi t} d\mu(t)$ and, for an almost periodic function $g$, we denote $\|g\|_{B_p} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} |g(t)|^p dt \right)^{1/p}$.

We shall use the notations $D_\varepsilon m(x, y) = m(\varepsilon x, \varepsilon y)$ and $\phi_\varepsilon(x) = \frac{1}{\varepsilon} \phi(\frac{\xi}{\varepsilon})$.

# Bilinear multipliers on $\mathbb{R} \times \mathbb{R}$

Let us start by reformulating the condition of $(p_1, p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$ using duality. The proof is straightforward and left to the reader.

**Lemma 2.1** Let $m(\xi, \eta)$ be a bounded measurable function on $\mathbb{R} \times \mathbb{R}$.

$m$ is a $(p_1, p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if there exists a constant $K$ so that

$$
| \int_\mathbb{R} \int_\mathbb{R} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\xi, \eta) d\xi d\eta | \leq K \|\hat{\phi}\|_{p_1} \|\hat{\psi}\|_{p_2} \|\hat{\nu}\|_{p_3}
$$

for all $\phi, \psi, \nu \in S$. 

3
Now we present some behavior of multipliers on \( \mathbb{R} \times \mathbb{R} \) with respect to convolution and dilation operators to be used later on.

**Lemma 2.2** Let \( m(\xi, \eta) \) be a bounded measurable function on \( \mathbb{R} \times \mathbb{R} \). If \( \Phi \in L^1(\mathbb{R}^2) \) and \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) then \( m * \Phi \) a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) and \( \|C_{\Phi \ast m}\| \leq \|\Phi\|_1 \|C_m\| \).

**Proof.** Let \( \phi, \psi, \nu \in \mathcal{S} \) and \( \|\hat{\phi}\|_{p_1} = \|\hat{\psi}\|_{p_2} = \|\hat{\nu}\|_{p'_{-1}} = 1 \). Applying Lemma 2.1 to \( \phi \ast_s, \psi \ast_t, \nu \ast_{s+t} \) where \( f_s(x) = f(x + s) \), we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\xi + s)\psi(\eta + t)\nu(\xi + \eta + s + t)m(\xi, \eta)d\xi d\eta \leq K
\]

for all \((s, t) \in \mathbb{R}^2\).

Therefore

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\xi)\psi(\eta)\nu(\xi + \eta)m * \Phi(\xi, \eta)d\xi d\eta
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\xi)\psi(\eta)\nu(\xi + \eta)(\int_{\mathbb{R}^2} m(\xi - s, \eta - t)\Phi(s, t)dsdt)d\xi d\eta
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}} \phi(\xi + s)\psi(\eta + t)\nu(\xi + \eta + s + t)m(\xi, \eta)\Phi(s, t)d\xi d\eta dsdt.
\]

This gives the result applying Lemma 2.1 again. \( \blacksquare \)

**Lemma 2.3** Let \( \varepsilon > 0 \) and \( m(\xi, \eta) \) be a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \). Then \( m(\varepsilon \xi, \varepsilon \eta) \) is also a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) and \( \|C_{m(\varepsilon \cdot \cdot \varepsilon)}\| \leq \|C_m\| \).

**Proof.** For \( \phi, \psi, \nu \in \mathcal{S} \) and \( \|\hat{\phi}\|_{p_1} = \|\hat{\psi}\|_{p_2} = \|\hat{\nu}\|_{p'_{-1}} = 1 \) we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\xi)\psi(\eta)\nu(\xi + \eta)m(\varepsilon \xi, \varepsilon \eta)d\xi d\eta
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon^{1/p_1}} \phi(\xi) \frac{1}{\varepsilon^{1/p_2}} \psi(\eta) \frac{1}{\varepsilon^{1/p'_{-1}}} \nu(\xi + \eta)m(\xi, \eta)d\xi d\eta
\]

where the functions now appearing in the integral are also norm 1 for each \( \varepsilon \). Use Lemma 2.1 again to finish the proof. \( \blacksquare \)
Theorem 2.4 Let $m(\xi, \eta)$ be a bounded continuous function on $\mathbb{R} \times \mathbb{R}$. The following are equivalent:

(i) $m$ is a $(p_1, p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$.

(ii) There exists a constant $K$ so that

$$ | \sum_{(t, s) \in \mathbb{R} \times \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) | \leq K \| \hat{\mu} \|_{B_{p_1}} \| \hat{\nu} \|_{B_{p_2}} \| \hat{\lambda} \|_{B_{p_3}} $$

for all measures $\mu, \nu, \lambda$ supported on a finite number of points.

Proof. (i) $\Rightarrow$ (ii) Assume that $m$ is a $(p_1, p_2)$-multiplier on $\mathbb{R} \times \mathbb{R}$. Denote by $\phi$ the gaussian function $\phi(x) = e^{-x^2/2}$ and take $0 < \alpha, \beta, \gamma$ such that $\alpha + \beta + \gamma = 2$.

Let us consider $\mu = \delta_a$, $\nu = \delta_b$ and $\lambda = \delta_c$ for $a, b, c \in \mathbb{R}$ and let us observe that

$$ \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon^2} \phi^\alpha(\xi - \frac{a}{\varepsilon}) \phi^\beta(\eta - \frac{b}{\varepsilon}) \phi^\gamma(\xi + \eta - \frac{c}{\varepsilon}) m(\xi, \eta) d\xi d\eta = $$

$$ = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma(\xi + \eta + \frac{a + b - c}{\varepsilon}) m(a + \varepsilon \xi, b + \varepsilon \eta) d\xi d\eta = $$

$$ = \int_{\mathbb{R}} \int_{\mathbb{R}} \mu * (\phi_\varepsilon)^\alpha(\xi) \nu * (\phi_\varepsilon)^\beta(\eta) \lambda * (\phi_\varepsilon)^\gamma(\xi + \eta) m(\xi, \eta) d\xi d\eta. $$

Since

$$ \lim_{\varepsilon \to 0} \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma(\xi + \eta + \frac{a + b - c}{\varepsilon}) m(a + \varepsilon \xi, b + \varepsilon \eta) = $$

$$ \delta_c(a + b) \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma(\xi + \eta) m(a, b), $$

the convergence Lebesgue theorem implies that

$$ \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon^2} \phi^\alpha(\xi - \frac{a}{\varepsilon}) \phi^\beta(\eta - \frac{b}{\varepsilon}) \phi^\gamma(\xi + \eta - \frac{c}{\varepsilon}) m(\xi, \eta) d\xi d\eta $$

$$ = C m(a, b) \delta_c(a + b) = C m(a, b) \mu(\{a\}) \nu(\{b\}) \lambda(\{a + b\}). $$

where $C = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma(\xi + \eta) d\xi d\eta.$

Therefore we have that

$$ \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \mu * (\phi_\varepsilon)^\alpha(\xi) \nu * (\phi_\varepsilon)^\beta(\eta) \lambda * (\phi_\varepsilon)^\gamma(\xi + \eta) m(\xi, \eta) d\xi d\eta $$

5
\[ = C \sum_{(t,s) \in \mathbb{R} \times \mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\}) \]

for all measures \(\mu, \nu, \lambda\) having their supports on finite sets of points.

On the other hand, from the assumption and Lemma 2.1 we have

\[
|\int_{\mathbb{R}} \int_{\mathbb{R}} \mu * (\phi_\varepsilon)^\alpha(\xi) \nu * (\phi_\varepsilon)^\beta(\eta) \lambda * (\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi d\eta| \\
\leq K\|\hat{\mu}(\phi_\varepsilon)^\alpha\|_{p_1} \|\hat{\nu}(\phi_\varepsilon)^\beta\|_{p_2} \|\hat{\lambda}(\phi_\varepsilon)^\gamma\|_{p_3}.
\]

Let us now choose \(\alpha = \frac{1}{p_1}, \beta = \frac{1}{p_2}\) and \(\gamma = \frac{1}{p_3}\). Since \((\phi_\varepsilon)^\alpha = \frac{\varepsilon^{1-\alpha}}{\alpha^{1/2}} \phi_{\varepsilon^{-1/2}}\), we get

\[
(\phi_\varepsilon)^\alpha(\xi) = C_\alpha \varepsilon^\frac{1}{p_1} e^{-\frac{\varepsilon^{\frac{3}{2}}}{2n}}, \quad (\phi_\varepsilon)^\beta(\xi) = C_\beta \varepsilon^\frac{1}{p_2} e^{-\frac{\varepsilon^{\frac{3}{2}}}{2n}} \quad \text{and} \quad (\phi_\varepsilon)^\gamma(\xi) = C_\gamma \varepsilon^\frac{1}{p_3} e^{-\frac{\varepsilon^{\frac{3}{2}}}{2n}}.
\]

Now taking into account that \(\int_{\mathbb{R}} e^{-\frac{\varepsilon^{\frac{3}{2}}}{2n}} d\xi = C\varepsilon^{-1}\) we have that

\[
\|\hat{\mu}(\phi_\varepsilon)^\alpha\|_{p_1} = C\left(\frac{1}{A(\varepsilon)}\right) \int_{\mathbb{R}} |\hat{\mu}(\xi)|^{p_1} e^{-\frac{\varepsilon^{\frac{3}{2}}}{2n}} d\xi)^{1/p_1},
\]

for \(A(\varepsilon) = \int_{\mathbb{R}} e^{-\frac{\varepsilon^{\frac{3}{2}}}{2n}} d\xi\). Hence \(C\|\hat{\mu}\|_{p_1} = \lim_{\varepsilon \to 0} \|\hat{\phi}_\varepsilon\|_{p_1}^\alpha\).

Applying similar procedure for \(\nu\) and \(\lambda\) we finish this implication.

(iii) \(\Rightarrow\) (i) From the assumption we can get that the same holds for all finite measures \(\mu, \nu, \lambda\) with countable support. Let us take \(\phi, \psi\) and \(\rho\) such that \(\hat{\phi}, \hat{\psi}\) and \(\hat{\rho}\) have compact support contain in \([-N/2, N/2]\) for \(N\) big enough.

Now consider \(\mu_N, \nu_N\) and \(\lambda_N\) the measures with support in \((1/N)\mathbb{Z}\) whose Fourier transform coincide with the periodic extensions of \(\phi, \psi\) and \(\rho\). In particular we have

\[
\mu_N\left(\frac{n}{N}\right) = \frac{1}{N} \phi\left(\frac{n}{N}\right), \nu_N\left(\frac{n}{N}\right) = \frac{1}{N} \psi\left(\frac{n}{N}\right) \text{ and } \lambda_N\left(\frac{n}{N}\right) = \frac{1}{N} \rho\left(\frac{n}{N}\right).
\]

Therefore we have

\[
\lim_{N \to \infty} N \sum_{(t,s) \in \mathbb{R} \times \mathbb{R}} m(t,s)\mu_N(\{t\})\nu_N(\{s\})\lambda_N(\{t+s\}) \\
= \lim_{N \to \infty} \sum_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} m(\frac{n}{N}, \frac{m}{N})\phi\left(\frac{n}{N}\right)\psi\left(\frac{m}{N}\right)\rho\left(\frac{n+m}{N}\right)\frac{1}{N^2} \\
= \int_{\mathbb{R} \times \mathbb{R}} m(\xi, \nu)\phi(\xi)\psi(\eta)\rho(\xi+\eta)d\xi d\eta.
\]
Now observe that \( \| \hat{\mu}_N \|_{B_{p_1}} = (\frac{1}{2N} \int_{-N}^{N} |\hat{\phi}(\xi)|^p d\xi)^{1/p} = (\frac{1}{2N})^{1/p} \| \hat{\phi} \|_{p_1} \) and the same for the others.

Using that \( \| \hat{\mu}_N \|_{B_{p_1}}, \| \hat{\nu}_N \|_{B_{p_2}}, \| \hat{\lambda}_N \|_{B_{p'}_3} = \frac{1}{2N} \) and passing to the limit we get the result. \( \square \)

Recall that a function \( m \) is called regulated if

\[
\lim_{\varepsilon \to 0} \frac{1}{4\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} m(x - s, y - t) ds dt = m(x, y)
\]

for all \((x, y) \in \mathbb{R}^2\).

**Theorem 2.5** Let \( m(\xi, \eta) \) be a bounded regulated function on \( \mathbb{R} \times \mathbb{R} \). \( m \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \) if and only if there exists a constant \( K \) so that

\[
| \sum_{(t, \xi) \in \mathbb{R} \times \mathbb{R}} m(t, \xi) \mu(t\{\{t\}\}) \nu(t\{\{s\}\}) \lambda(t\{t + s\}) | \leq K \| \hat{\mu} \|_{B_{p_1}} \| \hat{\nu} \|_{B_{p_2}} \| \hat{\lambda} \|_{B_{p'_3}}
\]

for all measures \( \mu, \nu, \lambda \) having their supports on finite sets of points.

**Proof.** Assume that \( m \) is \((p_1, p_2)\)-multiplier. Denote \( \Phi_\varepsilon(s, t) = \frac{1}{\varepsilon} \chi_{[-1, 1]}(s) \chi_{[-1, 1]}(t) \) and \( \Phi_\varepsilon(\xi, \eta) = \frac{1}{\varepsilon} \Phi_\varepsilon(\xi, \eta) \) for \( \varepsilon > 0 \). Now Lemma 2.2, Theorem 2.4 and the fact that \( m(x, y) = \lim_{\varepsilon \to 0} m * \Phi_\varepsilon(x, y) \) gives the direct implication.

Conversely, assume (2) for \( \mu, \nu, \lambda \) having finite supports,

\[
\sum_{(t, s) \in \mathbb{R} \times \mathbb{R}} m * \Phi_\varepsilon(t, s) \mu(t\{\{t\}\}) \nu(t\{\{s\}\}) \lambda(t\{t + s\})
\]

\[
= \int ( \sum_{(t, s) \in \mathbb{R} \times \mathbb{R}} m(t - u, s - v) \mu(t\{\{t\}\}) \nu(t\{\{s\}\}) \lambda(t\{t + s\}) ) \Phi_\varepsilon(u, v) dudv
\]

\[
= \int ( \sum_{(t, s) \in \mathbb{R} \times \mathbb{R}} m(t, s) \mu(t\{t + u\}) \nu(t\{s + v\}) \lambda(t\{t + s + u + v\}) ) \Phi_\varepsilon(u, v) dudv.
\]

This shows that \( m * \Phi_\varepsilon \) verifies (2) with uniform constant for all \( \varepsilon > 0 \). Now apply Theorem 2.4 to get that \( m * \Phi_\varepsilon \) are \((p_1, p_2)\)-multipliers with uniform norm.

Finally we have that for \( \phi, \psi, \nu \in \mathcal{S} \)

\[
| \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\xi, \eta) d\xi d\eta |
\]

\[
= | \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m * \Phi_\varepsilon(\xi, \eta) d\xi d\eta |
\]

\[
\leq C \| \hat{\phi} \|_{p_1} \| \hat{\psi} \|_{p_2} \| \hat{\nu} \|_{p'_3}.
\]
The result follows now from Lemma 2.1.

3 Transference theorems

Let us mention the formulations for \((p_1, p_2)\)-multipliers on the groups \(\mathbb{T} \times \mathbb{T}\) and \(\mathbb{Z} \times \mathbb{Z}\) which follows directly from duality.

**Lemma 3.1** Let \(\tilde{m}(t, s)\) be a bounded measurable function on \(\mathbb{T} \times \mathbb{T}\). \(m\) is a \((p_1, p_2)\)-multiplier on \(\mathbb{T} \times \mathbb{T}\) if and only if there exists a constant \(K\) so that
\[
\left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} P_a(t)P_b(t+s)\tilde{m}(t, s)dtds \right| \leq K\|a\|_{p_1}\|b\|_{p_2}\|c\|_{p_3}'
\]
for all finite sequences \((a(n))_n, (b(n))_n, (c(n))_n\) where \(P_a(t) = \sum_n a(n)e^{2\pi int}\).

**Lemma 3.2** Let \((m_{k, k'})\) be a bounded sequence on \(\mathbb{Z} \times \mathbb{Z}\). \(m\) is a \((p_1, p_2)\)-multiplier on \(\mathbb{Z} \times \mathbb{Z}\) if and only if there exists a constant \(K\) so that
\[
\left| \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} m_{k, k'} \hat{P}(k)\hat{Q}(k')\hat{R}(k + k') \right| \leq K\|P\|_{p_1}\|Q\|_{p_2}\|R\|_{p_3}'
\]
for all trigonometric polynomials \(P, Q\) and \(R\).

**Theorem 3.3** (See [9]) Let \(m(\xi, \eta)\) be a regulated bounded function on \(\mathbb{R} \times \mathbb{R}\). If \(m(\xi, \eta)\) is a \((p_1, p_2)\)-multiplier on \(\mathbb{R} \times \mathbb{R}\) then \((m(k, k'))_{k, k'}\) is a \((p_1, p_2)\)-multiplier on \(\mathbb{Z} \times \mathbb{Z}\).

**Proof.** According to Lemma 3.2 we have to show that there exists a constant \(K\) so that
\[
\left| \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} m(k, k') \hat{P}(k)\hat{Q}(k')\hat{R}(k + k') \right| \leq K\|P\|_{p_1}\|Q\|_{p_2}\|R\|_{p_3}'
\]
for all trigonometric polynomials \(P, Q\) and \(R\).

This follows by selecting in Theorem 2.5 the measures \(\mu, \nu, \lambda\) such that \(\hat{\mu} = P, \hat{\nu} = Q\) and \(\lambda = R\).
Theorem 3.4 Let \( m(\xi, \eta) \) be a bounded regulated function on \( \mathbb{R} \times \mathbb{R} \). The following are equivalent:

(i) \( m(\xi, \eta) \) is a \((p_1, p_2)\)-multiplier on \( \mathbb{R} \times \mathbb{R} \).

(ii) \( m(\varepsilon, \varepsilon) \chi_{[-\frac{1}{2}, \frac{1}{2}]^n} \chi_{[-\frac{1}{2}, \frac{1}{2}]^n} \) are uniformly bounded \((p_1, p_2)\)-multipliers on \( \mathbb{T} \times \mathbb{T} \).

Proof. (i)→(ii). Using Lemma 3.1, it suffices to show that for any finite sequences \((a(n))_n, (b(n))_n \) and \((c(n))_n \) with \( \|a\|_{p_1} = \|b\|_{p_2} = \|c\|_{p_1} = 1 \) there exists a constant \( K > 0 \) such that

\[
| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi, \eta) P_\alpha(\xi) P_\beta(\eta) P_c(\xi + \eta) d\xi d\eta | \leq K
\]

where \( P_\alpha(\xi) = \sum_n a(n) e^{2\pi in\xi} \).

Since \( P_\alpha(x) \chi_{[-1/2, 1/2]}(x) = \hat{\phi}_\alpha(x) \) where \( \hat{\phi}_\alpha(x) = \sum_n a(n) \frac{\sin(\pi(x-n))}{\pi(x-n)} \) and \( P_c(x) \chi_{[-1,1]}(x) = \hat{\psi}_c(x) \) where \( \hat{\psi}_c(x) = \sum_n c(n) \frac{\sin(2\pi(x-n))}{\pi(x-n)} \) we can write

\[
\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi, \eta) P_\alpha(\xi) P_\beta(\eta) P_c(\xi + \eta) d\xi d\eta = \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi, \eta) \hat{\phi}_\alpha(\xi) \hat{\psi}_\beta(\eta) \hat{\psi}_c(\xi + \eta) d\xi d\eta
\]

Using now the assumption and the known facts that \( \|\hat{\phi}_\alpha\|_{L^p(\mathbb{R})} \approx \|a\|_{\ell^p} \approx \|\hat{\psi}_c\|_{L^p(\mathbb{R})} \) for all \( 1 \leq p \leq \infty \) we obtain the desired inequality.

Now we apply Lemma 2.3 to get the result for each \( \varepsilon \).

(ii)→(i). Let us take \( \phi \) and \( \psi \) such that \( \text{supp} \phi \) and \( \text{supp} \psi \) are contained in \([-1/4, 1/4]\). For a fixed \( u \in [-1/2, 1/2] \) consider the periodic extension of the functions \( \hat{\phi}(\xi) e^{2\pi i\varepsilon \xi}, \hat{\psi}(\eta) e^{2\pi i\varepsilon \eta} \) to be denoted \( \hat{\phi}_\xi \) and \( \hat{\psi}_\eta \) respectively.

If \( a^n(u) = \int_{-1/2}^{1/2} \hat{\phi}_\xi(\xi) e^{-2\pi n \xi} d\xi, b^n(u) = \int_{-1/2}^{1/2} \hat{\psi}_\eta(\eta) e^{-2\pi n \eta} d\eta \) for all \( n \in \mathbb{Z} \) we have that if \( x = k + u \) for some \( k \in \mathbb{Z} \) and \( u \in [-1/2, 1/2] \)

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi, \eta) \hat{\phi}(\xi) \hat{\psi}(\eta) e^{2\pi i x(\xi + \eta)} d\xi d\eta = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi, \eta) \hat{\phi}_\xi(\xi) \hat{\psi}_\eta(\eta) e^{2\pi i k(\xi + \eta)} d\xi d\eta.
\]

Denote \( \hat{m}(\xi, \eta) = m(\xi, \eta) \chi_{[-1/2, 1/2]}(\xi) \chi_{[-1/2, 1/2]}(\eta) \). Hence for \( x = u + k \)

\[
C_m(\phi, \psi)(x) = D_m(a^n, b^n)(k).
\]
Now
\[
\int_{\mathbb{R}} |C_m(\phi, \psi)(x)|^p \, dx = \\
= \sum_k \int_{-1/2}^{1/2} |C_m(\phi, \psi)(k + u)|^p \, du \\
= \int_{-1/2}^{1/2} \sum_k |D_{\tilde{m}}(a^u, b^u)(k)|^p \, du \\
\leq \|D_{\tilde{m}}\|_{p^3} \int_{-1/2}^{1/2} \left( \sum_k |a^u(k)|^{p_1/p_3} \left( \sum_k |b^u(k)|^{p_2/p_3} \right) \right) \, du \\
\leq \|D_{\tilde{m}}\|_{p^3} \left( \int_{-1/2}^{1/2} \sum_k |a^u(k)|^{p_1/p_3} \left( \int_{-1/2}^{1/2} \sum_k |b^u(k)|^{p_2/p_3} \right) \, du \right) \\
= \|D_{\tilde{m}}\|_{p^3} \left( \int_{-1/2}^{1/2} \sum_k |\phi(u + k)|^{p_1/p_3} \left( \int_{-1/2}^{1/2} \sum_k |\psi(u + k)|^{p_2/p_3} \right) \, du \right) \\
= \|D_{\tilde{m}}\|_{p^3} \|\phi\|_{p_1} \|\psi\|_{p_2}^{p_3/p_2}.
\]

In the general case if \(\phi, \psi\) are such that \(\hat{\phi}, \hat{\psi}\) have compact support, then there exists \(\varepsilon > 0\) so that \(\hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon\) have their support in \([-1/4, 1/4]\). Now observe that
\[
C_m(\phi, \psi)(x) = \varepsilon^2 C_m(\varepsilon, \varepsilon)(\hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon)(\varepsilon x).
\]

Applying the previous case and the assumption we obtain
\[
\|C_m(\phi, \psi)\|_{p^3} = \varepsilon^{2-1/p_3} \|C_m(\varepsilon, \varepsilon)(\hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon)\|_{p^3} \\
\leq K \varepsilon^{2-1/p_3} \|\phi\|_{p_1} \|\psi\|_{p_2}^{p_3/p_2} \\
= K \varepsilon^{2-1/p_3} \|\phi\|_{p_1} \|\psi\|_{p_2}^{p_3/p_2} \\
= K \|\phi\|_{p_1} \|\psi\|_{p_2}.
\]

References


