VECTOR VALUED ANALYTIC FUNCTIONS OF BOUNDED MEAN OSCILLATION AND GEOMETRY OF BANACH SPACES.

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Introduction.

When dealing with vector-valued functions, sometimes is rather difficult to give non trivial examples, meaning examples which do not come from tensoring scalar-valued functions and vectors in the Banach space, belonging to certain classes. This is the situation for vector valued BMO. One of the objectives of this paper is to look for methods to produce such examples.

Our main tool will be the vector-valued extension of the following result on multipliers, proved in [MP], which says that the space of multipliers between H^1 and $BMOA$ can be identified with the space of Bloch functions $\mathcal{B}$, i.e. $(H^1, BMOA) = \mathcal{B}$ (see Section 3 for notation), which, in particular gives that $g * f \in BMOA$ whenever $f \in H^1$ and $g \in \mathcal{B}$.

Given two Banach spaces X, Y it is rather natural to define the convolution of an analytic function with values in the space of operators $\mathcal{L}(X,Y)$, say $F(z) = \sum_{n=0}^{\infty} T_n z^n$, and a function with values in X, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$, as the function given by $F * g(z) = \sum_{n=0}^{\infty} T_n(x_n) z^n$.

It is not difficult to see that the natural extension of the multipliers' result to the vector valued setting does not hold for general Banach spaces. To be able to get a proof of such a result we shall be using the analogue of certain inequalities, due to Hardy and Littlewood [HL3], in the vector valued setting, namely

$$\Big(\int_0^1 (1-r) M_1^2(f', r)\,dr\Big)^{\frac{1}{2}} \leq C||f||_{H^1}$$

and its dual formulation

$$||f||_* \leq C\Big(\int_0^1 (1-r) M_\infty^2(f', r)\,dr\Big)^{\frac{1}{2}}.$$

This lead to consider spaces where these inequalities hold when replacing the absolute value by the norm in the Banach space which turn out to be very closely related to notions as (Rademacher) cotype 2 and type 2.

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The paper is divided into six sections. We start with a section of preliminary character to recall the notions on geometry of Banach spaces to be used throughout the paper.

Section 1 contains the basic properties of vector valued analytic functions of bounded mean oscillation and functions in the vector valued Bloch space. It is presented a proof of the extension of Kintchine-Kahane's inequalities to vector valued BMOA.

In this Section 2 we characterize Hilbert spaces in terms of the equivalence between the norm in BMOA and the norm defined in terms of a Carleson measure condition. The rest of the section is devoted to give some sufficients conditions on the derivative of the function or on the Taylor coefficients of the function to assure that the function belongs to $BMOA(X)$. It is shown that one has that $M_p(f',r) = O((1-r)^{-\frac{1}{p'}})$ for some $1 \leq p < \infty$ or $||x_n|| = O(\frac{1}{n})$ (in the case of B-convex spaces) implies that $f(z) = \sum_{n=0}^{\infty} x_n z^n \in BMOA(X)$.

Section 3 deals with multipliers between spaces of vector valued functions defined on different Banach spaces X and Y. This is done by looking at functions with values in the space of operators $\mathcal{L}(X,Y)$ and considering the natural convolution mentioned above. We also introduce two new notions based on the the vector valued formulations of the Hardy-Littlewood inequalities previously pointed out, called (HL)-property and $(HL)^*$-property respectively. It is shown that under the assumptions of (HL)-property on X and $(HL)^*$-property on Y one has that $\big(H^1(X), BMOA(Y)\big) = \mathcal{B}\big(\mathcal{L}(X,Y)\big)$.

Section 4 is devoted to the study these properties. It is shown that they are related to Paley and type 2 and also that the natural duality between them holds for UMD spaces. We investigate Lebesgue spaces L^p and Schatten classes σ_p having such properties. The tools to deal with Schatten classes are the use of certain factorization and interpolation results holding for functions in Hardy spaces with values in Schatten classes.

Finally Section 5 is devoted to present several applications of different nature of the previous results.

Throughout the paper all spaces are assumed to be complex Banach spaces, D stands for the unit disc and $\mathbb{T}$ for its boundary. Given $1 \leq p < \infty$, we shall denote by $L^p(X)$ the space of X-valued Bochner p-integrable functions on the circle $\mathbb{T}$ and write $||f||_{p,X} = \left(\int_0^{2\pi} ||f(e^{it})||^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}$ and $M_{p,X}(F,r) = ||F_r||_{p,X} = \left(\int_0^{2\pi} ||F(re^{it})||^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}$ whenever F is any X-valued analytic function on D. We shall write $L^1(D,X)$ for the space of X-valued Bochner integrable functions on D with respect to the area measure $dA(z)$, and $H^p(X)$ (respec. $H^p_0(X)$) for the vector-valued Hardy spaces, i.e. space of functions in $L^p(X)$ whose negative (respec. non positive) Fourier coefficients vanish.

Of course Hardy spaces $H^p(X)$ (respec. $H^p_0(X)$) can be regarded as spaces of analytic functions on the disc. Actually they coincide with the closure of the X-valued polynomials, denoted by $\mathcal{P}(X)$ (respec. those which vanish at $z=0$, denoted by $\mathcal{P}_0(X)$,) under the norm given by $\sup_{0<r<1} M_{p,X}(f,r)$.

The reader shoud be aware that the analytic functions we are considering have boundary values a.e. on $\mathbb{T}$, but this in general does not hold (such a fact actually corresponds to the so called ARNP introduced in [BuD]).

Finally let us point out a notation to be used in the sequel. Whenever a scalar valued function ϕ is given we write $\phi_z(w) = \phi(zw)$ and look at $z \to \phi_z$ as a vector valued function.

As usual p' is the conjugate exponent of p when $1 \leq p \leq \infty$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ and C will stand for a constant that may vary from line to line.

§0 Preliminaries on Geometry of Banach spaces.

It is well known the connection between certain properties in geometry of Banach spaces and vector-valued Hardy spaces. Some of them, like ARNP [BuD], Paley [BP],..., were actually introduced to have certain theorems on Hardy spaces holding in the vector-valued setting, others, like UMD [Bu2], B-convexity [MPi] or Fourier type [Pee], were connected to this theory through the boundedness of classical operators like Hilbert transform, Paley projection or Fourier transform for vector-valued functions. In this section we shall recall those to be used in the sequel and give some references to get more information about them.

One of the more relevant properties in the vector-valued Fourier analysis is the so called UMD property. It was introduced in the setting of vector valued martingales, but was shown (see [Bu1, Bo1]) to be equivalent to the boundedness of the Hilbert transform on $L^p(X)$ for any $1 < p < \infty$. Because of this it is a natural assumption when dealing with vector valued Hardy spaces.

We shall say that a complex Banach space X is a UMD space if the Riesz projection $\mathcal{R}$, defined by $\mathcal{R}(f) = \sum_{n\geq 0} \hat{f}(n)e^{int}$, is bounded from $L^2(X)$ into $H^2(X)$.

One of the basic facts on this property that we shall use is that the vector valued version of the Fefferman's $H^1 - BMO$-duality theorem holds for UMD spaces (see for instance [Bo3, B2, RRT]). The reader is referred to the surveys [RF, Bu2] for information on the UMD property.

Another useful property for our purposes will be the notion of Fourier-type introduced by Peetre ([Pee]) which corresponds to spaces where the vector valued analogue of Hausdorff-Young's inequalies holds.

Let us recall that for $1 \leq p \leq 2$, a Banach space X is said to have Fourier type p if there exists a constant $C > 0$ such that

$$\left(\sum_{n=-\infty}^{\infty} ||\hat{f}(n)||^{p'}\right)^{\frac{1}{p'}} \leq C||f||_{L^p(X)}.$$

It is not hard to see that X has Fourier type p if and only if X^* has Fourier type p. Typical examples are L^r for $p \leq r \leq p'$ or those obtained by interpolation between any Banach space and a Hilbert space. The reader is referred to [Pee, GKT, K] for some equivalent formulations, connections with interpolation and several examples in the contex of function spaces.

Let us now recall two fundamental notions in geometry of Banach spaces associated to Kintchine's inequalities. Although they are defined in terms of the Rademacher functions, to be denoted r_n, we shall replace them by lacunary sequences $e^{i2^n t}$, which gives an equivalent definition ([MPi, Pi1]).

Given $1 \leq p \leq 2 \leq q \leq \infty$. A Banach space has cotype q (respec. type p) if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and for all $x_0, x_1, x_2, ...x_N \in X$ one has

$$\left(\sum_{k=0}^{N} ||x_k||^q\right)^{\frac{1}{q}} \leq ||\sum_{k=0}^{N} x_k e^{2^k it}||_{1,X}$$

and respectively

$$||\sum_{k=0}^{N} x_k e^{2^k it}||_{1,X} \leq C\left(\sum_{k=0}^{N} ||x_k||^p\right)^{\frac{1}{p}}.$$

A Banach space is called B-convex if it has (Rademacher)-type > 1.

The reader is referred to [LT, Pi2] for some applications of such notions to the Banach space theory.

Let us now state two fundamental theorems to be used in the sequel due to J. Bourgain and S. Kwapien respectively.

Theorem A. *([Bo4, Bo5]) Let X be a complex Banach space.*
X has Fourier type > 1 if and only if $X$is B-convex.

Theorem B. *([Kw]) Let X be a complex Banach space. X is isomorphic to a Hilbert space if and only if X has type 2 and cotype 2.*

Let us finish this section by recalling another property, stronger than cotype 2, to be used later on that was introduced in [BP] and depends upon the vector-valued analogue of Paley's inequality [Pa] for Hardy spaces. A complex Banach space X is said to be a Paley space if

$$\left(\sum_{k=0}^{\infty} ||x_{2^k}||^2\right)^{\frac{1}{2}} \leq C||f||_{1,X}$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

§1.- Basic definitions and properties on vector valued $\mathcal{B}$ and $BMOA$.

In this section we shall consider the vector valued version of analytic BMO and the space of Bloch functions $\mathcal{B}$. The reader is referred to [GR, G, Z] for scalar-valued theory on BMO and to [ACP, Z] for results on scalar-valued Bloch functions.

Definition 1.1. *Given a complex Banach space X, we shall denote by $BMOA(X)$ the space functions $f \in L^1(X)$ with $\hat{f}(n) = 0$ for $n < 0$ such that*

$$||f||_{*,X} = \sup_I \frac{1}{|I|} \int_I ||f(e^{it}) - f_I|| \frac{dt}{2\pi} < \infty,$$

where the supremum is taken over all intervals $I \in [0, 2\pi)$, $|I|$ stands for the normalized Lebesgue measure of I and $f_I = \frac{1}{|I|} \int_I f(e^{it}) \frac{dt}{2\pi}$.

The norm in the space is given by

$$||f||_{BMO(X)} = ||\int_0^{2\pi} f(e^{it})\frac{dt}{2\pi}|| + ||f||_{*,X}.$$

From John-Nirenberg lemma ([G, GF]), which holds in the vector valued setting, one can actually replace the L^1 norm in the definition for any other L^p norm, that is: For any $1 \leq p < \infty$,

$$||f||_{*,X} \approx \sup_I \left(\frac{1}{|I|}\int_I ||f(e^{it}) - f_I||^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}.$$

REMARK 1.1. Same technique as in the scalar-valued case allows us to replace the averaging over intervals by convolution with the Poisson kernel. According to this and the previous formulation for $p = 2$ one has that

$$||f||_{*,X} \approx \sup_{|z|<1} \left(\int_0^{2\pi} ||f(e^{it}) - f(z)||^2 P_z(e^{-it})\frac{dt}{2\pi}\right)^{\frac{1}{2}} \tag{1.1}$$

where P_z is the Poisson Kernel $P_z(w) = \frac{1-|z|^2}{|1-zw|^2}$ and $f(z) = \int_0^{2\pi} f(e^{it})P_z(e^{-it})\frac{dt}{2\pi}$.

Let us point out certain results on the duality to be used later on. Although most of the results on the duality $H^1 - BMO$ for vector valued functions (see [B2, Bo3, RRT]) are given for the space H^1 defined in terms of atoms, it is easy to deduce from the known results the following facts:

For any Banach space X one has that $BMOA(X^*)$ continuously embedds into $\left(H^1(X)\right)^*$. Actually if $f \in BMOA(X^*)$ and $g \in \mathcal{P}(X)$ then

$$|\int_0^{2\pi} < f(e^{it}), g(e^{-it}) > \frac{dt}{2\pi}| \leq ||f||_{BMOA(X^*)}||g||_{1,X}.$$

If X is a UMD space then we actually have the validity of Fefferman's duality result

$$\left(H^1(X)\right)^* = BMOA(X^*).$$

It is well known that Kintchine's inequalities hold for BMO functions, i.e.

$$\left(\sum_{k=0}^{\infty} |\alpha_k|^2\right)^{\frac{1}{2}} \approx ||\sum_{k=0}^{\infty} \alpha_k z^{2^k}||_{BMOA}. \tag{1.2}$$

Recall now that in the vector valued setting, although Kintchine's inequalites do not remain valid, at least one still has the so called Kahane's inequalities, i.e. for any $0 < p < \infty$

$$\int_0^{2\pi} ||\sum_{k=0}^{n} x_k e^{i2^k t}||\frac{dt}{2\pi} \approx \left(\int_0^{2\pi} ||\sum_{k=0}^{n} x_k e^{i2^k t}||^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}.$$

There exists an extension of Kahane-Kintchine inequalities to vector valued BMO which is part of the folklore. Let us present a proof based upon the following lemma.

Lemma A. *(see [Pe, Pi1]) Let X be a Banach space. Let $\lambda_k \in \mathbb{R}^+$ such that $\frac{\lambda_{k+1}}{\lambda_k} \geq C > 1$. Then for any $x_0, x_1, x_2, ..., x_n \in X$ there exist constants $K_1, K_2 > 0$, depending only on C, such that*

$$K_1 \int_0^{2\pi} ||\sum_{k=0}^n x_k e^{i2^k t}|| \frac{dt}{2\pi} \leq \int_0^{2\pi} ||\sum_{k=0}^n x_k e^{i\lambda_k t}|| \frac{dt}{2\pi} \leq K_2 \int_0^{2\pi} ||\sum_{k=0}^n x_k e^{i2^k t}|| \frac{dt}{2\pi}$$

Proposition 1.1. *Let X be a Banach space and $x_0, x_1, x_2, ..., x_n \in X$. Then*

$$||\sum_{k=0}^n x_k e^{i2^k t}||_{*,X} \approx \int_0^{2\pi} ||\sum_{k=0}^n x_k e^{i2^k t}|| \frac{dt}{2\pi}.$$

Proof. Let us write $f(e^{it}) = \sum_{k=0}^n x_k e^{i2^k t}$. Given an interval, say $J = \{e^{it} : |t - t_J| < 2\pi|J|\}$, then

$$\frac{1}{|J|} \int_J ||f(e^{it}) - f_J|| \frac{dt}{2\pi} \leq \frac{2}{|J|} \int_J ||f(e^{it})|| \frac{dt}{2\pi} = 2 \int_0^{2\pi} ||\sum_{k=1}^n x_k e^{i2^k (|J|t)}|| \frac{dt}{2\pi}.$$

Now applying Lemma A for $\lambda_k = 2^k |J|$ we get

$$\frac{1}{|J|} \int_J ||f(e^{it}) - f_J|| \frac{dt}{2\pi} \leq C \int_0^{2\pi} ||\sum_{k=0}^n x_k e^{i2^k t}|| \frac{dt}{2\pi}.$$

Taking now the supremum over J we get the direct inequality.
The converse inequality is trivial and the proof is finished. □

Let us now recall the formulation of BMO functions in terms of Carleson measures (see [G, Z]) that we shall use later on.

Definition 1.2. *Given an analytic function $f(z) = \sum_{k=0}^\infty x_k z^k$ we define*

$$||f||_{\mathcal{C},X} = \sup_{|z|<1} \Big(\int_D (1 - |w|) ||f'(w)||^2 P_z(\bar{w}) dA(w) \Big)^{\frac{1}{2}} \tag{1.3}$$

where P_z is the Poisson Kernel $P_z(w) = \frac{1-|z|^2}{|1-zw|^2}$.
We shall denote $BMOA_{\mathcal{C}}(X)$ the space of functions such that $||f||_{\mathcal{C},X} < \infty$.
$BMOA_{\mathcal{C}}(X)$ becomes a Banach space endowed with the norm

$$||f||_{BMOA_{\mathcal{C}}(X)} = ||f(0)|| + ||f||_{\mathcal{C},X}$$

We shall see in the next section that both notions only coincide for Hilbert spaces.
A simple and useful necessary condition for a function to belong to $BMOA_{\mathcal{C}}(X)$ is given in the following

Proposition 1.2. *Let f be a X-valued analytic function. If*

$$\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||^2 dr < \infty$$

then $f \in BMOA_{\mathcal{C}}(X)$.

Proof. For any $z \in D$ one has

$$\begin{aligned}
&\int_D \frac{(1-|z|^2)(1-|w|)||f'(w)||_X^2}{|1-\bar{w}z|^2} dA(w) \\
&\leq \int_0^1 (1-r) \sup_{|w|=r} ||f'(w)||_X^2 \Big(\int_0^{2\pi} \frac{1-r^2|z|^2}{|1-re^{-it}z|^2} \frac{dt}{2\pi} \Big) dr \\
&= \int_0^1 (1-r) \sup_{|w|=r} ||f'(w)||_X^2 dr.
\end{aligned}$$

Therefore

$$||f||_{\mathcal{C},X} \leq C \left(\int_0^1 (1-r) \sup_{|w|=r} ||f'(w)||_X^2 dr \right)^{\frac{1}{2}} < \infty. \quad \square$$

Next example shows that the same condition is not enough to get functions in $BMOA(X)$ for general Banach spaces.

Proposition 1.3. *Let $X = l^1$ and $f(z) = \big(\frac{1}{nlog(n+1)} z^n\big)_{n=0}^{\infty}$. Then*

$$\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||_{l^1}^2 dr < \infty$$

but $f \notin H^1(l^1)$.

Proof. Since $||f(z)||_{l^1} = \sum_{n=1}^{\infty} \frac{1}{nlog(n+1)} |z|^n$ then

$$\lim_{r\to 1} M_{1,l^1}(f,r) = \sum_{n=1}^{\infty} \frac{1}{nlog(n)} = \infty,$$

what gives that $f \notin H^1(l^1)$.

On the other hand, (see [L, page 93-96]),

$$||f'(z)||_{l^1} = \sum_{n=1}^{\infty} \frac{1}{log(n+1)} |z|^n \approx \frac{|z|}{(1-|z|)(log\frac{1}{1-|z|})}.$$

Therefore

$$\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||_{l^1}^2 dr \leq C \int_0^1 \frac{dr}{(1-r)(log\frac{1}{1-r})^2} < \infty. \quad \square$$

Observe that combining Propositions 1.2 and 1.3 one shows that in general $BMOA_{\mathcal{C}}(X)$ is not contained in $BMOA(X)$. We shall see in next section that this actually depends on the type 2 condition.

Let us now turn to some results on vector valued Bloch functions.

Definition 1.3. *Given a complex Banach space E we shall use the notation $\mathcal{B}(E)$ for the space of E- valued analytic functions on D, say $f(z)=\sum_{n=0}^\infty x_n z^n$, such that*

$$\sup_{|z|<1}(1-|z|)||f'(z)||<\infty.$$

We endow the space with the following norm

$$||f||_{\mathcal{B}(E)}=max\{||f(0)||,\sup_{|z|<1}(1-|z|)||f'(z)||\}.$$

REMARK 1.2. It follows clearly form the definition that for any Banach space E and $F(z)=\sum_{n=0}^\infty x_n z^n$ one has that $F\in\mathcal{B}(E)$ if and only if

$$F_{x^*}(z)=\sum_{n=0}^\infty <x^*,x_n> z^n\in\mathcal{B}$$

for any $x^*\in E^*$. Moreover

$$||F||_{\mathcal{B}(E)}=\sup_{||x^*||\leq 1}||F_{x^*}||_{\mathcal{B}}.$$

REMARK 1.3. Let $E=\mathcal{L}(X,Y)$, the space of bounded linear operators from X into Y and $(T_n)\subset\mathcal{L}(X,Y)$. It is elementary to see that $F(z)=\sum_{n=0}^\infty T_n z^n\in\mathcal{B}(\mathcal{L}(X,Y))$ if and only if the functions $F_{x,y^*}(z)=\sum_{n=0}^\infty <T_n(x),y^*> z^n\in\mathcal{B}$ for any $x\in X, y^*\in Y^*$. Moreover

$$||F||_{\mathcal{B}(\mathcal{L}(X,Y))}=\sup_{||x||\leq 1,||y^*||\leq 1}||F_{x,y^*}||_{\mathcal{B}}.$$

REMARK 1.4. In the case $E=l^\infty$ one can identify $\mathcal{B}(l^\infty)=l^\infty(\mathcal{B})$. Moreover if $f=(f_n)$

$$\sup_{n\in\mathbb{N}}||f_n||_{\mathcal{B}}=||f||_{\mathcal{B}(l^\infty)}.$$

EXAMPLE 1.1. Let $1\leq p\leq\infty$ and

$$f_p(z)=\sum_{n=1}^\infty n^{\frac{-1}{p}}e_n z^n,\qquad f_\infty(z)=\sum_{n=1}^\infty \frac{a_n}{n}z^n$$

where e_n stands for the canonic bases in l^p and $a_n=\sum_{k=1}^n e_k$. Then $f_p\in\mathcal{B}(l^p)$.

EXAMPLE 1.2. Let $1\leq p\leq\infty$ and

$$g_p(z)=\frac{1}{(1-z)^{\frac{1}{p}}},\qquad g_\infty(z)=log\frac{1}{1-z}.$$

Then $F_p(z)=(g_p)_z\in\mathcal{B}(H^p)$.

There are also other procedures to get X-valued Bloch functions that we state in the following propositions.

Proposition 1.4. *Let X be a Banach space and $T \in \mathcal{L}(L^1(D), X)$. Then $f(z) = T(K_z)$ is a X-valued Bloch function, where K_z denotes the Bergman Kernel $K_z(w) = \frac{1}{(1-zw)^2}$.*

Proof. Observe that $f(z) = \sum_{n=0}^\infty (n+1)T(u_n)z^n$ for $u_n(w) = w^n$.

Therefore $f'(z) = \sum_{n=1}^\infty n(n+1)T(u_n)z^{n-1} = T(\frac{-2w}{(1-wz)^3})$ and then we have

$$||f'(z)|| \leq ||T|| \int_0^1 \int_0^{2\pi} \frac{2r}{|1-rze^{it}|^3} \frac{dt}{2\pi} dr \leq C||T|| \int_0^1 \frac{2r}{(1-r|z|)^2} dr \leq C \frac{1}{1-|z|}. \quad \square$$

Proposition 1.5. *(see* [ACP, AS]*) Let E be a Banach space and $x_n \in E$.*

(i) If $\sup_{||x^*|| \leq 1} \sup_{n \geq 0} \sum_{k=2^n}^{2^{n+1}} | < x^*, x_k > | < \infty$ *then* $\sum_{n=0}^\infty x_n z^n \in \mathcal{B}(E)$.

(ii) $|| \sum_{n=0}^\infty x_n z^{2^n} ||_{\mathcal{B}(E)} \approx \sup_{n \geq 0} ||x_n||$.

Proof. (i) Note that for each $||x^*|| \leq 1$,

$$\begin{aligned} || \sum_{n=1}^\infty n < x^*, x_n > z^{n-1} || &\leq \sum_{n=0}^\infty \sum_{k=2^n}^{2^{n+1}} k | < x^*, x_k > | |z|^{k-1} \\ &\leq \Big(\sup_{n \geq 0} \sum_{k=2^n}^{2^{n+1}} | < x^*, x_k > | \Big) \Big(\sum_{n=0}^\infty 2^{n+1} |z|^{2^n - 1} \Big) \\ &\leq \frac{C}{1-|z|}. \end{aligned}$$

Hence $\sum_{n=0}^\infty < x^*, x_n > z^n \in \mathcal{B}$ uniformly in $||x^*|| \leq 1$.

(ii) Take $f(z) = \sum_{n=0}^\infty x_n z^{2^n}$. From (i) we have $|| \sum_{n=1}^\infty x_n z^{2^n} ||_{\mathcal{B}(E)} \leq C \sup_{n \geq 0} ||x_n||$.

The other estimate follows by taking $r = 1 - \frac{1}{2^n}$ in the following inequality

$$2^n ||x_n|| r^{2^n - 1} \leq \sup_{|z|=r} ||f'(z)|| \leq \frac{C}{1-r}. \quad \square$$

§2. Elementary properties and examples on $BMOA(X)$.

First of all let us establish the connection between $BMOA(X)$ and $BMOA_{\mathcal{C}}(X)$.

Theorem 2.1. *Let X be a complex Banach space.*

(i) If $BMOA(X) \subset BMOA_{\mathcal{C}}(X)$ then X has cotype 2.

(ii) If $BMOA_{\mathcal{C}}(X) \subset BMOA(X)$ then X has type 2.

Proof.

(i) Let us take $f(z) = \sum_{k=0}^n x_k z^{2^k}$. Assume first that $||f||_{\mathcal{C},X} \leq C||f||_{*,X}$.

Note that choosing $z=0$, and using Proposition 1.1, we have

$$\int_0^1 (1-s)M^2_{2,X}(f',s)ds \leq ||f||_{\mathcal{C},X} \leq C||f||_{1,X}.$$

Since $2^n||x_n||r^{2^n-1} \leq M_{2,X}(f',r)$ for $n \in \mathbb{N}$ then we can write

$$\begin{aligned}\left(\int_0^1 (1-r)M^2_{2,X}(f',r)dr\right)^{\frac{1}{2}} &\geq \left(\sum_{k=0}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} (1-r)2^{2k}||x_k||^2 r^{2(2^k-1)}dr\right)^{\frac{1}{2}} \\ &\geq C\left(\sum_{k=0}^{n} ||x_{2^k}||^2(1-2^{-k})^{2(2^k-1)}dr\right)^{\frac{1}{2}}.\end{aligned}$$

Using now the fact that $(1-2^{-k})^{2^k} \geq Ce^{-1}$ one gets the cotype 2 condition

$$\left(\sum_{k=0}^{n} ||x_{2^k}||^2\right)^{\frac{1}{2}} \leq C||f||_{1,X}.$$

Assume now that $||f||_{*,X} \leq C||f||_{\mathcal{C},X}$. Therefore,

$$||f||^2_{1,X} \leq C||f||^2_{*,X} \leq C \sup_{z\in D} \int_D (1-|w|)(\sum_{k=0}^{n} 2^k||x_k|||w|^{2^k-1})^2 P_z(\bar{w})dA(w).$$

From Cauchy-Schwarz inequality

$$\begin{aligned}(\sum_{k=0}^{n} 2^k||x_k|||w|^{2^k-1})^2 &\leq (\sum_{k=0}^{n} 2^k||x_k||^2|w|^{2^k-1})(\sum_{k=0}^{n} 2^k|w|^{2^k-1}) \\ &\leq (\sum_{k=0}^{n} 2^k||x_k||^2|w|^{2^k-1})(\frac{C}{1-|w|})\end{aligned}.$$

This gives that

$$\begin{aligned}||f||^2_{1,X} &\leq C\int_D \sum_{k=0}^{n} 2^k||x_k||^2|w|^{2^k-1}P_z(\bar{w})dA(w) \\ &= C\int_0^1 \sum_{k=0}^{n} 2^k||x_k||^2 r^{2^k-1}dr = C\sum_{k=0}^{n} ||x_{2^k}||^2. \quad \square\end{aligned}$$

As a consequence we get the following characterizacion of Hilbert spaces which is part of the folklore.

Corollary 2.1. *Let X be a complex Banach space. $BMOA(X) = BMOA_{\mathcal{C}}(X)$ (with equivalent norms) if and only if X is isomorphic to a Hilbert space.*

Proof. Recall that the classical proof ([G, Theorem 3.4]) can be reproduced because merely relies upon (1.1) and Plancherel's theorem, which is at our disposal in the Hilbert-valued case.

The converse follows by combining Theorem 2.1 with Theorem B. $\square$

Neither the definition nor the characterization (1.3) in the case of Hilbert spaces are easily checkable and this makes rather difficult to produce non trivial examples of vector valued $BMOA$ functions. We shall give some simple necessary conditions following [CP, BSS].

For such purpose we shall need some well known lemmas.

Lemma B.

Let $0 < p \leq q \leq \infty$ and g an X-valued analytic function. Then

$$M_{q,X}(g, r^2) \leq C(1-r)^{\frac{1}{q}-\frac{1}{p}} M_{p,X}(g, r) \qquad \text{(see [D, page 84])} \tag{2.1}$$

Let $\gamma > 1$ then

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^\gamma} = O((1-|z|)^{1-\gamma}) \quad \text{(see [D, page 65])} \tag{2.2}$$

Let $\gamma < \beta$ then

$$\int_0^1 \frac{(1-r)^{\gamma-1}}{(1-rs)^\beta} dr = O\left((1-s)^{\gamma-\beta}\right) \qquad \text{(see [SW, Lemma 6])} \tag{2.3}$$

Next result has an straightforward generalization to the vector valued setting.

Lemma C. *(Hardy-Littlewood,* [D,Theorem 5.4]*)*

Let $f : D \to X$ be analytic, $0 < p \leq \infty$ and $0 < \alpha < 1$.

If $M_{p,X}(f', r) = O\left(\frac{1}{(1-r)^{1-\alpha}}\right)$ $(r \to 1)$ then

$$\left(\int_0^{2\pi} ||f(e^{it}) - f(e^{i(t+h)})||^p dt\right)^{\frac{1}{p}} = O(|h|^\alpha), (h \to 0).$$

Theorem 2.2. *Let f be a X-valued analytic function. If there exists $0 < p < \infty$ such that*

$$M_{p,X}(f', r) = O\left((1-r)^{-\frac{1}{p'}}\right)$$

then $f \in BMOA_{\mathcal{C}}(X) \cap BMOA(X)$.

Proof. Notice that (2.1) implies that if there exists $0 < p_0 < \infty$ such that $M_{p_0,X}(f', r) = O\left((1-r)^{-\frac{1}{p'_0}}\right)$ then the same property holds for any $p \geq p_0$. Therefore it suffices to prove the result assuming $2 < p < \infty$.

Set then $q = \frac{p}{2}$ and take $z \in D$. Then using Hölder's inequality and (2.2) we have

$$\begin{aligned}
&\int_0^1 \int_0^{2\pi} \frac{(1-s)(1-|z|^2)||f'(se^{it})||^2}{|1-zse^{-it}|^2} \frac{dt}{2\pi} ds \\
&\leq \int_0^1 (1-s)(1-|z|^2) M^2_{p,X}(f',s) \left(\int_0^{2\pi} \frac{1}{|1-zse^{-it}|^{2q'}} \frac{dt}{2\pi} \right)^{\frac{1}{q'}} ds \\
&\leq C \int_0^1 \frac{(1-s)^{1-\frac{2}{p'}}(1-|z|^2)}{(1-|z|s)^{2-\frac{1}{q'}}} ds.
\end{aligned}$$

Applying now (2.3) for $\gamma = \frac{2}{p}$ and $\beta = 1 + \frac{2}{p}$ one gets

$$\int_0^1 \frac{(1-s)^{1-\frac{2}{p'}}}{(1-|z|s)^{2-\frac{1}{q'}}} ds \leq \frac{C}{1-|z|}.$$

This gives then $f \in BMOA_{\mathcal{C}}(X)$.

To see that $f \in BMOA(X)$ we can use Lemma C and the argument in [BSS, Theorem 2.5] that we include for sake of completeness.

Note that Lemma C implies $\int_{-\pi}^{\pi} ||f(e^{i(t-s)}) - f(e^{it})||^p \frac{dt}{2\pi} \leq C|s| \ \forall \delta$.

Assume $I = [-\delta, \delta]$ for some $0 < \delta < \frac{\pi}{2}$ (the general case follows by using translation invariance of the space).

$$\begin{aligned}
\frac{1}{|I|} \int_I ||f(e^{it}) - f_I||^p \frac{dt}{2\pi} &= \frac{1}{2\delta} \int_{-\delta}^{\delta} || \frac{1}{2\delta} \int_{-\delta}^{\delta} (f(e^{it}) - f(e^{is})) \frac{ds}{2\pi} ||^p \frac{dt}{2\pi} \\
&\leq \frac{1}{2\delta} \int_{-\delta}^{\delta} \frac{1}{2\delta} (\int_{-\delta}^{\delta} ||f(e^{it}) - f(e^{is})||^p \frac{ds}{2\pi}) \frac{dt}{2\pi} \\
&= \frac{1}{4\delta^2} \int_{-\delta}^{\delta} (\int_{t-\delta}^{t+\delta} ||f(e^{it}) - f(e^{i(t-s)})||^p \frac{ds}{2\pi}) \frac{dt}{2\pi} \\
&\leq \frac{1}{4\delta^2} \int_{-\pi}^{\pi} (\int_{-2\delta}^{2\delta} ||f(e^{it}) - f(e^{i(t-s)})||^p \frac{ds}{2\pi}) \frac{dt}{2\pi} \\
&= \frac{1}{4\delta^2} \int_{-2\delta}^{2\delta} (\int_{-\pi}^{\pi} ||f(e^{i(t-s)}) - f(e^{it})||^p \frac{dt}{2\pi}) \frac{ds}{2\pi} \\
&\leq C \frac{1}{4\delta^2} \int_{-2\delta}^{2\delta} |s| \frac{ds}{2\pi} \leq C. \quad \square
\end{aligned}$$

EXAMPLE 2.1. Let $(\alpha_n) \geq 0$ such that $\sum_{n=1}^{\infty} \alpha_n^p < \infty$ for some $1 < p < \infty$ and let s_n be an increasing sequence in $(0,1)$ with $\lim_{n\to\infty} s_n = 1$. Let $f_n(z) = log(\frac{1}{(1-s_n z)^{\alpha_n}})$ and $f(z) = \big(f_n(z)\big)_{n\in\mathbb{N}}$ then $f \in BMOA(l^p)$.

From Theorem 2.2 it suffices to see that $M_{p,l^p}(f',r) = O((1-r)^{-\frac{1}{p'}})$. Now using (2.2) we get

$$\begin{aligned} M^p_{p,l^p}(f',r) &= \sum_{n=1}^{\infty} M_p^p(f'_n, r) \\ &= \sum_{n=1}^{\infty} \alpha_n^p \int_0^{2\pi} \frac{s_n}{|1 - s_n r e^{-it})|^p} \frac{dt}{2\pi} \\ &\leq C \sum_{n=1}^{\infty} \alpha_n^p (1 - s_n r)^{1-p} \leq C(1-r)^{1-p}. \end{aligned}$$

Let us now go a bit further and find conditions on the sequence of Taylor coefficients x_n which guarantees that the corresponding analytic function belongs to $BMOA(X)$. Some conditions can easily be achieved for spaces of Fourier type p.

Corollary 2.2. *Let $1 < p \leq 2$ and let X be Banach space with Fourier type p and (x_n) a sequence in X such that*

$$\sum_{n=1}^{N} ||nx_n||^p = O(N)$$

then $f(z) = \sum_{n=1}^{\infty} x_n z^n \in BMOA(X)$.

Proof. Let us first observe that the assumption implies

$$\sup_{n\in\mathbb{N}} 2^{n(p-1)} \sum_{k=2^n}^{2^{n+1}} ||x_k||^p < \infty.$$

Let us now show that $M_{p',X}(f'r) = O\left((1-r)^{-\frac{1}{p}}\right)$ and then the result will follow from Theorem 2.2.

It is not difficult to see, using duality, that Fourier type p can be also formultated as

$$||f||_{p',X} \leq C \left(\sum_{n\in\mathbb{Z}} ||\hat{f}(n)||^p \right)^{\frac{1}{p}}.$$

Therefore, from the Fourier type p condition, it follows

$$\begin{aligned} M_{p',X}(f',r) &\leq C \left(\sum_{n=1}^{\infty} n^p ||x_n||^p r^{p(n-1)} \right)^{\frac{1}{p}} \\ &\leq C \left(\sum_{n=0}^{\infty} (\sum_{k=2^n}^{2^{n+1}} ||x_k||^p) 2^{pn} r^{p2^n} \right)^{\frac{1}{p}} \\ &\leq C \left(\sup_{n\in\mathbb{N}} 2^{n(p-1)} \sum_{k=2^n}^{2^{n+1}} ||x_k||^p \right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} 2^n r^{p2^n} \right)^{\frac{1}{p}} \leq \frac{C}{(1-r)^{\frac{1}{p}}}. \quad \square \end{aligned}$$

Corollary 2.3. *Let X be a B-convex space and $x_n \in X$.*
If $||x_n|| = O(\frac{1}{n})$ then $\sum_{n=1}^{\infty} x_n z^n \in BMOA(X)$.

Proof. We can invoke Theorem A to find $1 < p \leq 2$ such that X has Fourier type p. Now apply Corollary 2.2 for such a p. $\square$

EXAMPLE 2.2. Consider $f_1(z) = \big(\frac{z^n}{n}\big)_{n\in\mathbb{N}}$ and $f_2(z) = \sum_{n=1}^{\infty} \frac{r_n}{n} z^n$ (r_n are the Rademacher functions).

Observe that

$$||f_1(z)||_{l^1} = ||f_2(z)||_{L^\infty([0,1])} = \sum_{n=1}^{\infty} \frac{|z|^n}{n} = log\frac{1}{1-|z|}.$$

Hence $||\frac{e_n}{n}||_{l^1} = ||\frac{r_n}{n}||_{L^\infty([0,1])} = \frac{1}{n}$ but $f_1 \notin BMOA(l^1)$ and $f_2 \notin BMOA\big(L^\infty([0,1])\big)$ (because $f_i \notin H^1(X_i)$ for $X_1 = l^1, X_2 = L^\infty([0,1])$).

This shows that Corollary 2.3 does not hold for general Banach spaces.

§3.- Vector valued multipliers from $H^1(X)$ into $BMOA(Y)$

Let us denote by $\big(H^1, BMOA\big)$ the space of convolution multipliers between H^1 and $BMOA$, that is the set of functions $F(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ such that there exists a constant $C > 0$ for which

$$||\sum_{n=0}^{\infty} \lambda_n \alpha_n z^n||_{BMOA} \leq C||\sum_{n=0}^{\infty} \alpha_n z^n||_{H^1}.$$

It was proved in [MP] the following scalar-valued result

$$(*) \qquad \big(H^1, BMOA\big) = \mathcal{B}.$$

where $\mathcal{B}$ stands for the space of Bloch functions.

We shall be interested in this section in the vector valued formulation of this result. First of all we need to give sense to the notion of convolution multiplier acting between two different Banach spaces. We present here two possible interpretations.

Let us recall that given Banach spaces X, Y we denote by $X\hat{\otimes}Y$ the completion of $X \otimes Y$ endowed with the projective tensor norm, i.e. for $u \in X \otimes Y$

$$||u||_{X\hat{\otimes}Y} = inf\{\sum_{i=1}^{n} ||x_i||||y_i||\}$$

where the infimum goes over all possible representations of $u = \sum_{i=1}^{n} x_i \otimes y_i, x_i \in X, y_i \in Y$.

Definition 3.1. *Given $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$ and $g(z) = \sum_{n=0}^{\infty} y_n z^n \in H^1(Y)$ we shall define the $X\hat{\otimes}Y$-valued analytic function*

$$(3.1) \qquad f\hat{*}g(z) = \int_0^{2\pi} f(ze^{-it}) \otimes g(e^{it})\frac{dt}{2\pi} = \sum_{n=1}^{\infty} x_n \otimes y_n z^n.$$

It is rather simple to observe that $f\hat{*}g(z) \in H^1(X\hat{\otimes}Y)$.

Definition 3.2. *Let X, Y be complex Banach spaces and let $F(z) = \sum_{n=0}^{\infty} T_n z^n$ be a $\mathcal{L}(X,Y)$-valued analytic function and $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$. We define the Y-valued function*

$$F * f(z) = \sum_{n=0}^{\infty} T_n(x_n) z^n = \int_0^{2\pi} F(ze^{it}) \left(f(e^{-it})\right) \frac{dt}{2\pi}. \tag{3.2}$$

We shall denote by $\left(H^1(X), BMOA(Y)\right)$ the set of analytic functions $F : D \to \mathcal{L}(X,Y)$ such that $F * f \in BMOA(Y)$ for any $f \in H^1(X)$.

This becomes a closed subspace of $\mathcal{L}\left(H^1(X), BMOA(Y)\right)$.

Let us notice first that we have the following obvious extension.

Lemma 3.1. *Let X, Y be two complex Banach spaces. Then*

$$\left(H^1(X), BMOA(Y)\right) \subset \mathcal{B}\left(\mathcal{L}(X,Y)\right).$$

Proof. Given $F \in \left(H^1(X), BMOA(Y)\right)$ and $x \in X, y^* \in Y^*$ then $< F(z)(x), y^* > \in \left(H^1, BMOA\right)$. Hence, from the scalar-valued case (*),

$$|| < F(z)(x), y^* > ||_{\mathcal{B}} \leq ||F||_{(H^1(X), BMOA(Y))} ||x|| ||y^*||$$

what shows $F \in \mathcal{B}\left(\mathcal{L}(X,Y)\right)$ because of Remark 1.3. $\square$

Nevertheless let us first point out that there is no hope for the analogue of (*) to hold for general pairs of Banach spaces as the following remark shows.

REMARK 3.1. Let us assume $\mathcal{B}\left(\mathcal{L}(X,X)\right) \subset \left(H^1(X), BMOA(X)\right)$ then taking $T_n = I$, the identity operator, part (ii) in Proposition 1.5 shows that $F(z) = \sum_{n=0}^{\infty} T_n z^{2^n} \in \mathcal{B}\left(\mathcal{L}(X,X)\right)$ and then one should have

$$|| \sum_{n=0}^{\infty} x_{2^n} z^{2^n} ||_{*,X} = ||F * f||_{*,X} \leq C ||f||_{1,X}.$$

This cannot be true as soon as we take X being a cotype 2 space but not a Paley space (for instance $X = \frac{L^1}{H_0^1}$, see[BP]). In fact it will be shown later that actually under such an assumption X has to be isomorphic to a Hilbert space.

Definition 3.3. *Let X, Y be complex Banach spaces. The pair (X, Y) is said to have the $(H^1, BMOA)$-property if*

$$\left(H^1(X), BMOA(Y)\right) = \mathcal{B}\left(\mathcal{L}(X,Y)\right).$$

Let us now present various properties holding for pairs having $(H^1, BMOA)$-property.

Theorem 3.1. *Let (X, Y^*) have the $(H^1, BMOA)$-property.*
If $f \in H^1(X)$ and $g \in H^1(Y)$ then $(f \hat{*} g)' \in L^1(D, X \hat{\otimes} Y)$.

Proof. Let us recall that $\left(X \hat{\otimes} Y\right)^* = \mathcal{L}(X, Y^*)$ under the pairing

$$(T, \sum_{k=1}^{n} x_k \otimes y_k) = \sum_{k=1}^{n} < T(x_k), y_k >,$$

where $<,>$ stands for the pairing on (Y, Y^*).

On the other hand for any Banach space E one has $L^1(D, E) = L^1(D) \hat{\otimes} E$ what gives $\left(L^1(D, E)\right)^* = \mathcal{L}\left(L^1(D), E^*\right)$ under the pairing given by

$$[T, \sum_{k=1}^{n} e_k \phi_k] = \sum_{k=1}^{n} << T(\phi_k), e_k >>$$

for any $e_k \in E$ and $\phi_k \in L^1(D)$ where $<<,>>$ stands for the pairing on (E, E^*).

Assume now $f(z) = \sum_{n=0}^{m} x_n z^n$ and $g(z) = \sum_{n=0}^{m} y_n z^n$. Hence $(f \hat{*} g)'(z) = \sum_{n=0}^{m} n x_n \otimes y_n z^{n-1}$.

According to the previous dualities, and denoting $u_n(w) = w^n$, we can write

$$||(f \hat{*} g)'||_{L^1(D, X \hat{\otimes} Y)} = \sup\{|\sum_{n=1}^{m} n(T(u_{n-1}), x_n \otimes y_n)|\}$$

where the supremum is taken over $T \in \mathcal{L}\left(L^1(D), \mathcal{L}(X, Y^*)\right)$ with $||T|| = 1$.

Note that for each $T \in \mathcal{L}\left(L^1(D), \mathcal{L}(X, Y^*)\right)$ with $T_n = T(u_{n-1}) \in \mathcal{L}(X, Y^*)$ and $||T|| = 1$ we have

$$\sum_{n=1}^{m} n(T(u_{n-1}), x_n \otimes y_n) = \sum_{n=1}^{m} n < T_n(x_n), y_n > .$$

On the other hand observe that, denoting by $F(z) = \sum_{n=1}^{\infty} n T_n z^n$, we have that $F(z) = zT(K_z)$ and therefore, from Proposition 1.4, it is a $\mathcal{L}(X, Y^*)$-valued Bloch function with $||T(K_z)||_{\mathcal{B}(\mathcal{L}(X, Y^*))} \leq ||T||$.

Notice now that

$$\begin{aligned} |\sum_{n=1}^{m} n < T_n(x_n), y_n > | &= |\int_0^{2\pi} \int_0^{2\pi} < F(re^{i(t-s)})(f(e^{it})), g(e^{-is}) > \frac{dt}{2\pi} \frac{ds}{2\pi}| \\ &\leq ||F * f||_{BMOA(Y^*)} ||g||_{1,Y} \\ &\leq C ||F||_{\mathcal{B}(\mathcal{L}(X, Y^*))} ||f||_{1,X} ||g||_{1,Y}. \quad \square \end{aligned}$$

Corollary 3.1. *Let (X, Y^*) have the $(H^1, BMOA)$-property.*
If $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$ and $g(z) = \sum_{n=0}^{\infty} y_n z^n \in H^1(Y)$ then

$$\sum_{n=0}^{\infty} ||x_{2^n}||||y_{2^n}|| \leq C||f||_{1,X}||g||_{1,Y}.$$

Proof. Let $h(z) = (f \hat{*} g)'(z) \in L^1(D, X\hat{\otimes}Y)$. Obviously one has

$$n||x_n||||y_n||r^{n-1} \leq M_{1,X\hat{\otimes}Y}(h,r) \qquad (n \in \mathbb{N}).$$

Therefore

$$\begin{aligned}\int_0^1 M_{1,X\hat{\otimes}Y}(h,r)dr &\geq \sum_{k=0}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} 2^k||x_{2^k}||||y_{2^k}||r^{(2^k-1)}dr \\ &\geq C\sum_{k=0}^{\infty} ||x_{2^k}||||y_{2^k}||. \quad \square\end{aligned}$$

Corollary 3.2. *If $(\mathbb{C}, Y^*)$ have the $(H^1, BMOA)$-property then Y is a Paley space.*

Proof. Apply the Corollary 3.1 to $f(z) = \sum_{n=0}^{\infty} \alpha_n z^{2^n} \in H^1$ and $g \in H^1(Y)$ and recall that $||f||_1 \approx \left(\sum_{n=0}^{\infty} |\alpha_n|^2\right)^{\frac{1}{2}}$.

Lemma 3.2. *If (X, Y) has the $(H^1, BMOA)$-property then also $(X, \mathbb{C})$ and $(\mathbb{C}, Y)$ have the $(H^1, BMOA)$-property.*

Proof. (i) Let us take $F(z) = \sum_{n=0}^{N} x_n^* z^n \in \mathcal{B}(X^*)$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$. Let us fix $y \in Y$ with $||y|| = 1$ and consider $\hat{F}(z) = \sum_{n=0}^{N} T_n z^n$ where T_n are the operators in $\mathcal{L}(X, Y)$ defined by $T_n(x) =< x_n^*, x > y$. It is elementary to show that $\hat{F} \in \mathcal{B}(\mathcal{L}(X, Y))$ and $||\hat{F}||_{\mathcal{B}(\mathcal{L}(X,Y))} = ||\hat{F}||_{\mathcal{B}(X^*)}$.

Therefore

$$\begin{aligned}||\sum_{k=0}^{\infty} < x_k^*, x_k > z^k||_{BMOA} &= ||\sum_{k=0}^{\infty} T_k(x_k)z^k||_{BMOA(Y)} \\ &\leq C||\hat{F}||_{\mathcal{B}(\mathcal{L}(X,Y))}||\sum_{k=0}^{\infty} x_k z^k||_{1,X}.\end{aligned}$$

(ii) Let us take $F(z) = \sum_{n=0}^{\infty} y_n z^n \in \mathcal{B}(Y)$ and $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^1$. Let us fix $x_0 \in X$ and $x_0^* \in X^*$ with $||x_0|| = 1$ and $< x_0^*, x_0 >= 1$. Define $\hat{F}(z) = \sum_{n=0}^{\infty} T_n z^n$ where T_n are defined by $T_n(x) =< x_0^*, x > y_n$. It is elementary to show that $\hat{F} \in \mathcal{B}(\mathcal{L}(X, Y))$ and $||\hat{F}||_{\mathcal{B}(\mathcal{L}(X,Y))} = ||\hat{F}||_{\mathcal{B}(Y)}$. Observe that

$$\sum_{n=1}^{\infty} \alpha_n y_n z^n = \sum_{n=1}^{\infty} T_n(\alpha_n x_0)z^n = \hat{F} * f$$

where $f(z) = \phi(z)x_0$, then we have

$$\begin{aligned}||\sum_{n=1}^{\infty} y_n\alpha_n z||_{BMO(Y)} &\leq C||\hat{F}||_{\mathcal{B}(\mathcal{L}(X,Y))}||\sum_{n=0}^{\infty}\alpha_n x_0 z^n||_{1,X}\\ &\leq C||F||_{\mathcal{B}(Y)}||\sum_{n=0}^{\infty}\alpha_n z^n||_{H^1} \quad \square.\end{aligned}$$

Proposition 3.1. *Let X, Y be two complex Banach spaces.*
(i) If $(X, \mathbb{C})$ has the $(H^1, BMOA)$-property then X is a Paley space.
(ii) If $(\mathbb{C}, Y)$ has the $(H^1, BMOA)$-property then Y has type 2.

Proof.
(i) Let us take $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$ and choose $x_n^* \in X^*$ with $||x_n^*|| = 1$ and $< x_n^*, x_{2^n} >= ||x_{2^n}||$. Then, using (1.2),

$$\left(\sum_{k=1}^{\infty}||x_{2^k}||^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty}|< x_k^*, x_{2^k} >|^2\right)^{\frac{1}{2}} \approx ||\sum_{k=1}^{\infty}< x_k^*, x_{2^k} > z^{2^k}||_{BMOA}.$$

Let us observe that, from (ii) in Proposition 1.5, $F(z) = \sum_{n=1}^{\infty} x_n^* z^{2^n}$ belongs to $\mathcal{B}(X^*)$ and therefore

$$\begin{aligned}\left(\sum_{k=1}^{\infty}||x_{2^k}||^2\right)^{\frac{1}{2}} &\leq ||\sum_{k=1}^{\infty}< x_k^*, x_{2^k} > z^{2^k}||_{BMOA}\\ &\leq C||F||_{\mathcal{B}(X^*))}||\sum_{k=1}^{\infty}x_k z^k||_{1,X} \leq C||f||_{1,X}\end{aligned}$$

This shows that X is a Paley space.

(ii) Now given $y_0, y_1, y_2, ...y_N \in X$ with $y_j \neq 0$ we define $F(z) = \sum_{n=0}^{N}\frac{y_n}{||y_n||}z^{2^n}$ From Proposition 1.5 again we have $F \in \mathcal{B}(Y)$ and $||F||_{\mathcal{B}(Y)} \leq C$.

Observe that

$$\sum_{k=0}^{N} y_k z^{2^k} = \sum_{k=0}^{N}||y_k||\frac{y_k}{||y_k||}z^{2^k} = F * \phi$$

where $\phi(z) = \sum_{k=0}^{N}||y_k||z^{2^k}$, then we have

$$\begin{aligned}||\sum_{k=0}^{N} y_k z^{2^k}||_{1,Y} &\leq ||\sum_{k=0}^{N} y_k z^{2^k}||_{BMO(Y)}\\ &\leq C||F||_{\mathcal{B}(Y)}||\sum_{k=0}^{N}||y_k||z^{2^k}||_1\\ &\leq C\left(\sum_{k=0}^{N}||y_k||^2\right)^{\frac{1}{2}}.\end{aligned}$$

This shows that X has type 2. $\square$

We shall now introduce two new properties which are motivated by Proposition 1.2 and its dual formulation and will be adecuated to the $(H^1, BMOA)$-property.

Let us recall the notation $\mathcal{P}(X)$ and $\mathcal{P}_0(X)$ for the X-valued polynomials and those which vanish at $z = 0$ respectively.

Definition 3.4. *A complex Banach space X is said to have $(HL)^*$-property if there exists a constant $C > 0$ such that*

$$||f||_{*,X} \leq C \left(\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||^2 dr \right)^{\frac{1}{2}}. \tag{3.3}$$

for any $f \in \mathcal{P}(X)$.

Definition 3.5. *A complex Banach space X is said to have (HL)-property if there exists a constant $C > 0$ such that*

$$\left(\int_0^1 (1-r) M_{1,X}^2(f', r) dr \right)^{\frac{1}{2}} \leq C ||f||_{1,X} \tag{3.4}$$

for any $f \in \mathcal{P}_0(X)$.

REMARK 3.2. Observe that

$$\int_0^1 (1-r) M_{1,X}^2(f', r) dr = \sum_{k=0}^{\infty} \int_{r_k}^{r_{k+1}} (1-r) M_{1,X}^2(f', r) dr,$$

for $r_k = 1 - 2^{-k}$ and then, since $M_{1,X}(f, r)$ is increasing the inequalities (3.3) and (3.4) can be replaced by

$$||f||_{*,X} \leq C \left(\sum_{k=0}^{\infty} 2^{-2k} \sup_{|z|=r_k} ||f'(z)||^2 \right)^{\frac{1}{2}} \tag{3.5}$$

and

$$\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(f', r_k) \right)^{\frac{1}{2}} \leq C ||f||_{1,X}. \tag{3.6}$$

Therefore inequality (3.6) says that X has (HL)-property if and only if the operator $f \to \big(2^{-k} f'(r_k e^{it})\big)_k$ is bounded from $H_0^1(X)$ into $l^q\big(L^1(X)\big)$.

Theorem 3.2. *Let X, Y be a Banach spaces.*

If X has (HL)-property and Y has $(HL)^*$-property then (X,Y) has the $(H^1, BMOA)$-property.

Proof. From Lemma 3.1. we only have to prove

$$\mathcal{B}\left(\mathcal{L}(X,Y)\right) \subset \left(H^1(X), BMOA(Y)\right).$$

Let us take $F(z) = \sum_{n=0}^\infty T_n z^n \in \mathcal{B}\left(\mathcal{L}(X,Y)\right)$ and $f(z) = \sum_{n=0}^\infty x_n z^n \in H^1(X)$.

Now let us observe that

$$\begin{aligned} z(F*f)'(z^2) &= \sum_{n=1}^\infty nT_n(x_n)z^{2n-1} \\ &= \int_0^{2\pi} F'(ze^{it})\big(f(ze^{-it})\big)e^{it}\frac{dt}{2\pi} \\ &= 2\int_0^1\int_0^{2\pi} (\sum_{n=1}^\infty nT_n z^{n-1}s^{n-1}e^{i(n-1)t})(\sum_{n=1}^\infty nx_n s^{n-1}e^{-i(n-1)t})\frac{dt}{2\pi}sds \\ &= 2\int_0^1\int_0^{2\pi} F'(zse^{it})\big(f'(se^{-it})\big)se^{it}\frac{dt}{2\pi}ds. \end{aligned}$$

Therefore, since $F \in \mathcal{B}\left(\mathcal{L}(X,Y)\right)$, we have

$$\begin{aligned} ||z(F*f)'(z^2)|| &\le C||F||_{\mathcal{B}\left(\mathcal{L}(X,Y)\right)}\int_0^1 \frac{M_{1,X}(f_1, s|z|)}{(1-s|z|)}ds \\ &\le C||F||_{\mathcal{B}\left(\mathcal{L}(X,Y)\right)}\left(\int_0^1 \frac{ds}{(1-s|z|)^2}\right)^{\frac{1}{2}}\left(\int_0^{|z|} M^2_{1,X}(f', s)ds\right)^{\frac{1}{2}} \\ &\le \frac{C||F||_{\mathcal{B}\left(\mathcal{L}(X,Y)\right)}}{(1-|z|)^{\frac{1}{2}}}\left(\int_0^{|z|} M^2_{1,X}(f', s)ds\right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\sup_{|z|=r} ||z(F*f)'(z^2)|| \le \frac{C}{(1-r)^{\frac{1}{2}}}\left(\int_0^r M^2_{1,X}(f', s)ds\right)^{\frac{1}{2}}.$$

Now, using $(HL)^*$- property on Y and (HL)-property on X, we can estimate

$$\begin{aligned} ||F*f||^2_{*,Y} &\le \int_0^1 (1-r^2)\sup_{|z|=r^2} ||(F*f)'(z)||^2 rdr \\ &\le C\int_0^1 \Big(\int_0^r M^2_{1,X}(f', s)ds\Big)dr \\ &= C\int_0^1 (1-s)M^2_{1,X}(f', s)ds \le C||f||_{1,X}. \end{aligned}$$

Clearly $||\int_0^{2\pi} F*f(e^{it})\frac{dt}{2\pi}|| = ||T_0(x_0)|| \le C||f||_{1,X}$. This combined with the previous estimate finish the proof. $\square$

§4.- Lebesgue spaces and Schatten classes with (HL)-property

In this section we study these new properties and investigate the Lebesgue spaces and the Schatten classes having (HL)-property and $(HL)^*$-property.

Let us start with some general facts and their relations with the notions of type and cotype.

Proposition 4.1.

(i) If X has (HL)-property then X is a Paley space.

(ii) If X having $(HL)^*$-property then X has type 2.

Proof. Combine Theorem 3.2 together with Proposition 3.1.

Let us now establish the duality existing among both notions.

Theorem 4.1. *(Duality)*

(i) If X^* has $(HL)^*$-property then X has (HL)-property.

(ii) Let X be an UMD space. Then X^* has $(HL)^*$-property if and only if X has (HL)-property.

Proof. Let us take $f(z) = \sum_{n=1}^{\infty} x_n z^n \in H_0^1(X)$ with $||f||_{1,X} = 1$. Using the embedding

$$l^2(L^1(X)) \subseteq \big(l^2(C(X^*))\big)^*$$

we have, setting $r_k = 1 - 2^{-k}$,

$$\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(f', r_k)\right)^{\frac{1}{2}} = sup|\sum_{k=0}^{\infty} \int_0^{2\pi} < 2^{-k} f'(r_k e^{it}), g_k(e^{-it}) > \frac{dt}{2\pi}|,$$

where the supremum is taken over the set of sequences $(g_k)_{k\in\mathbb{N}} \subset C(X^*)$ such that $\sum_{k=0}^{\infty} ||g_k||_{\infty,X^*}^2 = 1$.

Denoting by

$$G_k(z) = \int_0^{2\pi} \frac{g_k(e^{it})}{(1 - ze^{-it})} \frac{dt}{2\pi}$$

we have, for $|z| = r$,

$$||G_k''(r_k z)||_{X^*} \leq ||g_k||_{\infty,X^*} \int_0^{2\pi} \frac{1}{|1 - zr_k e^{-it}|^3} \frac{dt}{2\pi} \leq C \frac{1}{(1 - r_k r)^2} ||g_k||_{\infty,X^*}.$$

Therefore for any sequence (g_k) with $\sum_{k=0}^{\infty}||g_k||^p_{\infty,X^*}=1$

$$
\begin{aligned}
&|\sum_{k=0}^{\infty}\int_0^{2\pi} < 2^{-k}f'(r_ke^{it}), g_k(e^{-it}) > \frac{dt}{2\pi}| \\
&= |\int_0^{2\pi} < f(e^{it}), \sum_{k=0}^{\infty} 2^{-k}G'_k(r_ke^{-it}) > \frac{dt}{2\pi}| \\
&\leq ||f||_{1,X}||\sum_{k=0}^{\infty} 2^{-k}G'_k(r_ke^{it})||_{*,X^*} \\
&\leq C\left(\int_0^1 (1-r)\sup_{|z|=r}||\sum_{k=0}^{\infty}2^{-k}G''_k(r_kz)||^2_{X^*}dr\right)^{\frac12} \\
&\leq C\left(\int_0^1 (1-r)\left(\sum_{k=0}^{\infty}2^{-k}\frac{||g_k||_{\infty,X^*}}{(1-r_kr)^2}\right)^2 dr\right)^{\frac12} = I
\end{aligned}
$$

Using Hölder's and the facts

$$\sum_{k=0}^{\infty}2^{-k}\frac{1}{(1-r_kr)^2}\approx\int_0^1\frac{ds}{(1-rs)^2},$$

$$\int_0^1\frac{ds}{(1-rs)^2}=\frac{1}{1-r},$$

then we can write

$$
\begin{aligned}
I &\leq C\left(\int_0^1(1-r)\left(\sum_{k=0}^{\infty}2^{-k}\frac{||g_k||^2_{\infty,X^*}}{(1-r_kr)^2}\right)\left(\sum_{k=0}^{\infty}2^{-k}\frac{1}{(1-r_kr)^2}\right)dr\right)^{\frac12} \\
&\leq C\left(\int_0^1(1-r)\left(\sum_{k=0}^{\infty}2^{-k}\frac{||g_k||^2_{\infty,X^*}}{(1-r_kr)^2}\right)\left(\int_0^1\frac{ds}{(1-rs)^2}\right)dr\right)^{\frac12} \\
&\leq C\left(\sum_{k=0}^{\infty}2^{-k}||g_k||^2_{\infty,X^*}\int_0^1\frac{dr}{(1-r_kr)^2}\right)^{\frac1p}\leq C.
\end{aligned}
$$

(ii) From part (i) we only have to show that if X is a UMD space having (HL)-property implies X^* has $(HL)^*$-property .

Given a X^*-valued polynomial, say $f(z)=\sum_{n=0}^{m}x^*_nz^n$, and using the duality $\left(H^1(X)\right)^*=BMOA(X^*)$, we have

$$||f||_{*,X^*}=sup\{\int_0^{2\pi} < f(e^{it}), g(e^{-it}) > \frac{dt}{2\pi} : g\in H^1_0(X), ||g||_{1,X}=1\}.$$

Now let us observe that for $g(z) = \sum_{n=1}^{\infty} x_n z^n$

$$\int_0^{2\pi} < f(e^{it}), g(e^{-it}) > \frac{dt}{2\pi} = \sum_{n=1}^{m} < x_n^*, x_n >$$
$$= 2\int_0^1 (1-r^2) \int_0^{2\pi} < \sum_{n=1}^{m} n x_n^* r^{n-1} e^{-i(n-1)t}, \sum_{n=1}^{\infty} (n+1) x_n r^n e^{i(n-1)t} > \frac{dt}{2\pi} dr$$
$$= 2\int_0^1 (1-r^2) \int_0^{2\pi} < f'(re^{it}), g_1'(re^{-it}) > e^{it} \frac{dt}{2\pi} dr$$

where $g_1(z) = zg(z)$. Hence

$$|\int_0^{2\pi} < f(e^{it}), g(e^{-it}) > \frac{dt}{2\pi}|$$
$$\leq \int_0^1 (1-r) M_{1,X}(g_1', r) \sup_{|z|=r} ||f'(z)||_{X^*} dr$$
$$\leq \left(\int_0^1 (1-r) M_{1,X}^2(g_1', r) dr\right)^{\frac{1}{2}} \left(\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||_{X^*}^2 dr\right)^{\frac{1}{2}}$$
$$\leq C||g_1||_{1,X} \left(\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||_{X^*}^2 dr\right)^{\frac{1}{2}}$$
$$\leq C||g||_{1,X} \left(\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||_{X^*}^2 dr\right)^{\frac{1}{2}}. \quad \square$$

Proposition 4.2. *Hilbert spaces have $(HL)^*$-property and (HL)- property.*

Proof. Using Corollary 2.1 and Proposition 1.2 one has that Hilbert spaces have $(HL)^*$-property. Now apply Theorem 4.1 to get (HL)-property.

Corollary 4.1. *([B2]) X is isomorphic to a Hilbert space if and only if (X, X) has the (H^1, BMO)-property.*

Proposition 4.3. *Let (Ω, Σ, μ) a measure space.*
If X has (HL)-property then $L^1(\mu, X)$ has (HL)-property.

Proof. Recall first that cotype 2 condition on $L^1(\mu)$ (cf. [LT]) means that

$$\left(\sum_{k=0}^{\infty} ||f_k||_{L^1(\mu)}^2\right)^{\frac{1}{2}} \leq C||(\sum_{k=0}^{\infty} |f_k(.)|^2)^{\frac{1}{2}}||_{L^1(\mu)}, \tag{4.1}$$

for any sequence $(f_k) \in L^1(\mu)$.

Now, given a $L^1(\mu, X)$-valued analytic polynomial, say $F(z) = \sum_{n=0}^m x_n z^n$ we have that for a.a. $\omega \in \Omega$ the X-valued polynomial $F(\omega)(z) = \sum_{n=0}^m x_n(\omega) z^n$ verifies

$$\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(F'(\omega), r_k)\right)^{\frac{1}{2}} \leq C \int_0^{2\pi} ||F(\omega)(e^{it})||_X \frac{dt}{2\pi} \qquad \omega \in \Omega$$

Now integrating over Ω,

$$||\big(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(F'(\omega), r_k)\big)^{\frac{1}{2}}||_{L^1(\mu)} \leq C||F||_{1,L^1(\mu,X)}.$$

On the other hand, from (4.1)

$$\begin{aligned}\Big(\sum_{k=0}^{\infty} 2^{-2k} M_{1,L^1(\mu,X)}^2(F', r_k)\Big)^{\frac{1}{2}} &= \Big(\sum_{k=0}^{\infty} ||2^{-k} M_{1,X}(F'(\omega), r_k)||_{L^1(\mu)}^2\Big)^{\frac{1}{2}} \\ &\leq ||\big(\sum_{k=0}^{\infty} 2^{-2k} ||F(., r_k)||_X^2\big)^{\frac{1}{2}}||_{L^1(\mu)} \leq ||F||_{1,L^1(\mu,X)}. \quad \square\end{aligned}$$

Proposition 4.4. *Let (Ω, Σ, μ) a measure space.*
(i) $L^p(\mu)$ has (HL)- property if and only if $1 \leq p \leq 2$.
(ii) $L^p(\mu)$ has $(HL)^*$-property if and only if $2 \leq p < \infty$.

Proof. (i) From Proposition 4.1 (HL)- property implies cotype 2 and then $1 \leq p \leq 2$.

On the other hand $L^1(\mu)$ has (HL)-property according to Proposition 4.3.

The case $1 < p \leq 2$ follows from the fact that L^p is isometrically isomorphic to a subspace of L^1 (see [R]).

(ii) Follows from (i) and Theorem 4.1. $\square$

Now let us investigate the $(HL)^*$-property and (HL)-property for the Schatten classes. Given $1 \leq p < \infty$ we shall denote by σ_p the Banach space of compact operators on l^2 such that

$$||A||_p = \Big(tr(A^*A)^{\frac{p}{2}}\Big)^{\frac{1}{p}} < \infty.$$

It is well known that σ_1 coincides with the space of nuclear operators on l^2 and σ_2 with the space of Hilbert-Schmidt operators on l^2. The reader is referred to [GK] for general properties on σ_p and to [TJ] for results on (Rademacher) type and cotype on these classes. The key point to deal with them is the use of factorization of analytic functions with values on theses classes. The reader is referred to [BP, L-PP, Pi3] for the use of factorization in related questions. Let us establish the result to be used later on.

Lemma D. *(Non commutative Factorization, see*[S]*) Let $f \in H^1(\sigma_1)$. Then there exist two functions $h_1, h_2 \in H^2(\sigma_2)$ such that*

$$f(e^{it}) = h_1(e^{it})h_2(e^{it}), \text{ and } ||f||_{1,\sigma_1} = ||h_1||_{2,\sigma_2}^2 = ||h_2||_{2,\sigma_2}^2.$$

Theorem 4.2. *σ_1 has (HL)-property.*

Proof. Given $f \in H^1(\sigma_1)$ take $h_1, h_2 \in H^2(\sigma_2)$ such that

$$f(e^{it}) = h_1(e^{it})h_2(e^{it}), \qquad ||h_1||^2_{2,\sigma_2} = ||h_2||^2_{2,\sigma_2} = ||f||_{1,\sigma_1}.$$

Note that for $i, j \in \{1, 2\}, i \neq j$

$$\int_0^{2\pi} ||h_i'(re^{it})h_j(re^{it})||_{\sigma_1} \frac{dt}{2\pi} \leq \int_0^{2\pi} ||h_i'(re^{it})||_{\sigma_2} ||h_j(re^{it})||_{\sigma_2} \frac{dt}{2\pi}$$

$$\leq \left(\int_0^{2\pi} ||h_i'(re^{it})||^2_{\sigma_2} \frac{dt}{2\pi}\right)^{\frac{1}{2}} \left(\int_0^{2\pi} ||h_j(re^{it})||^2_{\sigma_2} \frac{dt}{2\pi}\right)^{\frac{1}{2}}.$$

Therefore

$$M_{1,\sigma_1}(f', r) \leq M_{2,\sigma_2}(h_1', r)M_{2,\sigma_2}(h_2, r) + M_{2,\sigma_2}(h_1, r)M_{2,\sigma_2}(h_2', r)$$

This gives

$$\left(\int_0^{2\pi} (1-r)M^2_{1,\sigma_1}(f', r)dr\right)^{\frac{1}{2}} \leq ||f||^{\frac{1}{2}}_{1,\sigma_1} \sum_{i=1}^{2} \left(\int_0^{2\pi} (1-r)M^2_{2,\sigma_2}(h_i', r)dr\right)^{\frac{1}{2}}.$$

Since σ_2 is a Hilbert space we have, using Plancherel,

$$\int_0^{2\pi} (1-r)M^2_{2,\sigma_2}(h_i', r)dr = \sum_{n=1}^{\infty} ||\hat{h}_i(n)||^2_{\sigma_2} n^2 \int_0^{2\pi} (1-r)r^{2n-2}dr \leq C||h_i||^2_{\sigma_2}$$

This shows

$$\left(\int_0^{2\pi} (1-r)M^2_{1,\sigma_1}(f', r)dr\right)^{\frac{1}{2}} \leq C||f||_{1,\sigma_1}. \quad \square$$

To cover other values of p we shall use some of the recent advances on interpolation of vector-valued Hardy spaces. It is known (see [BX]) that interpolation spaces by complex or real method, $\big(H^{p_1}(X_1), H^{p_2}(X_2)\big)_\theta$ or $\big(H^{p_1}(X_1), H^{p_2}(X_2)\big)_{\theta,p}$ do not coincide, in general, with $H^{p_\theta}(X_\theta)$ or $H^{p_\theta}(X_{\theta,p})$, but nevertheless there are some positive results that still can be used to find out the (HL)-property of certain spaces.

For some particular spaces, like L^p in the conmutative and non-conmutative versions, the expected result remains true (see[X1, X2 BX, Pi4]):

If $0 < \theta < 1$ and $\frac{1}{p} = 1 - \frac{\theta}{2}$ then

$$\Big(H^1\big(L^1(\mu)\big), H^1\big(L^2(\mu)\big)\Big)_\theta = H^1\big(L^p(\mu)\big). \tag{4.2}$$

$$\big(H^1(\sigma_1), H^1(\sigma_2)\big)_\theta = H^1(\sigma_p). \tag{4.3}$$

$$\big(H^1(L^1(\mu)), H^1(L^2(\mu))\big)_{\theta,1} = H^1\big(L^{p,1}(\mu)\big). \tag{4.4}$$

where $L^{p,1}(\mu)$ stands for the corresponding Lorentz space.

Proposition 4.5. *Let X_i (i=1,2) be spaces having (HL)-property and assume*

$$\big(H^1(X_1), H^1(X_2)\big)_\theta = H^1\big((X_1, X_2)_\theta\big).$$

Then $\big(X_1, X_2\big)_\theta$ has (HL)-property.

Proof. Since

$$T(f) = \big(2^{-k} f'(r_k e^{it})\big)_k$$

defines a bounded operator $T : H^1_0(X_i) \to l^2(L^1(\mathbb{T}, X_i))$ for $i = 1, 2$, then the assumption together with the well known result of interpolation

$$\big(l^2(L^1(X_1)), l^2(L^1(X_2))\big)_\theta = l^2(L^1((X_1, X_2)_\theta))$$

shows that T is also bounded from $H^1_0\big((X_1, X_2)_\theta\big)$ into $l^2\Big(L^1\big((X_1, X_2)_\theta\big)\Big)$ what gives that $\big(X_1, X_2\big)_\theta$ has (HL)-property. $\square$

Combining the results (4.3), (4.2) and the previous proposition we easily obtain the following corollary.

Proposition 4.6. *Let $1 \le p < \infty$. Then*
(i) σ_p has (HL)-property if and only if $1 \le p \le 2$.
(ii) σ_p has $(HL)^*$-property if and only if $2 \le p < \infty$.
(iii) $L^{p,1}(\mu)$ has (HL)-property for $1 \le p \le 2$.

REMARK 4.1. Some of the previous ideas appeared already in [BP]. Proposition 4.6 gives an alternative proof of the Paley property of σ_p for $1 \le p \le 2$ and then the cotype 2 condition (see [TJ]). Another approach was also obtained in [L-PP].

§5. Applications

Let us start this section with some new examples of vector-valued $BMOA$ functions. Observe that Theorem 3.2 actually provides a procedure to find functions in $BMOA(X)$ for spaces with $(HL)^*$-property.

Proposition 5.1. *Let $0 < \alpha \le \frac{1}{2}$ and $p = \frac{1}{\alpha}$. Define*

$$I_\alpha(\phi)(z) = \int_0^{2\pi} \frac{\phi(e^{-it})}{(1 - ze^{it})^\alpha} \frac{dt}{2\pi}.$$

Then the operator given by $\phi \to f_\alpha(z) = I_\alpha(\phi)_z$ is bounded from H^1 to $BMOA(L^p)$.

Proof. Use Theorem 3.2 applied to the L^p- valued Bloch function g_p provided by Example 1.2. $\square$

Proposition 5.2. *Let* $(\mathbb{C}, X)$ *have* $(H^1, BMOA)$*-property and* $T \in \mathcal{L}(L^1(D), X)$*. Then* $f(z) = T(\phi'_z) \in BMOA(X)$ *for any* $\phi \in H^1$*.*

Proof. From Proposition 1.4 one has that $T(K_z) \in \mathcal{B}(X)$. Now for any $\phi \in H^1$

$$T(K_z) * \phi(z) = \int_0^{2\pi} T(K_{ze^{it}}\phi(e^{-it}))\frac{dt}{2\pi} = T\left(\int_0^{2\pi} K_{ze^{it}}\phi(e^{-it})\frac{dt}{2\pi}\right) = T(\phi'_z)$$

what gives $f(z) \in BMOA(X)$ from Theorem 3.2. $\square$

Let us now get some information about Taylor coefficients of vector valued Bloch functions.

It is well known (see [D, page 103]) that the space of multipliers (H^1, H^2) can be identified with sequences (λ_n) such that

$$\sup_{n\in\mathbb{N}} \sum_{k=2^n}^{2^{n+1}} |\lambda_k|^2 < \infty.$$

Therefore, since $BMOA \subset H^2$, then one has the following:

If $f(z) = \sum_{n=0}^{\infty} x_n z^n \in \mathcal{B}(X)$ then $< f(z), x^* >\in (H^1, BMOA) \subset (H^1, H^2)$. Therefore

$$\sup_{||x^*||=1} \sup_{n\in\mathbb{N}} \sum_{k=2^n}^{2^{n+1}} | < x^*, x_k > |^2 < \infty.$$

Next result give some necessary conditions on $||x_n||$ when dealing with L^p-spaces for $2 \leq p < \infty$. We shall need the following lemma.

Lemma 5.1. *Let* X *has* $(HL)^*$*-property and* $f \in \mathcal{B}(X)$*. Denoting* $f_r(z) = f(rz)$ *then*

$$||f_r||_{BMOA(X)} \leq C log\frac{1}{1-r}||f||_{\mathcal{B}(X)}.$$

Proof. It is a simple consequence of Theorem 3.2 and the fact

$$\int_0^{2\pi} \frac{1}{|1-re^{it}|}\frac{dt}{2\pi} \approx log\frac{1}{1-r}. \quad \square$$

Proposition 5.3. *Let* X *has* $(HL)^*$*-property and assume* X *has Fourier type* p*.*

If $f(z) = \sum_{n\in\mathbb{N}} x_n z^n \in \mathcal{B}(X)$ *then*

$$\sup_{n\in\mathbb{N}} n^{-p'} \sum_{k=2^n}^{2^{n+1}} ||x_k||^{p'} < \infty.$$

Proof. From Lemma 5.1 we have

$$||f_r||_{p,X} \leq Clog\frac{1}{1-r}||f||_{\mathcal{B}(X)}.$$

Applying now the Fourier type condition

$$\left(\sum_{n\in\mathbb{N}} ||x_n||^{p'} r^{np'}\right)^{\frac{1}{p'}} \leq Clog\frac{1}{1-r}||f||_{\mathcal{B}(X)}.$$

Take now $r = 1 - \frac{1}{N}$. This implies

$$\sum_{n=1}^{N} ||x_n||^{p'} \leq C\sum_{n=1}^{N} ||x_n||^{p'}(1-\frac{1}{N})^{np'} \leq C(logN)^{p'}||f||^{p'}_{\mathcal{B}(X)}.$$

The result now follows by choosing $N = 2^n$. □

Proposition 5.4. *Let X have (HL)-property and a sequence (x_n^*) in X^*. If*

$$\sup_{||x||=1}\sup_{n\in\mathbb{N}}\sum_{k=2^n}^{2^{n+1}} |< x_k^*, x >| < \infty$$

then

$$\sum_{n\in\mathbb{N}} |< x_n^*, x_n >|^2 < \infty$$

for any sequence x_n such that $\sum_{n\in\mathbb{N}} x_n z^n \in H^1(X)$

Proof. Note that it follows from (i) in Proposition 1.5 and Remark 1.3 that for any sequence $\varepsilon_n \in \{0,1\}$ we have $\sum_{n\in\mathbb{N}} \varepsilon_n x_n^* z^n \in \mathcal{B}(X^*)$ with norm bounded by a constant independent of the choice of ε_n. Then, from Theorem 3.2, if $f(z) = \sum_{n\in\mathbb{N}} x_n z^n \in H^1(X)$ we have

$$||\sum_{n\in\mathbb{N}} \varepsilon_n < x_n^*, x_n > z^n||_{BMOA} \leq C||\sum_{n\in\mathbb{N}} \varepsilon_n x_n^* z^n||_{\mathcal{B}(X^*)}||f||_{1,X}$$

This shows that for any $t \in [0,1]$

$$||\sum_{n\in\mathbb{N}} r_n(t) < x_n^*, x_n > z^n||_{H^1} \leq C||f||_{1,X}.$$

Therefore

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}} |< x_n^*, x_n >|^2\right)^{\frac{1}{2}} &\approx \int_0^1\int_0^{2\pi} |\sum_{n\in\mathbb{N}} r_n(t) < x_n^*, x_n > e^{in\theta}|dt\frac{d\theta}{2\pi} \\ &= \int_0^1 ||\sum_{n\in\mathbb{N}} r_n(t) < x_n^*, x_n > z^n||_{H^1}dt \\ &\leq C||f||_{1,X} < \infty. \quad \square\end{aligned}$$

Let us now give a couple of applications to sequences of scalar valued functions.

Note that if (f_n) is a sequence of functions in H^1 such that $\sum_{n\in\mathbb{N}} ||f_n||_1 < \infty$ and (g_n) is a sequence of Bloch functions such that $\sup_{n\in\mathbb{N}} ||g_n||_{\mathcal{B}} < \infty$ then $\sum_{n\in\mathbb{N}} f_n * g_n$ is absolutely convergent in $BMOA$. This shows that if $f = (f_n) \in H^1(l^1)$ and $g = (g_n) \in \mathcal{B}(l^\infty)$ then $f * g \in BMOA$.

We now produce an extension of this result to other values of p differents from 1.

Proposition 5.5. *Let $1 < p \leq 2$. Let (f_n) be a sequence of functions in H^1 such that $\left(\sum_{n\in\mathbb{N}} |f_n(e^{it})|^p\right)^{\frac{1}{p}} \in L^1$, and (g_n) be a sequence of Bloch functions such that $\left(\sum_{n\in\mathbb{N}} |g_n'(z)|^{p'}\right)^{\frac{1}{p'}} = O(\frac{1}{1-|z|})$. Then $\sum_{n\in\mathbb{N}} f_n * g_n$ converges in BMOA.*

Proof. Note that $f = (f_n) \in H^1(l^p)$ and $g = (g_n) \in \mathcal{B}(l^{p'})$. Since l^p has (HL)-property then we can apply Theorem 3.2 to $(l^p, \mathbb{C})$ to get $f * g = \sum_{n\in\mathbb{N}} f_n * g_n \in BMOA$. $\square$

Proposition 5.6. *Let $\phi \in H^1$ and let (g_n) be a sequence of Bloch functions such that $\left(\sum_{n\in\mathbb{N}} |g_n'(z)|^2\right)^{\frac{1}{2}} = O(\frac{1}{1-|z|})$. Then $d\mu(z) = (1-|z|)\sum_{n\in\mathbb{N}} |(g_n * \phi)'(z)|^2 dA(z)$ is a Carleson measure on D.*

Proof. From Corollary 2.1 this is actually equivalent to show that $(g_n * \phi)_{n\in\mathbb{N}} \in BMOA(l^2)$.

This now follows again from Theorem 3.2 applied to $(\mathbb{C}, l^2)$. $\square$

Let us recall that the Paley projection $\mathcal{P}$ stands for the operator $\mathcal{P}(\sum_{n=0}^\infty x_n z^n) = \sum_{n=0}^\infty x_{2^n} z^{2^n}$. We now give an application to spaces X where the Paley projection is bounded in $H^1(X)$.

Regarding $\mathcal{B}$ as the subspace of $\mathcal{B}(\mathcal{L}(X, X))$ given by tensoring with the identity operator we notice that as soon as we have $\mathcal{B} \subset \left(H^1(X), BMOA(X)\right)$, the fact that $\sum_{n=0}^\infty z^{2^n} \in \mathcal{B}$ implies that the Paley projection $\mathcal{P}$ has to be bounded in $H^1(X)$. Our technique gives an alternative proof of the following result due to F. Lust-Picard and G. Pisier.

Proposition 5.7. *([L-PP]) Let $1 < p < \infty$ and consider X_p either L^p or σ_p. Then $\mathcal{P}$ is a bounded operator on $H^1(X_p)$.*

Proof. For $2 \leq q < \infty$ we can apply Theorem 3.2. For $1 < p \leq 2$, it follows from duality and Proposition 1.1,

$$\begin{aligned} ||\mathcal{P}(f)||_{1,\sigma_p} &\approx ||\mathcal{P}(f)||_{BMOA(\sigma_p)} \\ &= \sup\{| < \mathcal{P}(f), g > | : ||g||_{1,\sigma_{p'}} = 1\} \\ &= \sup\{| < f, \mathcal{P}(g) > | : ||g||_{1,\sigma_{p'}} = 1\} \\ &\leq ||f||_{1,\sigma_p}\{\sup ||\mathcal{P}(g)||_{BMOA(\sigma_{p'})} : ||g||_{1,\sigma_{p'}} = 1\} \\ &\leq C||f||_{1,\sigma_p}. \quad \square \end{aligned}$$

REMARK 5.1. Proposition 5.7 is also a consequence of the B-convexity of the space X_p, because of one observation due to Pisier (see [BP, Proposition 4.2]). Also the case $p = 1$ in the Proposition 5.7 holds true. It does not follow from our arguments but the case L^1 is rather elementary and the case σ_1 was proved in [L-PP] using Lemma D.

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