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COMMUTATORS OF LINEAR AND BILINEAR HILBERT TRANSFORMS.

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ABSTRACT. Let $\alpha \in \mathbb{R}$ and let $H_\alpha(f,g)(x) = \frac{1}{\pi} p.v. \int f(x-t)g(x-\alpha t)\frac{dt}{t}$ and $Hf(x) = \frac{1}{\pi} p.v. \int f(x-t)\frac{dt}{t}$ denote the bilinear and linear Hilbert transforms respectively. It is proved that, for $1 < p < \infty$ and $\alpha_1 \neq \alpha_2$, $H_{\alpha_1} - H_{\alpha_2}$ maps $L^p \times BMO$ into L^p and it maps $BMO \times L^p$ into L^p if and only if $sign(\alpha_1) = sign(\alpha_2)$. It is also shown that, for $\alpha \leq 1$ the commutator $[H_{\alpha,f}, H]$ is bounded on L^p for $1 < p < \infty$ if and only if $f \in BMO$, where $H_{\alpha,f}(g) = H_\alpha(f,g)$.

1. Introduction.

Recall that the (linear) Hilbert transform over $\mathbb{R}$ is defined by

$$H(f)(x) = \frac{1}{\pi} p.v. \int f(x-t)\frac{dt}{t}, \tag{1.1}$$

and it is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$.

Analogue bilinear operators are defined, for each $\alpha \in \mathbb{R}$, by

$$H_\alpha(f,g)(x) = \frac{1}{\pi} p.v. \int f(x-t)g(x-\alpha t)\frac{dt}{t}. \tag{1.2}$$

Note that the case $\alpha = 0$ corresponds to $H_0(f,g) = H(f)g$. Hence it is bounded from $L^{p_1} \times L^{p_2} \to L^{p_3}$ whenever $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p_3$, while the case $\alpha = 1$ corresponds to $H_1(f,g) = H(fg)$ and it is bounded only in the case $1 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p_3$ and $1 < p_3 < \infty$.

The case $\alpha = -1$ corresponds to the so-called Bilinear Hilbert transform,

$$H_{-1}(f,g)(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} f(x-t)g(x+t)\frac{dt}{t},$$

whose boundedness for $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1$ was conjectured by A.P. Calderón.

This last result was shown to be true by M. Lacey and C. Thiele. Actually they proved first the case $p_3 > 1$ and then they were able to push their techniques to get up to $p_3 > 2/3$.

Theorem A *(See* [9] *and* [11] *or* [16]*) For each triple* (p_1, p_2, p_3) *such that* $1 < p_1, p_2 \leq \infty$, $1/p_1 + 1/p_2 = 1/p_3$ *and* $p_3 > 2/3$ *and each* $\alpha \in \mathbb{R} \setminus \{0, 1\}$ *there exists* $C(\alpha, p_1, p_2) > 0$ *for which*

$$\|H_\alpha(f,g)\|_{p_3} \leq C(\alpha, p_1, p_2)\|f\|_{p_1}\|g\|_{p_2}.$$

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This was improved by L. Grafakos and X. Li (see [6],[7]) showing that actually there exists $C > 0$ such that

$$\int_0^1 \|H_\alpha(f,g)\|_2 d\alpha \leq C\|f\|_2\|g\|_\infty.$$

which is the result that Calderón conjectured when he was working with the classical Hilbert transform defined over Lipschitz curves.

The aim of this paper is to understand better the connection between H and H_α and to see the interplay of the Bilinear Hilbert transform with BMO.

Recall that the space BMO happens to be connected with the Hilbert transform H for different reasons. By means of the duality with $Re(H^1) = \{f \in L^1(\mathbb{R}) : Hf \in L^1(\mathbb{R})\}$, or using the commutator theorem, due to R. Coifman, R. Rochberg and W. Weiss (see [4]), which says that $[M_g, H]$, where $M_g(f) = gf$, is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ if and only if $f \in BMO$.

We start by noticing that $(H_0 - H_1)(f,g) = gH(f) - H(fg) = [M_g, H]$ and then it maps $L^p \times BMO$ into L^p for $1 < p < \infty$. We shall see that this property holds true in general, by showing the following theorem

Theorem 1.1. *Let $\alpha_1 \neq \alpha_2 \in \mathbb{R}$ and $1 < p < \infty$. Then $H_{\alpha_1} - H_{\alpha_2}$ is bounded from $L^p(\mathbb{R}) \times BMO$ into $L^p(\mathbb{R})$.*

Now it also makes sense to ask ourselves about the boundedness from $BMO \times L^p(\mathbb{R})$ into $L^p(\mathbb{R})$. We shall see the following result.

Theorem 1.2. *Let $\alpha_1 \neq \alpha_2 \in \mathbb{R}$ and $1 < p < \infty$. Then $H_{\alpha_1} - H_{\alpha_2}$ is bounded from $BMO \times L^p(\mathbb{R})$ into $L^p(\mathbb{R})$if and only if $sign(\alpha_1) = sign(\alpha_2)$.*

Let us denote by $H_{\alpha,f}(g) = H_\alpha(f,g)$ and use the well-known formula

$$HfHg - fg = H(gHf + fHg)$$

to get

$$[H_{1,f}, H](g) = H(fHg) + fg = HfHg - H(gHf) = [H_{0,f}, H](g).$$

Therefore $[H_{1,f}, H](g) = [H_{0,f}, H](g) = [H, M_f](Hg) = [M_{Hf}, H](g)$ which, by the commutator theorem, implies that $f \in BMO$ if and only if $[H_{1,f}, H]$ or $[H_{0,f}, H]$ are bounded on $L^p(\mathbb{R})$.

We shall prove the following formula (see Corollary 2.16 below)

$$[H_{\alpha,f}, H](g) = [H_{1,f}, H](g) + (1 - sign(1-\alpha))(H_{\alpha,f} - H_{1,f})(Hg).$$

Hence for $\alpha \leq 1$ we get

$$[H_{1,f}, H] = [H_{\alpha,f}, H],$$

and, in particular, for $\alpha \leq 1$ and $1 < p < \infty$, $f \in BMO$ if and only if the commutator $[H_{\alpha,f}, H]$ is bounded on $L^p(\mathbb{R})$.

Using the previous formula and Theorem 1.1 we obtain the following corollary:

Corollary 1.3. *Let $\alpha \in \mathbb{R}$ and $1 < p < \infty$. If $f \in BMO$ then the commutator $[H_{\alpha,f}, H] = H_{\alpha,f}H - HH_{\alpha,f}$ is bounded on $L^p(\mathbb{R})$.*

2. Basic formulas

We first mention the formula for the operator using Fourier transform, whose elementary proof is left to the reader. If $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{S}$ then

$$H_\alpha(f,g)(x) = -i \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta) sign(\xi + \alpha\eta) e^{2\pi i x(\xi+\eta)} d\xi d\eta. \tag{2.1}$$

From (2.1) one easily gets, if $\alpha \neq 0$,

$$H_\alpha(f,g) = sign(\alpha) H_{1/\alpha}(g,f). \tag{2.2}$$

For the duality pairing $\langle f, g\rangle = \int f(x)g(x)dx$ for real functions f and g, using (2.1), we have

$$\langle H_\alpha(f,g), h\rangle = -i \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(-\xi-\eta) sign(\xi + \alpha\eta) d\xi d\eta$$

and hence

$$\langle H_\alpha(f,g), h\rangle = -\langle H_{1-\alpha}(h,g), f\rangle \tag{2.3}$$

and, if $\alpha \neq 1$, we get

$$\langle H_\alpha(f,g), h\rangle = sign(1-\alpha)\langle H_{\frac{\alpha}{\alpha-1}}(f,h), g\rangle. \tag{2.4}$$

Let us start now giving a formula which shows the interplay between bilinear and linear Hilbert transforms.

Note that the boundedness of H_α is equivalent to the one of

$$(H_\alpha - H_0)(f,g)(x) = \frac{1}{\pi} p.v. \int f(x-t)(g(x-\alpha t) - g(x))\frac{dt}{t}. \tag{2.5}$$

Nevertheless this last one has the following extra property.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$ and f and g belonging to S. Then*

$$H_\alpha(f,g) - H_0(f,g) = sign(\alpha)(H_\alpha(Hf, Hg) - H_0(Hf, Hg)). \tag{2.6}$$

Proof. The case $\alpha = 0$ is obvious.

In the case $\alpha = 1$, (2.6) becomes the well-known formula

$$H(fg - HfHg) = fHg + gHf.$$

Since

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta) e^{2\pi i x(\xi+\eta)} d\xi d\eta = f(x)g(x),$$

then (2.1) gives

$$H_\alpha(f_1, f_2) = -i f_1 f_2 \tag{2.7}$$

in the case $\alpha > 0$ for functions such that $supp(\hat{f}_1)$ and $supp(\hat{f}_2)$ contained in $[0,\infty)$ or in the case $\alpha < 0$ for functions with $supp(\hat{f}_1) \subset [0,\infty)$ and $supp(\hat{f}_2) \subset (-\infty, 0]$

Let f and g be real valued Schwarz functions. Applying (2.7) for $f_1 = f + iHf$ and $f_2 = g + isign(\alpha)Hg$ we obtain

$$H_\alpha(f + iHf, g + isign(\alpha)Hg) = -i(f + iHf)(g + isign(\alpha)Hg)$$

and thus

$$H_\alpha(f,g) - sign(\alpha)H_\alpha(Hf, Hg) = Hfg + sign(\alpha)fHg \tag{2.8}$$

and

$$H_\alpha(Hf, g) + sign(\alpha)H_\alpha(f, Hg) = sign(\alpha)HfHg - fg \tag{2.9}$$

and the first is the formula we wanted to show. ■

From formulas (2.8) and (2.9) we easily get

Corollary 2.2. *If $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ $\alpha \neq 0$ and $sign(\alpha_1)sign(\alpha_2) > 0$ then*

$$(H_{\alpha_1} - H_{\alpha_2})(Hf, Hg) = sign(\alpha_1)(H_{\alpha_1} - H_{\alpha_2})(f, g). \tag{2.10}$$

$$(H_{\frac{1}{\alpha}} + H_\alpha)(f, Hf) = (Hf)^2 - sign(\alpha)f^2 \tag{2.11}$$

$$H_{-1}(f, Hf) = \frac{f^2 + (Hf)^2}{2}. \tag{2.12}$$

Let us obtain an estimate from below for the norm of the bilinear Hilbert transform in the case $p_1 = p_2$. This shows, in particular that Theorem A can not be proved without restrictions on p_3.

Proposition 2.3. *Let $1 < p < \infty$ and let $A_{p,\alpha}$ denote the norm of the operator $H_\alpha : L^p \times L^p \to L^{p/2}$ and C_p the norm of $H : L^p \to L^p$. Then*

$$\frac{C_p^\beta - C_p^{-\beta}}{2} \leq A_{p,\alpha}^\beta,$$

where $\beta = \beta(p) = \min\{1, p/2\}$.

Proof. For $\alpha \in \mathbb{R}$ we use (2.8) to write

$$(Hf)^2 = H_\alpha(f, Hf) + sign(\alpha)H_\alpha(Hf, f) + sign(\alpha)f^2$$

and then we have

$$\|(Hf)^2\|_{p/2}^\beta \leq \|H_\alpha(f, Hf)\|_{p/2}^\beta + \|H_\alpha(Hf, f)\|_{p/2}^\beta + \|f^2\|_{p/2}^\beta.$$

Therefore

$$\|Hf\|_p^{2\beta} \leq 2A_{p,\alpha}^\beta \|f\|_p^\beta \|Hf\|_p^\beta + \|f\|_p^{2\beta}$$

which implies

$$C_p^{2\beta} \leq 2A_{p,\alpha}^\beta C_p^\beta + 1.$$

■

Clearly we have that $A_{p,0} = C_p$. We see that for negative values of α we get better estimates than the ones in the previous proposition.

Proposition 2.4. *Let $1 < p < \infty$ and denote by A_p and C_p the norm of $H_{-1} : L^p(\mathbb{R}) \times L^p(\mathbb{R}) \to L^{p/2}(\mathbb{R})$ and $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ respectively. Then*

$$4^{\frac{-1}{\min(p,p')}}(C_p^{-1} + C_p) \leq A_p.$$

Proof. Using (2.12) we have that

$$\|H_{-1}(f, Hf)\|_{p/2} = \frac{1}{2}\Big(\int_{\mathbb{R}} (f(x)^2 + Hf(x)^2)^{p/2} dx\Big)^{2/p}.$$

Since for all $a, b, r > 0$

$$\min(1, 2^{r-1})(a^r + b^r) \leq (a + b)^r \leq \max(2^{r-1}, 1)(a^r + b^r)$$

we have

$$
\begin{aligned}
\|H_{-1}(Hf,f)\|_{p/2} &\geq \frac{1}{2}\Big(\min(1,2^{\frac{p}{2}-1})\int_{\mathbb{R}}|f(x)|^p+|Hf(x)|^p dx\Big)^{2/p} \\
&= \frac{1}{2}\min(1,2^{\frac{p}{2}-1})^{\frac{2}{p}}\Big(\|f\|_p^p+\|Hf\|_p^p\Big)^{2/p} \\
&\geq \frac{1}{2}\min(1,2^{1-\frac{2}{p}})\min(1,2^{\frac{2}{p}-1})\Big(\|f\|_p^2+\|Hf\|_p^2\Big) \\
&\geq \frac{1}{2}\min(2^{1-\frac{2}{p}},2^{\frac{2}{p}-1})(C_p^{-2}+1)\|Hf\|_p^2 \\
&= 4^{\frac{-1}{\min(p,p')}}(C_p^{-2}+1)\|Hf\|_p^2.
\end{aligned}
$$

Now estimating from above we get

$$4^{\frac{-1}{\min(p,p')}}(C_p^{-2}+1)\|Hf\|_p^2 \leq A_p\|Hf\|_p\|f\|_p.$$

Therefore the proof is finished. ■

Remark 2.1. Since (2.8) gives for $\alpha<0$

$$H_\alpha(f,Hf)-H_\alpha(Hf,f)=(Hf)^2+f^2$$

we can repeat the previous argument and obtain

$$\frac{1}{4}4^{\frac{1}{\max(p,p')}}(C_p^{-1}+C_p)\leq A_{p,\alpha} \text{ for } \alpha<0.$$

Another interesting formula relating linear and bilinear Hilbert transforms is the following:

Theorem 2.5. *Let $\alpha\in\mathbb{R}$ and f and g belonging to $\mathcal{S}$. Then*

$$H(H_\alpha(f,g)-H_\alpha(Hf,Hg))=H_\alpha(f,Hg)+H_\alpha(Hf,g). \tag{2.13}$$

Proof. Observe that $H_\alpha(f+iHf,g+iHg)$ for real valued functions coincides with

$$H_\alpha(f,g)-H_\alpha(Hf,Hg)+i(H_\alpha(f,Hg)+H_\alpha(Hf,g)).$$

For $z\in\mathbb{C}$ we can define

$$\tilde{H}_\alpha(f_1,f_2)(z)=-i\int_{\mathbb{R}}\int_{\mathbb{R}}\hat{f}_1(\xi)\hat{f}_2(\eta)sign(\xi+\alpha\eta)e^{2\pi i(\xi+\eta)z}d\xi d\eta.$$

Since

$$|\tilde{H}_\alpha(f_1,f_2)(z)|\leq\int_{\mathbb{R}}\int_{\mathbb{R}}|\hat{f}_1(\xi)||\hat{f}_2(\eta)|e^{-2\pi(\xi+\eta)Im(z)}d\xi d\eta$$

this function is well defined when $(\xi+\eta)Im(z)>0$ and turns out to be holomorphic in $Im(z)>0$ when $supp(\hat{f}_1)$ and $supp(\hat{f}_2)\subset[0,\infty)$. Hence $\tilde{H}_\alpha(f+iHf,g+iHg)(z)$ is holomorphic in $Imz>0$. Now, denoting $F_y(x)=F(x+iy)$, we have that, for $y>0$,

$$H((H_\alpha(f,g)-H_\alpha(Hf,Hg))_y)=(H_\alpha(f,Hg)+H_\alpha(f,Hg))_y.$$

Finally taking limits as y goes to zero we get the result .■

It follows from Theorem A and Theorem 2.5 the following corollary.

Corollary 2.6. *If $p>1$, $f\in L^p(\mathbb{R})$ and $g\in L^{p'}(\mathbb{R})$ then $H_\alpha(f,g)-H_\alpha(Hf,Hg)$ and $H_\alpha(Hf,g)+H_\alpha(f,Hg)\in Re(H^1)$.*

Next result exhibits the commutative behaviour of the operator

$$(H_\alpha - H_1)(f,g)(x) = \frac{1}{\pi} p.v. \int f(x-t)(g(x-\alpha t) - g(x-t))\frac{dt}{t} \tag{2.14}$$

and the Hilbert transform.

Theorem 2.7. *Let $\alpha \in \mathbb{R}$ and f and g belonging to $\mathcal{S}$. Then*

$$H(H_\alpha(f,g) - H_1(f,g)) = sign(1-\alpha)(H_\alpha(f,Hg) - H_1(f,Hg)).$$

Proof. For $\alpha = 1$ it is obvious.

Assume $\alpha \neq 1$. The equality is true if and only if

$$H(fg - sign(1-\alpha)H_\alpha(f,Hg)) = sign(1-\alpha)fHg + H_\alpha(f,g).$$

Note that

$$\begin{aligned} & fg - sign(1-\alpha)H_\alpha(f,Hg) + i(sign(1-\alpha)fHg + H_\alpha(f,g)) \\ & = f(g + isign(1-\alpha)Hg) + iH_\alpha(f, g + isign(1-\alpha)Hg) \\ = & \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)(g + isign(1-\alpha)Hg)\hat{}(\eta)(1 + sign(\xi + \alpha\eta))e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ & = 2\int\int_{(1-\alpha)\eta>0} \hat{f}(\xi)\hat{g}(\eta)(1 + sign(\xi+\alpha\eta))e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ & = 4\int\int_{\xi+\alpha\eta>0,(1-\alpha)\eta>0} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(\xi+\eta)x} d\xi d\eta. \end{aligned}$$

Since $\xi + \eta > (1-\alpha)\eta > 0$ in the domain of integration, the function

$$4\int\int_{\xi+\alpha\eta>0,(1-\alpha)\eta>0} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(\xi+\eta)z} d\xi d\eta$$

is well defined and holomorphic in $Im(z) > 0$. Taking limits at the boundary we get the desired result.■

Corollary 2.8. *Let $\alpha \neq 1$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and let $f \in L^{p_1}$ and $g \in L^{p_2}$. Then $H_\alpha(f,g) \in Re(H^1)$ if and only if $fH(g) \in Re(H^1)$.*

Proof. Note that Theorem 2.7 shows that

$$H(H_\alpha(f,g)) = -sign(1-\alpha)H(fHg) - fg + sign(1-\alpha)H_\alpha(f,Hg).$$

Hence $H(H_\alpha(f,g) + sign(1-\alpha)fHg) \in L^1$. This gives the conclusion. ■

Let us now point out some results about commutators coming from the previous section. Theorem 2.5 and Theorem 2.7 give the following results.

Corollary 2.9. *Let $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{S}$. Then*

$$[H_{\alpha,f}, H](g) = [H_{\alpha,Hf}, H](Hg) \tag{2.15}$$

$$[H_{\alpha,f}, H](g) = [H_{1,f}, H](g) + (1 - sign(1-\alpha))(H_{\alpha,f} - H_{1,f})(Hg) \tag{2.16}$$

In particular $[H_{\alpha,f}, H] = [H_{1,f}, H]$ for $\alpha \leq 1$.

3. Proof of the main theorems

Proof of Theorem 1.1.

Proof. Assume first that $\alpha_i \neq 1$ for $i = 1, 2$. To prove that $H_{\alpha_1} - H_{\alpha_2}$ maps $L^p \times BMO$ into L^p it suffices to see, by duality (2.4), that $sign(1-\alpha_1)H_{\frac{\alpha_1}{\alpha_1 - 1}} - sign(1-\alpha_2)H_{\frac{\alpha_2}{\alpha_2 - 1}}$ maps $L^p \times L^{p'}$ into $Re(H^1)$.

Since $\frac{\alpha_1}{\alpha_1-1} \neq 1, \frac{\alpha_2}{\alpha_2-1} \neq 1$ then Theorem A implies that $sign(1-\alpha_i)H_{\frac{\alpha_i}{\alpha_i-1}}(f,g) \in L^1$ for $f \in L^p$ and $g \in L^{p'}$.

Now, using Theorem 2.7 we have

$$H(sign(1-\alpha_1)H_{\frac{\alpha_1}{\alpha_1-1}}(f,g) - sign(1-\alpha_2)H_{\frac{\alpha_2}{\alpha_2-1}}(f,g)) =$$

$$-(sign(1-\alpha_1) - sign(1-\alpha_2))fg + H_{\frac{\alpha_1}{\alpha_1-1}}(f,Hg) - H_{\frac{\alpha_2}{\alpha_2-1}}(f,Hg) \in L^1.$$

To cover the case where $\alpha_i = 1$ for some $i = 1, 2$, note that $H_\alpha - H_1$ maps $L^p \times BMO$ into L^p because $H_\alpha(f,g) - H_1(f,g) = (H_\alpha - H_0)(f,g) + [M_g, H](f)$ and we can use the previous case and the commutator theorem.■

Proof of Theorem 1.2.

Proof. Assume that $sign(\alpha_1) = sign(\alpha_2)$. To prove that $H_{\alpha_1} - H_{\alpha_2}$ maps $BMO \times L^p$ into L^p it suffices to see, by duality (2.3), that $H_{1-\alpha_1} - H_{1-\alpha_2}$ maps $L^{p'} \times L^p$ into $Re(H^1)$. Now, since $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, then Theorem A gives

$$H_{1-\alpha_1}(f,g) - H_{1-\alpha_2}(f,g) \in L^1.$$

On the other hand, Theorem 2.7 implies that

$$H(H_{1-\alpha_1}(f,g) - H_{1-\alpha_2}(f,g)) = sign(\alpha_1)(H_{1-\alpha_1}(f,Hg) - H_{1-\alpha_2}(f,Hg)) \in L^1.$$

Conversely, assume that $H_{\alpha_1} - H_{\alpha_2}$ maps $BMO \times L^p$ into L^p. Observe first that if $f \in L^{p'}(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ then

$$H(H_{1-\alpha_1}(f,g) - H_{1-\alpha_2}(f,g)) \in L^1(\mathbb{R}).$$

Indeed, if $h \in L^\infty(\mathbb{R})$ (hence $Hh \in BMO$) we have

$$\langle H(H_{1-\alpha_1}(f,g) - H_{1-\alpha_2}(f,g)), h\rangle = \langle H_{\alpha_1}(Hh,g) - H_{\alpha_2}(Hh,g), f\rangle.$$

Let us see now that $\alpha_i \neq 0$. Assume that $\alpha_j = 0$ for some $j = 1, 2$ and so $\alpha_i \neq 0$ for $i \neq j$. Hence, combining the previous statement with Theorem 2.7, we get

$$H_1(f,Hg) = H_{\alpha_i - 1}(f,Hg) - sign(\alpha_i)H(H_{1-\alpha_i}(f,g) - H_1(f,g)) \in L^1(\mathbb{R})$$

which is contradictory.

Applying one more time Theorem 2.7 we have

$$\begin{aligned}(sign(\alpha_2) - sign(\alpha_1))H(fHg) &= H(H_{1-\alpha_1}(f,g) - H_{1-\alpha_2}(f,g)) \\ &- sign(\alpha_1)H_{1-\alpha_1}(f,Hg) + sign(\alpha_2)H_{1-\alpha_2}(f,Hg)\end{aligned}$$

for all $f \in L^{p'}$ and $g \in L^p$.

Then $(sign(\alpha_2) - sign(\alpha_1))H(fHg) \in L^1$ for all $f \in L^{p'}(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ which implies $sign(\alpha_2) - sign(\alpha_1) = 0$ and the proof is finished. ■

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