q-CONCAVITY AND RELATED PROPERTIES ON SYMMETRIC SEQUENCE SPACES.

OSCAR BLASCO AND TERESA SIGNES

ABSTRACT. We introduce a new property between the q-concavity and the lower q-estimate of a Banach lattice and we get a general method to construct maximal symmetric sequence spaces that satisfies this new property but fails to be q-concave. In particular this gives examples of spaces with the Orlicz property but without cotype 2.

1. INTRODUCTION.

The reader is referred to [LT2] for the following notions from the theory of Banach lattices.

Let $1 \leq q < \infty$. A Banach lattice X is said to be q-concave if there exists a constant $C > 0$ such that

$$(\sum_{k=1}^{n} \|x_k\|^q)^{\frac{1}{q}} \leq C \|(\sum_{k=1}^{n} |x_k|^q)^{\frac{1}{q}}\|$$

for every choice of elements $x_1, x_2, ..., x_n$ in X.

A Banach lattice X is said to satisfy a lower q-estimate if there exists a constant $C > 0$ such that

$$(\sum_{k=1}^{n} \|x_k\|^q)^{\frac{1}{q}} \leq C \|\sum_{k=1}^{n} |x_k|\|$$

for every choice of elements $x_1, x_2, ..., x_n$ in X.

Obviously the q-concavity implies the lower q-estimate. The converse is false. For example, the Lorentz spaces $L_{q,p}$ with $1 \leq p < q$ satisfies a lower q-estimate (see [Cre] Prop. 3.2) but is not q-concave (see [Cre] Prop. 3.1). The first example of a Banach lattice satisfying a lower q-estimate, $q \geq 2$, but not being q-concave is due to G. Pisier (see [LT2], examples 1.f.19 and 1.f.20). The reader is referred to [CT] and [KMP]

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for more information on the case $p < 1$ or more general Lorentz spaces respectively.

Two related concepts from the theory of general Banach spaces are the following:

A Banach space X is said to have cotype q if there exists a constant $C > 0$ such that

$$(\sum_{k=1}^{n} \|x_k\|^q)^{\frac{1}{q}} \leq C \int_0^1 \|\sum_{k=1}^{n} x_k r_k(t)\| dt$$

for every choice of elements $x_1, x_2, ..., x_n$ in X, where r_k stand for the Rademacher functions.

X is said to have the q-Orlicz property if $id : X \to X$ is $(q,1)$-summing, that is, there exists a constant $C > 0$ such that

$$(\sum_{k=1}^{n} \|x_k\|^q)^{\frac{1}{q}} \leq C \sup_{\epsilon_k=\pm 1} \|\sum_{k=1}^{n} \epsilon_k x_k\|$$

for every choice of elements $x_1, x_2, ..., x_n$ in X.

Let us recall that Kintchine's inequalities (see [DJT] 1.16) tell us that only trivial spaces have cotype $q < 2$ and that, using an extension of the Dvoretzky-Rogers Theorem (see [DJT], Thm. 10.5), we get that only finite dimensional Banach spaces have q-Orlicz property for $1 \leq q < 2$.

Let us mention the relationship between all these notions.

On the one hand, taking into account that

$$\sup_{t\in[0,1]} \|\sum_{k=1}^{n} x_k r_k(t)\| \approx \sup_{\epsilon_k=\pm 1} \|\sum_{k=1}^{n} \epsilon_k x_k\|,$$

one actually has that cotype q implies the q-Orlicz property and that the q-Orlicz property implies a lower q-estimate.

On the other hand Banach lattices X which are q-concave for some $1 \leq q < \infty$ satisfy the so-called Maurey-Kintchine inequalities (see [DJT], Thm. 16.11)

$$\|(\sum_{k=1}^{n} |x_k|^2)^{\frac{1}{2}}\| \approx \int_0^1 \|\sum_{k=1}^{n} x_k r_k(t)\| dt.$$

Using this, one can easily sees that a Banach lattice X is 2-concave if and only if it has cotype 2. Hence we have the following chain of implications

2-concavity $\Leftrightarrow$ cotype 2 $\Rightarrow$ 2-Orlicz property $\Rightarrow$ lower 2-estimate.

The converses of the two last implications are false. In the setting of Banach lattices M. Talagrand (see [T2]) constructed an example with the 2-Orlicz property but without cotype 2. Actually this author (see [T3]) was even able to construct a symmetric sequence space with the 2-Orlicz property which is not 2-concave. The reader is referred to [BS] for some modifications of [T3].

It is rather interesting to mention that in the setting of rearrangement invariant (r.i.) spaces defined over $[0,1]$ (see [LT1] for definitions) the notions of cotype 2 and the 2-Orlicz properties coincide. This result is due to E. M. Semenov and A. M. Shteinberg (see [SS]). They actually showed that the Lorentz space $L_{2,1}([0,1])$ satisfies a lower 2-estimate but fails to have the 2-Orlicz property.

The situation for $2 < q < \infty$ is a bit different. B. Maurey (see [M] or [DJT], Cor. 16.7) showed that if a Banach lattice has the q-Orlicz property then it also has cotype q. Actually he proved (see [M] or [DJT], Cor. 16.15) that X satisfies a lower q-estimate if and only if X has cotype q. Some years later M. Talagrand (see [T2]) showed that the equivalence between q-Orlicz property and cotype q for $2 < q < \infty$ holds true for any Banach space.

Therefore for Banach lattices and $2 < q < \infty$ we have that

q-concavity $\Rightarrow$ cotype q $\Leftrightarrow$ q-Orlicz property $\Leftrightarrow$ lower q-estimate

The aim of this paper is to introduce, in the setting of symmetric sequence spaces, a property between the q-concavity and the lower q-estimate, which will allow us to analyze all the cases $1 < q < \infty$ in a unified way. We shall get a general method, introduced by Talagrand in [T3], to construct maximal symmetric sequence spaces which are not q-concave, but still have this new property. The definition is as follows.

Definition 1.1. *Let $1 \leq q < \infty$ and let X, X_1 be two Banach lattices such that $X \subset X_1$ (with continuous inclusion). X is said to be q-concave with respect to $X_1(l^1)$ if there exists a constant $C > 0$ such that*

$$(\sum_{k=1}^{n} \|x_k\|_X^q)^{\frac{1}{q}} \leq C \max\{\|\sum_{k=1}^{n} |x_k|\|_{X_1}, \|(\sum_{k=1}^{n} |x_k|^q)^{\frac{1}{q}}\|_X\}$$

for every choice of elements $x_1, x_2, ..., x_n$ in X.

Obviously if X is q-concave then X is q-concave with respect to $X_1(l^1)$ for any X_1 such that $X \subset X_1$ and if there exists X_1 such that X is q-concave with respect to $X_1(l^1)$ then X satisfies a lower q-estimate.

Let us recall that a maximal symmetric sequence space $(X, \|.\|)$ (see [J] [LT2]) is a Banach space of sequences such that

(a) $l^1 \subset X \subset l^\infty$, and $\|x\|_\infty \leq \|x\| \leq \|x\|_1$,

(b) $|x| \leq |y|$, $y \in X \Longrightarrow x \in X$, and $\|x\| \leq \|y\|$,

(c) $y \in X, \sigma \in \Pi(\mathbb{N}) \Longrightarrow x.\sigma \in X,$ and $\|x.\sigma\| = \|x\|,$

(d) $\|x\| = \sup_{n\in\mathbb{N}} \|P_n(x)\|$ where $P_n(x) = \sum_{k=1}^n x_k e_k$ if $x = (x_k)$.

Using that $\| \sum_{k=1}^n |x_k| \|_\infty = \sup_{\epsilon_k = \pm 1} \| \sum_{k=1}^n \epsilon_k x_k \|_\infty$, one has that a maximal symmetric sequence space X is q-concave with respect to $\ell^\infty(l^1)$ if and only if there exists a constant $C > 0$ such that

$$(\sum_{k=1}^n \|x_k\|^q)^{\frac{1}{q}} \leq C \max\{ \sup_{\epsilon_k=\pm 1} \| \sum_{k=1}^n \epsilon_k x_k \|_\infty, \|(\sum_{k=1}^n |x_k|^q)^{\frac{1}{q}} \| \}$$

for every choice of elements $x_1, x_2, ..., x_n$ in X. In particular, for $2 \leq q < \infty$ the q-concavity with respect to $\ell^\infty(l^1)$ implies the q-Orlicz property.

We shall consider the following method of constructing maximal symmetric sequence spaces generated by a family of sequences.

Let $\mathcal{F}$ be a family of non-negative sequences in the unit ball of l^∞ with the following properties:

(i) If $f \in \mathcal{F}$ and $0 \leq g \leq f$ then $g \in \mathcal{F}$.

(ii) If $f \in \mathcal{F}$ and $\sigma \in \Pi(\mathbb{N})$ then $f \cdot \sigma \in \mathcal{F}$.

(iii) There exists $f \in \mathcal{F}$ such that $\max_{i\in\mathbb{N}} f(i) = 1$.

We call it a generating family.

Let h be a non-negative, non-increasing sequence in $c_0(\mathbb{N})$, $h(1) = 1$ and denote by $\mathcal{H} = \{h.\sigma, \sigma \in \Pi(\mathbb{N})\}$. For each $m \in \mathbb{N}$ we write

$$\mathcal{H}_m = \{0 \leq f \leq \sum_{l\geq 0} 2^{-l} \sum_{j\leq m^l} \alpha_{j,l} h_{j,l} : \sum_{j\leq m^l} \alpha_{j,l} \leq 1, \forall\, l \geq 0,\ h_{j,l} \in \mathcal{H}\}.$$

Given an increasing sequence $(m_k) \subseteq \mathbb{N}$, $m_0 = 1$, we define

$$\mathcal{F} = \mathcal{F}(h, (m_k)) = \{f : 0 \leq f \leq \sum_{r=0}^\infty \beta_r f_r, \ \sum_{r=0}^\infty \beta_r \leq 1, \\ \|f_r\|_\infty \leq 2^{-r}, f_r \in \mathcal{H}_{m_r}\}.$$

Let H be a non-decreasing sequence of positive real numbers and let us denote by ℓ_H the space of sequences $(x(n))$ such that

$$\sum_{n\in A} |x(n)| \leq H(card(A))$$

for all finite sets $A \subseteq \mathbb{N}$.

Starting with a fixed sequence h and a fixed sequence (m_k) we shall produce a way to find families $\mathcal{F} \subseteq \ell_H$ where $H(n) = \sum_{k=1}^n h(k)$.

Lemma 1.2. *Let h, H and (m_k) be as above.Then $\mathcal{F} = \mathcal{F}(h, (m_k))$ is a generating family and $\mathcal{F} \subseteq \ell_H$.*

Proof. The properties (i), (ii) and (iii) in the definition are immediate. To see that $\mathcal{F} \subseteq \ell_H$ it suffices to see that $\mathcal{H} \subseteq \ell_H$ and this follows from

$$\sum_{k\in A} h(k) \leq \sum_{i=1}^{card(A)} h(i) = H(card(A)).$$

□

Lemma 1.3. *Let $m, n \in \mathbb{N}$ and $f \in \mathcal{H}_m$. Then there exists a set $B \subseteq \mathbb{N}$ with card $(B) = n$ and $\|f\chi_{B^c}\|_\infty \leq \frac{H(n)}{n}$.*

Proof. Take i_1 such that $f(i_1) = max_{i\in\mathbb{N}} f(i)$ (this exists since $f \in c_0(\mathbb{N})$) and, inductively, choose i_k so that $f(i_k) = \max_{i\in\mathbb{N}\setminus\{i_1,\dots,i_{k-1}\}} f(i)$. Let $B = \{i_1, \dots, i_n\}$. Now it is clear that if $i \notin B$ then $f(i) \leq f(i_k)$, $k = 1, \dots, n$. Hence

$$n \sup_{i\notin B} f(i) \leq \sum_{i\in B} f(i) \leq H(n).$$

□

Given $1 < q < \infty$ let $X_q = X_q(\mathcal{F})$ be the space of sequences such that

$$\|x\|_{X_q} = \sup_{f\in\mathcal{F}} \langle |x|, f^{\frac{1}{q'}} \rangle < \infty$$

where $\langle x, f\rangle$ means $\sum_{i\in\mathbb{N}} x(i)f(i)$.

It is easy to see that X_q is a maximal symmetric sequence space.

Our main theorem can be now stated as follows

Theorem 1.4. *Given $1 < q < \infty$. There exists a generating family $\mathcal{F}_q$ such that $X_q(\mathcal{F}_q)$ is q-concave with respect to $\ell^\infty(l^1)$ but is not q-concave.*

As a corollary we have that $X_q(\mathcal{F}_q)$, for $2 < q < \infty$, are examples of spaces of cotype q which are not q-concave, $X_2(\mathcal{F}_2)$ satisfies the 2-Orlicz property but is not of cotype 2, and $X_q(\mathcal{F}_q)$, for $1 < q < 2$, satisfies a lower q-estimate but fails to be q-concave.

2. Construction of spaces which are not q-concave

With the notation in the above section, we can find the following conditions to get maximal symetric sequence spaces $X_q(\mathcal{F})$ which are not q-concave.

Theorem 2.1. *Let $h \in c_0(\mathbb{N})$, $h \geq 0$ non-increasing, $h(1) = 1$ and such that there exists a convex subsequence $(n_k) \subset \mathbb{N}$, i.e. $2n_k \leq n_{k+1} + n_{k-1}$, $n_0 = 0$, for which*

$$\sup_k \frac{H(n_k)}{kH(n_k - n_{k-1})} = \infty.$$

Let $1 < q < \infty$, $(m_k) \subseteq \mathbb{N}$, $m_0 = 1$ and $\mathcal{F} = \mathcal{F}(h, (m_k))$. Then $X_q = X_q(\mathcal{F})$ is not a q-concave space.

Proof. Let us fix $\tau \in \mathbb{N}$ and $(x(k)) = (h^{\frac{1}{q}}(k))_{k \leq n_\tau}$. Taking $N = n_\tau - n_{\tau-1}$ we consider $\sigma \in \Pi(\mathbb{N})$ given by

$$\begin{cases} \sigma(n_k) = n_{k-1} + 1, & k \in \mathbb{N} \\ \\ \sigma(p) = p + 1 & \textit{otherwise.} \end{cases}$$

Let us define $x_j = x.\sigma^j$, $j = 1, 2 \ldots, N$ and denote $D_k = (n_{k-1}, n_k] \cap \mathbb{N}$, $k \in \mathbb{N}$.

A simple computation shows that

$$\begin{aligned} (\sum_{j=1}^N |x_j|^q)^{\frac{1}{q}} &\leq \sum_{k=1}^{\tau} \Big(([\frac{N}{card(D_k)}] + 1) \sum_{i \in D_k} x^q(i) \Big)^{\frac{1}{q}} \chi_{D_k} \\ &\leq 2^{\frac{1}{q}} N^{\frac{1}{q}} \sum_{k=1}^{\tau} \Big(\frac{1}{card(D_k)} \sum_{i \in D_k} x^q(i) \Big)^{\frac{1}{q}} \chi_{D_k}. \end{aligned}$$

Hence

$$(\sum_{j=1}^N |x_j|^q)^{\frac{1}{q}} \leq C N^{\frac{1}{q}} \sum_{k=1}^{\tau} (\frac{1}{card(D_k)} \sum_{i \in D_k} h(i))^{\frac{1}{q}} \chi_{D_k}.$$

To show that $X_q(\mathcal{F}(h, (m_k)))$ is not q-concave it is enough to see that

$$\sup_\tau \frac{(\sum_{j=1}^N \|x_j\|^q)^{\frac{1}{q}}}{\|(\sum_{j=1}^N |x_j|^q)^{\frac{1}{q}}\|} = \infty.$$

Since $h \in \mathcal{F}$then we have

$$\|x_j\| = \|x\| \geq \langle |x|, h^{\frac{1}{q'}} \rangle = H(n_\tau)$$

for all $j = 1, 2, \ldots, N$.

On the other hand, if $y = \sum_{k=1}^{\tau} (\frac{1}{card(D_k)} \sum_{i \in D_k} h(i))^{\frac{1}{q}} \chi_{D_k}$ then, for any $f \in \mathcal{F}$, we have

$$\begin{array}{lll}
\langle y, f^{\frac{1}{q'}}\rangle & = & \sum_{k=1}^{\tau}(\frac{H(n_k)-H(n_{k-1})}{card(D_k)})^{\frac{1}{q}} \sum_{i\in D_k} f^{\frac{1}{q'}(i)} \\
\text{(Hölder)} & \leq & \sum_{k=1}^{\tau}\Big(H(n_k)-H(n_{k-1})\Big)^{\frac{1}{q}}\Big(\sum_{i\in D_k} f(i)\Big)^{\frac{1}{q'}} \\
\text{(Lemma 1.2)} & \leq & \sum_{k=1}^{\tau}\Big(H(n_k)-H(n_{k-1})\Big)^{\frac{1}{q}}\Big(H(n_k-n_{k-1})\Big)^{\frac{1}{q'}} \\
\text{(Hölder)} & \leq & \Big(H(n_\tau-n_{\tau-1})\Big)^{\frac{1}{q'}}(H(n_\tau))^{\frac{1}{q}}.\tau^{\frac{1}{q'}}.
\end{array}$$

Hence

$$\begin{aligned}
\frac{(\sum_{j=1}^{N}\|x_j\|^q)^{\frac{1}{q}}}{\|(\sum_{j=1}^{N}|x_j|^q)^{\frac{1}{q}}\|} \geq \frac{\|x\|}{C\|y\|} \ &\geq \ \frac{H(n_\tau)}{C(H(n_\tau))^{\frac{1}{q}}(H(n_\tau-n_{\tau-1}))^{\frac{1}{q'}}\tau^{\frac{1}{q'}}} \\
&= \ \frac{1}{C}(\frac{H(n_\tau)}{\tau H(n_\tau-n_{\tau-1})})^{\frac{1}{q'}}.
\end{aligned}$$

This finishes the proof. □

3. Construction of spaces which are q-concave with respect to $\ell^\infty(l^1)$.

Theorem 3.1. *Let $1<q<\infty$, (m_k) such that $m_k \geq k$, $m_0 = 1$ and*

$$\sum_{s=1}^{\infty} k_s m_{s-1}^{\frac{-1}{q'}} < \infty.$$

where $k_p = \min\{i : \frac{H(i)}{i} \leq 2^{-p}\}$, $p \geq 0$.

If $\mathcal{F} = \mathcal{F}(h, m_k)$ then $X_q = X_q(\mathcal{F})$ is q-concave with respect to $\ell^\infty(l^1)$.

Proof. Let us take N elements $(x_k)_{k=1}^N$ such that

$$\sup_{i\in\mathbb{N}}\sum_{k=1}^{N}|x_k(i)| \leq 1, \|(\sum_{k=1}^{N}|x_k|^q)^{\frac{1}{q}}\| \leq 1.$$

First choose $f_k \in \mathcal{F}$, $k = 1, 2, \ldots, N$ such that

$$\|x_k\| \leq \frac{4}{3}\langle |x_k|, f_k^{\frac{1}{q'}}\rangle.$$

Let us write $f_k = \sum_{r\geq 0} \beta_{k,r} f_{k,r}$ where $\sum_{r\geq 0}\beta_{k,r} \leq 1$, $\|f_{k,r}\|_\infty \leq 2^{-r}$ and

$$f_{k,r} = \sum_{l\geq 0} 2^{-l} \sum_{j\leq m_r^l} \alpha_{j,l,k,r} h_{j,l,k,r}$$

with $\sum_{j\leq m_r^l} \alpha_{j,l,k,r} \leq 1$ for all $l \geq 0$, $r \geq 0$.

For each $k \in \{1, 2, \ldots, N\}$, take $s(k)$ so that $k \in [m_{s(k)-1}, m_{s(k)})$ and denote

$$f_k' = \sum_{r\geq s(k)} \beta_{k,r} f_{k,r} \quad \text{and} \quad f_k'' = \sum_{r<s(k)} \beta_{k,r} f_{k,r}.$$

Let us assume that $\|x_k\|$ is decreasing and let us write $S_N^q = \sum_{k=1}^N \|x_k\|^q$. Hence $\|x_k\| \leq S_N k^{\frac{-1}{q}}$.

Denoting $I_N = [1, N] \cap \mathbb{N}$ and $I_{s,N} = [m_{s-1}, m_s) \cap I_N$, for all $s \geq 1$ we can write

$$\begin{aligned} S_N^q = \sum_{k=1}^N \|x_k\|^q &\leq \frac{4}{3} \sum_{k=1}^N \|x_k\|^{q-1} \langle |x_k|, f_k^{\frac{1}{q'}} \rangle \\ &\leq \frac{4}{3} \sum_{k=1}^N \langle |x_k|, \sqrt[q']{f_k' \|x_k\|^q} \rangle \\ &+ \frac{4}{3} \sum_{s=1}^{s(N)} \sum_{k\in I_{s,N}} \|x_k\|^{q-1} \langle |x_k|, \sqrt[q']{f_k''} \rangle \\ &= (I) + (II). \end{aligned}$$

We are going to show that (I) and (II) are bounded by CS_N^{q-1} which will imply $S_N \leq C$.

To deal with the first term we use that

$$\sum_{k=1}^N \langle |x_k|, \sqrt[q']{f_k' \|x_k\|^q} \rangle \leq \langle (\sum_{k=1}^N |x_k|^q)^{\frac{1}{q}}, (\sum_{k=1}^N \|x_k\|^q f_k')^{\frac{1}{q'}} \rangle.$$

Now observe that

$$\begin{aligned}\sum_{k=1}^{N}\|x_k\|^q f'_k &= \sum_{k=1}^{N}\sum_{r\geq s(k)}\|x_k\|^q\beta_{k,r}f_{k,r}\\ &= \sum_{s=1}^{s(N)-1}\sum_{m_{s-1}\leq k<m_s}(\sum_{r\geq s}\beta_{k,r}f_{k,r})\|x_k\|^q\\ &+ \sum_{m_{s(N)-1}\leq k\leq N}(\sum_{r\geq s}\beta_{k,r}f_{k,r})\|x_k\|^q\\ &= \sum_{s=1}^{s(N)}\sum_{r\geq s}\sum_{k\in I_{s,N}}\beta_{k,r}f_{k,r}\|x_k\|^q\\ &= \sum_{r=1}^{\infty}\sum_{k\in[1,m_r]\cap I_N}\|x_k\|^q\beta_{k,r}f_{k,r}\end{aligned}$$

Denoting by $\gamma_{k,r}=\frac{\|x_k\|^q\beta_{k,r}}{\sum_{k\in[1,m_r]\cap I_N}\|x_k\|^q\beta_{k,r}}$ and $g_r=\sum_{k\in[1,m_r]\cap I_N}\gamma_{k,r}f_{k,r}$ we get that

$$\sum_{k=1}^{N}\|x_k\|^q f'_k=\sum_{r=1}^{\infty}(\sum_{k\in[1,m_r]\cap I_N}\|x_k\|^q\beta_{k,r})g_r.$$

Since $\frac{1}{2}g_r\in\mathcal{H}_{m_r}$ then

$$\sum_{k=1}^{N}\|x_k\|^q f'_k\leq 2S_N^q g\quad\text{for}\quad g\in\mathcal{F}.$$

This shows that $(I)\leq CS_N^{q-1}$.

To deal with (II) observe first that for each $s\in\mathbb{N}$

$$card(\{(k,r):m_{s-1}\leq k<m_s,r<s\})\leq m_s^2$$

which gives

$$\sum_{k\in I_{s,N}}\sum_{r<s}\beta_{k,r}f_{k,r}\|x_k\|^q=(\sum_{r<s}\sum_{k\in I_{s,N}}\beta_{k,r}\|x_k\|^q)4h_s=4\gamma_s h_s \tag{3.1}$$

where $h_s\in\mathcal{H}_{m_s}$ and $\gamma_s=\sum_{r<s}\sum_{k\in I_{s,N}}\beta_{k,r}\|x_k\|^q$.

Applying now Lemma 1.3 to $n=k_s$ and $m=m_s$ we get a set $B_s\subset\mathbb{N}$ with $card(B_s)=k_s$ and $\|h_s\chi_{B_s^c}\|_\infty\leq 2^{-s}$. This allows us to split (II)

into two pieces as follows

$$\begin{aligned}(II) &\leq \frac{4}{3}\sum_{s=1}^{s(N)}\sum_{k\in I_{s,N}}\|x_k\|^{q-1}\langle |x_k|, (f_k''\chi_{B_s^c})^{\frac{1}{q'}}\rangle \\ &+ \frac{4}{3}\sum_{s=1}^{s(N)}\sum_{k\in I_{s,N}}\|x_k\|^{q-1}\langle |x_k|\chi_{B_s}, (f_k'')^{\frac{1}{q'}}\rangle \\ &= (II)' + (II)''.\end{aligned}$$

Hence

$$(II)' \leq \frac{4}{3}\sum_{s=1}^{s(N)}\langle (\sum_{k\in I_{s,N}}|x_k|^q)^{\frac{1}{q}}, (\sum_{k\in I_{s,N}}\|x_k\|^q f_k''\chi_{B_s^c})^{\frac{1}{q'}}\rangle$$

Applying Hölder again and using (3.1)

$$\begin{aligned}(II)' \leq \frac{4}{3}\langle (\sum_{k=1}^{N}|x_k|^q)^{\frac{1}{q}}, (\sum_{s=1}^{s(N)}4\gamma_s h_s\chi_{B_s^c})^{\frac{1}{q'}}\rangle \\ = \frac{4}{3}(\sum_{s=1}^{s(N)}4\gamma_s)^{\frac{1}{q'}}\langle (\sum_{k=1}^{N}|x_k|^q)^{\frac{1}{q}}, (\sum_{s=1}^{s(N)}\gamma_s' h_s')^{\frac{1}{q'}}\rangle\end{aligned}$$

where $h_s' = h_s\chi_{B_s^c} \in \mathcal{H}_{m_s}$, $\|h_s'\|_\infty \leq 2^{-s}$ and $\sum_s \gamma_s' \leq 1$.

Therefore

$$(II)' \leq C(\sum_{s=1}^{s(N)}\sum_{r<s}\sum_{k\in I_{s,N}}\beta_{k,r}\|x_k\|^q)^{\frac{1}{q'}} \leq CS_N^{q-1}.$$

Finally to deal with $(II)''$ we use that $\ell^1 \subset X$ with inclusion norm 1 to obtain

$$\begin{aligned}&\sum_{s=1}^{s(N)}\sum_{k\in I_{s,N}}\|x_k\|^{q-1}\langle |x_k|\chi_{B_s}, (f_k'')^{\frac{1}{q'}}\rangle \\ &\leq \sum_{s=1}^{s(N)}(\max_{k\in I_{s,N}}\|x_k\|^{q-1})\sum_{k\in I_{s,N}}\|x_k\chi_{B_s}\| \\ &\leq \sum_{s=1}^{s(N)}S_N^{q-1}m_{s-1}^{\frac{-1}{q'}}\sum_{k\in I_{s,N}}\sum_{i\in B_s}|x_k(i)| \leq S_N^{q-1}\sum_{s=1}^{\infty}k_s m_{s-1}^{\frac{-1}{q'}}.\end{aligned}$$

The proof is now finished using the extra assumption on (m_s). □

Proof of the main theorem. Take $h(n) = \frac{log(n)}{n}$, $n \geq 2$, $h(1) = 1$. This gives $H(n) \sim (log(n))^2$ and, using that $\frac{p^2}{e^p} \leq 2^{-p}$ (for p big enough) one gets $k_p \leq e^{-p}$. Then it suffices to take $m_k = k^{2q'} e^{kq'}$ and $n_k = e^{k^2}$ which satisfy the assumptions in Theorem 2.1 and Theorem 3.1 to obtain an example where $X_q(\mathcal{F})$ is q-concave with respect $\ell^\infty(\ell^1)$ but not q-concave. . $\square$

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Address of the authors:

Oscar Blasco
Departamento de Análisis Matemático
Universidad de Valencia
Doctor Moliner, 50
E-46100 Burjassot (Valencia), Spain

Oscar.Blasco@uv.es

Teresa Signes
Departamento de Análisis Matemático
Universidad Complutense de Madrid
28040 Madrid, Spain
teresass@eucmos.sim.ucm.es