A bilinear version of Orlicz-Pettis theorem.

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Abstract

Given three Banach spaces X, Y and Z and a bounded bilinear map $\mathscr{B} : X \times Y \to Z$, a sequence $\boldsymbol{x} = (x_n)_n \subseteq X$ is called $\mathscr{B}$-absolutely summable if $\sum_{n=1}^{\infty} \|\mathscr{B}(x_n, y)\|_Z < \infty$ for any $y \in Y$. Connections of this space with $\ell^1_{\text{weak}}(X)$ are presented. A sequence $\boldsymbol{x} = (x_n)_n \subseteq X$ is called $\mathscr{B}$-unconditionally summable if $\sum_{n=1}^{\infty} |\langle \mathscr{B}(x_n, y), z^* \rangle| < \infty$

for any $y \in Y$ and $z^* \in Z^*$ and for any $M \subseteq \mathbb{N}$ there exists $x_M \in X$ for which $\sum_{n\in M}\langle \mathscr{B}(x_n, y), z^*\rangle = \langle \mathscr{B}(x_M, y), z^*\rangle$ for all $y \in Y$ and $z^* \in Z^*$. A bilinear version of Orlicz-Pettis theorem is given in this setting and some applications are presented.

Key words: 40F05 Absolute and strong summability, 46A45 Sequence spaces, 46B45 Banach sequence spaces.

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1 Notation and preliminaries.

Throughout this paper X, Y and Z denote Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\mathscr{B} : X \times Y \to Z$ is a bounded bilinear map. As usual $\mathscr{L}(X, Y)$ denotes the set consisting of all linear and continuous maps T defined from X into Y, B_X denotes the closed unit ball of X and X^* the topological dual $X^* = \mathscr{L}(X, \mathbb{K})$.

We use the notations $\ell^1(X)$ and $\ell^1_{\text{weak}}(X)$ for the spaces of all sequences $\boldsymbol{x} = (x_n)_n \subseteq X$ such that

$$\|\boldsymbol{x}\|_{\ell^1(X)} = \Big\|(\|x_n\|_X)_n\Big\|_{\ell^1} = \sum_{n=1}^{\infty}\|x_n\|_X < \infty,$$

$$\|\boldsymbol{x}\|_{\ell^1_{\text{weak}}(X)} = \sup_{x^*\in \mathrm{B}_{X^*}}\Big\|(\langle x_n, x^*\rangle)_n\Big\|_{\ell^1} = \sup_{x^*\in \mathrm{B}_{X^*}}\sum_{n=1}^{\infty}|\langle x_n, x^*\rangle| < \infty.$$

The sequences in $\ell^1(X)$ and $\ell^1_{\text{weak}}(X)$ are called *absolutely summable* and *weakly absolutely summable* sequences respectively.

A sequence $\boldsymbol{x} = (x_n)_n$ is called *unconditionally summable* if the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, i.e. $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is convergent for each permutation $\sigma : \mathbb{N} \to \mathbb{N}$. Among other things —see (11)— the unconditional summability of a sequence is equivalent to

(a) $\sum_{n=1}^{\infty}\varepsilon_n x_n$ converges for any choice of $\varepsilon_n = \pm 1$.

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(b) $\sum_{n=1}^{\infty} x_{n_k}$ converges for any increasing $(n_k)_k \subseteq \mathbb{N}$.
(c) For any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ so that $\|\sum_{k\in M} x_k\| < \varepsilon$ whenever $\min M \geq N_\varepsilon$.

The set consisting of these sequences will be denoted by $\mathrm{UC}(X)$. It is well known the fact that if X is a normed space:

$$X \text{ is complete if and only if } \ell^1(X) \subseteq \mathrm{UC}(X)\ .$$

A sequence $\boldsymbol{x} = (x_n)_n \subseteq X$ is called *weakly unconditionally summable* if the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is weakly convergent for each permutation $\sigma : \mathbb{N} \to \mathbb{N}$. Equivalently if we have that $\boldsymbol{x} \in \ell^1_{\text{weak}}(X)$ and for all $M \subseteq \mathbb{N}$ there is an $x_M \in X$ such that

$$\sum_{n\in M} \langle x_n, x^* \rangle = \langle x_M, x^* \rangle, \qquad \text{for all } x^* \in X^*.$$

The set consisting of those sequences will be denoted by $\mathrm{wUC}(X)$.

Of course we have the following chain of inclusions for any Banach space X:

$$\ell^1(X) \subseteq \mathrm{UC}(X) \subseteq \mathrm{wUC}(X) \subseteq \ell^1_{\text{weak}}(X).$$

Clearly for finite dimensional Banach spaces X one has $\ell^1(X) = \ell^1_{\text{weak}}(X)$ but, in the general, both spaces are different. Actually the so called weak Dvoretzky-Rogers theorem —see (11, p. 50)— asserts that

$$\text{A Banach space } X \text{ has finite dimension if and only if } \ell^1(X) = \ell^1_{\text{weak}}(X).$$

In fact using the Dvoretzky-Rogers theorem —see for instance (11, p. 2)— which says that in each infinite dimensional Banach space X for each $(\lambda_n) \in \ell^2$ it is possible to find sequences $\boldsymbol{x} = (x_n)_n \subseteq X$ which are unconditionally summable and $\|x_n\| = \lambda_n$ one obtains that

$$X \text{ is finite dimensional if and only if } \ell^1(X) = \mathrm{UC}(X).$$

On the other hand, in general, $\ell^1_{\text{weak}}(X)$ and $\mathrm{UC}(X)$ are different. For instante take $X = c_0$ and $\boldsymbol{x} = (e_n)_n \subseteq c_0$ —as usual e_n is the canonical basis— which is clearly in $\ell^1_{\text{weak}}(c_0)$, but $\boldsymbol{x} \notin \mathrm{UC}(c_0)$ since $\lim_n \|e_n\|_{c_0} = 1 \neq 0$. Actually we have the following important result that characterizes when $\ell^1_{\text{weak}}(X) = \mathrm{UC}(X)$. This goes back to 1958 and it is due to Bessaga and Pelczyński —see for instance (11, p. 22)—:

$$X \text{ does not contain copies of } c_0 \text{ if and only } \ell^1_{\text{weak}}(X) = \mathrm{UC}(X).$$

The classical Orlicz-Pettis theorem —see for instance (11, p. 7)— states that weakly unconditional convergence is equivalent to unconditional convergence.

$$\mathrm{wUC}(X) = \mathrm{UC}(X) \text{ for any Banach space } X.$$

The Orlicz-Pettis theorem is one of the most celebrated theorems concerning series in Banach spaces. It has been used in many different situations in functional analysis —see for instance (10) for applications in integration theory—. The main objective of this paper is to give a more general version of the Orlicz-Pettis theorem in the setting summability with respect to bounded bilinear maps.

We plan to develop the previous notions of summability in a general setting adapted to a given bounded bilinear map $\mathscr{B} : X \times Y \to Z$ where Y and Z are also Banach spaces. We say that a vector sequence $\boldsymbol{x} = (x_n)_n \subseteq X$ is *$\mathscr{B}$-absolutely summable* if the Z-valued sequence $(\mathscr{B}(x_n, y))_n$ belongs to $\ell^1(Z)$ for all $y \in Y$. The set of this sequences will be denoted by $\ell^1_{\mathscr{B}}(X)$.

We need to impose some conditions on the bilinear map $\mathscr{B} : X \times Y \to Z$ for the basic theory to be developed. Let us denote

$$\phi_{\mathscr{B}} : X \to \mathscr{L}(Y, Z), \quad \text{given by} \quad \phi_{\mathscr{B}}(x) = \mathscr{B}(x, \cdot) = \mathscr{B}_x.$$

We say that $\mathscr{B}$ *is admissible* if $\phi_{\mathscr{B}}$ is injective. This assumption gives that

$$\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = \sup_{y \in \mathrm{B}_Y} \left\|(\mathscr{B}(x_n, y))_n\right\|_{\ell^1(Z)}$$

is a norm in the space $\ell^1_{\mathscr{B}}(X)$. In fact if there exists $k > 0$ such that

$$\|x\| \leq k \|\mathscr{B}_x\|_{\mathscr{L}(Y,Z)}, \qquad \text{for all } x \in X,$$

the space X *is said to be $\mathscr{B}$-normed*. This concept is basic to get, among other things, that $\ell^1_{\mathscr{B}}(X)$ is complete. These notions have recently been considered when handling problems in integration with respect to a bounded bilinear map —see (1; 2)— or developing a theory of Fourier Analysis with respect to a bounded bilinear map —see (3)—.

We say that a sequence $\boldsymbol{x} = (x_n)_n \subseteq X$ is $\mathscr{B}$-unconditionally summable if for all $y \in Y$ and $z^* \in Z^*$ we have that $(\langle \mathscr{B}(x_n, y), z^* \rangle)_n \in \ell^1$ and for all $M \subseteq \mathbb{N}$ there is $x_M \in X$ such that

$$\sum_{n \in M} \langle \mathscr{B}(x_n, y), z^* \rangle = \langle \mathscr{B}(x_M, y), z^* \rangle, \qquad \text{for all } y \in Y, z^* \in Z^*.$$

We use the notation $\mathscr{B} - \mathrm{UC}(X)$ for the space of $\mathscr{B}$-unconditionally summable sequences.

From this point of view we have that, using the notation $\mathcal{B}, \mathscr{D}$ and $\mathscr{D}_1$, for the standard bilinear maps

$$\mathcal{B} : X \times \mathbb{K} \to X, \quad \text{given by} \quad \mathcal{B}(x, \alpha) = \alpha x, \tag{1}$$

$$\mathscr{D} : X \times X^* \to \mathbb{K}, \quad \text{given by} \quad \mathscr{D}(x, x^*) = \langle x, x^* \rangle, \tag{2}$$

$$\mathscr{D}_1 : X^* \times X \to \mathbb{K}, \quad \text{given by} \quad \mathscr{D}_1(x^*, x) = \langle x, x^* \rangle, \tag{3}$$

the spaces become

$$\ell^1_{\mathcal{B}}(X) = \ell^1(X), \ell^1_{\mathscr{D}}(X) = \ell^1_{\text{weak}}(X) \text{ and } \ell^1_{\mathscr{D}_1}(X^*) = \ell^1_{\text{weak}^*}(X^*).$$

Note that a sequence in $\ell^1_{\mathscr{D}_1}(X^*)$ is always $\mathscr{D}_1$-unconditionally summable, i.e. $\ell^1_{\mathscr{D}_1}(X^*) = \mathscr{D}_1 - \mathrm{UC}(X^*)$. However, by considering $X = \ell^\infty$ and the standard canonical sequence $\boldsymbol{x} = (e_n)_n$ one sees that $\boldsymbol{x} = (e_n)_n$ is $\mathscr{D}_1$-unconditionally summable but not unconditionally summable. Hence Orlicz-Pettis theorem does not hold for $\mathscr{B} = \mathscr{D}_1$.

On the other hand both $\mathcal{B}$-unconditional summability and $\mathscr{D}$-unconditional summability correspond to the weak unconditional summability. Then the classical Orlicz-Pettis theorem can be rewritten as:

$$\mathscr{D} - \mathrm{UC}(X) = \mathrm{UC}(X) \textit{ or } \mathcal{B} - \mathrm{UC}(X) = \mathrm{UC}(X) \textit{ for any Banach space } X.$$

The question that we would like to address is the validity of Orlicz-Pettis theorem for bilinear maps: Given $\mathscr{B} : X \times Y \to Z$ an admissible bounded bilinear map,

Under which conditions does one have $\mathscr{B} - \mathrm{UC}(X) = \mathrm{UC}(X)$*?*

The key point to understand the difference between $\mathscr{D}$ and $\mathscr{D}_1$ in the corresponding version of the Orlicz-Pettis theorem is the observation that X embeds not only into $\mathscr{L}(X^*, \mathbb{K})$ but actually into the weak$*$-norm continuous operators in $\mathscr{L}(X^*, \mathbb{K})$. So to present our main result we need then to consider the Banach space $\mathscr{W}^*(X^*, Y)$ consisting of all bounded linear maps from X^* into Y that are weak$*$-norm continuous. The reader may consult to (12) for information on this space.

We are now ready to state the main result of the paper.

Theorem 1 (Bilinear Orlicz-Pettis) *Let $\mathscr{B} : X \times Y^* \to Z$ be a bounded bilinear map such that*

(a) X is $\mathscr{B}$-normed,
(b) Y is w^*-sqcu, i.e., B_{Y^*} is weak$*$ sequentially compact,
(c) $\phi_{\mathscr{B}}(X) \subseteq \mathscr{W}^*(Y^*, Z)$.

Then every $\mathscr{B}$-unconditionally summable sequence in X is unconditionally summable.

The paper consists of two more sections: In the first one we introduce the spaces under consideration, present some particular bilinear maps and deal with the inclusions between $\ell^1_{\mathscr{B}}(X)$ and $\ell^1_{\rm weak}(X)$. In particular, it is shown that the inclusion $\ell^1_{\mathscr{B}}(X) \subseteq \ell^1_{\rm weak}(X)$ holds if and only if X is $\mathscr{B}$-normed and the inclusion $\ell^1_{\rm weak}(X) \subseteq \ell^1_{\mathscr{B}}(X)$ is described in terms of absolutely summing operators. The last section contains the proof of the bilinear version of Orlicz-Pettis theorem and provides some applications.

2 $\mathscr{B}$-summability of sequences.

2.1. Absolute summability with respect to the bilinear maps.

We start this section with the definitions of the spaces to be used throughout the paper.

Let X, Y and Z be Banach spaces and $\mathscr{B} : X \times Y \to Z$ be a bounded bilinear map. Denote

$$\phi_{\mathscr{B}} : X \to \mathscr{L}(Y, Z), \quad \text{given by} \quad \phi_{\mathscr{B}}(x) = \mathscr{B}(x, \cdot) = \mathscr{B}_x, \tag{4}$$

and

$$\psi_{\mathscr{B}} : Y \to \mathscr{L}(X, Z), \quad \text{given by} \quad \psi_{\mathscr{B}}(y) = \mathscr{B}(\cdot, y) = \mathscr{B}^y. \tag{5}$$

We also denote $\mathscr{B}^*$ the adjoint bilinear map

$$\mathscr{B}^* : X \times Z^* \to Y^*, \quad \text{given by} \quad \langle \mathscr{B}^*(x, z^*), y\rangle = \langle \mathscr{B}(x, y), z^*\rangle. \tag{6}$$

In other words $\mathscr{B}^*_x = (\mathscr{B}_x)^*$.

Definition 2 *Let $\mathscr{B} : X \times Y \to Z$ be a bounded bilinear map. We say that a vector sequence $\boldsymbol{x} = (x_n)_n \subseteq X$ is $\mathscr{B}$-summable if the Z-valued sequence $(\mathscr{B}(x_n, y))_n$ belongs to $\ell^1(Z)$ for all $y \in Y$. The set of these sequences will be denoted by $\ell^1_{\mathscr{B}}(X)$ and we write*

$$\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = \sup_{y \in \mathrm{B}_Y} \left\|(\mathscr{B}(x_n, y))_n\right\|_{\ell^1(Z)} = \sup_{y \in \mathrm{B}_Y} \sum_{n=1}^{\infty} \|\mathscr{B}(x_n, y)\|_Z. \tag{7}$$

Remark 3 *One might think on defining $\ell^1_{\mathscr{B},\rm weak}(X)$ as the vector space consisting of all sequences $\boldsymbol{x} = (x_n)_n \subseteq X$ verifying that the $(\mathscr{B}(x_n, y))_n \in \ell^1_{\rm weak}(Z)$ for all $y \in Y$. That is defined by the condition*

$$\sum_{n=1}^{\infty} |\langle \mathscr{B}(x_n, y), z^*\rangle| < \infty, \quad \textit{for all} \quad y \in Y, z^* \in Z^*.$$

However this notion is actually the same as above for a different bilinear map. Indeed, for any $\mathscr{B} : X \times Y \to Z$ we can define a bounded bilinear map

$$\widetilde{\mathscr{B}} : X \times (Y \widehat{\otimes}_\pi Z^*) \to \mathbb{K}, \quad \textit{given by} \quad \widetilde{\mathscr{B}}(x, y \otimes z^*) = \langle \mathscr{B}(x, y), z^*\rangle,$$

where $Y\widehat{\otimes}_\pi Z^*$ stands for the projective tensor norm. Clearly $\ell^1_{\mathscr{B},\mathrm{weak}}(X) = \ell^1_{\widetilde{\mathscr{B}}}(X)$.

Note that for $x_0^* \in X^* \setminus \{0\}$, we can define the bounded bilinear map

$$\begin{array}{rccc} \mathscr{B}\colon & X \times X & \to & \mathbb{K} \\ & (x^1, x^2) & \mapsto & x_0^*(x^1)x_0^*(x^2) \ . \end{array}$$

Taking $\boldsymbol{x} = (x_n)_n \subseteq \ker(x_0^*)$ then $\mathscr{B}(x_n, x) = 0$ for all $x \in X$ and $n \in \mathbb{N}$. Thus $\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = 0$ but $\boldsymbol{x} \neq 0$.

This difficulty leads us to restrict ourselves to the following class of bilinear maps.

Definition 4 *We say that a bounded bilinear map $\mathscr{B} : X \times Y \to Z$ is admissible for X if*

$$\mathscr{B}(x, y) = 0 \textit{ for all } y \in Y \textit{ implies that } x = 0.$$

Proposition 5 *Let $\mathscr{B} : X \times Y \to Z$ be a bounded bilinear map. The space $(\ell^1(X), \|\cdot\|_{\ell^1_{\mathscr{B}}(X)})$ is normed if and only if $\mathscr{B} : X \times Y \to Z$ is admissible*

Proof. Given $\boldsymbol{x} = (x_n)_n \in \ell^1_{\mathscr{B}}(X)$ we define the linear operator

$$\begin{array}{rccc} T_{\mathscr{B},\boldsymbol{x}}\colon & Y & \to & \ell^1(Z) \\ & y & \mapsto & (\mathscr{B}(x_n, y))_n \ . \end{array} \tag{8}$$

Observe first that this operator has closed graph. Let $(y_k, T_{\mathscr{B},\boldsymbol{x}}(y_k))_k$ be a convergent sequence in $Y \times \ell^1(Z)$ and write $(y, \boldsymbol{z})$ for its limit. The continuity of $\mathscr{B}$ provides that, for every $n \in \mathbb{N}$, the sequence $(\mathscr{B}(x_n, y_k))_k$ converges to $\mathscr{B}(x_n, y)$ in Z. Thus, for each $n \in \mathbb{N}$,

$$\begin{aligned} \|z_n - T_{\mathscr{B},\boldsymbol{x}}(y)_n\|_Z &\leq \|z_n - \mathscr{B}(x_n, y_k)\|_Z + \|\mathscr{B}(x_n, y_k) - T_{\mathscr{B},\boldsymbol{x}}(y)_n\|_Z \\ &\leq (\sum_{n=1}^{\infty} \|z_n - \mathscr{B}(x_n, y_k)\|_Z) + \|\mathscr{B}(x_n, y_k) - \mathscr{B}(x_n, y)\|_Z \\ &\leq \|T_{\mathscr{B},\boldsymbol{x}}(y_k) - \boldsymbol{z}\|_{\ell^1(Z)} + \|\mathscr{B}\| \cdot \|x_n\|_X \|y_k - y\|_Y . \end{aligned}$$

Taking limits when $k \to \infty$ we obtain that $T_{\mathscr{B},\boldsymbol{x}}$ has closed graph. Hence using Closed Graph theorem the operator defined in (8) is continuous and

$$\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = \sup_{y \in \mathrm{B}_Y} \sum_{n=1}^{\infty} \|\mathscr{B}(x_n, y)\|_Z$$

is finite.

Clearly the expression (7) verifies that $\|\boldsymbol{x}+\boldsymbol{y}\|_{\ell^1_{\mathscr{B}}(X)} \leq \|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} + \|\boldsymbol{y}\|_{\ell^1_{\mathscr{B}}(X)}$ and $\|\alpha\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = |\alpha| \cdot \|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}$ for every $\boldsymbol{x}, \boldsymbol{y} \in \ell^1_{\mathscr{B}}(X)$ and $\alpha \in \mathbb{K}$.

Observe that the condition $\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = 0$ *implies* $\boldsymbol{x} = 0$ is actually equivalent to $\phi_{\mathscr{B}}$ being injective, which corresponds to the notion of admissibility. ■

Let us now study the completeness of the spaces $\ell^1_{\mathscr{B}}(X)$. The following example shows that in general the space $\ell^1_{\mathscr{B}}(X)$ is not complete.

Proposition 6 *Let $T : X \to Z$ be a bounded linear map such that $T(X)$ is not a closed subspace of Z (for example the inclusion map defined in ℓ^1 into c_0). Define the bounded bilinear map*

$$\mathscr{I} : X \times \mathbb{K} \to Z, \quad \textit{given by} \quad \mathscr{I}(x, \alpha) = \alpha T(x). \tag{9}$$

Then $\ell^1_{\mathscr{I}}(X)$ is not complete.

Proof. Observe that $\boldsymbol{x} \in \ell^1_{\mathscr{I}}(X)$ means $\sum_{n=1}^{\infty} \|T(x_n)\|_Z < \infty$. Let us take a sequence $\boldsymbol{x^0} = (x^0_m)_m \subseteq X$ verifying that $(T(x^0_m))_m$ converges to $z \in Z \backslash T(X)$. Now define, for each $m \in \mathbb{N}$,

$$\boldsymbol{x^m} = \left(\frac{x^0_m}{2^n}\right)_n .$$

Using that for all $m, k \in \mathbb{N}$ we have that $\|\boldsymbol{x^m} - \boldsymbol{x^k}\|_{\ell^1_{\mathscr{I}}(X)} = \|T(x^0_m - x^0_k)\|_Z$ so $(\boldsymbol{x^m})_m$ is a Cauchy sequence in $\ell^1_{\mathscr{I}}(X)$. On the other hand for each $\boldsymbol{x}$ in $\ell^1_{\mathscr{I}}(X)$ then

$$\frac{1}{2}\|T(x^0_m) - T(2x_1)\|_Z = \|T(\frac{x^0_m}{2} - x_1)\|_Z \leq \sum_{n=1}^{\infty} \|T(\frac{x^0_m}{2^n} - x_n)\|_Z = \|\boldsymbol{x^m} - \boldsymbol{x}\|_{\ell^1_{\mathscr{I}}(X)}.$$

Hence if $\boldsymbol{x}^m$ converges to $\boldsymbol{x}$ in $\ell^1_{\mathscr{I}}(X)$ then $z = T(2x_1) \in T(X)$. Thus $\boldsymbol{x}^m$ does not converges in $\ell^1_{\mathscr{I}}(X)$. ■

Let us mention an elementary but useful fact.

Proposition 7 *Let $\mathscr{B} : X \times Y \to Z$ be a bounded bilinear map, Y_1, Z_1 Banach spaces and let $R : Z \to Z_1$, $S : Y_1 \to Y$ two bounded linear maps. Consider*

$$\mathscr{B}_{R,S} : X \times Y_1 \to Z_1, \quad \textit{given by} \quad \mathscr{B}_{R,S}(x, y_1) = R(\mathscr{B}(x, Sy_1)).$$

Then $\ell^1_{\mathscr{B}}(X)$ is continuously embedded in $\ell^1_{\mathscr{B}_{R,S}}(X)$ and

$$\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}_{R,S}}(X)} \leq \|R\| \cdot \|S\| \cdot \|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}, \ \textit{for all } \boldsymbol{x} \in \ell^1_{\mathscr{B}}(X).$$

Remark 8 *If $\mathscr{B}_1 : X \times Y_1 \to Z_1$ and $\mathscr{B}_2 : X \times Y_2 \to Z_2$ are bounded bilinear maps, we say that $\mathscr{B}_1 < \mathscr{B}_2$ if there exist $R : Z_2 \to Z_1$ and $S : Y_1 \to Y_2$ such that $\mathscr{B}_1(x, y_1) = R(\mathscr{B}_2(x, Sy_1))$, i.e. $(\mathscr{B}_2)_{R,S} = \mathscr{B}_1$.*

Proposition 7 says that $\mathscr{B}_1 < \mathscr{B}_2$ implies $\ell^1_{\mathscr{B}_2}(X) \subseteq \ell^1_{\mathscr{B}_1}(X)$ and there is $C > 0$ verifying that $\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}_1}(X)} \leq C\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}_2}(X)}$ for any $\boldsymbol{x} \in \ell^1_{\mathscr{B}_2}(X)$.

We present now some general admissible bounded bilinear maps naturally defined for any Banach space X and that generalize those given in (1), (2) and (3).

Example 9

(a) $\pi_Y : X \times Y \to X\widehat{\otimes}_\pi Y$ *given by*

$$\pi_Y(x, y) = x \otimes y, \tag{10}$$

where $X\hat{\otimes}_\pi Y$ is the projective tensor norm.
Note that $\pi_{\mathbb{K}} = \mathcal{B}$ given in (1). Clearly $\ell^1_{\pi_Y}(X) = \ell^1(X) \subseteq \ell^1_{\mathrm{weak}}(X)$.

(b) $\widetilde{\mathscr{O}}_Y : X \times \mathscr{L}(X, Y) \to Y$ *given by*

$$\widetilde{\mathscr{O}}_Y(x, T) = Tx. \tag{11}$$

In this case $\widetilde{\mathscr{O}}_{\mathbb{K}} = \mathscr{D}$ given in (2), $\pi_Y^* = \tilde{\mathscr{O}}_{Y^*}$ and

$$\ell^1_{\widetilde{\mathscr{O}}_Y}(X) = \{\ \boldsymbol{x} = (x_n)_n : \sup_{T \in \mathrm{B}_{\mathscr{L}(X,Y)}} \sum_{n=1}^{\infty} \|T(x_n)\| < \infty\}.$$

Note also that $\ell^1_{\widetilde{\mathscr{O}}_Y}(X) \subseteq \ell^1_{\mathrm{weak}}(X)$. Indeed, given $\boldsymbol{x} \in \ell^1_{\widetilde{\mathscr{O}}_Y}(X)$ and fixing $\|y\| = 1 = \|x^*\|$ we can consider the bounded linear map $y \otimes x^* : X \to Y$ given by $y \otimes x^*(x) = \langle x, x^* \rangle y$. Then

$$\begin{aligned}\|\boldsymbol{x}\|_{\ell^1_{\mathrm{weak}}(X)} &= \sup_{\substack{x^* \in \mathrm{B}_{X^*} \\ y \in \mathrm{B}_Y}} \Big\| \Big((y \otimes x^*)(x_n) \Big)_n \Big\|_{\ell^1(Y)} \\ &\leq \sup_{T \in \mathrm{B}_{\mathscr{L}(X,Y)}} \Big\| (T(x_n))_n \Big\|_{\ell^1(Y)} = \|\boldsymbol{x}\|_{\ell^1_{\tilde{\mathscr{O}}_Y}(X)}.\end{aligned}$$

(c) For spaces of operators, we consider $\mathscr{O} : \mathscr{L}(X, Y) \times X \to Y$ given by

$$\mathscr{O}(T, x) = Tx. \tag{12}$$

Now if $Y = \mathbb{K}$ then $\mathscr{O} = \mathscr{D}_1$ given in (3) and

$$\ell^1_{\mathscr{O}}(\mathscr{L}(X, Y)) = \{\ \boldsymbol{T} = (T_n)_n \subseteq \mathscr{L}(X, Y) : \sup_{x \in \mathrm{B}_X} \sum_{n=1}^{\infty} \|T_n x\| < \infty\}.$$

This space was studied in (4) *where it was denoted by $\ell^1_s(X, Y)$ and shown to satisfy*

$$\ell^1(\mathscr{L}(X, Y)) \subsetneq \ell^1_{\mathscr{O}}(\mathscr{L}(X, Y)) \subsetneq \ell^1_{\mathrm{weak}}(\mathscr{L}(X, Y)).$$

Using vector-valued continuous functions, we can consider other natural admissible bilinear maps.

Example 10

(a) $\mathcal{C} : X \times \mathrm{C}([0,1], X^*) \to \mathrm{C}[0,1]$ *given by*

$$\mathcal{C}(x, \boldsymbol{f}) = \langle x, \boldsymbol{f} \rangle. \tag{13}$$

(b) Let (Ω, Σ, μ) *be a positive measure space and* $1 \le q < \infty$*, we can consider* $\mathcal{V}_q : X \times L^q(\mu, X^*) \to L^q(\mu)$ *given by*

$$\mathcal{V}_q(x, \boldsymbol{f}) = \langle x, \boldsymbol{f} \rangle. \tag{14}$$

For more concrete Banach spaces, there are also natural admissible bilinear maps

Example 11

(a) Let $X(\mu)$ *be a function space of measurable functions in a* σ*-finite measure space, and* $X(\mu)'$ *its associate space. Let us define* $\mathcal{A} : X(\mu) \times X(\mu)' \to L^1(\mu)$ *given by*

$$\mathcal{A}(f, g) = fg. \tag{15}$$

(b) Let (Ω, Σ, μ) *be a* σ*-finite measure space,* $1 \le p \le \infty$ *and* $X = L^p(\mu)$*. For* $1 \le q \le \infty$ *and* $1/p + 1/q = 1/r$*, we can consider* $\mathcal{H}_q : L^p(\mu) \times L^q(\mu) \to L^r(\mu)$ *given by*

$$\mathcal{H}_q(f, g) = fg. \tag{16}$$

(c) Let $1 \le p \le \infty$ *and* $X = L^p(\mathbb{R}^n)$*. For* $1 \le q \le \infty$ *and* $1/p + 1/q = 1/r - 1$*, we can consider* $\mathcal{Y}_q : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ *given by*

$$\mathcal{Y}_q(f, g) = f * g. \tag{17}$$

Finally also mention in the case X is a Banach space vector-valued functions, the natural admissible bilinear maps:

Example 12 *Let* μ *be a finite measure space,* $1 \le p \le \infty$ *and* $X = L^p(\mu, Y)$*.*

(a) $\mathcal{W}_p : L^p(\mu, Y) \times Y^* \to L^p(\mu)$ *given by*

$$\mathcal{W}_p(\boldsymbol{f}, y^*) = \langle \boldsymbol{f}, y^* \rangle. \tag{18}$$

(b) $\widetilde{\mathcal{W}}_p : L^p(\mu, Y) \times L^{p'}(\mu) \to L^1(\mu, Y)$ *given by*

$$\widetilde{\mathcal{W}}_p(\boldsymbol{f}, \phi) = \boldsymbol{f}\phi. \neq \tag{19}$$

It is well-known —see for instance (11)— that the space $\ell^1_{\text{weak}}(X)$ can be identified with $\mathscr{L}(c_0, X)$ or $\mathscr{L}(X^*, \ell^1)$. Let us investigate the analogues for $\ell^1_{\mathscr{B}}(X)$.

Proposition 13 *Let $\mathscr{B} : X \times Y \to Z$ be an admissible bounded bilinear map.*

(a) $\ell^1_{\mathscr{B}}(X)$ is isometrically isomorphic to a subspace of $\mathscr{L}(Y, \ell^1(Z))$.
(b) $\ell^1_{\mathscr{B}}(X)$ is isometrically isomorphic to a subspace of $\ell^1_s(Y, Z)$.
(c) $\ell^1_{\mathscr{B}}(X)$ is isometrically isomorphic to a subspace of $\mathscr{L}(c_0(Z^*), Y^*)$.
(d) $\ell^1_{\mathscr{B}}(X)$ is isometrically isomorphic to a subspace of $\mathscr{L}(\ell^\infty(Z^*), \ell^1_{\text{weak}^*}(Y^*))$.

Proof. *(a)* follows using the embedding $\boldsymbol{x} \to T_{\mathscr{B},\boldsymbol{x}}$ given in (8).
(b) Since $X \subseteq \mathscr{L}(Y, Z)$ using $x \to \mathscr{B}_x$ we can embed $\ell^1_{\mathscr{B}}(X)$ in $\ell^1_{\mathscr{O}}(\mathscr{L}(Y, Z)) = \ell^1_s(Y, Z)$ as follows. Given the linear operator $\phi_{\mathscr{B}}$ defined in (4) the correspondence

$$\widetilde{\phi}_{\mathscr{B}}((x_n)_n) = (\phi_{\mathscr{B}}(x_n))_n$$

induces a linear and continuous operator from $\ell^1_{\mathscr{B}}(X)$ into $\ell^1_{\mathscr{O}}(\mathscr{L}(Y, Z))$. Moreover, $\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = \|\widetilde{\phi}_{\mathscr{B}}(\boldsymbol{x})\|_{\ell^1_{\mathscr{O}}(\mathscr{L}(Y,Z))}$ for any $\boldsymbol{x} \in \ell^1_{\mathscr{B}}(X)$.
(c) Note that, given $\boldsymbol{x} = (x_n)_n$ and $y \in Y$, $N, M \in \mathbb{N}$, one has

$$\begin{aligned}
\sum_{n=N}^{M} \|\mathscr{B}(x_n, y)\| &= \sup_{z_n^* \in \mathrm{B}_{Z^*}} \sum_{n=N}^{M} |\langle \mathscr{B}(x_n, y), z_n^* \rangle| \\
&= \sup_{\substack{z_n^* \in \mathrm{B}_{Z^*} \\ \varepsilon_n \in \mathrm{B}_{\mathbb{K}}}} |\langle \sum_{n=N}^{M} \mathscr{B}^*(x_n, \varepsilon_n z_n^*), y \rangle| \\
&= \sup_{z_n^* \in \mathrm{B}_{Z^*}} |\langle \sum_{n=N}^{M} \mathscr{B}^*(x_n, z_n^*), y \rangle|.
\end{aligned}$$

This shows that

$$\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = \sup_{\substack{z_n^* \in \mathrm{B}_{Z^*} \\ N \in \mathbb{N}}} \|\sum_{n=1}^{N} \mathscr{B}^*(x_n, z_n^*)\|_{Y^*}.$$

This allows to show that if $\boldsymbol{x} = (x_n)_n \in \ell^1_{\mathscr{B}}(X)$ and $(z_n^*)_n \in c_0(Z^*)$ then $\sum_{n=1}^\infty \mathscr{B}^*(x_n, z_n^*) \in Y^*$ and the map $\boldsymbol{x} = (x_n)_n \to \Phi_{\boldsymbol{x}}$ where $\Phi_{\boldsymbol{x}} : c_0(Z^*) \to Y^*$ is given by

$$\Phi_{\boldsymbol{x}}((z_n^*)_n) = \sum_{n=1}^{\infty} \mathscr{B}^*(x_n, z_n^*) \in Y^*$$

defines an isometric embedding from $\ell^1_{\mathscr{B}}(X)$ into $\mathscr{L}(c_0(Z^*), Y^*)$.
(d) Given $\boldsymbol{x} = (x_n)_n \in \ell^1_{\mathscr{B}}(X)$ let us consider the linear map

$$\widetilde{\Phi}_{\boldsymbol{x}} : \ell^\infty(Z^*) \to \ell^1_{\text{weak}^*}(Y^*), \quad \text{given by} \quad \widetilde{\Phi}_{\boldsymbol{x}}(\boldsymbol{z^*}) = (\mathscr{B}^*(x_n, z_n^*))_n.$$

Using the duality $\ell^1(Z)^* = \ell^\infty(Z^*)$ we obtain that

$$\begin{aligned}
\|\widetilde{\Phi}_{\boldsymbol{x}}\| &= \sup_{\substack{\boldsymbol{z^*}\in \mathrm{B}_{\ell^\infty(Z^*)}\\ y\in \mathrm{B}_Y}} \sum_{n=1}^{\infty} |\langle y, \mathscr{B}^*(x_n, z_n^*)\rangle| = \sup_{\substack{\boldsymbol{z^*}\in \mathrm{B}_{\ell^\infty(Z^*)}\\ y\in \mathrm{B}_Y}} \sum_{n=1}^{\infty} |\langle \mathscr{B}(x_n, y), z_n^*\rangle| \\
&= \sup_{\substack{\boldsymbol{z^*}\in \mathrm{B}_{\ell^\infty(Z^*)}\\ y\in \mathrm{B}_Y, |\varepsilon_n|=1}} |\sum_{n=1}^{\infty} \langle \mathscr{B}(x_n, y), \varepsilon_n z_n^*\rangle| = \sup_{y\in \mathrm{B}_Y} \sum_{n=1}^{\infty} \|\mathscr{B}(x_n, y)\|_Z \\
&= \|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}.
\end{aligned}$$

■

2.2. Relations between the spaces $\ell^1_{\mathscr{B}}(X)$ and $\ell^1_{\text{weak}}(X)$.

It is trivial that $\ell^1(X) \subseteq \ell^1_{\mathscr{B}}(X)$ for any bounded bilinear map $\mathscr{B} : X \times Y \to Z$, and

$$\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} \leq \|\mathscr{B}\| \cdot \|\boldsymbol{x}\|_{\ell^1(X)}, \qquad \text{for all } \boldsymbol{x} \in \ell^1(X). \tag{20}$$

Clearly the containment can be strict. For example if we take the bounded bilinear map $\mathscr{D}$ defined in (2) then $\ell^1(X) \subsetneq \ell^1_{\text{weak}}(X) = \ell^1_{\mathscr{D}}(X)$. Thus, in the general case $\ell^1(X) \subsetneq \ell^1_{\mathscr{B}}(X)$. A natural question now is,

What is the relation between $\ell^1_{\mathscr{B}}(X)$ and $\ell^1_{\text{weak}}(X)$?

For instance, the bounded bilinear maps $\mathcal{B}$, π_Y, $\mathscr{D}$, $\mathscr{D}_1$, $\tilde{\mathscr{O}}_Y$ and $\mathscr{O}$ are examples of bilinear maps verifying that the inclusion operator $\mathsf{i} : \ell^1_{\mathscr{B}}(X) \to \ell^1_{\text{weak}}(X)$ are continuous.

For $\mathscr{B} = \mathcal{C}$ or $\mathscr{B} = \mathcal{V}_q$ in Example 10 it suffices to select $f(t) = x^*\mathbf{1}_{[0,1]}$ or $f = x^*\mathbf{1}_\Omega$ for each $x^* \in X^*$ to show the continuous inclusion $\ell^1_{\mathscr{B}}(X) \subseteq \ell^1_{\text{weak}}(X)$.

Also for $\mathscr{B} = \mathcal{A}$ or $\mathscr{B} = \mathcal{H}_q$ in Example 11 it easily follows from Proposition 7 to see $\ell^1_{\mathscr{B}}(L^p(\mu)) \subseteq \ell^1_{\text{weak}}(L^p(\mu))$.

To produce examples where $\ell^1_{\text{weak}}(X) \subseteq \ell^1_{\mathscr{B}}(X)$ it suffices to work with the case $Z = \mathbb{K}$. Indeed, take $T \in \mathscr{L}(Y, X^*)$ and define $\mathscr{B}_T(x, y) = \langle x, Ty\rangle$. Clearly $\boldsymbol{x} = (x_n)_n \in \ell^1_{\mathscr{B}_T}(X)$ if

$$\sup_{y\in \mathrm{B}_Y} \sum_{n=1}^{\infty} |\langle x_n, Ty\rangle| < \infty,$$

and $\boldsymbol{x} = (x_n)_n \in \ell^1_{\text{weak}}(X)$ if

$$\sup_{x^*\in \mathrm{B}_{X^*}} \sum_{n=1}^{\infty} |\langle x_n, x^*\rangle| < \infty.$$

Hence $\ell^1_{\text{weak}}(X) \subseteq \ell^1_{\mathscr{B}_T}(X)$ and $\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}_T}(X)} \leq \|T\| \cdot \|\boldsymbol{x}\|_{\ell^1_{\text{weak}}(X)}$. Moreover $\ell^1_{\text{weak}}(X) = \ell^1_{\mathscr{B}_T}(X)$ is equivalent to $\|T^*x\| \approx \|x\|$ for any $x \in X$.

Example 14 *Consider the bounded bilinear map*

$$\mathscr{A}_2 : \ell^2 \times \ell^2 \to \mathbb{R}, \quad \textit{given by} \quad \mathscr{A}_2(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \beta_n. \tag{21}$$

Note that this corresponds to $\mathscr{B}_T$ *for* $T : \ell^2 \to \ell^2$ *given by* $T(\alpha) = (\frac{1}{n}\alpha_n)_n$.

Let $\boldsymbol{x} = (e_k)_k$ *where* e_k *is the canonical basis. It is clear that* $\boldsymbol{x} \in \ell^1_{\mathscr{A}_2}(\ell^2) \setminus \ell^1_{\text{weak}}(\ell^2)$.

Hence, in general $\ell^1_{\mathscr{B}}(X)$ is not continuously embedded into $\ell^1_{\text{weak}}(X)$. However we always have that $\ell^1_{\mathscr{B}}(X) \subseteq \ell^1_{\text{weak}}(\mathscr{L}(Y, Z))$. Indeed using that $\ell^1_{\text{weak}}(\mathscr{L}(Y, Z)) = \mathscr{L}(c_0, \mathscr{L}(Y, Z))$, one has that for each $\boldsymbol{x} \in \ell^1_{\mathscr{B}}(X)$ we can consider the linear map $S_{\mathscr{B},\boldsymbol{x}} : c_0 \to \mathscr{L}(Y, Z)$, given by $S_{\mathscr{B},\boldsymbol{x}}(\boldsymbol{\alpha})(y) = \sum_{n=1}^{\infty} \mathscr{B}(x_n, y)\alpha_n$. Duality gives

$$\|S_{\mathscr{B},\boldsymbol{x}}\| \le \sup_{\substack{y \in \mathrm{B}_Y \\ \boldsymbol{\alpha} \in \mathrm{B}_{c_0}}} \sum_{n=1}^{\infty} \|\mathscr{B}(x_n, y)\|_Z |\alpha_n| \le \|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}.$$

Remark 15 *Observe that Proposition 7 gives another general inclusion, is that* $\ell^1_{\mathscr{B}}(X) \subseteq \ell^1_{\mathscr{B}_{z^*}}(X)$ *for every* $z^* \in Z^*$ *where*

$$\mathscr{B}_{z^*} : X \times Y \to \mathbb{K}, \quad \textit{given by} \quad \mathscr{B}_{z^*}(x, y) = \langle \mathscr{B}(x, y), z^* \rangle.$$

Moreover $\sup\{\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}_{z^*}}(X)} : z^* \in \mathrm{B}_{Z^*}\} \le \|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}$ *for each* $\boldsymbol{x} \in \ell^1_{\mathscr{B}}(X)$.

So the natural question now is,

When is $\ell^1_{\mathscr{B}}(X)$ *continuously included into* $\ell^1_{\text{weak}}(X)$*?*

The answer of this question relies upon the notion of $(Y, Z, \mathscr{B})$-normed space X.

Definition 16 (see (1; 2)) *Let* $\mathscr{B} : X \times Y \to Z$ *be a bounded bilinear map. We say that a Banach space* X *is* $(Y, Z, \mathscr{B})$*-normed —or simply* $\mathscr{B}$*-normed space— if there exists a constant* $k > 0$ *such that*

$$\|x\|_X \le k \|\mathscr{B}_x\|_{\mathscr{L}(Y,Z)}, \quad \textit{for all} \quad x \in X. \tag{22}$$

The following result characterizes when a Banach space is $\mathscr{B}$-normed.

Theorem 17 *Let* $\mathscr{B} : X \times Y \to Z$ *be a bounded bilinear map admissible for* X*. The following assertions are equivalent:*

- *(a) The inclusion* $\mathsf{i} : \ell^1_{\mathscr{B}}(X) \to \ell^1_{\text{weak}}(X)$ *is continuous.*
- *(b)* X *is* $(Y, Z, \mathscr{B})$*-normed.*

(c) *There exists a constant $k > 0$ such that for each $x^* \in X^*$ there exists a functional $\varphi_{x^*} \in \mathscr{L}(Y,Z)^*$ verifying $\|\varphi_{x^*}\| \le k\|x^*\|$ and*

$$\langle x, x^*\rangle = \varphi_{x^*}(\mathscr{B}_x), \qquad \textit{for all } x \in X.$$

PROOF. *(a)⇒(b)* Fix $x \in X$ and consider the sequence $\boldsymbol{x} = (x, 0, 0, \ldots)$. Apply the assumption to $\boldsymbol{x}$ to obtain $\|\boldsymbol{x}\|_{\ell^1_{\text{weak}}(X)} = \|x\|_X$ and $\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)} = \|\mathscr{B}_x\|_{\mathscr{L}(Y,Z)}$.
(b)⇒(c) Let us assume that X is $(Y, Z, \mathscr{B})$-normed and we denote by $\hat{X} = \{\mathscr{B}_x : x \in X\} \subseteq \mathscr{L}(Y,Z)$. According to the assumption $\hat{X}$ is a closed subspace of $\mathscr{L}(Y,Z)$. Given $x^* \in X^*$ one has that

$$|\langle x^*, x\rangle| \le k\|x^*\| \cdot \|\mathscr{B}_x\|, \quad \text{for all } x \in X.$$

Hence the map $\hat{x}^* : \mathscr{B}_x \to \langle x, x^*\rangle$ is bounded and linear in $\hat{X}^*$ with $\|\hat{x}^*\| \le k\|x^*\|$. Therefore, by Hahn-Banach theorem, there is an extension φ_{x^*} to $\mathscr{L}(Y,Z)^*$ such that $\|\varphi_{x^*}\| \le k\|x^*\|$ where $k > 0$ is the constant in (22).
(c)⇒(a) For each $\boldsymbol{x} \in \ell^1_{\mathscr{B}}(X)$ using that $\ell^1(N)^* = \ell^\infty(N)$ for all $N \in \mathbb{N}$ we have that

$$\begin{aligned}
\|\boldsymbol{x}\|_{\ell^1_{\text{weak}}(X)} &= \sup_{\substack{x^* \in \mathrm{B}_{X^*} \\ N \in \mathbb{N}}} \sum_{n=1}^N |\langle x_n, x^*\rangle| = \sup_{\substack{\boldsymbol{\alpha} \in \mathrm{B}_{\ell^\infty(N)} \\ x^* \in \mathrm{B}_{X^*}, N \in \mathbb{N}}} \left|\sum_{n=1}^N \langle x_n, x^*\rangle \alpha_n\right| \\
&= \sup_{\substack{\boldsymbol{\alpha} \in \mathrm{B}_{\ell^\infty(N)} \\ x^* \in \mathrm{B}_{X^*}, N \in \mathbb{N}}} \left|\varphi_{x^*}(\mathscr{B}_{\sum_{n=1}^N \alpha_n x_n})\right| \\
&\le k \sup_{\substack{\boldsymbol{\alpha} \in \mathrm{B}_{\ell^\infty(N)} \\ y \in \mathrm{B}_Y, N \in \mathbb{N}}} \|\mathscr{B}_{\sum_{n=1}^N \alpha_n x_n}(y)\|_Z \\
&\le k \sup_{\substack{\boldsymbol{\alpha} \in \mathrm{B}_{\ell^\infty(N)} \\ y \in \mathrm{B}_Y, N \in \mathbb{N}}} \sum_{n=1}^N \|\mathscr{B}(x_n, y)\|_Z |\alpha_n| \\
&\le k \sup_{\substack{y \in \mathrm{B}_Y \\ N \in \mathbb{N}}} \sum_{n=1}^N \|\mathscr{B}(x_n, y)\|_Z = k\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}.
\end{aligned}$$

■

Remark 18 *Of course X is a $(Y, Z, \mathscr{B})$-normed means that the norms $\|\cdot\|_X$ and $\|\mathscr{B}.\|_{\mathscr{L}(Y,Z)}$ are equivalent and therefore $\mathscr{B} : X \times Y \to Z$ is admissible for X (see Definition 4) and consequently $\|\cdot\|_{\ell^1_{\mathscr{B}}(X)}$ defines a norm in the space $\ell^1_{\mathscr{B}}(X)$.*

Observe that X is a $(Y, Z, \mathscr{B})$-normed space if the bounded linear map

$$\phi^t_{\mathscr{B}} : \mathscr{L}(Y,Z)^* \to X^*, \tag{23}$$

is surjective.

Corollary 19 *Let $\mathscr{B} : X \times Y \to \mathbb{K}$ be a scalar bounded bilinear map. The following are equivalent:*

(a) *X is $(Y, \mathbb{K}, \mathscr{B})$-normed.*
(b) *There is a constant $C_1 > 0$ such that $C_1\|x\| \leq \|\phi_{\mathscr{B}}(x)\| \leq \|\mathscr{B}\| \cdot \|x\|$ for all $x \in X$.*
(c) *$\ell^1_{\text{weak}}(X) = \ell^1_{\mathscr{B}}(X)$ and the norms $\|\cdot\|_{\ell^1_{\text{weak}}(X)}$ and $\|\cdot\|_{\ell^1_{\mathscr{B}}(X)}$ are equivalent.*

Let us point out that in many cases it is possible to give a explicit definition for the functional φ_{x^*} appearing in Theorem 17.*(c)* when a Banach space X is $(Y, Z, \mathscr{B})$-normed.

Example 20

(a) *For the bounded bilinear map $\mathcal{B}$ and $x^* \in X^*$ take*

$$\varphi_{x^*} : \mathscr{L}(\mathbb{K}, X) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(T) = \langle T(1), x^* \rangle.$$

(b) *For the bounded bilinear map π_Y and $x^* \in X^*$, select $y_0 \in Y$ and $y_0^* \in Y^*$ verifying that $\langle y_0, y_0^* \rangle = 1$ and take*

$$\varphi_{x^*} : \mathscr{L}(Y, X \hat{\otimes}_\pi Y) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(T) = \sum_n \langle x_n, x^* \rangle \langle y_n, y_0^* \rangle$$

where $T(y_0) = \sum_n x_n \otimes y_n$.

(c) *For the bilinear map $\mathscr{D}$ just take for every $x^* \in X^*$*

$$\varphi_{x^*} : \mathscr{L}(X^*, \mathbb{K}) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(x^{**}) = \langle x^*, x^{**} \rangle.$$

(d) *For the bilinear map $\mathscr{D}_1$ just take for every $x^{**} \in X^{**}$*

$$\varphi_{x^{**}} : \mathscr{L}(X, \mathbb{K}) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^{**}}(x^*) = \langle x^*, x^{**} \rangle.$$

(e) *For the bilinear map $\mathscr{O}$ and $x^* \in (\mathscr{L}(X, Y))^*$,*

$$\varphi_{x^*} : \mathscr{L}(X, Y) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(T) = \langle T, x^* \rangle.$$

(f) *For $\widetilde{\mathscr{O}}_Y$ for each $x^* \in X^*$, select $y_0 \in Y$ and $y_0^* \in Y^*$ such that $\langle y_0, y_0^* \rangle = 1$ and take the functional*

$$\varphi_{x^*} : \mathscr{L}(\mathscr{L}(X, Y), Y) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(T) = \langle T(y_0 \otimes x^*), y_0^* \rangle.$$

(g) *For the bilinear map $\mathcal{C}$ and $x^* \in X^*$, let us fix $t_0 \in [0, 1]$ and $f_0 \in \mathrm{C}[0, 1]$ verifying $f_0(t_0) = 1$ we take*

$$\varphi_{x^*} : \mathscr{L}(\mathrm{C}([0, 1], X^*), \mathrm{C}[0, 1]) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(T) = T(f_0 \otimes x^*)(t_0).$$

(h) *For the bilinear map $\mathcal{V}_q$ and $x^* \in X^*$, let us consider $E \in \Sigma$ such that $\mu(E) > 0$ and take*

$$\varphi_{x^*} : \mathscr{L}(L^q(\mu, X^*), L^q(\mu)) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(T) = \int_\Omega T(\frac{x^* \mathbf{1}_E}{\mu(E)}) d\mu.$$

(i) *For the bilinear map $\mathcal{A}$ in the case $X(\mu)' = X(\mu)^*$ and $x^* \in X^*$, let us consider*

$$\varphi_{x^*} : \mathscr{L}(X(\mu)', L^1(\mu)) \to \mathbb{K}, \quad \textit{given by} \quad \varphi_{x^*}(T) = \int_\Omega T(x^*) d\mu.$$

Theorem 21 *Let $\mathscr{B} : X \times Y \to Z$ be a bounded bilinear map such that X is $(Y, Z, \mathscr{B})$-normed. Then $(\ell^1_{\mathscr{B}}(X), \|\cdot\|_{\ell^1_{\mathscr{B}}(X)})$ is a Banach space.*

Proof. Let $(\boldsymbol{x^m})_m$ be a Cauchy sequence in $\ell^1_{\mathscr{B}}(X)$ and let us fix $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that for each $m, k \geq k_0$ we have that $\|\boldsymbol{x^m} - \boldsymbol{x^k}\|_{\ell^1_{\mathscr{B}}(X)} \leq \varepsilon$. In particular for every $y \in \mathrm{B}_Y$ and $m, k \geq k_0$ we have that

$$\sum_{n=1}^{\infty} \|\mathscr{B}(x_n^m - x_n^k, y)\|_Z \leq \varepsilon. \tag{24}$$

This implies that $\|\mathscr{B}_{x_n^m - x_n^k}\|_{\mathscr{L}(Y,Z)} \leq \varepsilon$ for all $n \in \mathbb{N}$ and $m, k \geq k_0$. Hence using that X is a $(Y, Z, \mathscr{B})$-normed space we conclude that $\|x_n^m - x_n^k\|_X \leq c\varepsilon$ for some $c > 0$ and all $n \in \mathbb{N}$ and $m, k \geq k_0$. This means that for all $n \in \mathbb{N}$ the sequence $(x_n^k)_k$ is a Cauchy sequence in the Banach space X. Then $(x_n^k)_k$ converges in X to a certain element —say x_n—. Consider then the sequence $\boldsymbol{x} = (x_n)_n$. Taking limits when $m \to \infty$ in expression (24) we have that for all $y \in \mathrm{B}_Y$ and $k \geq k_0$

$$\sum_{n=1}^{\infty} \|\mathscr{B}(x_n - x_n^k, y)\|_Z \leq \varepsilon.$$

This means that $\boldsymbol{x} - \boldsymbol{x^k} \in \ell^1_{\mathscr{B}}(X)$ and thus $\boldsymbol{x} = (\boldsymbol{x} - \boldsymbol{x^k}) + \boldsymbol{x^k} \in \ell^1_{\mathscr{B}}(X)$. In addition we have that the sequence $(\boldsymbol{x^k})_k$ converges to $\boldsymbol{x}$ in $\ell^1_{\mathscr{B}}(X)$. ■

Let us now analyze the converse question:

When does $\ell^1_{\mathrm{weak}}(X)$ is continuously embedded into $\ell^1_{\mathscr{B}}(X)$?

Recall that a linear map $T : X \to Y$ is called *absolutely summing* if there is a constant $k > 0$ verifying that for every finite family $x_1, \ldots, x_n \in X$ we have that

$$\sum_{i=1}^{n} \|T(x_i)\|_X \leq k \sup_{x^* \in \mathrm{B}_{X^*}} \sum_{i=1}^{n} |\langle x_i, x^* \rangle|. \tag{25}$$

The vector space of those bounded linear maps is denoted by $\Pi(X, Y)$ or $\Pi_1(X, Y)$. Endowed with the norm

$$\pi(T) = \inf\{k > 0 : \text{ the inequality (25) holds }\},$$

the space $\Pi_1(X,Y)$ is a Banach subspace of $\mathscr{L}(X,Y)$. We recall here that any bounded linear operator from $L^1(\mu)$ into a Hilbert space H is absolutely summing by Grothendieck's theorem —see (11, p. 15)—, i.e.

$$\mathscr{L}(L^1(\mu),H)=\Pi_1(L^1(\mu),H). \tag{26}$$

A Banach space X is called a *GT-space* if $\mathscr{L}(X,H)=\Pi_1(X,H)$ for any Hilbert space H. The reader is referred to (11) for information about the class of absolutely summing operators and properties related to them.

Proposition 22 *Let $\mathscr{B}:X\times Y\to Z$ be an admissible bounded bilinear map. The following are equivalent:*

(a) $\ell^1_{\mathrm{weak}}(X)$ is continuously embedded into $\ell^1_{\mathscr{B}}(X)$.
(b) $\psi_{\mathscr{B}}(Y)\subseteq\Pi_1(X,Z)$ and there exists $C>0$ such that $\pi(\mathscr{B}_y)\le C\|y\|$ for all $y\in Y$.

Proof. *(a) $\Rightarrow$ (b)* For each $y\in Y$ and $\boldsymbol{x}=(x_n)_n\in\ell^1_{\mathrm{weak}}(X)$ we have that for each $y\in Y$

$$\sum_{n=1}^{\infty}\|\mathscr{B}^y(x_n)\|_Z\le\|y\|\cdot\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}\le C\|y\|\cdot\|\boldsymbol{x}\|_{\ell^1_{\mathrm{weak}}(X)}.$$

This shows that $\mathscr{B}^y$ belongs to $\Pi_1(X,Z)$ and $\pi(\mathscr{B}^y)\le C\|y\|$ for every $y\in Y$.
(b) $\Rightarrow$ (a) Given $\boldsymbol{x}=(x_n)_n\in\ell^1_{\mathscr{B}}(X)$

$$\|\boldsymbol{x}\|_{\ell^1_{\mathscr{B}}(X)}=\sup_{y\in\mathrm{B}_Y}\sum_{n=1}^{\infty}\|\mathscr{B}^y(x_n)\|_Z\le\sup_{y\in\mathrm{B}_Y}\pi(\mathscr{B}^y)\|\boldsymbol{x}\|_{\ell^1_{\mathrm{weak}}(X)}\le C\|\boldsymbol{x}\|_{\ell^1_{\mathrm{weak}}(X)}.$$

■

Corollary 23 $\ell^1_{\widetilde{\mathscr{O}}_Y}(X)=\ell^1_{\mathrm{weak}}(X)\Longleftrightarrow\mathscr{L}(X,Y)=\Pi_1(X,Y)$.

In particular, $\ell^1_{\widetilde{\mathscr{O}}_{\ell^2}}(X)=\ell^1_{\mathrm{weak}}(X)$ if and only if X is a GT-space.

From the previous results we observe that not only for $Z=\mathbb{K}$ (or even finite dimensional spaces Z) one can obtain that $\ell^1_{\mathrm{weak}}(X)\subsetneq\ell^1_{\mathscr{B}}(X)$ whenever X is not $\mathscr{B}$-normed, but it is possible to have $\ell^1_{\mathrm{weak}}(X)\subsetneq\ell^1_{\mathscr{B}}(X)$ for infinite dimensional spaces Z.

Corollary 24 *Let $X=L^1(\mathbb{R})$ and consider the bilinear map*

$$\mathcal{Y}_2:L^1(\mathbb{R})\times L^2(\mathbb{R})\to L^2(\mathbb{R}),\quad \textit{given by}\quad \mathcal{Y}_2(f,g)=f*g. \tag{27}$$

Then $\ell^1_{\mathrm{weak}}(L^1(\mathbb{R}))\subsetneq\ell^1_{\mathcal{Y}_2}(L^1(\mathbb{R}))$.

PROOF. Let us see that $L^1(\mathbb{R})$ is not $(L^2(\mathbb{R}), L^2(\mathbb{R}), \mathcal{Y}_2)$-normed. Assume the contrary, i.e. there exists $k > 0$ with

$$\|f\|_{L^1(\mathbb{R})} \leq k \sup_{g \in \mathrm{B}_{L^2(\mathbb{R})}} \|f * g\|_{L^2(\mathbb{R})}.$$

Then

$$\|f\|_{L^1(\mathbb{R})} \leq k \sup_{h \in \mathrm{B}_{L^2(\mathbb{R})}} \|\hat{f} h\|_{L^2(\mathbb{R})} = k \|\hat{f}\|_{L^\infty(\mathbb{R})},$$

which is clearly false in general.

Therefore combining Theorem 17, Proposition 22 and (26) one concludes $\ell^1_{\text{weak}}(L^1(\mathbb{R})) \subsetneq \ell^1_{\mathcal{Y}_2}(L^1(\mathbb{R}))$. ■

Example 25 *Let Y be an infinite dimensional , $X = L^1([0,1], Y)$ and consider the bilinear map*

$$\mathcal{W}_1 : L^1([0,1], Y) \times Y^* \to L^1([0,1]), \quad \textit{given by} \quad \mathcal{W}_1(f, y^*) = \langle f, y^* \rangle. \tag{28}$$

Then $\ell^1_{\text{weak}}(L^1([0,1], Y)) \subsetneq \ell^1_{\mathcal{W}_1}(L^1([0,1], Y))$.

PROOF. In order to see the inclusion just observe that —since $(L^1([0,1], Y))^* = L(L^1([0,1]), Y^*)$— given $y^* \in Y^*$ then the function $y^* \mathbf{1}_\Omega \in L(L^1([0,1], Y^*)$ defines an element of $(L^1([0,1], Y))^*$.
Let us see now that $L^1([0,1], Y)$ is not $(Y^*, L^1([0,1]), \mathcal{W}_1)$-normed. Assume the contrary, i.e. there exists $k > 0$ with

$$\|f\|_{L^1([0,1],Y)} \leq k \sup_{y^* \in \mathrm{B}_{Y^*}} \int_0^1 \langle f(t), y^* \rangle dt.$$

Now it suffices to take $(y_n)_n \in \ell^1_{\text{weak}}(Y) \setminus \ell^1(Y)$ and define

$$f = \sum_{n=1}^\infty 2^{n+1} y_n \mathbf{1}_{[2^{-(n+1)}, 2^{-n}[}$$

to get a contradiction. Hence Theorem 17 gives that the inclusion is strict. ■

3 The proof of the theorem and consequences.

Definition 26 *We say that a sequence $\boldsymbol{x} = (x_n)_n \subseteq X$ is $\mathscr{B}$-unconditionally summable if for all $y \in Y$ and $z^* \in Z^*$ we have that $(\langle \mathscr{B}(x_n, y), z^* \rangle)_n \in \ell^1$ and for all $M \subseteq \mathbb{N}$ there is $x_M \in X$ such that*

$$\sum_{n \in M} \langle \mathscr{B}(x_n, y), z^* \rangle = \langle \mathscr{B}(x_M, y), z^* \rangle, \qquad \textit{for all } y \in Y, z \in Z^*.$$

Let us observe that using classical Orlicz-Pettis this is equivalent to the following: for all $y \in Y$ and $z^* \in Z^*$ we have that $(\langle \mathscr{B}(x_n, y), z^* \rangle)_n \in \ell^1$ and for all $M \subseteq \mathbb{N}$ there is $x_M \in X$ such that

$$\sum_{n \in M} \mathscr{B}(x_n, y) = \mathscr{B}(x_M, y), \quad \text{for all } y \in Y. \tag{29}$$

There is several ways to prove the classical Orlicz-Pettis theorem. For instance it is possible to prove this result using the Bochner integral —see for instance (9)—. Another possibility is based in the use of Schur theorem and Mazur theorem —see (11)—. We shall use the approach in this last reference.

Theorem 27 (Bilinear Orlicz-Pettis) *Let $\mathscr{B} : X \times Y^* \to Z$ be a bounded bilinear map such that*

(a) X is $\mathscr{B}$-normed,
(b) Y is w^*-sqcu, i.e., B_{Y^*} is weak$*$ sequentially compact,
(c) $\phi_{\mathscr{B}}(X) \subseteq \mathscr{W}^*(Y^*, Z)$.

Then every $\mathscr{B}$-unconditionally summable sequence in X is unconditionally summable.

Proof. Let $\boldsymbol{x} = (x_n)_n \subseteq X$ be a $\mathscr{B}$-unconditionally summable sequence and define the bilinear map

$$\mathscr{S} : Y^* \times Z^* \to \ell^1, \quad \text{given by} \quad \mathscr{S}(y^*, z^*) = (\langle \mathscr{B}(x_n, y^*), z^* \rangle)_n. \tag{30}$$

Step 1: *$\mathscr{S}$ is bounded.* Note that, since $\boldsymbol{x}$ is $\mathscr{B}$-unconditionally summable, then for all $y^* \in Y^*$ and every $z^* \in Z^*$

$$\sum_{n=1}^{\infty} |\langle \mathscr{B}(x_n, y^*), z^* \rangle| < \infty,$$

so $\mathscr{S}$ is well defined. Now, using Closed Graph theorem it is easy to see that the two linear maps

$$\mathscr{S}^{y^*} : Z^* \to \ell^1, \quad \text{given by} \quad \mathscr{S}^{y^*}(z^*) = \mathscr{S}(y^*, z^*), \quad \text{for all} \quad y^* \in Y^*,$$

$$\mathscr{S}^{z^*} : Y^* \to \ell^1, \quad \text{given by} \quad \mathscr{S}^{z^*}(y^*) = \mathscr{S}(y^*, z^*), \quad \text{for all} \quad z^* \in Z^*,$$

are bounded and hence $\mathscr{S}$ is separately continuous and thus continuous.

Step 2: *$\mathscr{S}$ is compact.* Let $(y_n^*, z_n^*)_n$ be a sequence in $\mathrm{B}_{Y^*} \times \mathrm{B}_{Z^*}$. In particular, since $(y_n^*)_n \subseteq \mathrm{B}_{Y^*}$ and B_{Y^*} is weak$*$ sequentially compact there exists a subsequence $(y_{n_k}^*)_k$ convergent to a certain y_0^* in the weak$*$ topology of Y^*, i.e.,

$$(\langle y, y_{n_k}^* \rangle)_k \text{ converges to } \langle y, y_0^* \rangle, \text{ for all } y \in Y.$$

Using now that $\phi_{\mathscr{B}}(X) \subseteq \mathscr{W}^*(Y^*, Z)$ since $(y^*_{n_k})_k$ is weak$*$ convergent then for every $x \in X$

$$(\mathscr{B}(x, y^*_{n_k}))_k \text{ converges to } \mathscr{B}(x, y^*_0), \text{ in the norm topology of } Z. \tag{31}$$

Consider now the separable subspace D of Z given by $D = \overline{\text{span}(\mathscr{B}(x_n, y^*_0))_n}$. According to Alauglu's theorem, B_{D^*} is a weak$*$ compact. For each $k \in \mathbb{N}$ denote by $\widetilde{z}^*_{n_k}$ the restriction of $z^*_{n_k}$ to D. Since $\|z^*_{n_k}\| \leq 1$ then $\widetilde{z}^*_{n_k}$ belongs to the compact B_{D^*}. This allows us to extract a subsequence of $\widetilde{z}^*_{n_k}$ —denoted also, by simplicity, by $\widetilde{z}^*_{n_k}$— convergent in the weak$*$ topology of Z^* to an element in Z^* that we call $\widetilde{z}^*_0$. That is

$$(\langle \widetilde{z}, \widetilde{z}^*_{n_k} \rangle)_k \text{ converges to } \langle \widetilde{z}, \widetilde{z}^*_0 \rangle, \text{ for all } \widetilde{z} \in D. \tag{32}$$

Since $\widetilde{z}^*_0 \in D^*$ using Hahn-Banach theorem there exists z^*_0 a continuous extension of $\widetilde{z}^*_0$ to Z^* —with the same norm—. On the other hand, there is $x_0 \in X$ verifying that for all $z^* \in Z^*$

$$\lim_{N \to \infty} \sum_{n=1}^{N} \langle \mathscr{B}(x_n, y^*_0), z^* \rangle = \sum_{n=1}^{\infty} \langle \mathscr{B}(x_n, y^*_0), z^* \rangle = \langle \mathscr{B}(x_0, y^*_0), z^* \rangle.$$

This means that $\mathscr{B}(x_0, y^*_0)$ belongs to $\overline{D}^w$ so by Mazur's theorem $\mathscr{B}(x_0, y^*_0) \in \overline{D} = D$. Replacing in (32) we obtain that

$$(\langle \mathscr{B}(x_0, y^*_0), \widetilde{z}^*_{n_k} \rangle)_k \text{ converges to } \langle \mathscr{B}(x_0, y^*_0), \widetilde{z}^*_0 \rangle. \tag{33}$$

To prove the compactness of $\mathscr{S}$ it remains to show that $\mathscr{S}((y_{n_k}, z^*_{n_k}))_k$ converges in ℓ^1. But using Schur's theorem all we need to show is the convergence in the weak topology of ℓ^1. The continuity of $\mathscr{S}$ allows us to show

$$\langle (\mathscr{S}(y^*_{n_k}, z^*_{n_k}))_k, \boldsymbol{\alpha} \rangle \text{ converges to } \langle \mathscr{S}(y_0, z^*_0), \boldsymbol{\alpha} \rangle, \tag{34}$$

for all $\boldsymbol{\alpha}$ in some norm dense subset of ℓ^∞. By linearity to prove (34) it suffices to take $\boldsymbol{\alpha} = \mathbf{1}_M$ for every $M \subseteq \mathbb{N}$. Fixing $k \in \mathbb{N}$ take then an arbitrary $M \subseteq \mathbb{N}$

$$\langle \mathscr{S}(y^*_{n_k}, z^*_{n_k}), \mathbf{1}_M \rangle = \sum_{n \in M} \langle \mathscr{B}(x_n, y^*_{n_k}), z^*_{n_k} \rangle = \langle \mathscr{B}(x_M, y^*_{n_k}), z^*_{n_k} \rangle. \tag{35}$$

On the other hand

$$\langle \mathscr{S}(y^*_0, z^*_0), \mathbf{1}_M \rangle = \sum_{n \in M} \langle \mathscr{B}(x_n, y^*_0), z^*_0 \rangle = \langle \mathscr{B}(x_M, y^*_0), z^*_0 \rangle. \tag{36}$$

Replacing (35) and (36) in (34) we obtain that in order to finish the proof it is enough to prove that for all $M \subseteq \mathbb{N}$

$$(\langle \mathscr{B}(x_M, y^*_{n_k}), z^*_{n_k} \rangle)_k \text{ converges to } \langle \mathscr{B}(x_M, y^*_0), z^*_0 \rangle.$$

But for every $k \in \mathbb{N}$ and $M \subseteq \mathbb{N}$ we have that

$$\begin{aligned}
&|\langle \mathscr{B}(x_M, y^*_{n_k}), z^*_{n_k}\rangle - \langle \mathscr{B}(x_M, y^*_0), z^*_0\rangle| \\
&\qquad \leq |\langle \mathscr{B}(x_M, y^*_{n_k}), z^*_{n_k}\rangle - \langle \mathscr{B}(x_M, y^*_0), z^*_{n_k}\rangle| \\
&\qquad +|\langle \mathscr{B}(x_M, y^*_0), z^*_{n_k}\rangle - \langle \mathscr{B}(x_M, y^*_0), z^*_0\rangle| \\
&\qquad = |\langle \mathscr{B}(x_M, y^*_{n_k}) - \mathscr{B}(x_M, y^*_0), z^*_{n_k}\rangle| \\
&\qquad +|\langle \mathscr{B}(x_M, y^*_0), z^*_{n_k}\rangle - \langle \mathscr{B}(x_M, y^*_0), z^*_0\rangle| \\
&\qquad \leq \|\mathscr{B}(x_M, y^*_{n_k}) - \mathscr{B}(x_M, y^*_0)\|_Z \ \|z^*_{n_k}\| \\
&\qquad +|\langle \mathscr{B}(x_M, y^*_0), z^*_{n_k}\rangle - \langle \mathscr{B}(x_M, y^*_0), z^*_0\rangle|.
\end{aligned}$$

Using (31), (32) and (33) we have that $\mathscr{S}((y_{n_k}, z^*_{n_k}))_k$ converges in the weak topology of ℓ^1 and by Schur theorem also converges in the topology of the norm of ℓ^1. Thus $\mathscr{S}$ is compact.

STEP 3: *$\boldsymbol{x} = (x_n)_n$ is unconditionally summable.* Recall that a set K is relatively compact in ℓ^1 if and only if $\lim_n \sup \left\{\sum_{k\geq n} |a_k| : (a_k)_k \in \mathrm{K}\right\} = 0$. In particular, since $\mathscr{S}(\mathrm{B}_{Y^*} \times \mathrm{B}_{Z^*})$ is a relatively compact in ℓ^1 then

$$\lim_{n\to\infty} \sup_{\substack{y^*\in \mathrm{B}_{Y^*} \\ z^*\in \mathrm{B}_{Z^*}}} \sum_{k=n}^{\infty} |\,\langle \mathscr{B}(x_k, y^*), z^*\rangle\,| = 0. \tag{37}$$

Let $(n_s)_s$ be an increasing sequence in $\mathbb{N}$. Using that X is $(Y, Z, \mathscr{B})$-normed there exists a constant $k > 0$ such that for every $N \in \mathbb{N}$

$$\left\|\sum_{s=1}^{N-1} x_{n_s} - x_0\right\|_X \leq k \left\|\mathscr{B}_{\sum_{s=1}^{N-1} x_{n_s} - x_0}\right\|_{\mathscr{L}(Y^*,Z)} \leq k \sup_{\substack{y^*\in \mathrm{B}_{Y^*} \\ z^*\in \mathrm{B}_{Z^*}}} \sum_{s=N}^{\infty} |\langle \mathscr{B}\left(x_{n_s}, y^*\right), z^*\rangle| \,.$$

Taking limits when $N \to \infty$ and using (37) we have that $\boldsymbol{x}$ is what it is called *subseries summable* and this is equivalent —see (11)— to have unconditional summability. ■

Remark 28 *Recall that a linear map T from a Banach space X into a Banach space Y is called completely continuous if it takes weakly null sequences in X to norm null sequences in Y, or, equivalently, if T maps every weakly convergent sequence in X into a norm convergent sequence in Y. The set consisting of those maps are denoted by $\mathscr{W}(X,Y)$ —see the notation $\mathscr{V}(X,Y)$ in (11)—. We can also state a result when the space Y is not necessarily a dual space. The reader can check that our proof can easily be adapted —using reflexivity and completely continuous operators— by replacing the above assumptions by*

(a) X is $\mathscr{B}$-normed,
(b) Y is reflexive,
(c) $\phi_{\mathscr{B}}(X) \subseteq \mathscr{W}(Y,Z)$,

to get the same conclusion.

Remark 29 *We would like to point out that the classical Orlicz-Pettis theorem is not needed to get this bilinear version and actually it follows as a corollary. Note that we can use (29) in the case $Z = \mathbb{K}$ when assuming weak-unconditionality. Now observe that X is $\mathscr{D}$-normed and we can assume that X is separable — since all the actions happens inside the weakly closed linear span of x_n— and consequently X is w^*-sqcu. Finally $\phi_{\mathscr{D}}(X) \subseteq \mathscr{W}^*(X^*, \mathbb{K})$. Hence we obtain* $\mathrm{wUC}(X) = \mathrm{UC}(X)$ *for any Banach space.*

Corollary 30 *Let $\mathscr{B} : X \times Y^* \to \ell^1$ be a bounded bilinear map such that X is $\mathscr{B}$-normed and Y is reflexive. Then every $\mathscr{B}$-unconditionally summable sequence is an unconditionally summable sequence.*

Proof. Note first that every bounded linear map from a Banach space into ℓ^1 is completely continuous, so $\phi_{\mathscr{B}}(Y^*) \subseteq \mathscr{W}(Y^*, \ell^1)$. Since every reflexive space is w^*-sqcu and satisfies that every weakly$*$ convergent sequence in Y^* is also weakly convergent, in particular —see (12)— we have that

$$\mathscr{W}(Y^*, Z) \subseteq \mathscr{K}(Y^*, Z) \subseteq \mathscr{W}^*(Y^*, Z)$$

where $\mathscr{K}(X, Y)$ stands for the compact operators. Hence all the assumptions in Theorem 27 are satisfied. ■

Let X be a Banach space and (Ω, Σ, μ) a finite space. Given $1 \leq p < \infty$ we denote by p' the (extended) real number given by $\frac{1}{p} + \frac{1}{p'} = 1$. Let us denote by $P^p(\mu, X)$ the completion of simple functions on the space of strongly measurable Pettis p-integrable functions, that is, the space consisting of all strongly measurable functions $f : \Omega \to X$ verifying that $\langle f, x^* \rangle \in L^p(\mu)$ for all $x^* \in X^*$ and for any $E \in \Sigma$ there exists $x_E \in X$ such that

$$\int_E \langle f, x^* \rangle d\mu = \langle x_E, x^* \rangle, \qquad \text{for all } x^* \in X^*.$$

We set the norm

$$\|f\|_{P^p(\mu,X)} = \sup_{x^* \in \mathrm{B}_{X^*}} \Big(\int_\Omega |\langle f, x^* \rangle|^p d\mu\Big)^{\frac{1}{p}}.$$

Corollary 31 *Let X be a reflexive Banach space, $1 \leq p < \infty$, (Ω, Σ, μ) a finite space and $(f_n)_n \in P^p(\mu, X)$. Assume that for any $x^* \in X^*, \phi \in L^{p'}(\mu)$*

$$\sum_{n=1}^{\infty} \int_\Omega |\langle f_n, x^* \rangle \phi| d\mu < \infty$$

and there exists $f \in P^p(\mu, X)$ such that

$$\sum_{n=1}^{\infty} \int_\Omega \langle f_n, x^* \rangle \phi \, d\mu = \int_\Omega \langle f, x^* \rangle \phi \, d\mu$$

for any $x^* \in X^*, \phi \in L^{p'}(\mu)$. Then $\sum_n f_n$ converges unconditionally in $P^p(\mu, X)$.

Proof. We may assume that X is separable (because the f_n has essentially separable range for any $n \in \mathbb{N}$). Take the bilinear map $\mathscr{B} : P^p(\mu, X) \times X^* \to L^p(\mu)$ defined by $\mathscr{B}(f, x^*) = \langle f, x^* \rangle$. It is $\mathscr{B}$-normed and $P^p(\mu, X) \subseteq \mathscr{L}(X^*, L^p(\mu))$ satisfies that $P^p(\mu, X) \subseteq \mathscr{K}(X^*, L^p(\mu)) \subseteq \mathscr{W}^*(X^*, L^p(\mu))$. The assumption means that $(f_n)_n$ is $\mathscr{B}$-unconditionally summable. Then apply the bilinear Orlicz-Pettis theorem to conclude the result. ■

Let $m : \Sigma \to X$ be a (countable) additive vector measure defined on a σ-algebra of subsets Σ of a nonempty set Ω. A measurable function $f : \Omega \to \mathbb{R}$ is called *weakly integrable (with respect to m)* if $f \in L^1(|\langle m, x^* \rangle|)$ for every $x^* \in X^*$. The space $L^1_w(m)$ of all (equivalence classes of) weakly integrable functions (with respect to m) becomes a Banach space when it is endowed with the norm

$$\|f\|_{1,m} = \sup_{x^* \in B_{X^*}} \int_\Omega |f| d|\langle m, x^* \rangle|.$$

We say that a weakly integrable function f is *integrable (with respect to m)* if for every $E \in \Sigma$ there is $x_E \in X$ such that

$$\int_E f d(\langle m, x^* \rangle) = \langle x_E, x^* \rangle, \quad \text{for all} \quad x^* \in X^*.$$

The vector x_E is unique and it is denoted by $\int_E f dm$. The space of all (equivalence classes of) integrable functions (with respect to m) is denoted by $L^1(m)$ and is a closed subspace of $L^1_w(m)$. The integral operator is the bounded linear map

$$I_m^{(1)} : L^1(m) \to X, \quad \text{given by} \quad I_m^{(1)}(f) = \int_\Omega f dm.$$

For $1 < p < \infty$ denote by p' the conjugate index of p —that is the real number given by $\frac{1}{p} + \frac{1}{p'} = 1$—. The function f is p-integrable with respect to m (resp. weakly p-integrable with respect to m) if $|f|^p \in L^1(m)$ (resp. $|f|^p \in L^1_w(m)$). The space $L^p(m)$ (resp. $L^p_w(m)$) of (equivalence classes of) p-integrable functions with respect to m (resp. weakly p-integrable with respect to m) is a Banach space with the norm

$$\|f\|_{p,m} = \sup_{x^* \in B_{X^*}} \Big(\int_\Omega |f|^p d|\langle m, x^* \rangle| \Big)^{\frac{1}{p}}.$$

See for instance (5; 6; 7) for the unexplained information. It is known that $L^p(m)$ need not be reflexive for $p > 1$. However if X is weakly sequentially complete $L^p(m)$ is reflexive for all $p > 1$ and $L^p(m) = L^p_w(m)$ —see (6, Corollary 3.10)—.

Corollary 32 *Let X be a weakly sequentially complete Banach space and let $m : \Sigma \to X$ be a (countable) additive vector measure verifying that the integration map $I_m^{(1)} : L^1(m) \to X$ is completely continuous. Given $1 < p < \infty$*

and $(f_n)_n \in L^p(m)$ *let us assume that*

$$\sum_{n=1}^{\infty} \Big\| \int_\Omega f_n g dm \Big\|_X < \infty, \quad \textit{for all} \quad g \in L^{p'}(m)$$

and that there exists a function $f \in L^p(m)$ *such that*

$$\sum_{n=1}^{\infty} \int_\Omega f_n g dm = \int_\Omega f g dm, \quad \textit{for all} \quad g \in L^{p'}(m).$$

Then $\sum_{n=1}^{\infty} f_n$ *converges unconditionally in* $L^p(m)$.

Proof. Let us consider the bounded bilinear map

$$\mathscr{M}_m^{(p)} : L^p(m) \times L^{p'}(m) \to X, \quad \text{given by} \quad \mathscr{M}_m^{(p)}(f,g) = \int_\Omega f g dm.$$

Following (7, Proposition 8) it is not difficult to prove that

$$\|f\|_{p,m} = \sup_{g \in \mathrm{B}_{L^{p'}(m)}} \Big\| \int_\Omega f g dm \Big\|_X.$$

Hence $L^p(m)$ is $\mathscr{M}_m^{(p)}$-normed. Also the weak sequential completeness of X implies that $L^p(m)$ is reflexive. On the other hand fixing $f \in L^p(m)$ then (8, Theorem 7) gives that the multiplication operator $M_m^{(p')} : L^{p'}(m) \to L^1(m)$ given by $M_m^{(p')}(g) = fg$ is weakly compact. But by the assumption the integration map $I_m^{(1)} : L^1(m) \to X$ is completely continuous. Hence for every $f \in L^p(m)$

$$(\mathscr{M}_m^{(p)})_f = I_m^{(1)} \circ (M_m^{(p')})_f$$

is a compact operator so $\phi_{\mathscr{M}_m^{(p)}}(L^p(m)) \subseteq \mathscr{K}(L^{p'}(m), X) \subseteq \mathscr{W}(L^{p'}(m), X)$. The result is then a consequence of the theorem. ■

Remark 33 *There are many situations for which the hypotheses of the previous result are fulfilled. We present some of them —see (5) for more information—.*

(a) *Given* $0 < (\alpha_n)_n \in \ell^1 = X$ *let us take*

$$m : 2^{\mathbb{N}} \to \ell^1, \quad m(A) = (\alpha_n)_n \mathbf{1}_A.$$

In this case $L^1(m) = \frac{1}{(\alpha_n)_n}\ell^1$ *and the integration map* $I_m^{(1)} : L^1(m) \to \ell^1$ *is completely continuous.*

(b) *Let* $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. *Given* λ *a non zero measure on the Borel* σ*-algebra* $\mathcal{B}(\mathbb{T})$ *verifying that the Fourier Stieljes Transform* $\widehat{\lambda} : \mathbb{Z} \to \mathbb{C}$ *belongs to* $c_0(\mathbb{Z})$ *consider the* $L^1(\mathbb{T})$*-valued measure given by the convolution*

$$v_\lambda : \mathcal{B}(\mathbb{T}) \to L^1(\mathbb{T}), \quad v_\lambda(A) = \mathbf{1}_A * \lambda.$$

In this case $L^1(v_\lambda) = L^1(|v_\lambda|) = L^1(\mathbb{T})$ *and the integration map is* $I^{(1)}_{v_\lambda}(f) = f * \mathbf{1}_A$ *for every* $f \in L^1(\mathbb{T})$ *which it is also completely continuous.*

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