# Transference of bilinear multiplier operators on Lorentz spaces.

Oscar Blasco, Francisco Villarroya*

AMS Class (2000) 42A45

**Abstract**

Let $m(\xi, \eta)$ be a bounded continuous function in $\mathbb{R} \times \mathbb{R}$, $0 < p_i, q_i < \infty$ for $i = 1, 2$ and $0 < p_3, q_3 \leq \infty$ where $1/p_1 + 1/p_2 = 1/p_3$. It is shown that

$$C_m(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i x(\xi+\eta)}d\xi d\eta$$

is a bounded bilinear operator from $L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R})$ into $L^{p_3,q_3}(\mathbb{R})$ if and only if

$$P_{D_{\varepsilon^{-1}}m}(f,g)(\theta) = \sum_{k\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} \hat{f}(k)\hat{g}(k')m(\varepsilon k, \varepsilon k')e^{2\pi i\theta(k+k')}$$

are bounded bilinear operators from $L^{p_1,q_1}(\mathbb{T}) \times L^{p_2,q_2}(\mathbb{T})$ into $L^{p_3,q_3}(\mathbb{T})$ with norm bounded by uniform constant for all $\epsilon > 0$.

## 1 Introduction.

Let $m(\xi_1, \xi_2, ..., \xi_n)$ be a bounded measurable function in $\mathbb{R}^n$ and define

$$C_m(f_1, f_2, ..., f_n)(x) = \int_{\mathbb{R}^n} \hat{f}_1(\xi_1)...\hat{f}_n(\xi_n)m(\xi_1, \xi_2, ..., \xi_n)e^{2\pi i x(\xi_1+\xi_2+...+\xi_n)}d\xi$$

for Schwartz test functions $f_i$ in $\mathcal{S}$ for $i = 1, ..., n$.

*Both authors have been partially supported by grants DGESIC PB98-0146 and BMF 2002-04013

Given now $0 < p_i \leq \infty$ for $i = 1, ..., n$ and $1/q = 1/p_1 + 1/p_2 + ...1/p_n$. The function $m$ is said to be a multilinear multiplier of strong type $(p_1, p_2, ..., p_n)$ (respect. weak type $(p_1, p_2, ..., p_n)$) if $C_m$ extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R}) \times ... \times L^{p_n}(\mathbb{R})$ into $L^q(\mathbb{R})$ (respect. to $L^{q,\infty}(\mathbb{R})$).

The study of such multilinear multipliers was started by R. Coifman and Y. Meyer (see [4, 5, 6]) for smooth symbols. However, in the last years people got interested in them after the results proved by M. Lacey and C. Thiele ([21, 22, 23]) which establish that $m(\xi, \nu) = sign(\xi + \alpha\nu)$ are multipliers of strong type $(p_1, p_2)$ for $1 < p_1, p_2 \leq \infty$, $p_3 > 2/3$ and each $\alpha \in \mathbb{R} \setminus \{0, 1\}$.

New results for non-smooth symbols, extending the ones given by the bilinear Hilbert transform, have been achieved by J.E. Gilbert and A.R. Nahmod (see [10, 11, 12]) and by C. Muscalu, T. Tao and C. Thiele (see [20]).

We refer the reader to [18, 17, 9, 13] for several results on bilinear multipliers and related topics.

The first transference methods for linear multipliers were given by K. Deleeuw. It is known that if $m$ is continuous then

$$T_m(f)(x) = \int_{\mathbb{R}} \hat{f}(\xi) m(\xi) e^{2\pi i x \xi} d\xi$$

(defined for $f \in S(\mathbb{R})$) is bounded on $L^p(\mathbb{R})$ if and only if

$$\tilde{T}_{m_\varepsilon}(f)(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}(k) m(\varepsilon k) e^{2\pi i \theta k}$$

(defined for trigonometric polynomials $f$) are uniformly bounded on $L^p(\mathbb{T})$ for all $\varepsilon > 0$ (see [8], [29] page 264).

Although the results in the paper hold true for multilinear multipliers, for simplicity of the notation we restrict ourselves to bilinear multipliers and only state and prove the theorems in such a situation.

Let $(m_{k,k'})$ be a bounded sequence we use the notation

$$P_m(f, g)(\theta) = \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} a(k) b(k') m_{k,k'} e^{2\pi i \theta (k + k')}$$

for $f(t) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n t}$ and $g(t) = \sum_{n \in \mathbb{Z}} b(n) e^{2\pi i n t}$.

Let $0 < p_1, p_2 \leq \infty$ and $p_3$ such that $1/p_1 + 1/p_2 = 1/p_3$. We write $P_{D_{t^{-1}} m}$ when the symbol is $m(tk, tk')$ and say that $m(tk, tk')$ is a bounded multiplier of strong (respect. weak) type $(p_1, p_2)$ on $\mathbb{Z} \times \mathbb{Z}$ if the corresponding $P_{D_{t^{-1}}m}$ is bounded from $L^{p_1}(\mathbb{T}) \times L^{p_1}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ (respect. $L^{p_3,\infty}(\mathbb{T})$).

In a recent paper (see [9]) D. Fan and S. Sato have shown certain DeLeeuw type theorems for transferring multilinear operators on Lebesgue and Hardy spaces from $\mathbb{R}^n$ to $\mathbb{T}^n$. They show that the multilinear version of the transference between $\mathbb{R}$ and $\mathbb{Z}$ holds true, namely that for continuous functions $m(\xi, \eta)$ one has that $m$ is a multiplier of strong (respect. weak) type $(p_1, p_2)$ on $\mathbb{R} \times \mathbb{R}$ if and only if $(D_{\varepsilon^{-1}}m)_{k,k'} = (m(\varepsilon k, \varepsilon k'))_{k,k'}$ are uniformly bounded multipliers of strong (respect. weak) type $(p_1, p_2)$ on $\mathbb{Z} \times \mathbb{Z}$.

The first author (see [1]) has shown a Deleeuw type theorem to transfer bilinear multipliers from $L^p(\mathbb{R})$ to bilinear multipliers acting on $\ell_p(\mathbb{Z})$.

The aim of this paper is to get an extension of those results in [9] for bilinear multipliers acting on Lorentz spaces (see [9], Remark 3).

We shall show that if $m$ is a bounded continuous function on $\mathbb{R}^2$ then $C_m$ defines a bounded bilinear map from $L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R})$ into $L^{p_3,q_3}(\mathbb{R})$ if and only if the $P_{D_{t^{-1}}m}$, the restriction to $m(tk, tk')$ for $k, k' \in \mathbb{Z}$, define bilinear maps from $L^{p_1,q_1}(\mathbb{T}) \times L^{p_2,q_2}(\mathbb{T})$ into $L^{p_3,q_3}(\mathbb{T})$ uniformly bounded for $t > 0$.

Throughout the paper $|A|$ denotes the Lebesgue measure of $A$ and we identify functions $f$ on $\mathbb{T}$ and periodic functions on $\mathbb{R}$ with period 1 defined on $[-\frac{1}{2}, \frac{1}{2})$, that is $f(x) = f(e^{2\pi i x})$ and $\int_{\mathbb{T}} f(z)dm(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)dt$. For $0 < p \leq \infty$, we write $D_t^p f(x) = t^{-\frac{1}{p}} f(t^{-1}x)$ (with the notation $D_t = D_t^\infty$), $M_y f(x) = f(x)e^{2\pi i y x}$ and $T_y f(x) = f(x - y)$ for the dilation, modulation and translation operators. In this way $(D_t^q f)\hat{} = D_{t^{-1}}^{q'} \hat{f}$ where, as usual, $q'$ stands for the conjugate exponent of $q$.

Adknowledgement: We want to thank the referee for his or her carefull reading.

# 2 Preliminaries

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite and complete measure space. Given a complex-valued measurable function $f$ we shall denote the distribution function of $f$ by $\mu_f(\lambda) = \mu(E_\lambda)$ for $\lambda > 0$ where $E_\lambda = \{w \in \Omega : |f(w)| > \lambda\}$, the nonincreasing rearrangement of $f$ by $f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$ and $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds$.

Now the Lorentz space $L^{p,q}$ consists of those measurable functions $f$ such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} \left\{ \dfrac{q}{p} \displaystyle\int_0^\infty t^{\frac{q}{p}} f^*(t)^q \frac{dt}{t} \right\}^{\frac{1}{q}}, & 0 < p < \infty,\ 0 < q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & 0 < p \le \infty,\ q = \infty. \end{cases}$$

It is well known that

$$||f||_{p\infty} = \sup_{\lambda>0} \lambda \mu_f(\lambda)^{1/p}.$$

Here we shall use the following fact: If $0 < p, q < \infty$ and $f$ is a measurable function then

$$\|f\|_{pq}^* = \left( q \int_0^\infty \lambda^{q-1} \mu_f(\lambda)^{\frac{q}{p}} d\lambda \right)^{1/q}. \tag{1}$$

(This can be easily checked for simple functions).

Let us recall some facts about these spaces. Simple functions are dense in $L^{p,q}$ for $q \neq \infty$, $(L^{p,1})^* = L^{p',\infty}$ for $1 \le p < \infty$, and $(L^{p,q})^* = L^{p',q'}$ for $1 < p, q < \infty$ as well. Replacing $f^*$ by $f^{**}$ and putting $\|f\|_{pq} = \|f^{**}\|_{pq}^*$ then we get a functional equivalent to $\|\cdot\|_{pq}^*$ (for $1 < p < \infty$) for which $L^{1,1}$ and $L^{p,q}$ for $1 < p \le \infty$, $1 \le q \le \infty$ are Banach spaces.

The reader is referred to [19], [2], [29] or [25] for the basic information on Lorentz spaces. We only condider $\mu$ to be either the Lebesgue measure on $\mathbb{R}$ or the normalized Lebesge measure on $\mathbb{T}$ and the distribution function will be denoted $m_f$ in both cases.

**Definition 2.1** *Let $m$ be a bounded measurable function on $\mathbb{R}^2$. Let $0 < p_i, q_i \le \infty$ for $i = 1, 2, 3$. For $t > 0$ we define*

$$C_{D_{t^{-1}}m}(f,g)(x) = C_t(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta) m(t\xi, t\eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta$$

*for $f, g \in S(\mathbb{R})$ .*

*We say that $m$ is a bilinear multipier in $(L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$ if there exists $C > 0$ such that*

$$\|C_1(f,g)\|_{L^{p_3,q_3}(\mathbb{R})} \le C \|f\|_{L^{p_1,q_1}(\mathbb{R})} \|g\|_{L^{p_2,q_2}(\mathbb{R})}$$

*for all $f, g \in S(\mathbb{R})$ .*

**Definition 2.2** *Let $(m_{k_1,k_2})_{k_1\in\mathbb{Z},k_2\in\mathbb{Z}}$ be a bounded sequence. Let $p_i, q_i > 0$ such that $p_3^{-1} = p_1^{-1} + p_2^{-1}$. We define*

$$P_m(f,g)(x) = \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} a_{k_1} b_{k_2} m_{k_1,k_2} e^{2\pi i(k_1+k_2)x}$$

*for all trigonometric polynomials $f(x) = \sum_{|k|\le N} a_k e^{2\pi ikx}$, $g(x) = \sum_{|k|\le M} b_k e^{2\pi ikx}$ and $N, M \in \mathbb{N}$.*

*We say that $m_{k,k'}$ is a bilinear multiplier in $(L^{p_1,q_1}(\mathbb{T})\times L^{p_2,q_2}(\mathbb{T}), L^{p_3,q_3}(\mathbb{T}))$ if there exists $C > 0$ such that*

$$\|P_m(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} \le C\|f\|_{L^{p_1,q_1}(\mathbb{T})}\|g\|_{L^{p_2,q_2}(\mathbb{T})}$$

*for all trigonometric polynomials $f$ and $g$.*

**Remark 2.1** *$m$ is a multiplier in $(L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$ if and only if $D_{t^{-1}}m(\xi,\eta) = m(t\xi, t\eta)$ is also a multiplier for each $t > 0$.*

*Note that for each $t > 0$ we have $m_{D_tf}(\lambda) = tm_f(\lambda)$. Hence*

$$||D_tf||_{L^{p,q}(\mathbb{R})} = t^{1/p}||f||_{L^{p,q}(\mathbb{R})}. \tag{2}$$

*for $0 < p, q \le \infty$.*

*Now the remark follows easily from the formula*

$$C_t(f,g) = D_tC_1(D_{t^{-1}}f, D_{t^{-1}}g).$$

*Actually we have $\|C_t\| = \|C_1\|$ for all $t > 0$.*

Let us start by recalling some facts to be used in the sequel.

**Definition 2.3** *If $f$ is a measurable function on $\mathbb{R}$ such that $\max\{|f(x)|, |\hat{f}(x)|\} \le A/(1+|x|)^\alpha$ for some $A > 0$ and $\alpha > 0$ then $\tilde{f}$ stands for the well-defined periodic function (see [29], pages 250-253)*

$$\tilde{f}(x) = \sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx}.$$

**Lemma 2.4** *Let $0 < p < \infty$ and $0 < q \le \infty$. If $f \in S(\mathbb{R})$ we have*

$$4^{-1}\|f\|_{L^{p,q}(\mathbb{R})} \le \liminf_{t\to 0} t^{-\frac{1}{p}}\|\widetilde{D_tf}\|_{L^{p,q}(\mathbb{T})} \le \limsup_{t\to 0} t^{-\frac{1}{p}}\|\widetilde{D_tf}\|_{L^{p,q}(\mathbb{T})} \le 4\|f\|_{L^{p,q}(\mathbb{R})}$$

*where $\widetilde{D_tf}(x) = \sum_{k\in\mathbb{Z}} D_tf(x+k)$ is defined on $\mathbb{T}$.*

*Proof.* Assume first that $f$ has compact support. For $t > 0$ small enough we have $supp(D_t f) \subset [-\frac{1}{2}, \frac{1}{2}]$. This gives that

$$\widetilde{D_t f}\chi_{[-\frac{1}{2},\frac{1}{2}]} = D_t f \chi_{[-\frac{1}{2},\frac{1}{2}]} = D_t f$$

In particular, for such $t$ we have

$$m_{\widetilde{D_t f}}(\lambda) = |\{x \in [-\frac{1}{2}, \frac{1}{2}]/|D_t f(x)| > \lambda\}| = |\{x \in \mathbb{R}/|f(t^{-1}x)| > \lambda\}| = t m_f(\lambda)$$

and

$$(\widetilde{D_t f})^*(x) = D_t(f^*)(x), \qquad x > 0.$$

Hence

$$\|\widetilde{D_t f}\|^q_{L^{p,q}(\mathbb{T})} = \frac{q}{p}\int_0^1 \left(x^{\frac{1}{p}}(\widetilde{D_t f})^*(x)\right)^q \frac{dx}{x}$$

$$= \frac{q}{p}\int_0^1 \left(x^{\frac{1}{p}} f^*(t^{-1}x)\right)^q \frac{dx}{x} = t^{\frac{q}{p}}\frac{q}{p}\int_0^{t^{-1}} \left(x^{\frac{1}{p}} f^*(x)\right)^q \frac{dx}{x}$$

and therefore

$$\lim_{t\to 0} t^{-\frac{1}{p}}\|\widetilde{D_t f}\|_{L^{p,q}(\mathbb{T})} = \lim_{t\to 0}\Big(\frac{q}{p}\int_0^{t^{-1}} \left(x^{\frac{1}{p}} f^*(x)\right)^q \frac{dx}{x}\Big)^{\frac{1}{q}} = \|f\|_{L^{p,q}(\mathbb{R})}.$$

The case $q = \infty$ is simpler.

For the general case, take $f_n = f\chi_{[-n,n]}$. Observe that, for $|x| < 1/2$

$$\begin{aligned}\widetilde{D_t f}(x) - \widetilde{D_t f_n}(x) &= \sum_{k\in\mathbb{Z}} f(t^{-1}(x+k)) - f_n(t^{-1}(x+k)) \\ &= \sum_{|k+x|>tn} f(t^{-1}(x+k))\end{aligned}$$

Hence, for any $m > 0$, we have that

$$|\widetilde{D_t f}(x) - \widetilde{D_t f_n}(x)| \le \sum_{|k+x|>tn} \frac{C_m}{(1+t^{-1}|x+k|)^m} \le t^m \sum_{|k+x|>tn} \frac{C_m}{|x+k|^m} \le C_m t^m.$$

This shows that, selecting $m > 1/p$, we have

$$\lim_{t\to 0} t^{-1/p}||\widetilde{D_t f_n} - \widetilde{D_t f}||_{L^\infty(\mathbb{T})} \le C_m \lim_{t\to 0} t^{m-1/p} = 0.$$

Given $\epsilon > 0$, choose $n \in \mathbb{N}$ such that

$$(1-\epsilon)||f||_{L^{p,q}(\mathbb{R})} \leq ||f_n||_{L^{p,q}(\mathbb{R})} \leq ||f||_{L^{p,q}(\mathbb{R})}$$

Now since $\|\cdot\|_{L^{p,q}(\mathbb{R})}$ is a quasi-norm with constant $C = 2^{\frac{1}{p}} \max(2^{\frac{1}{q}-1}, 1)$ we have by the Aoki-Rolewic theorem [26] that $\|\cdot\|_{L^{p,q}(\mathbb{R})}$ is equivalent to a $r$-norm, namely $|\cdot|$, for $r = \log_2^{-1}(2C)$. More precisely we have

$$|f| \leq \|f\|_{L^{p,q}(\mathbb{R})} \leq 4^{\frac{1}{r}}|f|$$

and thus we can write a triangular inequality to the $r$-power in the following way

$$\|f+g\|^r_{L^{p,q}(\mathbb{R})} \leq 4(\|f\|^r_{L^{p,q}(\mathbb{R})} + \|g\|^r_{L^{p,q}(\mathbb{R})}).$$

Hence, using this triangular inequality for $||.||^r_{L^{p,q}(\mathbb{T})}$ for the corresponding power $r \leq 1$, according to different values of $p$ and $q$, and the previous case we get the desired formula. ■

**Lemma 2.5** *Let $0 < p, q \leq \infty$, $\varphi = \chi_{[-\frac{1}{2},\frac{1}{2}]}$, $f \in L^{p,q}(\mathbb{T})$ and $k \in \mathbb{N}$. Then*

$$\|f\|_{L^{p,q}(\mathbb{T})} = \|fD^p_k\varphi\|_{L^{p,q}(\mathbb{R})}$$

*Proof.* Using that $f$ is periodic we get

$$\begin{aligned} m_{fD^p_k\varphi}(\lambda) &= |\{x \in \mathbb{R} : |f(x)k^{-\frac{1}{p}}\chi_{[-\frac{1}{2},\frac{1}{2}]}(k^{-1}x)| > \lambda\}| \\ &= |\{x \in [-\frac{k}{2}, \frac{k}{2}] : |f(x)| > k^{\frac{1}{p}}\lambda\}| \\ &= k|\{x \in [-\frac{1}{2}, \frac{1}{2}] : |f(x)| > k^{\frac{1}{p}}\lambda\}| = km_f(k^{\frac{1}{p}}\lambda). \end{aligned}$$

Hence

$$(fD^p_k\varphi)^*(t) = \inf\{\lambda > 0 : km_f(k^{\frac{1}{p}}\lambda) < t\}$$

$$= k^{-\frac{1}{p}} \inf\{\lambda > 0 : m_f(\lambda) < k^{-1}t\} = D^p_k f^*(t) = (D^p_k f)^*(t)$$

Therefore

$$\|fD^p_k\varphi\|^q_{L^{p,q}(\mathbb{R})} = \frac{q}{p}\int_0^\infty t^{\frac{q}{p}}(fD^p_k\varphi)^*(t)^q \frac{dt}{t}$$

$$= \frac{q}{p}\int_0^\infty t^{\frac{q}{p}} k^{-\frac{q}{p}} f^*(k^{-1}t)^q \frac{dt}{t} = \frac{q}{p}\int_0^\infty t^{\frac{q}{p}} f^*(t)^q \frac{dt}{t} = \|f\|^q_{L^{p,q}(\mathbb{T})}.$$

■

**Lemma 2.6** *Let $0 < p < \infty$ and $f \in L^{p,\infty}(\mathbb{T})$. If $\varphi \in S(\mathbb{R})$ is radial and decreasing then*

$$\limsup_{\epsilon \to 0} \|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,\infty}(\mathbb{R})} \leq \|\varphi\|_{L^p(\mathbb{R})}\|f\|_{L^{p,\infty}(\mathbb{T})}.$$

*Proof.* Note that for each $\epsilon > 0$ and $\lambda > 0$ we have

$$\begin{aligned}
& |\{x \in \mathbb{R} : |f(x)\varphi(\epsilon x)| > t\}| = |\{|x| \leq 2^{-1}\lambda\epsilon^{-1} : |f(x)\varphi(\epsilon x)| > t\}| \\
& + \sum_{n=0}^{\infty} |\{2^{n-1}\lambda\epsilon^{-1} < |x| \leq 2^n\lambda\epsilon^{-1} : |f(x)\varphi(\epsilon x)| > t\}| \\
\leq \; & |\{|x| \leq 2^{-1}\lambda\epsilon^{-1} : |f(x)| > t\varphi(0)^{-1}\}| \\
& + \sum_{n=0}^{\infty} |\{2^{n-1}\lambda\epsilon^{-1} < |x| \leq 2^n\lambda\epsilon^{-1} : |f(x)| > t\varphi(\lambda 2^{n-1})^{-1}\}| \\
\leq \; & |\{|x| \leq 2^{-1}([\lambda\epsilon^{-1}] + 1) : |f(x)| > t\varphi(0)^{-1}\}| \\
& + \sum_{n=0}^{\infty} |\{2^{n-1}[\lambda\epsilon^{-1}] < |x| \leq 2^n([\lambda\epsilon^{-1}] + 1) : |f(x)| > t\varphi(\lambda 2^{n-1})^{-1}\}| \\
= \; & ([\lambda\epsilon^{-1}] + 1)|\{x \in \mathbb{T} : |f(x)| > t\varphi(0)^{-1}\}| \\
& + \sum_{n=0}^{\infty} (2^{n+1}([\lambda\epsilon^{-1}] + 1) - 2^n[\lambda\epsilon^{-1}])|\{x \in \mathbb{T} : |f(x)| > t\varphi(\lambda 2^{n-1})^{-1}\}| \\
\leq \; & (\lambda\epsilon^{-1} + 1)|\{x \in \mathbb{T} : |f(x)| > t\varphi(0)^{-1}\}| \\
& + \sum_{n=0}^{\infty} 2^n(\lambda\epsilon^{-1} + 2)|\{x \in \mathbb{T} : |f(x)| > t\varphi(\lambda 2^{n-1})^{-1}\}|.
\end{aligned}$$

Hence we get

$$m_{fD_{\epsilon^{-1}}\varphi}(t) \leq (\lambda\epsilon^{-1}+1)m_f(t\varphi(0)^{-1}) + (\lambda\epsilon^{-1}+2)\sum_{n=0}^{\infty} 2^n m_f(t\varphi(\lambda 2^{n-1})^{-1}). \quad (3)$$

Therefore, using that $m_f(t) \leq \frac{||f||^p_{p\infty}}{t^p}$, we get

$$\begin{aligned}
m_{fD^p_{\epsilon^{-1}}\varphi}(s) \; &= \; m_{fD_{\epsilon^{-1}}\varphi}(s\epsilon^{-1/p}) \\
&\leq \; (\lambda\epsilon^{-1} + 1)\epsilon s^{-p}\varphi(0)^p||f||^p_{L^{p,\infty}(\mathbb{T})} \\
&+ \; \sum_{n=0}^{\infty} 2^n(\lambda\epsilon^{-1} + 2)\epsilon s^{-p}\varphi(\lambda 2^{n-1})^p||f||^p_{L^{p,\infty}(\mathbb{T})}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; s^{-p}(\lambda+\epsilon)|\varphi(0)|^p||f||^p_{L^{p,\infty}(\mathbb{T})} \\
&+ \; s^{-p}\sum_{n=0}^{\infty} 2^n(\lambda+2\epsilon)\varphi(\lambda 2^{n-1})^p||f||^p_{L^{p,\infty}(\mathbb{T})}
\end{aligned}
$$

Hence, if $\varphi_\lambda = \varphi(0)\chi_{[-\lambda 2^{-1},\lambda 2^{-1}]} + \sum_{n\geq 0}\varphi(\lambda 2^{n-1})\chi_{[-\lambda 2^n,\lambda 2^n]\setminus[-\lambda 2^{n-1},\lambda 2^{n-1}]}$ we have

$$
\limsup_{\epsilon\to 0} ||fD^p_{\epsilon^{-1}}\varphi||^p_{L^{p,\infty}(\mathbb{R})} \leq \|\varphi_\lambda\|^p_{L^p(\mathbb{R})}||f||^p_{L^{p,\infty}(\mathbb{T})}.
$$

Now pass to the limit as $\lambda$ goes to zero to get the result. ■

**Lemma 2.7** *Let $0<p,q<\infty$ and $f\in L^{p,q}(\mathbb{T})$. If $\varphi\in S(\mathbb{R})$ is radial and decreasing then*

$$
C_{p,s}\|\varphi\|_{L^{p,s}(\mathbb{R})}\|f\|_{L^{p,q}(\mathbb{T})} \leq \liminf_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})}
$$

$$
\leq \limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \leq C_{p,r}\|\varphi\|_{L^{p,r}(\mathbb{R})}\|f\|_{L^{p,q}(\mathbb{T})}
$$

*where $C_{p_1,p_2} = (2^{\frac{p_2}{p_1}}-1)^{-\frac{1}{p_2}}$, $r=\min(p,q)$ and $s=\max(p,q)$ .*

*Proof.* Use (1) to write

$$
\begin{aligned}
\|fD^p_{\epsilon^{-1}}\varphi\|^q_{L^{p,q}(\mathbb{R})} &= \int_0^\infty qt^{q-1}(m_{fD_{\epsilon^{-1}}}(\epsilon^{-1/p}t))^{\frac{q}{p}}dt \\
&= \int_0^\infty qt^{q-1}(\epsilon m_{fD_{\epsilon^{-1}}}(t))^{\frac{q}{p}}dt
\end{aligned}
$$

Using the estimate in the previous lemma we have

$$
\epsilon m_{fD_{\epsilon^{-1}}\varphi}(t) \leq (\lambda+\epsilon)m_f(t\varphi(0)^{-1}) + (\lambda+2\epsilon)\sum_{n=0}^{\infty}2^n m_f(t\varphi(\lambda 2^{n-1})^{-1}).
$$

Now we see that for $r=\min(p,q)$ we have

$$
\limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \leq \Big(\lambda^{\frac{r}{p}}\varphi(0)^r + \sum_{n=0}^{\infty}(\lambda 2^n)^{\frac{r}{p}}\varphi(\lambda 2^{n-1})^r\Big)^{\frac{1}{r}}\|f\|_{L^{p,q}(\mathbb{T})}. \quad (4)
$$

If $q \le p$ then, for every $\lambda$, we have

$$
\begin{aligned}
&\|fD^p_{\epsilon^{-1}}\varphi\|^q_{L^{p,q}(\mathbb{R})} = \int_0^\infty qt^{q-1}\Big(\epsilon|\{x\in\mathbb{R}|f(x)\varphi(\epsilon x)|>t\}|\Big)^{\frac{q}{p}}dt\\
\le\;& \int_0^\infty qt^{q-1}\Big((\lambda+\epsilon)m_f(t\varphi(0)^{-1})+(\lambda+2\epsilon)\sum_{n=0}^\infty 2^n m_f(t\varphi(\lambda 2^{n-1})^{-1})\Big)^{\frac{q}{p}}dt\\
\le\;& \int_0^\infty qt^{q-1}(\lambda+\epsilon)^{\frac{q}{p}}m_f(t\varphi(0)^{-1})^{\frac{q}{p}}dt\\
&+\int_0^\infty qt^{q-1}(\lambda+2\epsilon)^{\frac{q}{p}}\sum_{n=0}^\infty 2^{n\frac{q}{p}}m_f(t\varphi(\lambda 2^{n-1})^{-1})^{\frac{q}{p}}dt\\
=\;& (\lambda+\epsilon)^{\frac{q}{p}}\varphi(0)^q\int_0^\infty qt^{q-1}m_f(t)^{\frac{q}{p}}dt\\
&+(\lambda+2\epsilon)^{\frac{q}{p}}\sum_{n=0}^\infty 2^{n\frac{q}{p}}\varphi(\lambda 2^{n-1})^q\int_0^\infty qt^{q-1}m_f(t)^{\frac{q}{p}}dt\\
=\;& \Big((\lambda+\epsilon)^{\frac{q}{p}}|\varphi(0)|^q+(\lambda+2\epsilon)^{\frac{q}{p}}\sum_{n=0}^\infty 2^{n\frac{q}{p}}\varphi(\lambda 2^{n-1})^q\Big)\|f\|^q_{L^{p,q}(\mathbb{T})}
\end{aligned}
$$

Therefore

$$
\limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})}\le\Big(\lambda^{\frac{q}{p}}\varphi(0)^q+\sum_{n=0}^\infty(\lambda 2^n)^{\frac{q}{p}}\varphi(\lambda 2^{n-1})^q\Big)^{\frac{1}{q}}\|f\|_{L^{p,q}(\mathbb{T})},
$$

which gives (4).

In the case $q > p$ we can use Minkowski and get

$$
\begin{aligned}
\|fD^p_{\epsilon^{-1}}\varphi\|^p_{L^{p,q}(\mathbb{R})} &= \Big(\int_0^\infty(q^{\frac{p}{q}}t^{p(1-\frac{1}{q})}\epsilon|\{x\in\mathbb{R}:|f(x)\varphi(\epsilon x)|>t\}|)^{\frac{q}{p}}dt\Big)^{\frac{p}{q}}\\
&\le \Big(\int_0^\infty\Big(q^{\frac{p}{q}}t^{p(1-\frac{1}{q})}(\lambda+\epsilon)m_f(t\varphi(0)^{-1})\\
&+ (\lambda+2\epsilon)\sum_{n=0}^\infty 2^n q^{\frac{p}{q}}t^{p(1-\frac{1}{q})}m_f(t\varphi(\lambda 2^{n-1})^{-1})\Big)^{\frac{q}{p}}dt\Big)^{\frac{p}{q}}\\
&\le (\lambda+\epsilon)\Big(\int_0^\infty\Big(q^{\frac{p}{q}}t^{p(1-\frac{1}{q})}m_f(t\varphi(0)^{-1})\Big)^{\frac{q}{p}}dt\Big)^{\frac{p}{q}}\\
&+ (\lambda+2\epsilon)\sum_{n=0}^\infty 2^n\Big(\int_0^\infty\Big(q^{\frac{p}{q}}t^{p(1-\frac{1}{q})}m_f(t|\varphi(\lambda 2^{n-1})|^{-1})\Big)^{\frac{q}{p}}dt\Big)^{\frac{p}{q}}\\
&= (\lambda+\epsilon)\varphi(0)^p\Big(\int_0^\infty qt^{q-1}m_f(t)^{\frac{q}{p}}dt\Big)^{\frac{p}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (\lambda+2\epsilon)\sum_{n=0}^{\infty}2^n\varphi(\lambda 2^{n-1})^p\Big(\int_0^\infty qt^{q-1}m_f(t)^{\frac{q}{p}}dt\Big)^{\frac{p}{q}}\\
&= \quad \Big((\lambda+\epsilon)\varphi(0)^p+(\lambda+2\epsilon)\sum_{n=0}^{\infty}2^n\varphi(\lambda 2^{n-1})^p\Big)\|f\|^p_{L^{p,q}(\mathbb{T})}
\end{aligned}
$$

Therefore

$$
\limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})}\leq\Big(\lambda\varphi(0)^p+\sum_{n=0}^{\infty}\lambda 2^n\varphi(\lambda 2^{n-1})^p\Big)^{\frac{1}{p}}\|f\|_{L^{p,q}(\mathbb{T})},
$$

and (4) is proved.

If $\varphi_\lambda=\varphi(0)\chi_{[-\lambda 2^{-1},\lambda 2^{-1}]}+\sum_{n\geq 0}\varphi(\lambda 2^{n-1})\chi_{[-\lambda 2^n,\lambda 2^n]\setminus[-\lambda 2^{n-1},\lambda 2^{n-1}]}$ then clearly we have that

$$
||\varphi_\lambda||_p=\Big(\lambda\varphi(0)^p+\sum_{n=0}^{\infty}\lambda 2^n\varphi(\lambda 2^{n-1})^p\Big)^{\frac{1}{p}}.
$$

Since $\varphi$ and $\varphi_\lambda$ are radial and decreasing then $\varphi^*_\lambda(t)=\varphi_\lambda(2t)$ for $t>0$ and

$$
||\varphi_\lambda||^*_{pr}=\Big(\lambda^{\frac{r}{p}}\varphi(0)^r+(2^{\frac{r}{p}}-1)\sum_{n=0}^{\infty}(\lambda 2^n)^{\frac{r}{p}}\varphi(\lambda 2^{n-1})^r\Big)^{\frac{1}{r}}.
$$

Hence, using that $r\leq p$, we have

$$
\Big(\lambda^{\frac{r}{p}}\varphi(0)^r+\sum_{n=0}^{\infty}(\lambda 2^n)^{\frac{r}{p}}\varphi(\lambda 2^{n-1})^r\Big)^{\frac{1}{r}}\leq(2^{\frac{r}{p}}-1)^{-\frac{1}{r}}\|\varphi_\lambda\|_{L^{p,r}(\mathbb{R})}.
$$

Finally taking limits as $\lambda\to 0$ give

$$
\limsup_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})}\leq\lim_{\lambda\to 0}\Big(\lambda^{\frac{r}{p}}\varphi(0)^r+\sum_{n=0}^{\infty}(\lambda 2^n)^{\frac{r}{p}}\varphi(\lambda 2^{n-1})^r\Big)^{\frac{1}{r}}
$$

$$
\leq(2^{\frac{r}{p}}-1)^{-\frac{1}{r}}\limsup_{\lambda\to 0}\|\varphi_\lambda\|_{L^{p,r}(\mathbb{R})}=(2^{\frac{r}{p}}-1)^{-\frac{1}{r}}\|\varphi\|_{L^{p,r}(\mathbb{R})}.
$$

This gives one of the inequalities of the Lemma.

To get the other inequality, we use estimates from below to obtain

$$
\liminf_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})}\geq\Big(\lambda^{\frac{s}{p}}\varphi(\lambda 2^{-1})^s+\sum_{n=0}^{\infty}(\lambda 2^n)^{\frac{s}{p}}\varphi(\lambda 2^n)^s\Big)^{\frac{1}{s}}\|f\|_{L^{p,q}(\mathbb{T})}
$$

where $s=\max(p,q)$.

Using now that $s \geq p$, we get, arguing as above, that

$$\Big(\lambda^{\frac{s}{p}}\varphi(\lambda 2^{-1})^s + \sum_{n=0}^{\infty}(\lambda 2^n)^{\frac{s}{p}}\varphi(\lambda 2^n)^s\Big)^{\frac{1}{s}} \geq (2^{\frac{s}{p}}-1)^{-\frac{1}{s}}\|\varphi^\lambda\|_{L^{p,s}(\mathbb{R})}$$

where $\varphi^\lambda = \varphi(\lambda 2^{-1})\chi_{[-\lambda 2^{-1},\lambda 2^{-1}]} + \sum_{n\geq 0}\varphi(\lambda 2^n)\chi_{[-\lambda 2^n,\lambda 2^n]\setminus[-\lambda 2^{n-1},\lambda 2^{n-1}]}$.

Hence

$$\liminf_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^{p,q}(\mathbb{R})} \geq (2^{\frac{s}{p}}-1)^{-\frac{1}{s}}\|\varphi\|_{L^{p,s}(\mathbb{R})}.$$

Then proof is then completed.■

**Corollary 2.8** *Let $0<p<\infty$ and $f\in L^p(\mathbb{T})$. If $\varphi\in S(\mathbb{R})$ is radial and decreasing then*

$$\|\varphi\|_{L^p(\mathbb{R})}\|f\|_{L^p(\mathbb{T})} = \lim_{\epsilon\to 0}\|fD^p_{\epsilon^{-1}}\varphi\|_{L^p(\mathbb{R})}.$$

*In particular for $p=1$ and the periodized function $f=\widetilde{\chi_A}$ where $A\subset[-\frac{1}{2},\frac{1}{2}]$ we get*

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} f(x)D^1_{\epsilon^{-1}}\varphi(x)dx = \lim_{\epsilon\to 0}\int_{\mathbb{R}} D_\epsilon f(x)\varphi(x)dx = m(A)\int_{\mathbb{R}}\varphi(x)dx.$$

Now we are ready to proof our main theorem.

**Theorem 2.9** *Let $m$ be a bounded continuous function on $\mathbb{R}^2$. Let $0<p_i,q_i<\infty$ for $i=1,2$, and $0<p_3,q_3\leq\infty$ where $1/p_1+1/p_2=1/p_3$.*

*Then $m$ is a multiplier in $(L^{p_1,q_1}(\mathbb{R})\times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$ if and only if $(D_{t^{-1}}m)_{t>0}$ restricted to $\mathbb{Z}^2$ are uniformly bounded multipliers in $(L^{p_1,q_1}(\mathbb{T})\times L^{p_2,q_2}(\mathbb{T}), L^{p_3,q_3}(\mathbb{T}))$,i.e, denoting $P_t = P_{(D_{t^{-1}}m)_{k,k'}}$ where $(D_{t^{-1}}m)_{k,k'} = m(tk,tk')$, there exists $C>0$ such that*

$$\|C_1(f,g)\|_{L^{p_3,q_3}(\mathbb{R})} \leq C\|f\|_{L^{p_1,q_1}(\mathbb{R})}\|g\|_{L^{p_2,q_2}(\mathbb{R})}$$

*for $f,g\in S(\mathbb{R})$ if and only if there exists $C'>0$ such that*

$$\|P_t(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} \leq C'\|f\|_{L^{p_1,q_1}(\mathbb{T})}\|g\|_{L^{p_2,q_2}(\mathbb{T})}$$

*uniformly in $t>0$ for all trigonometric polynomials $f,g$.*

*Proof.* ($\Rightarrow$) Let $\varphi = \chi_{[-\frac{1}{2},\frac{1}{2}]}$ and $\psi(x) = \pi^{-1/2}e^{-x^2}$ Let $t > 0$ and let $f(x) = \sum_k a_k e^{2\pi i x}$ and $g(x) = \sum_k b_k e^{2\pi i x}$.

Since $m$ is continuous we can write

$$P_t(f,g)(x) = \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} a_{k_1}b_{k_2}m(tk_1,tk_2)e^{2\pi i(k_1+k_2)x}$$

$$= \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} a_{k_1}b_{k_2}\lim_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} D^1_\epsilon\psi(k_1-r)D^1_\epsilon\psi(k_2-s)m(tr,ts)e^{2\pi i(r+s)x}drds$$

$$= \lim_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k_1\in\mathbb{Z}} a_{k_1}D^1_\epsilon\psi(r-k_1)\ \sum_{k_2\in\mathbb{Z}} b_{k_2}D^1_\epsilon\psi(s-k_2)\ m(tr,ts)\ e^{2\pi i(r+s)x}\ drds.$$

That is

$$P_t(f,g)(x) = \lim_{\epsilon\to 0} C_t(f_\epsilon,g_\epsilon)(x) \tag{5}$$

where

$$\hat{f}_\epsilon = \sum_{k_1\in\mathbb{Z}} a_{k_1}T_{k_1}D^1_\epsilon\psi, \qquad \hat{g}_\epsilon = \sum_{k_2\in\mathbb{Z}} b_{k_2}T_{k_2}D^1_\epsilon\psi$$

or, in other words,

$$f_\epsilon(x) = \sum_{k_1\in\mathbb{Z}} a_{k_1}M_{k_1}D^\infty_{\epsilon^{-1}}\check{\psi}(x) = \sum_{k_1\in\mathbb{Z}} a_{k_1}\check{\psi}(\epsilon x)e^{2\pi i k_1 x} = \check{\psi}(\epsilon x)f(x),$$

and similar formula for $g_\epsilon$. Moreover, this the convergence is uniform since

$$|P_t(f,g)(x) - C_t(f_\epsilon,g_\epsilon)(x)|$$

$$\leq \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} |a_{k_1}||b_{k_2}|\int_{\mathbb{R}}\int_{\mathbb{R}} |m(tk_1,tk_2) - m(tk_1-\epsilon r, tk_2-\epsilon s)|\psi(r)\psi(s)drds$$

which tends to zero uniformly in $x \in \mathbb{R}$ because the continuity of $m$.

Thus

$$P_t(f,g) = \lim_{n\to\infty} C_t(f_n,g_n) \tag{6}$$

where $f_n(x) = \check{\psi}(n^{-1}x)f(x)$ and $g_n(x) = \check{\psi}(n^{-1}x)g(x)$ with uniform convergence and from Lemma 2.5 for $k \in \mathbb{N}$ we also have

$$\|P_t(f,g)\|_{L^{p_3,q_3}(\mathbb{T})} = \|P_t(f,g)D^{p_3}_k\varphi\|_{L^{p_3,q_3}(\mathbb{R})} \tag{7}$$

get

Combining these two facts we write

$$\|P_t(f,g)\|_{L^{p_3}(\mathbb{T})} = \|P_t(f,g)D_n^{p_3}\varphi\|_{L^{p_3}(\mathbb{R})}$$

$$\leq \|C_t(f_n,g_n)D_n^{p_3}\varphi\|_{L^{p_3}(\mathbb{R})} + \|D_{n^{-1}}(P_t(f,g) - C_t(f_n,g_n))\varphi\|_{L^{p_3}(\mathbb{R})}$$

For the first sumand

$$\begin{aligned} \|C_t(f_n,g_n)D_n^{p_3}\varphi\|_{L^{p_3,q_3}(\mathbb{R})} &= \|D_n^{p_3}(\varphi D_{n^{-1}}C_t(f_n,g_n))\|_{L^{p_3,q_3}(\mathbb{R})} \\ &= \|\varphi D_{n^{-1}}C_t(f_n,g_n)\|_{L^{p_3,q_3}(\mathbb{R})} \\ &\leq \|D_{n^{-1}}C_t(f_n,g_n)\|_{L^{p_3,q_3}(\mathbb{R})}\|\varphi\|_{L^\infty(\mathbb{R})} \\ &= n^{-\frac{1}{p_3}}\|C_t(f_n,g_n)\|_{L^{p_3,q_3}(\mathbb{R})} \\ &\leq n^{-\frac{1}{p_3}}C\|f_n\|_{L^{p_1,q_1}(\mathbb{R})}\|g_n\|_{L^{p_2,q_2}(\mathbb{R})} \\ &= Cn^{-\frac{1}{p_1}}\|f_n\|_{L^{p_1,q_1}(\mathbb{R})}n^{-\frac{1}{p_2}}\|g_n\|_{L^{p_2,q_2}(\mathbb{R})} \end{aligned}$$

where, using Lemmas 2.6 and 2.7, we know

$$\lim_{n\to\infty} n^{-\frac{1}{p_1}}\|f_n\|_{L^{p_1,q_1}(\mathbb{R})} \leq (2^{\frac{r_1}{p_1}} - 1)^{-\frac{1}{r_1}}\|f\|_{L^{p_1,q_1}(\mathbb{T})}\|\check{\psi}\|_{L^{p_1,r_1}(\mathbb{R})}$$

and

$$\lim_{n\to\infty} n^{-\frac{1}{p_2}}\|g_n\|_{L^{p_2,q_2}(\mathbb{R})} \leq (2^{\frac{r_2}{p_2}} - 1)^{-\frac{1}{r_2}}\|g\|_{L^{p_2,q_2}(\mathbb{T})}\|\check{\psi}\|_{L^{p_2,r_2}(\mathbb{R})}$$

with $r_i = \min(p_i, q_i)$ for $i = 1, 2$.

Thus

$$\|P_t(f,g)\|_{L^{p_3}(\mathbb{T})} \leq \lim_{n\to\infty} \|C_t(f_n,g_n)D_n^{p_3}\varphi\|_{L^{p_3,q_3}(\mathbb{R})}$$

$$+ \lim_{n\to\infty} \|P_t(f,g) - C_t(f_n,g_n)\|_{L^\infty(\mathbb{R})} = A(p_1,p_2)\|f\|_{L^{p_1,q_1}(\mathbb{T})}\|g\|_{L^{p_2,q_2}(\mathbb{T})}$$

and the proof of this implication is completed.

($\Leftarrow$) Assume $D_{t^{-1}}m$ restricted to $\mathbb{Z}^2$ are uniformly bounded multipliers on $\mathbb{Z}^2$ and let $f, g \in S(\mathbb{R})$ such that $\hat{f}$ and $\hat{g}$ have compact support contained in $K$.

Using Poisson formula

$$t\sum_{k_1} \hat{f}(tk_1)e^{2\pi i k_1 x} = \sum_{k_1}(D_t f)\hat{\ }(k_1)e^{2\pi i k_1 x} = \sum_{k_1} D_t f(x + k_1) = \widetilde{D_t f}(x)$$

Therefore, since $m$ is continuous, we can write

$$\begin{aligned} C_1(f,g)(x) &= \iint_{K\times K} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \lim_{t\to 0} t^2 \sum_{k_1}\sum_{k_2} \hat{f}(tk_1)\hat{g}(tk_2)m(tk_1,tk_2)e^{2\pi it(k_1+k_2)x} \\ &= \lim_{t\to 0} P_t(\widetilde{D_t f},\widetilde{D_t g})(tx) \end{aligned}$$

Note that

$$\begin{aligned} |\{x\in \mathbb{R} : |C_1(f,g)(x)|>\lambda\}| &\leq \liminf_{t\to 0} |\{|x|\leq t^{-1}/2 : |P_t(\widetilde{D_t f},\widetilde{D_t g})(tx)|>\lambda\}| \\ &\leq \liminf_{t\to 0} t^{-1}|\{|x|\leq 1/2 : |P_t(\widetilde{D_t f},\widetilde{D_t g})(x)|>\lambda\}| \end{aligned}$$

Therefore, formula (1) and Fatou's lemma give

$$||C_1(f,g)||^{p_3}_{L^{p_3,q_3}(\mathbb{R})} \leq C \liminf_{t\to 0} t^{-1}||P_t(\widetilde{D_t f},\widetilde{D_t g})||^{p_3}_{L^{p_3,q_3}(\mathbb{T})}.$$

An application of the assumption and Lemma 2.4 lead to

$$\begin{aligned} ||C_1(f,g)||_{L^{p_3,q_3}(\mathbb{R})} &\leq C \liminf_{t\to 0} t^{-1/p_3}||\widetilde{D_t f}||_{L^{p_1,q_1}(\mathbb{T})}||\widetilde{D_t g}||_{L^{p_2,q_2}(\mathbb{T})} \\ &\leq C||f||_{L^{p_1,q_1}(\mathbb{R})}||g||_{L^{p_2,q_2}(\mathbb{R})}. \end{aligned}$$

This finishes the proof $\blacksquare$.

It is known that transference theorems can be extended to symbols more general than continuous (see [8], [7], [9]). Actually a bounded measurable function $m_1$ defined on $\mathbb{R}$ is called regulated if

$$\lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} m_1(x+t)dt = m_1(x)$$

for all $x\in\mathbb{R}$.

It is pointed out in [8] (see Corollary 2.5 ) that if $m_1$ is regulated and $\phi$ is non-negative, symmetric, smooth with compact support and $\int_{\mathbb{R}}\phi(t)dt = 1$ then

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}} m_1(x-\epsilon t)\phi(t)dt = \lim_{\epsilon\to 0} m_1 * D^1_\epsilon\phi(x) = m_1(x)$$

for all $x\in\mathbb{R}$.

This acually implies that

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}} m_1(x-\epsilon t)\psi(t)dt = \lim_{\epsilon\to 0} m_1 * D^1_\epsilon\psi(x) = m_1(x) \tag{8}$$

where $\psi$ is non-negative symmetric, smooth and $\int_{\mathbb{R}}\psi(t)dt = 1$.

Indeed, given $\psi$ take non-negative, symmetric, smooth functions $\phi_n$ with compact support and $\int_{\mathbb{R}}\phi_n(t)dt = 1$ such that $\lim_{n\to\infty}||\psi-\phi_n||_1 = 0$ and observe that

$$|\int_{\mathbb{R}}(m_1(x-\epsilon t)-m_1(x))\psi(t)dt| \leq 2||m_1||_\infty\int_{\mathbb{R}}|D^1_\epsilon\psi(t)-D^1_\epsilon\phi_n(t)|dt$$

$$+|\int_{\mathbb{R}}(m_1(x-\epsilon t)-m_1(x))\phi_n(t)dt|$$

$$= 2||m_1||_\infty||\psi-\phi_n||_1 + |\int_{\mathbb{R}}(m_1(x-\epsilon t)-m_1(x))\phi_n(t)dt|.$$

**Definition 2.10** *Let $G(t,s) = \pi^{-1}e^{-(t^2+s^2)}$. A bounded measurable function $m$ defined on $\mathbb{R}^2$ is $G$-regulated if*

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2} m(x-\epsilon t, y-\epsilon s)G(t,s)dtds = \lim_{\epsilon\to 0} m * D^1_\epsilon G(x,y) = m(x,y)$$

*for all $(x,y)\in\mathbb{R}^2$.*

A look at the proof of the previous theorem shows that $m$ needs not be continuous but only $G$-regulated for the argument to work.

**Theorem 2.11** *Let $m$ be a bounded $G$-regulated function on $\mathbb{R}^2$, $0 < p_i, q_i < \infty$ for $i = 1,2$ and $0 < p_3, q_3 \leq \infty$ where $1/p_1 + 1/p_2 = 1/p_3$.*

*If $m$ is a multiplier in $(L^{p_1,q_1}(\mathbb{R})\times L^{p_2,q_2}(\mathbb{R}), L^{p_3,q_3}(\mathbb{R}))$ then $m$ restricted to $\mathbb{Z}^2$ is a bounded multiplier in $(L^{p_1,q_1}(\mathbb{T})\times L^{p_2,q_2}(\mathbb{T}), L^{p_3,q_3}(\mathbb{T}))$.*

Now we can apply this result to transfer results for the bilinear Hilbert transform because of the following remark.

**Remark 2.2** *If $m_1$ be a regulated function defined in $\mathbb{R}$ then $m_\alpha(x,y) = m_1(x+\alpha y)$ is $G$-regulated in $\mathbb{R}^2$.*

*In particular, $m(x,y) = sign(x+\alpha y)$ is $G$-regulated.*

Indeed, observe that

$$\begin{aligned}\int_{\mathbb{R}^2} m_1(x-t+\alpha(y-s))D^1_\epsilon G(t,s)dtds &= \int_{\mathbb{R}}\int_{\mathbb{R}} m_1(x+\alpha y-\epsilon(t+\alpha s))G(t,s)dtds \\ &= \int_{\mathbb{R}} m_1(x+\alpha y-\epsilon t)(\int_{\mathbb{R}} G(t-\alpha s,s)ds)dt \\ &= \int_{\mathbb{R}} m_1(x+\alpha y-\epsilon t)\psi_\alpha(t)dt\end{aligned}$$

where $\psi_\alpha(t)=\int_{\mathbb{R}} G(t-\alpha s,s)ds$. Hence we have, from (8), that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2} m_\alpha(x-t,y-s)D^1_\epsilon G(t,s)dtds = m_\alpha(x,y).$$

# References


[1] O. Blasco *Bilinear multipliers and Transference*, Preprint

[2] C. Bennet, R. Sharpley, *Interpolation of operators*, Pure and applied mathematics vol. 129, Academic Press, Inc., New York (1988).

[3] Auscher,P. and Carro, M.J., *On relations between operators on* $\mathbb{R}^n, \mathbb{T}^n, \mathbb{Z}^n$. Studia Math., **101** [1992] pp. 165-182.

[4] Coifman R.R., Meyer Y. *Commutateurs d' integrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble),bf 28, pp.177-202 [1978]

[5] Coifman R.R., Meyer Y. *Fourier analysis of multilinear convolutions, Calderón's theorem and analysis of Lipschitz curves*, Euclidean harmonic analysis (Proc. Sem. Univ. Maryland, College Park, Md.), pp.104-122, Lecture Notes in Math., 779 [1979]

[6] Coifman R.R., and Meyer Y. *Ondelettes and opérateurs III, Operatéurs multilinéaires*, Actualités Mathematiques, Hermman, Paris [1991]

[7] Coifman R.R., and Weiss, G. *Transference methods in Analysis*, Memoirs Amer. Math. Soc., 31 [1977]

[8] DeLeeuw, K. *On $L_p$-multipliers* . Ann. Math. **91** [1965] pp. 364-379.

[9] Fan, D. and Sato, S. , *Transference on certain multilinear multiplier operators* , J. Austral. Math. Soc. **70**, [2001], pp. 37-55.

[10] Gilbert J., Nahmod A., *Boundedness of bilinear operators with non-smooth symbols* Math. Res. Lett. **7** [2000] pp. 767-778

[11] Gilbert J., Nahmod A., *Bilinear operators with non-smooth symbols* J. Fourier Anal. Appl. **7** [2001] pp. 437-469.

[12] Gilbert J., Nahmod A., $L^p$ *boundedness for time-frecuency paraproducts* J. Fourier Anal. Appl. (To appear)

[13] Grafakos L., Torres, R., *Multilinear Calderón-Zygmund theory* Adv. in Math. **165** [2002] pp. 124-164.

[14] Grafakos L., Weiss, G., *Transference of multilinear operators.* Illinois J. Math. **40** [1996] pp. 344-351.

[15] Hewitt, E, Ross, K. A., *Abstract Harmonic Analysis II* Springer-Verlag, Berlin, [1970].

[16] Kenig, C. E. and Thomas, P. A., *Maximal operators defined by Fourier multipliers* Studia Math. **68** [1980] pp. 79-83.

[17] Kenig, C. E. and Stein, E. M., *Multilinear estimates and fractional integration* Math. Res. Ledtt. **6** [1999] pp. 1-15.

[18] Kalton, N.J. and Grafakos, L., *The Marcinkiewicz multiplier condition for bilinear operators* Studia Math. **146** [2001] pp. 115-156.

[19] R.A. Hunt, On $L(p,q)$ spaces, *L'Enseignement mathèm., t. XII, fasc. 4* (1966) 249-276.

[20] Muscalu, C., Tao, T. and Thiele, C *Multilinear operators given by singular multipliers* J. Amer. Math. Soc. **15** [2002] pp. 469-496.

[21] Lacey M., Thiele C., $L^p$ *bounds on the bilinear Hilbert transform for* $2 < p < \infty$ Ann. Math. 146, [1997], pp. 693-724.

[22] M. Lacey, C. M. Thiele, *Weak bounds for the bilinear Hilbert tranform on* $L^p$. Documenta Mathematica, extra volume ICM 1-1000, [1997].

[23] Lacey M., Thiele C., *On Calderón's conjecture.* Ann. Math. **149** nº2 [1999] pp. 475-496

[24] J. Lindenstrauss, L. Tzafriri,*Classical Banach spaces II. Function spaces,* Springer-Verlag, Berlin, (1979).

[25] G.G. Lorentz, *Some new functional spaces*, Ann. Math., Vol 51 (1950), 37-55.

[26] Rolewicz, S. *Metric linear spaces* MM 56 PWN, Warsaw [1972]

[27] Stein E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, [1970].

[28] Stein E. M., *Harmonic Analysis: real-variable methods, orthogonality and oscilatory integrals*, Princeton Univ. Press, [1993].

[29] Stein E. M. and Weiss G., *Introduction to Fourier Analysis on Euclidean Spaces* , Princeton Univ. Press, [1971].