VECTOR–VALUED HARDY INEQUALITIES AND B–CONVEXITY

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Abstract. Inequalities of the form $\sum_{k=0}^{\infty}\frac{|\hat{f}(m_k)|}{k+1}\leq C\,\|f\|_1$ for all $f\in H^1$, where $\{m_k\}$ are special subsequences of natural numbers, are investigated in the vector-valued setting. It is proved that Hardy's inequality and the generalized Hardy inequality are equivalent for vector valued Hardy spaces defined in terms of atoms and that they actually characterize B-convexity. It is also shown that for $1<q<\infty$ and $0<\alpha<\infty$ the space $X=H(1,q,\alpha)$ consisting of analytic functions on the unit disc such that $\int_0^1(1-r)^{q\alpha-1}M_1^q(f,r)dr<\infty$ happens to satisfy the previous inequality for vector valued functions in $H^1(X)$, defined as the space of X-valued Bochner integrable functions on the torus whose negative Fourier coefficients vanish, for the case $\{m_k\}=\{2^k\}$ but not for $\{m_k\}=\{k^a\}$ for any $a\in\mathbb{N}$.

Introduction.

In this paper we shall deal with the vector-valued formulation of certain inequalities in the theory of Hardy spaces. The first one, due to G. H. Hardy ([Du], page 48), reads

$$\sum_{n=0}^{\infty}\frac{|\hat{f}(n)|}{n+1}\leq C\,\|f\|_1 \qquad \text{for all } f\in H^1 \tag{H}$$

where $H^1=\{f\in L^1(\mathbb{T}):\hat{f}(n)=0 \text{ for } n<0\}$ and, as usual, $\mathbb{T}$ stands for the unit circle and $\hat{f}(n)=\int_{-\pi}^{\pi}f(t)e^{-int}\frac{dt}{2\pi}$ for $n\in\mathbb{Z}$.

Recently K. M. Dyakonov [D] considered the following generalized Hardy inequality:

There exists a constant $C>0$ such that, for any increasing subsequence $\{n_k\}$ of $\mathbb{N}$ satisfying

$$\delta=\inf_{k\in\mathbb{N}}k\frac{n_{k+1}-n_k}{n_k}>0 \tag{$*$}$$

one has

$$\sum_{k=0}^{\infty}\frac{|\hat{f}(n_k)|}{k+1}\leq C(1+\frac{1}{\delta})\,\|f\|_1 \qquad \text{for all } f\in H^1. \tag{GH}$$

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In particular, besides the classical Hardy's inequality (H), we have the cases $n_k = k^a$ for any $a \in \mathbb{N}$ or $n_k = 2^k$ (or any other lacunary sequence), these last cases being also a consequence of Paley's inequality (see [Du], page 104).

All of these facts can be regarded as special cases of multiplier inequalities between H^1 and l^1. Recall that a sequence $\{m_n\}$ is an $(H^1 - l^1)$-multiplier, to be denoted by $\{m_n\} \in (H^1 - l^1)$, if $T_{m_n}(f) = (\hat{f}(n)\, m_n)$ defines a bounded operator from H^1 into l^1.

The $(H^1 - l^1)$-multipliers were characterized by C. Fefferman in the following way (see [AS] and [SW,SS] for a proof):

$$(H^1 - l^1) = \{\{m_n\} : \sup_{s\geq 1} \left(\sum_{k\geq 1} (\sum_{j=ks+1}^{(k+1)s} |m_j|)^2 \right)^{1/2} < \infty\}. \qquad (**)$$

The proof of this fundamental result depends upon the atomic decomposition of functions in H^1.

In [BP] the vector valued analogues of several classical inequalities in the theory of Hardy spaces were investigated. Here we use several techniques from that paper and from [B2] to deal with the properties corresponding to the vector valued version of (GH) and some of its particular cases.

A complex Banach space X is said to satisfy the vector valued Hardy inequality (for short X is an (HI)-space) if there exists a constant $C > 0$ such that

$$\sum_{n=0}^{\infty} \frac{\|\hat{f}(n)\|}{n+1} \leq C\,\|f\|_1 \qquad \text{for all } f \in H^1(X) \qquad (H)$$

where $H^1(X) = \{f \in L^1(\mathbb{T}, X) : \hat{f}(n) = 0 \text{ for } n < 0\}$.

A complex Banach space X is said to have $(H^1 - l^1)$-Fourier type if for any $\{m_n\} \in (H^1 - l^1)$ there exists a constant $C > 0$ such that

$$\sum_{n=O}^{\infty} \|\hat{f}(n)\| |m_n| \leq C\,\|f\|_1 \qquad \text{for all } f \in H^1(X). \qquad (F)$$

Given $2 \leq q < \infty$ a complex Banach space X is said to be a q-Paley space if there exists a constant $C > 0$ such that

$$(\sum_{k=0}^{\infty} \|\hat{f}(2^k)\|^q)^{1/q} \leq C\,\|f\|_1 \qquad \text{for all } f \in H^1(X). \qquad (P)_q$$

The reader is referred to [BP] for examples of spaces with or without these properties and for their connection with other well known properties in the theory of Banach spaces.

Let us now introduce the vector-valued extension of (GH) and some of its particular cases.

Definition 1.1. *Let $a \in \mathbb{N}$. A complex Banach space X is said to satisfy the vector valued Hardy inequality for $\{n_k\} = \{k^a\}$ (for short X is an $(HI)_a$-space) if*

$$\sum_{n=0}^{\infty} \frac{\|\hat{f}(n^a)\|}{n+1} \leq C\,\|f\|_1 \qquad \text{for all } f \in H^1(X). \tag{$(H)_a$}$$

Definition 1.2. *A complex Banach space X is said to satisfy the vector valued Hardy inequality for $n_k = 2^k$ (for short X is a $(HI)_{lac}$-space) if there exists a constant $C > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{\|\hat{f}(2^n)\|}{n+1} \leq C\,\|f\|_1 \qquad \text{for all } f \in H^1(X). \tag{$(H)_{lac}$}$$

Definition 1.3. *A complex Banach space X is said to verify generalized vector valued Hardy inequality (for short X is an (GHI)-space) if there exists $C > 0$ such that for any $\{n_k\}$ verifying (*)*

$$\sum_{k=0}^{\infty} \frac{\|\hat{f}(n_k)\|}{k+1} \leq C\,(1 + \frac{1}{\delta})\|f\|_1 \qquad \text{for all } f \in H^1(X). \tag{(GH)}$$

Using (**) it is easy to see that any space of $(H^1 - l^1)$-Fourier type must be a 2-Paley (hence q-Paley for any $q \geq 2$) and an (HI)-space (see [BP]).

Actually repeating the proof in [D] one sees that any space of $(H^1 - l^1)$-Fourier type must be an (GHI)-space.

It should be noted that now the use of vector-valued atoms is still at our disposal but the spaces $H^1(X)$ and $H^1_{at}(X)$ (see definition below) are not isomorphic. The aim of this paper is to make it clear that actually one can get the generalized Hardy inequality for $H^1_{at}(X)$ using only the classical Hardy inequality for $H^1_{at}(X)$.

Let me now recall the following definitions (see [B1], [Bo1])

Definition 1.4. *Given a complex Banach space X, we denote by $H^1_{max}(X)$ the space of functions $f \in L^1(\mathbb{T}, X)$ such that $P^*(f)(t) = \sup_{0<r<1} \|P_r * f(t)\| \in L^1(\mathbb{T})$ where P_r stands for the Poisson kernel.*

We endow this space with the norm $\|f\|_{max,X} = \|P^*(f)\|_1$.

Definition 1.5. *Given a complex Banach space X, we denote by $H^1_{at}(X)$ the space of functions $f \in L^1(\mathbb{T}, X)$ such that $f = \sum_{n\in\mathbb{N}} \lambda_n a_n$ (in the sense of distributions) where $\sum_{n\in\mathbb{N}} |\lambda_n| < \infty$ and a_n are X-valued atoms, that is a_n is either a constant function or it has the following three properties:*

(i) $a_n \in L^\infty(\mathbb{T}, X)$ and $supp(a_n) \subset I_n$ for some interval I_n,

(ii) $\int_{I_n} a_n(t)dt = 0$,

(iii) $\|a_n\|_\infty \leq \frac{1}{|I_n|}$ where $|I_n|$ stands for the normalized Lebesgue measure on $\mathbb{T}$.

As usual the norm is given by $\|f\|_{at,X} = \inf\{\sum_{n\in\mathbb{N}} |\lambda_n|\}$ where the infimum is taken over all possible decompositions.

The facts that $H^1_{max}(X) = H^1_{at}(X)$ and $\|f\|_{at,X} \sim \|f\|_{max,X}$ can be established by repeating the scalar-valued proof in [CW].

It is also well known (see [B1]) that $H^1(X) \subset H^1_{max}(X)$ but they are not the same space unless X has the so-called UMD property.

Because of this, it makes sense to consider the analogs of (H), $(H)_a$ and $(H)_{lac}$ with $H^1_{at}(X)$ in place of $H^1(X)$. The arising "atomic" properties, denoted by $(H)^{at}$, $(H)^{at}_a$, $(H)^{at}_{lac}$, are stronger than their respective counterparts discussed above.

Let us now recall some fundamental notions in geometry of Banach spaces to be used in the sequel. Although they are usually defined in terms of the Rademacher functions we shall replace these by lacunary sequences $e^{i2^n t}$, which gives an equivalent definition ([MPi, Pi]).

Given $1 \le p \le 2 \le q \le \infty$, a Banach space X has cotype q (respectively type p) if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and for all $x_0, x_1, x_2, ...x_N \in X$ one has

$$\left(\sum_{k=0}^{N} ||x_k||^q\right)^{\frac{1}{q}} \le C|| \sum_{k=0}^{N} x_k e^{2^k it}||_1$$

(respectively

$$|| \sum_{k=0}^{N} x_k e^{2^k it}||_1 \le C \left(\sum_{k=0}^{N} ||x_k||^p\right)^{\frac{1}{p}} .)$$

A Banach space is called B-convex if it has type > 1.

Given a complex Banach space X and a function $f(z) = \sum_{n=0}^{\infty} x_n z^n$ with $x_n \in X$ we write $\mathcal{P}(f) = \sum_{n=0}^{\infty} x_{2^n} z^{2^n}$ for the vector-valued version of the Paley projection acting on f.

It is well known that X is B-convex space if and only if the Paley projection bounded on $H^p(X)$ for some (or any) $1 < p < \infty$. This can be extended to $p = 1$. It is a result due to Pisier (see [BP], Proposition 4.2) that X is B-convex if and only if $\|\mathcal{P}(f)\|_{at,X} \le C\|f\|_{at,X}$ for all $f \in H^1_{at}(X)$.

Definition 1.6. *We say that X satisfies the Paley projection property (for short $X \in (PP)$) if the Paley projection is bounded in $H^1(X)$.*

In [L-PP] this property was studied for the case of Schatten classes.

Remark 1.1. Observe that, since $X = c_0 \notin (PP)$, then the inclusion $X \in (PP)$ implies that X has finite cotype.

Remark 1.2. If $X \in (PP)$ and X has cotype q for some $2 \le q < \infty$ then $X \in (P)_q$.

Combining both remarks one easily gets the following result.

Proposition 1.1. *If X has the Paley projection property then it also has q-Paley property for some $2 \le q < \infty$ and satisfies the Hardy inequality for $n_k = 2^k$.*

We shall be also using the notion of Fourier-type introduced by J. Peetre ([Pee]). Let us recall that for $1 \le p \le 2$, a Banach space X is said to have Fourier type p if there exists a constant $C > 0$ such that

$$\left(\sum_{n=-\infty}^{\infty} ||\hat{f}(n)||^{p'}\right)^{\frac{1}{p'}} \leq C||f||_{L^p(X)} \text{ for all } f \in L^p(\mathbb{T}, X).$$

It is not hard to see that X has Fourier type p if and only if X^* has Fourier type p. Typical examples are the spaces L^r for $p \leq r \leq p'$ or those obtained by interpolation between any Banach space and a Hilbert space.

Let us now state the fundamental theorem, due to J. Bourgain, which connects the last two properties.

Theorem A. *(* [Bo2, Bo3]*) Let X be a complex Banach space. Then X has Fourier type bigger than 1 if and only if X is B-convex.*

Throughout the paper $L_p(\mu, Y)$ (respectively $L^p(Y)$) stands for the space of Y-valued strongly measurable functions on a σ-finite measure space (Ω, Σ, μ) (respectively $(\mathbb{T}, \mathcal{B}, \frac{dt}{2\pi})$) such that $\|f\| \in L_p(\mu)$ and we denote by $H^p(Y)$ the subspace of $L^p(Y)$ consisting of functions such that $\hat{f}(n) = \int_{-\pi}^{\pi} f(t)e^{-int}\frac{dt}{2\pi} = 0$ for $n < 0$. We write $H^p(\mathbb{D}, X)$ for the space of analytic functions f from $\mathbb{D}$ into X such that $\sup_{0<r<1} M_p(f, r) < \infty$ where $M_{p,X}(f, r) = (\int_{-\pi}^{\pi} \|f(re^{it})\|^p \frac{dt}{2\pi})^{\frac{1}{p}}$.

Clearly if $f \in H^p(X)$ then $f(re^{it}) = f * P_r(t) \in H^p(\mathbb{D}, X)$, but in general the space $H^p(\mathbb{D}, X)$ can not be identified with $H^p(X)$ or, in other words, the functions in $H^p(\mathbb{D}, X)$ do not necessarily have radial boundary limits.

A complex Banach space X for which any function in $H^\infty(\mathbb{D}, X)$ has radial boundary limits a.e. is said to have the analytic Radon-Nikodym property, for short $X \in (ARNP)$. This was first introduced in [BuD] where it was shown, among other things, that $L^1(\mu) \in (ARNP)$.

2.- Hardy type inequalities for $H^1(X)$.

We shall first show an extension to the vector valued setting of one inequality by Hardy and Littlewood (see [Du, HL]). Our proof follows ideas in [F] and uses the Marcinkiewicz interpolation theorem.

Theorem 2.1. *Let X be a Banach space and let $1 < p < \infty$. If $f \in H^1(X)$ then*

$$\int_0^1 (1-r)^{-\frac{1}{p}} M_{p,X}(f, r) dr \leq C||f||_1.$$

Proof. Let us first recall that if $0 < p \leq q \leq \infty$ and g is an X-valued analytic function then (see [Du, page 84])

$$M_{q,X}(g, r^2) \leq (1-r)^{\frac{1}{q}-\frac{1}{p}} M_{p,X}(g, r). \tag{2.1}$$

To prove the result let us first fix $p_1 < 1 < p_2 < p$. For $i = 1, 2$, using (2.1) one has

$$(1-r)^{-\frac{1}{p}} M_{p,X}(f, r) \leq (1-r)^{-\frac{1}{p_i}} ||f||_{H^{p_i}(\mathbb{D}, X)}.$$

Hence

$$|\{r \in [0,1] : (1-r)^{-\frac{1}{p}} M_{p,X}(f,r) > \lambda\}| \leq C \frac{||f||^{p_i}_{H^{p_i}(\mathbb{D},X)}}{\lambda^{p_i}}.$$

This actually gives that

$$f \to F(r, e^{it}) = (1-r)^{-\frac{1}{p}} f(re^{it})$$

defines a bounded operator from $H^{p_i}(\mathbb{D}, X)$ into $L^{p_i,\infty}(dr, L^p(X))$ where $L^{p_i,\infty}(dr, L^p(X))$ stands for the corresponding vector valued Lorenz space.

Now using the standard real method of interpolation for $\theta \in (0,1)$ such that $\frac{1-\theta}{p_1} + \frac{\theta}{p_2} = 1$ we have (see [BL])

$$\Big(L^{p_1,\infty}\big(dr, L^p(X)\big), L^{p_2,\infty}\big(dr, L^p(X)\big)\Big)_{\theta,1} = L^1\big(dr, L^p(X)\big).$$

On the other hand, since the Banach space X is the same for both indices, it is not difficult to extend the scalar-valued proof (see [BX] and references there) to get

$$(H^{p_1}(\mathbb{D}, X), H^{p_2}(\mathbb{D}, X))_{\theta,1} = H^1(\mathbb{D}, X).$$

Hence the operator is bounded from $H^1(X)$ to $L^1(dr, L^p(X))$, that is

$$\int_0^1 (1-r)^{-\frac{1}{p}} M_p(f,r)dr \leq C||f||_1. \quad \square$$

Corollary 2.1. *([BP], [Bo3]) If X is a B-convex space then X satisfies the vector valued Hardy inequality, i.e.*

$$\sum_{n=0} \frac{||x_n||}{n+1} \leq C||\sum_{n=0}^{\infty} x_n e^{int}||_1.$$

Proof. From Theorem A we have that X has Fourier type p for some $p > 1$. Then applying Hölder's inequality and Theorem 2.1 for such a p, one has

$$\begin{aligned}\sum_{n=0}^{m} \frac{||x_n||}{n+1} &= \int_0^1 \sum_{n=0}^{m} ||x_n|| r^n dr \\ &\leq \int_0^1 \left(\sum_{n=0}^{m} ||x_n||^{p'} r^{np'}\right)^{\frac{1}{p'}} \left(\sum_{n=0}^{m} r^{np}\right)^{\frac{1}{p}} dr \\ &\leq C \int_0^1 (1-r)^{-\frac{1}{p}} M_{p,X}(f,r)dr \leq C||f||_1. \quad \square\end{aligned}$$

The following is a simple modification of a proof in [B2] regarding Paley spaces (corresponding to 2-Paley property).

Lemma 2.1. *Let $2 \le q < \infty$ and $1 \le p \le q$. If Y is a q-Paley space then so is $L_p(\mu, Y)$.*

Proof. Put $r = \left(\frac{q}{p}\right)' = \frac{q}{q-p}$. Let us take $f(t) = \sum_{n\ge 0} x_n\, e^{int}$ where $x_n \in L_p(\mu, Y)$. Then we have

$$
\begin{aligned}
\left(\sum_{k\ge 0} \|x_{2^k}\|^q_{L_p(\mu,Y)}\right)^{1/q} &= \left(\sum_{k\ge 0}\left(\int_\Omega \|x_{2^k}(w)\|^p_Y \, d\mu(w)\right)^{q/p}\right)^{1/q} \\
&= \sup_{\sum |\alpha_k|^r = 1}\left(\sum_{k\ge 0}\int_\Omega \|x_{2^k}(w)\|^p_Y \,\alpha_k \, d\mu(w)\right)^{1/p} \\
&\le \left(\int_\Omega \left(\sum_{k\ge 0} \|x_{2^k}(w)\|^q_Y\right)^{p/q} d\mu(w)\right)^{1/p} \\
&\le C \left(\int_\Omega \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\sum_{n\ge 0} x_n(w) e^{int}\|_Y dt\right)^p d\mu(w)\right)^{1/p} \\
&= C \sup_{\|h\|_{p'}=1} \int_\Omega \int_{-\pi}^{\pi} \|\sum_{n\ge 0} x_n(w) e^{int}\|_Y h(w) dt\, d\mu(w) \\
&\le C \int_{-\pi}^{\pi} \|\sum_{n\ge 0} x_n(w) e^{int}\|_{L_p(\mu,Y)} dt \\
&= C \int_{-\pi}^{\pi} \|f(t)\|_{L_p(\mu,Y)} dt. \quad \square
\end{aligned}
$$

Theorem 2.2. *Let $2 \le q < \infty$. Then $L_{p_1}(\mu, L_{p_2}(\nu))$ is a q-Paley space if and only if $1 \le p_1, p_2 \le q$.*

Proof. It is clear from the definition that a q-Paley space must have cotype q. Now the cotype q condition forces the values of p_1, p_2 to be in the required range.

To get the converse, observe that the classical Paley inequality together with Lemma 2.1 for $Y = \mathbb{C}$ gives that L_{p_2} is a q-Paley space for $1 \le p_2 \le q$. Now apply Lemma 2.1 again. $\square$

Lemma 2.2. *Let $1 \le p < \infty$ and $X \in (PP)$. Then $L_p(\mu, X) \in (PP)$.*

Proof. Using Kahane's inequality we can write

$$\begin{aligned}\|\sum_{n=0}^{\infty} x_{2^n} z^{2^n}\|_{H^1(L^p(\mu,X))} &\sim (\int_\Omega \|\sum_{n=0}^{\infty} x_{2^n}(\omega) z^{2^n}\|^p_{H^1(X)} d\mu(\omega))^{\frac{1}{p}} \\ &\leq C(\int_\Omega \|\sum_{n=0}^{\infty} x_n(\omega) z^n\|^p_{H^1(X)} d\mu(\omega))^{\frac{1}{p}} \\ &\leq C\int_{-\pi}^{\pi} (\int_\Omega \|\sum_{n=0}^{\infty} x_n(\omega) e^{int}\|^p_X d\mu(\omega))^{\frac{1}{p}} \frac{dt}{2\pi}. \quad \square\end{aligned}$$

Theorem 2.3. *Let $1 \leq p, q \leq \infty$. Then $L_p(\mu, L_q(\nu))$ is an $(HI)_{lac}$-space if and only if $1 \leq p, q < \infty$*

Proof. Observe first that c_0 is not an $(HI)_{lac}$-space (take the canonical example $f_N(z) = \sum_{n=1}^N e_n z^n$ to check this fact). Consequently, if X is an $(HI)_{lac}$-space then it must have finite cotype.

Assume that $L_p(\mu, L_q(\nu))$ is a $(HI)_{lac}$-space. Now the cotype condition forces the values of p, q to be finite.

To get the converse, observe that the classical Paley inequality gives that $Y = \mathbb{C} \in (PP)$. Now, applying Lemma 2.2 twice, one has that $L_p(\mu, L_q(\nu)) \in (PP)$ for $1 \leq p, q < \infty$. Finally apply Proposition 1.1 to finish the proof. $\square$

Now we shall consider some classes of analytic functions that will serve us to get examples of spaces satisfying $(HI)_{lac}$ but failing to satisfy $(HI)_a$. The reader is referred to [B2] for the fact that $l^p(H^1)$ fails to satisfy (HI) for $1 < p \leq 2$ but it is a 2-Paley space.

Let us recall that, given $1 \leq p, q \leq \infty$ and $0 < \alpha < \infty$, $H(p, q, \alpha)$ stands for the space of analytic functions on the unit disc such that

$$(\int_0^1 (1-r)^{\alpha q - 1} M_p^q(f, r) dr)^{\frac{1}{q}} = \|f\|_{p,q,\alpha} < \infty.$$

Theorem 2.4. *Let $1 < q < \infty$, $0 < \alpha < \infty$ and $a \in \mathbb{N}$. Then $H(1, q, \alpha)$ is an $(HI)_{lac}$-space but fails to be an $(HI)_a$-space.*

Proof. That $H(1, q, \alpha)$ is an $(HI)_{lac}$-space follows from Theorem 2.3, since

$$\|f\|_{1,q,\alpha} = \|g\|_{L^q(\frac{dr}{1-r}, L^1(\mathbb{T}))}$$

where $g(r, \theta) = (1-r)^\alpha f(re^{i\theta})$.

To see that it does not satisfy $(H)_a$, let us consider the function

$$\phi(z) = \frac{1}{(1-z)^{\alpha+1}} \frac{z}{\log \frac{1}{1-z}} = \sum_{n=0}^{\infty} a_n z^n.$$

It is known (see [L],page 93-96) that

$$a_n \sim \frac{n^\alpha}{\log n} \qquad (n \to \infty) \tag{2.2}$$

and

$$M_1(\phi, r) \sim \frac{(1-r)^{-\alpha}}{\log \frac{1}{1-r}} \qquad (r \to 1). \tag{2.3}$$

Consider now $f(z)(w) = \phi(zw) = \sum_{n=0}^\infty a_n\, w^n z^n$ and write $x_n(w) = a_n w^n$.
Since we have

$$\|x_n\|_{1,q,\alpha} = |a_n| \|w^n\|_{1,q,\alpha} = |a_n| B^{\frac{1}{q}}(\alpha q, qn+1)$$

then, (2.2) together with the estimate $B(\beta, m) \sim m^{-\beta}$ as $m \to \infty$, give

$$\|x_n\|_{1,q,\alpha} \sim \frac{1}{\log(n)} \tag{2.4}$$

This allows us to say that $f(z) = \sum_{n=0}^\infty x_n\, z^n$ is an analytic function on the open unit disc with values in $H(1, q, \alpha)$.

Using now (2.3) and the assumption $q > 1$ we have that

$$\begin{aligned} \|f(z)\|_{1,q,\alpha} &= \Big(\int_0^1 (1-r)^{\alpha q - 1} M_1^q(\phi, |z|r)\,dr\Big)^{\frac{1}{q}} \\ &\le C\Big(\int_0^1 (1-r)^{\alpha q-1} \frac{(1-r)^{-\alpha q}}{\log^q \frac{1}{1-r}}\,dr\Big)^{\frac{1}{q}} \\ &\le C\Big(\int_0^1 \frac{1}{(1-r)\log^q \frac{1}{1-r}}\,dr\Big)^{\frac{1}{q}} < \infty. \end{aligned}$$

Therefore $f \in H^\infty(\mathbb{D}, H(1, q, \alpha))$. Using the fact that $H(1, q, \alpha)$ has the analytic Radon-Nikodym property (recall that $L^q(\frac{dr}{1-r}, L^1(\mathbb{T})) \in (ARNP)$) we can show that the radial limits exist almost everywhere and hence, in particular $f \in H^1(H(1, q, \alpha))$.

On the other hand, from (2.4) we have $\sum_{n=0}^\infty \frac{\|x_{n^a}\|_{1,q,\alpha}}{n+1} = \infty$. $\square$.

3.- Generalized Hardy inequalities for $H^1_{at}(X)$

Let us start by showing the differences appearing when dealing with the vector-valued versions $H^1_{at}(X)$ and $H^1(X)$.

It follows rather easily, using Fubini's theorem and the scalar-valued result by K. Dyakonov, that $L^1(\mu)$ verifies the generalized Hardy's inequality, i.e. $L^1(\mu) \in (GHI)$. Actually the same argument shows that $L^1(\mu)$ even has $(H^1 - l^1)$-Fourier type (see [BP]).

Nevertheless $L^1(\mathbb{T})$ fails to have $(HI)^{at}_a$ for any value of $a \in \mathbb{N}$ as the following proposition shows.

Proposition 3.1. *Let $a \in \mathbb{N}$. Then $L^1(\mathbb{T})$ is not an $(HI)_a^{at}$-space.*

Proof. It is well known (see [Zy]) that $\phi(t) = \sum_{|n|\geq 2} \frac{e^{int}}{log(|n|)} \in L^1(\mathbb{T})$.

Now consider $f : \mathbb{T} \to L^1$ given by $f(t)(s) = \phi(e^{i(t+s)})$. It is clear that $f \in L^\infty(L^1)$ and that

$$P^*(f)(t) = \sup_{0<r<1} \int_{-\pi}^{\pi} |\sum_{n\neq 0} \frac{r^{|n|} e^{in(t+s)}}{log(|n|)}| \frac{ds}{2\pi} \leq \|\phi\|_1.$$

Therefore, in particular, $f \in H^1_{max}(L^1)$.

On the other hand $\hat{f}(n)(t) = \frac{1}{log(n)} e^{int}$ for $n \in \mathbb{N}$, which gives $\sum_{n\in\mathbb{N}} \frac{\|\hat{f}(k^a)\|_1}{k} = \infty$ for all $a \in \mathbb{N}$. $\square$

Let us now prove a couple of lemmas to be used later on.

Lemma 3.1. *Let $f \in L^1(X)$ and let J be an interval in $\mathbb{Z}$. If $g(t) = f(t)(1 - e^{-it})$ then*

$$\sup_{j\in J} \|\hat{f}(j)\| \leq \frac{1}{card(J)} \sum_{j\in J} \|\hat{f}(j)\| + \sum_{j\in J} \|\hat{g}(j)\|. \tag{3.1}$$

Proof. Let us fix $j \in J$. For any $k \in J$ we can write

$$\begin{aligned} \|\hat{f}(j)\| &\leq \|\hat{f}(k)\| + \|\hat{f}(k) - \hat{f}(j)\| \\ &\leq \|\hat{f}(k)\| + \sum_{l=min(J)}^{max(J)-1} \|\hat{f}(l) - \hat{f}(l+1)\|. \end{aligned}$$

Averaging over $k \in J$ we get

$$\|\hat{f}(j)\| \leq \frac{1}{card(J)} \sum_{k\in J} \|\hat{f}(k)\| + \sum_{l=min(J)}^{max(J)-1} \|\hat{f}(l) - \hat{f}(l+1)\|.$$

Finally, taking into account that $\hat{f}(l) - \hat{f}(l+1) = \hat{g}(l)$, we get (3.1). $\square$

Lemma 3.2. *Let $M \in \mathbb{N}$ and let $\{n_k\}$ be an increasing sequence in $\mathbb{N} \cup \{0\}$ such that there exists a constant $A > 0$ for which*

$$\frac{n_k - n_{k-1}}{n_k} \geq \frac{A}{k} \qquad (k \in \mathbb{N}).$$

Then for any $l \in \mathbb{N}$

$$\sum_{\{k:lM<n_k\leq lM+M\}} \frac{1}{k} \leq (1 + \frac{1}{A}) \frac{1}{l}. \tag{3.2}$$

Proof. For a fixed $l \in \mathbb{N}$ we may assume that there exists $n_k \in (lM, lM + M]$. Let k_l be the smallest index with this property, so that $n_{k_{l-1}} \leq lM < n_{k_l}$.

Observe now that

$$\sum_{\{k:lM<n_k\leq lM+M\}} \frac{1}{k} \leq \frac{1}{k_l} + \frac{1}{A}\sum_{k=k_l+1}^{k_{l+1}} \frac{n_k - n_{k-1}}{n_k}$$
$$\leq \frac{1}{k_l} + \frac{1}{An_{k_l}}(n_{k_{l+1}} - n_{k_l}).$$

Since $k_l \geq l$, $n_{k_l} \geq lM$ and $n_{k_{l+1}} - n_{k_l} \leq M$ we get (3.2). □

Theorem 3.1. *Let X be a Banach space. The following statements are equivalent:*
(1) There exists a constant $C > 0$ such that

$$\sum_{n=1}^{\infty} \frac{\|\hat{f}(n)\|}{n} \leq C\,\|f\|_{at,X} \qquad \text{for all } f \in H^1_{at}(X).$$

(2) For any increasing sequence $\{n_k\}$ in $\mathbb{N}\cup\{0\}$ satisfying that

$$\frac{n_k - n_{k-1}}{n_k} \geq \frac{A}{k} \qquad (k \in \mathbb{N}) \tag{3.3}$$

for some $A > 0$ there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} \frac{\|\hat{f}(n_k)\|}{k} \leq C(1+\frac{1}{A})\,\|f\|_{at,X} \qquad \text{for all } f \in H^1_{at}(X)$$

Proof. Obviously (2) implies (1).

To see that (1) implies (2) let us fix a sequence $\{n_k\}$ satisfying (3.3). It suffices to show that there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} \frac{\|\hat{a}(n_k)\|}{k} \leq C(1+\frac{1}{A})$$

for any $M \in \mathbb{N}$ and any X-valued atom a supported on $(-\frac{\pi}{M}, \frac{\pi}{M})$.

Take such an X-atom, say a. Given $n \in \mathbb{N}$, using that a has zero mean, we have

$$\|\hat{a}(n)\| = \|\int_{\frac{-\pi}{M}}^{\frac{\pi}{M}} a(t)(e^{-int}-1)\frac{dt}{2\pi}\| \leq n\|a\|_\infty \int_{\frac{-\pi}{M}}^{\frac{\pi}{M}} |t|\frac{dt}{2\pi} \leq C\frac{n}{M}. \tag{3.4}$$

Let us write $F = \{k : n_k \leq M\}$ and $G = \{k : n_k > M\}$.

On the one hand

$$\sum_{k\in F} \frac{\|\hat{a}(n_k)\|}{k} \leq \frac{C}{M}\sum_{k\in F}\frac{n_k}{k} \leq \frac{C}{MA}\sum_{k\in F}(n_k - n_{k-1}) \leq \frac{C}{A}. \tag{3.5}$$

On the other hand, denoting $G_l = \{k : lM < j \leq lM + M\}$, we have

$$\sum_{k\in G} \frac{\|\hat{a}(n_k)\|}{k} = \sum_{l=1}^{\infty} \sum_{k\in G_l} \frac{\|a_{n_k}\|}{k} \leq \sum_{l=1}^{\infty} \sup_{lM<j\leq lM+M} \|\hat{a}(j)\| \sum_{k\in G_l} \frac{1}{k}.$$

Now applying Lemmas 3.1 and 3.2 and denoting $b_1(t) = a(t)(1 - e^{-it})$ we have

$$\begin{aligned}\sum_{k\in G} \frac{\|\hat{a}(n_k)\|}{k} &\leq (1 + \frac{1}{A}) \sum_{l=1}^{\infty} \frac{1}{lM} \sum_{j=lM+1}^{lM+M} \|\hat{a}(j)\| + (1 + \frac{1}{A}) \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=lM+1}^{lM+M} \|\hat{b}_1(j)\| \\ &\leq (1 + \frac{1}{A}) \sum_{l=1}^{\infty} \sum_{j=lM+1}^{lM+M} \frac{\|\hat{a}(j)\|}{j} + (1 + \frac{1}{A}) \sum_{l=1}^{\infty} \sum_{j=lM+1}^{lM+M} \frac{\|M\hat{b}_1(j)\|}{j} \\ &\leq (1 + \frac{1}{A}) \sum_{j=M+1}^{\infty} \frac{\|\hat{a}(j)\|}{j} + (1 + \frac{1}{A}) \sum_{j=M+1}^{\infty} \frac{\|M\hat{b}_1(j)\|}{j}.\end{aligned}$$

To finish the proof note that $b(t) = M(b_1(t) - \int_{\frac{-\pi}{M}}^{\frac{\pi}{M}} b_1(t) \frac{dt}{2\pi})$ is also an X-valued atom and $\hat{b}(j) = M\hat{b}_1(j)$.

Therefore applying Hardy's inequality to a and b we get

$$\sum_{k\in G} \frac{\|\hat{a}(n_k)\|}{k} \leq 2(1 + \frac{1}{A})C.$$

We now finish the proof by combining this last estimate with the one in (3.5). $\square$

Corollary 3.1. *(see* [BP, Bo3]*) Let X be a Banach space. The following statements are equivalent.*

(1) X is B-convex.
(2) X is an $(HI)^{at}$-space.
(3) X is an $(GHI)^{at}$-space.

Proof. (1) $\Rightarrow$ (2). As in the previous theorem it suffices to show that there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} \frac{\|\hat{a}(n)\|}{n} \leq C$$

for any $M \in \mathbb{N}$ and any X-valued atom a supported at $(-\frac{\pi}{M}, \frac{\pi}{M})$.

Using Theorem A we may assume that X has Fourier type $p > 1$. It is clear that if a is such an X-valued atom then $\|a\|_p \leq M^{\frac{1}{p'}}$. Hence, in view of (3.4),

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\|\hat{a}(n)\|}{n} &= \sum_{n=1}^{M} \frac{\|\hat{a}(n)\|}{n} + \sum_{n=M+1}^{\infty} \frac{\|\hat{a}(n)\|}{n} \\ &\leq C + (\sum_{n=M+1}^{\infty} \|\hat{a}(n)\|^{p'})^{\frac{1}{p'}} (\sum_{n=M+1}^{\infty} \frac{1}{n^p})^{\frac{1}{p}} \\ &\leq C + C\|a\|_p \frac{1}{M^{\frac{1}{p'}}} \leq C.\end{aligned}$$

(2) $\Rightarrow$ (3). It follows from Theorem 3.1.

(3) $\Rightarrow$ (1). By a finite representability argument (see Prop. 2.6 in [BP]) it is enough to show that L^1 fails $(GHI)^{at}$. This now follows from Proposition 3.1. $\square$

As a corollary of our previous theorems we get some improvements of results by H. König and V. Tarieladze (see [K] and Prop. 3 in [T]).

Corollary 3.2. *Let X be a B-convex Banach space and let $a \in \mathbb{N}$. If $f \in \cup_{p>1} L^p(X)$ with $\hat{f}(0) = 0$ and $F(t) = \int_0^t f(s) \frac{ds}{2\pi}$ then*

$$\sum_{n \in \mathbb{Z}} n^{a-1} \|\hat{F}(n^a)\| < \infty.$$

Proof. Since $f \in H^1_{at}(X)$ and $\|\hat{f}(n)\| = |n| \|\hat{F}(n)\|$ we can apply Corollary 3.1 for $n_k = k^a$ to the function f. $\square$

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