q-CONCAVITY AND q-ORLICZ PROPERTY ON SYMMETRIC SEQUENCE SPACES.

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Abstract. We give a general method for constructing symmetric sequence spaces that for $1 < q < 2$ satisfy a lower q-estimate but fail to be q-concave and, for $2 \leq q < \infty$ have the q-Orlicz property but fail to be q-concave. In particular this gives examples of spaces with the 2-Orlicz property but without cotype 2.

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1. Introduction.

Let $1 \leq q < \infty$. A Banach lattice X is said to be q-concave if there exists a constant $C \geq 0$ such that

$$\Big(\sum_{k=1}^{n} \|x_k\|_X^q\Big)^{\frac{1}{q}} \leq C\Big\|\Big(\sum_{k=1}^{n} |x_k|^q\Big)^{\frac{1}{q}}\Big\|_X$$

for every choice of elements $x_1, \ldots, x_n$ in X.

A Banach lattice X is said to satisfy a lower q-estimate if there exists a constant $C \geq 0$ so that, for every choice of elements $x_1, \ldots, x_n$ in X, we have

$$\Big(\sum_{k=1}^{n} \|x_k\|_X^q\Big)^{\frac{1}{q}} \leq C\Big\|\Big(\sum_{k=1}^{n} |x_k|\Big)\Big\|_X.$$

Obviously q-concavity implies lower q-estimate and both notions are the same when $q = 1$. On the other hand, there are Banach lattices that satisfy a lower q-estimate but fail to be q-concave (see [1, Prop. 3.1], [4, Ex. 1.f.19 and 1.f.20]).

Two related concepts from the theory of Banach spaces are the following:

A Banach space X is said to have cotype q, $2 \leq q < \infty$, if there exists a constant $C \geq 0$ so that

$$\Big(\sum_{k=1}^{n} \|x_k\|_X^q\Big)^{\frac{1}{q}} \leq C \int_0^1 \Big\|\sum_{k=1}^{n} r_k(t) x_k\Big\|_X dt$$

for every choice of elements $x_1, \ldots, x_n$ in X, where r_k stands for the Rademacher functions.

X is said to have the q-Orlicz property if the identity operator $id : X \longrightarrow X$ is $(q, 1)$-summing. That is, if there exists a constant $C \geq 0$ such that regardless of the choice of $x_1, \ldots, x_n$ in X we have

$$\Big(\sum_{k=1}^{n} \|x_k\|_X^q\Big)^{\frac{1}{q}} \leq C \sup_{|\epsilon_k|=1} \Big\|\sum_{k=1}^{n} \epsilon_k x_k\Big\|_X.$$

Let us observe that every Banach space with cotype q has the q-Orlicz property, $2 \leq q < \infty$. The converse was an open problem for some time and was solved by Talagrand in [7] and [8]. Indeed, Talagrand in [8] showed that if a Banach space has the q-Orlicz property for $2 < q < \infty$ then it also has cotype q. Also, he proved in [7] that the situation for

$q = 2$ is a bit different. He constructed an example with the 2-Orlicz property but without cotype 2.

There are many connections between all these notions. The reader is referred to [2] or [4] for the following chain of implications.

For $2 < q < \infty$ we have that

q-concavity $\Rightarrow$ cotype q $\Leftrightarrow$ q-Orlicz property $\Leftrightarrow$ lower q-estimate.

The examples mentioned above show that the converse of the first implication fails.

For $q = 2$ we have that

2-concavity $\Leftrightarrow$ cotype 2 $\Rightarrow$ 2-Orlicz property $\Rightarrow$ lower 2-estimate.

The converse of the two last implications fail. E. M. Semenov and A. M. Shteinberg [6] showed that the Lorentz space $L_{2,1}([0,1])$ satisfies a lower 2-estimate but fails to have the 2-Orlicz property. As we said before M. Talagrand in [7] constructed an example with the 2-Orlicz property but without cotype 2. Moreover in [9] he was even able to construct a counterexample in the setting of symmetric sequence spaces.

The aim of this paper is to continue the study of the relationship between all this notions and to give a general method, which is inspired by Talagrand's techniques in [9], to construct symmetric sequence spaces that satisfy a lower q-estimate but fail to be q-concave, $1 < q < 2$, and that have the q-Orlicz property but fail to be q-concave for $2 \leq q$.

Let us recall that a symmetric sequence space $(X, \|\cdot\|)$ is a Banach space of sequences such that

1. If $x \in X$ and $|y(i)| \leq |x(i)|$ for all $i \in \mathbb{N}$, then $y \in X$ and $\|y\| \leq \|x\|$.
2. If $x \in X$ and $\sigma \in \Pi(\mathbb{N})$, then $x\sigma \in X$ and $\|x\sigma\| = \|x\|$.

We shall consider the following method to construct symmetric sequence spaces generated by a family of sequences.

Let $\mathcal{F}$ be a family sequences in ℓ_∞ with the following properties:

(i) (Solid) If $f \in \mathcal{F}$ and $|g(i)| \leq |f(i)|$, for all $i \in \mathbb{N}$, then $g \in \mathcal{F}$.

(ii) (Symmetric) If $f \in \mathcal{F}$ and $\sigma \in \Pi(\mathbb{N})$ then $f\sigma \in \mathcal{F}$.

(iii) (Bounded) There exists a constant $C \geq 0$ such that

$$\sup_{f\in\mathcal{F}} \|f\|_{\ell_\infty} \leq C.$$

In this case, it will be called a generating family.

Given $1 < q < \infty$ we consider $X_q(\mathcal{F})$ the space of sequences such that

$$\|x\|_{X_q(\mathcal{F})} = \sup_{f\in\mathcal{F}} \langle |x|, |f|^{\frac{1}{q'}}\rangle < \infty$$

where $\langle x, f\rangle$ means $\sum_{i=1}^\infty x(i)f(i)$.

It is easy to see that $X_q(\mathcal{F})$ is a symmetric sequence space and

$$\ell_1 \hookrightarrow X_q(\mathcal{F}) \hookrightarrow \ell_\infty$$

with

$$\|x\|_{\ell_\infty}(\sup_{f\in\mathcal{F}} \|f\|_{\ell_\infty})^{1/q'} \leq \|x\| \leq \|x\|_{\ell_1} \sup_{f\in\mathcal{F}} \|f\|_{\ell_\infty}^{1/q'}.$$

Our main theorem can now be stated as follows.

Theorem 1.1. *Let $1 < q < \infty$. There exists a generating family $\mathcal{F}$ such that $X_q(\mathcal{F})$ satisfies a lower q-estimate but is not q-concave.*

As a corollary we have that $X_q(\mathcal{F})$, for $2 < q < \infty$, are examples of spaces of cotype q which are not q-concave and $X_2(\mathcal{F})$ satisfies the 2-Orlicz property but is not of cotype 2.

2. Families generated by a function

In this section we give the main construction for our families.

Let $(k_s)_{s=0}^\infty$ be a strictly increasing sequence of natural numbers with $k_0 = k_1 := 1$ and let $(\alpha_s)_{s=0}^\infty$ be a sequence in $\mathbb{R}^+$ with $\alpha_0 = \alpha_1$, such that the sequence $(\alpha_s/k_s)_{s=1}^\infty$ is decreasing and

$$\lim_{s\to\infty} \frac{\alpha_s}{k_s} = 0. \tag{1}$$

Step 1.

We start with a single function on $\mathbb{N}$

$$h = \sum_{s=2}^{\infty} \frac{\alpha_s}{k_s} \chi_{[k_{s-1}, k_s)}$$

and the set of functions

$$\mathcal{H} = \{h\sigma : \sigma \in \Pi(\mathbb{N})\}.$$

By (1) we know that $h \in c_o(\mathbb{N})$ and so $\mathcal{H} \subseteq c_o(\mathbb{N})$. Observe also that $\mathcal{H}$ is symmetric and bounded by α_2/k_2.

Proposition 2.1. *The following properties hold:*

1. $\sum_{i\leq k_s} h(i) \leq \sum_{\ell=2}^{s} \alpha_\ell$ *for* $s \geq 2$.
2. *If* $h' \in \mathcal{H}$ *and* $A \subseteq \mathbb{N}$ *with card*$(A) \leq k_s$, $s \geq 2$, *then*
$$\sum_{i\in A} h'(i) \leq \sum_{\ell=2}^{s} \alpha_\ell.$$
3. *Let* $h' \in \mathcal{H}$ *and* $s \geq 0$. *Then, there exists* $A \subseteq \mathbb{N}$ *such that card*$(A) = k_s$ *and* $\|h'\chi_{A^c}\|_{\ell_\infty} \leq \alpha_{s+1}/k_{s+1}$.
4. *Let* $h' \in \mathcal{H}$ *and* $s \geq 0$. *Then, there exist* h'_1 *and* h'_2 *functions on* $\mathbb{N}$ *such that*
$$h' = h'_1 + h'_2 \quad \textit{with} \begin{cases} \textit{card}(\textit{supp } h'_1) = k_s, \\ \|h'_2\|_{\ell_\infty} \leq \dfrac{\alpha_{s+1}}{k_{s+1}}. \end{cases}$$

Proof. 1) Let $s \geq 2$. Then

$$\sum_{i\leq k_s} h(i) \leq \sum_{\ell=2}^{s} \frac{\alpha_\ell}{k_\ell}(k_\ell - k_{\ell-1}) + \frac{\alpha_{s+1}}{k_{s+1}} \leq \sum_{\ell=2}^{s-1} \frac{\alpha_\ell}{k_\ell} k_\ell + \frac{\alpha_s}{k_s}(k_s - k_{s-1} + 1) \leq \sum_{\ell=2}^{s} \alpha_\ell.$$

3) Suppose that $h' = h\sigma$, $\sigma \in \Pi(\mathbb{N})$, and let $A = \sigma^{-1}([1, k_s])$. If $i \notin A$ then $h'(i) = h(j)$ with $j > k_s$ ($j = \sigma(i)$), hence $h'(i) = h(j) \leq \alpha_{s+1}/k_{s+1}$.

2) and 4) follows from 1) and 3), respectively. □

Step 2.

For each $m \in \mathbb{N}$ we consider the family:

$$co_m(\mathcal{H}) = \Big\{ \sum_{j=1}^{m} \zeta_j h_j \ : \ h_j \in \mathcal{H} \ , \ \zeta_j \in \mathbb{R}^+ , \ \sum_{j=1}^{m} \zeta_j = 1 \ \Big\}.$$

The family $co_m(\mathcal{H})$ is symmetric, bounded by α_2/k_2.

Let $(m_r)_{r=1}^{\infty}$ be a strictly increasing sequence of natural numbers, $m_1 \geq 2$. Then, for $r \in \mathbb{N}$, we define

$$\mathcal{G}_r = \Big\{ f : \mathbb{N} \longrightarrow \mathbb{R}^+ \ : \ f \leq \sum_{\ell=0}^{\infty} 2^{-\ell} f_\ell \text{ with } f_\ell \in co_{m_r^\ell}(\mathcal{H}) \Big\}.$$

Again $\mathcal{G}_r \subseteq c_o(\mathbb{N})$ and $\mathcal{H} \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \ldots \subseteq \mathcal{G}_r \subseteq \mathcal{G}_{r+1} \subseteq \ldots$.

Proposition 2.2. *Let $r \in \mathbb{N}$, $f \in \mathcal{G}_r$ and $s \geq 2$. Then,*

1. $\sum_{i \in A} f(i) \leq \sum_{\ell=2}^{s} \alpha_\ell$ *for every $A \subseteq \mathbb{N}$ with $card(A) \leq k_s$.*
2. *There exists $A \subseteq \mathbb{N}$ such that $card(A) = k_s$ and*
$$\| f \chi_{A^c} \|_{\ell_\infty} \leq \frac{\sum_{\ell=2}^{s} \alpha_\ell}{k_s}.$$
3. *There exist f_1 and f_2 functions on $\mathbb{N}$ such that*
$$f = f_1 + f_2 \quad \textit{with} \quad \begin{cases} card(supp\ f_1) = k_s \\ \\ \|f_2\|_{\ell_\infty} \leq \dfrac{\sum_{\ell=2}^{s} \alpha_\ell}{k_s}. \end{cases}$$

Proof. It suffices to show the result for functions in $co_m(\mathcal{H})$ for a fixed $m \in \mathbb{N}$.

1) is immediate. To prove 2) let $f \in co_m(\mathcal{H}) \subseteq c_o(\mathbb{N})$. Then there exists $i_1 \in \mathbb{N}$ such that $f(i_1) \geq f(i)$ for all $i \in \mathbb{N}$. We consider now $N_1 = \mathbb{N} \setminus \{i_1\}$. Since $f \in c_o(N_1)$, then there exists $i_2 \in N_1$ such that $f(i_2) \geq f(i)$ for all $i \in N_1$. Hence we can find $A = \{i_1, \ldots, i_{k_s}\}$ such that $f(j) \leq f(i)$ if $i \in A$ and $j \notin A$. Therefore

$$k_s \sup_{j \notin A} f(j) \leq \sum_{i \in A} f(i) \leq \sum_{\ell=2}^{s} \alpha_\ell.$$

3) follows from 2). □

The family $\mathcal{G}_r$ is a generating family which is almost convex.

Lemma 2.3. *Let $r \in \mathbb{N}$ and let $(f_j)_{j \leq m_r}$ be functions in $\mathcal{G}_r$. Let $\xi_j \in \mathbb{R}^+$, $j = 1, \ldots, m_r$, such that $\sum_{j \leq m_r} \xi_j = 1$. Then*

$$\frac{1}{2} \sum_{j \leq m_r} \xi_j f_j \in \mathcal{G}_r.$$

Proof. Since $f_j \in \mathcal{G}_r$ it holds

$$f_j \leq \sum_{\ell=0}^{\infty} 2^{-\ell} \sum_{s \leq m_r^\ell} \gamma_{\ell,s,j} h_{\ell,s,j}$$

with $h_{\ell,s,j} \in \mathcal{H}$, $\gamma_{\ell,s,j} \geq 0$ and $\sum_{s \leq m_r^\ell} \gamma_{\ell,s,j} = 1$ for all ℓ, j. Hence

$$\frac{1}{2} \sum_{j \leq m_r} \xi_j f_j \leq \sum_{\ell=0}^{\infty} 2^{-(\ell+1)} \sum_{\substack{s \leq m_r^\ell \\ j \leq m_r}} \xi_j \gamma_{\ell,s,j} h_{\ell,s,j}$$

and the point is that there are at most $m_r^{\ell+1}$ terms in the last summation. □

Finally we glue the families $\mathcal{G}_r$ as follows:

$$\mathcal{G} = \Big\{ 0 \leq f \leq \sum_{r=1}^{\infty} \gamma_r f_r : f_r \in \mathcal{G}_r, \gamma_r \geq 0, \sum_{r=1}^{\infty} \gamma_r = 1 \Big\}.$$

The family $\mathcal{G}$ is again a generating family with the following convexity property.

Lemma 2.4. *Let $(g_\ell)_{\ell \leq N}$ be a finite collection of functions in $\mathcal{G}$ and let $\xi_\ell \in \mathbb{R}^+$, $\ell = 1, \ldots, N$, such that $\sum_{\ell \leq N} \xi_\ell = 1$. Then*

$$\frac{1}{8} \sum_{\ell=1}^{N} \xi_\ell g_\ell \in \mathcal{G}.$$

Proof. Let us write $g_\ell = \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r}$ with $f_{\ell,r} \in \mathcal{G}_r$, $\gamma_{\ell,r} \in \mathbb{R}^+$ and $\sum_{r=1}^{\infty} \gamma_{\ell,r} = 1$ for all $\ell \leq N$. We let $I_N = [1, N] \cap \mathbb{N}$ and for each $r \geq 1$ we set

$$g'_r = \sum_{\ell \in [1,m_r] \cap I_N} \xi_\ell \gamma_{\ell,r} f_{\ell,r} \quad \text{and} \quad \nu_r = \sum_{\ell \in [1,m_r] \cap I_N} \xi_\ell \gamma_{\ell,r}.$$

By Lemma 2.3 we have that $g'_r \in 2\nu_r \mathcal{G}_r$. On the other hand, if we fix r and take $s \leq r$ we can show that

$$\sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} f_{\ell,s} \in 2 w_s \mathcal{G}_{r+1} \tag{2}$$

where $w_s = \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s}$. Indeed, for all $s \leq r$, $f_{\ell,s} \in \mathcal{G}_s$ and $\mathcal{G}_s \subseteq \mathcal{G}_r$ so that $f_{\ell,s} \in \mathcal{G}_{r+1}$, by Lemma 2.3 we get (2). We take now

$$g''_r = \sum_{s \leq r} \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} f_{\ell,s} \quad \text{and} \quad \delta_r = \sum_{s \leq r} w_s.$$

Then by Lemma 2.3 we have that $g''_r \in 4\delta_r\mathcal{G}_{r+1}$, since $r \le m_r$. Now observe that

$$\sum_{r=1}^{\infty}(\nu_r + \delta_r) = \sum_{r=1}^{\infty}\sum_{\ell=1}^{N}\xi_\ell\gamma_{\ell,r} = 1$$

because

$$\begin{aligned}\sum_{r=1}^{\infty}\delta_r &= \sum_{r=1}^{\infty}\sum_{s\le r}\sum_{\ell\in(m_r,m_{r+1}]\cap I_N}\xi_\ell\gamma_{\ell,s} = \sum_{r=1}^{\infty}\sum_{\ell\in(m_r,m_{r+1}]\cap I_N}\sum_{s\le r}\xi_\ell\gamma_{\ell,s}\\ &= \sum_{r=1}^{\infty}\sum_{\ell\in(m_r,N]\cap I_N}\xi_\ell\gamma_{\ell,r}.\end{aligned}$$

Therefore, using Lemma 2.3 one more time we know that the function $g = \sum_{r\ge1} g'_r + g''_r$ belongs to $8\mathcal{G}$. Now we are going to see that $g = \sum_{\ell=1}^{N}\xi_\ell g_\ell$, so that $\sum_{\ell=1}^{N}\xi_\ell g_\ell \in 8\mathcal{G}$. Indeed,

$$\begin{aligned}\sum_{\ell=1}^{N}\xi_\ell g_\ell &= \sum_{r=1}^{\infty}\sum_{\ell=1}^{N}\xi_\ell\gamma_{\ell,r}f_{\ell,r} = \sum_{r=1}^{\infty}\Big(\sum_{\ell\in[1,m_r]\cap I_N}\xi_\ell\gamma_{\ell,r}f_{\ell,r} + \sum_{\ell\in(m_r,N]\cap I_N}\xi_\ell\gamma_{\ell,r}f_{\ell,r}\Big)\\ &= \sum_{r=1}^{\infty}\Big(g'_r + \sum_{\ell\in(m_r,N]\cap I_N}\xi_\ell\gamma_{\ell,r}f_{\ell,r}\Big).\end{aligned}$$

But

$$\sum_{r=1}^{\infty}\sum_{\ell\in(m_r,N]\cap I_N}\xi_\ell\gamma_{\ell,r}f_{\ell,r} = \sum_{r=1}^{\infty}\sum_{\ell\in(m_r,m_{r+1}]\cap I_N}\sum_{s\le r}\xi_\ell\gamma_{\ell,s}f_{\ell,s} = \sum_{r=1}^{\infty}g''_r.$$

Therefore,

$$\sum_{\ell=1}^{N}\xi_\ell g_\ell = \sum_{r=1}^{\infty}g'_r + g''_r.$$

□

Our first result about concavity of these spaces is the following.

Theorem 2.5. *Let $1 < q < \infty$. Then, the space $X_q(\mathcal{G})$ is q-concave.*

Proof. Let $x_1, \dots, x_N$ be a finite number of elements in $X_q(\mathcal{G})$. We set $S^q = \sum_{\ell=1}^{N}\|x_\ell\|^q$ and $\xi_\ell = \frac{\|x_\ell\|^q}{S^q}$. Then $\sum_{\ell=1}^{N}\xi_\ell = 1$.

For each ℓ take $f_\ell \in \mathcal{G}$ such that

$$\|x_\ell\| \le \frac{4}{3}\langle|x_\ell|, \sqrt[q']{|f_\ell|}\rangle.$$

Hence,

$$\begin{aligned} S^q &\leq \frac{4}{3}\sum_{\ell=1}^{N} \|x_\ell\|^{(q-1)}\langle |x_\ell|, \sqrt[q']{|f_\ell|}\rangle = \frac{4}{3}\sum_{\ell=1}^{N} S^{q/q'} \sqrt[q']{\xi_\ell}\,\langle |x_\ell|, \sqrt[q']{|f_\ell|}\rangle \\ &= \frac{4}{3}S^{q-1}\sum_{\ell=1}^{N}\sum_{i=1}^{\infty} |x_\ell(i)|\,\sqrt[q']{|\xi_\ell f_\ell(i)|}. \end{aligned}$$

Using Hölder's inequality and Lemma 2.4 we have that $\sum_{\ell\leq N}|\xi_\ell f_\ell| \in 8\mathcal{G}$. Now

$$S^q \leq \frac{4}{3}S^{q-1}\sum_{i=1}^{\infty}\Big(\sum_{\ell=1}^{N}|x_\ell(i)|^q\Big)^{\frac{1}{q}}\Big(\sum_{\ell=1}^{N}|\xi_\ell f_\ell(i)|\Big)^{\frac{1}{q'}} \leq \frac{1}{6}S^{q-1}\Big\|\Big(\sum_{\ell=1}^{N}|x_\ell|^q\Big)^{\frac{1}{q}}\Big\|.$$

This implies

$$\Big(\sum_{\ell=1}^{N}\|x_\ell\|^q\Big)^{\frac{1}{q}} \leq \frac{1}{6}\Big\|\Big(\sum_{\ell=1}^{N}|x_\ell|^q\Big)^{\frac{1}{q}}\Big\|$$

and the proof is complete. □

Step 3. For each $r \geq 1$ we write

$$\mathcal{F}_r = \Big\{f \in \mathcal{G}_r :\ \|f\|_{\ell_\infty} \leq \frac{\alpha_{r-1}}{k_{r-1}}\Big\}.$$

Again $\mathcal{F}_r \subseteq c_o(\mathbb{N})$ and $\mathcal{F}_r$ are generating families with $\mathcal{F}_1 \subseteq \mathcal{F}_2$ but now, for $r \geq 2$, $\mathcal{F}_r \not\subset \mathcal{F}_{r+1}$.

Finally we define the generating family

$$\mathfrak{F} = \Big\{0 \leq f \leq \sum_{r=1}^{\infty}\gamma_r f_r : f_r \in \mathcal{F}_r, \gamma_r \geq 0, \sum_{r=1}^{\infty}\gamma_r = 1\Big\}.$$

We have to observe that the family $\mathfrak{F}$ depends on the sequences $(k_s)_{s=0}^{\infty}$, $(\alpha_s)_{s=0}^{\infty}$ and $(m_r)_{r=1}^{\infty}$.

3. q-Orlicz property and lower q-estimate

In this section we prove under suitable conditions on $\mathfrak{F}$ that the space $X_q(\mathfrak{F})$ satisfies a lower q-estimate for $1 < q < \infty$ and has the q-Orlicz property for $2 \leq q < \infty$ (the reader should notice that this is stronger only for $q = 2$).

We begin with some lemmas to be used in the sequel. The first one follows from Lemma 2.3.

Lemma 3.1. *Let $r \in \mathbb{N}$, let $(f_j)_{j\le m_r}$ functions in $\mathcal{F}_r$ and let $\xi_j \in \mathbb{R}^+$, $j = 1, \dots, m_r$, such that $\sum_{j\le m_r} \xi_j = 1$. Then*

$$\frac{1}{2}\sum_{j\le m_r} \xi_j f_j \in \mathcal{F}_r.$$

From here on we will assume another property on the sequence $(\alpha_s)_{s=0}^{\infty}$.

$(*)$ There exists a constant $C \ge 1$ such that $\sum_{\ell=2}^{s} \alpha_\ell \le C\alpha_s$ for all $s \ge 2$.

Lemma 3.2. *Let s, $r \in \mathbb{N}$ with $s \le r$, let $(f_j)_{j\le m_{r+1}}$ be a collection of functions in $\mathcal{F}_s$ and let $\xi_j \in \mathbb{R}^+$, $j = 1, \dots, m_{r+1}$, such that $\sum_{j\le m_{r+1}} \xi_j = 1$. If the sequence $(\alpha_s)_{s=0}^{\infty}$ satisfies $(*)$, then there exists $A_{s,r} \subseteq \mathbb{N}$ with $card(A_{s,r}) = k_r$ such that*

$$\chi_{A_{s,r}^c} \frac{1}{2C} \sum_{j\le m_{r+1}} \xi_j f_j \in \mathcal{F}_{r+1}.$$

Proof. If $r = s = 1$ we only have to notice that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Assume that $r \ge 2$. We define $g = \frac{1}{2}\sum_{j\le m_{r+1}} \xi_j f_j$. If we show that $g \in \mathcal{G}_{r+1}$ and that $\|\frac{1}{C} g\chi_{A_{s,r}^c}\|_{\ell_\infty} \le \alpha_r/k_r$ for a set $A_{s,r}$ of integers then the proof will be finished.

By hypothesis $f_j \in \mathcal{G}_s \subseteq \mathcal{G}_r \subseteq \mathcal{G}_{r+1}$ for all $j \le m_{r+1}$, so by Lemma 2.3 $g \in \mathcal{G}_{r+1}$. On the other hand, by (2) in Proposition 2.2 and $(*)$ we can find $A_{s,r} \subseteq \mathbb{N}$ with $card(A_{s,r}) = k_r$ such that

$$\left\|\frac{1}{C} g\chi_{A_{s,r}^c}\right\|_{\ell_\infty} \le \frac{\sum_{\ell=2}^{r} \alpha_\ell}{Ck_r} \le \frac{\alpha_r}{k_r}.$$

□

Our next result shows a convexity property of the family $\mathfrak{F}$.

Theorem 3.3. *Let $(g_\ell)_{\ell\le N}$ a finite collection of functions in $\mathfrak{F}$ given by*

$$g_\ell \le \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r},$$

where $f_{\ell,r} \in \mathcal{F}_r$, $\gamma_{\ell,r} \in \mathbb{R}^+$ and $\sum_{r=1}^{\infty} \gamma_{\ell,r} = 1$ for all $\ell \le N$. Let $\xi_\ell \in \mathbb{R}^+$ such that $\sum_{\ell\le N} \xi_\ell = 1$ and assume that the sequence $(\alpha_s)_{s=0}^{\infty}$ satisfies

(∗). *Then, there exists* $B_r \subseteq \mathbb{N}$ *with* $card(B_r) \leq rk_r$, $r \geq 1$, *such that the functions defined by*

$$f'_\ell = \chi_{B^c_{r(\ell)}} \sum_{r=1}^{r(\ell)} \gamma_{\ell,r} f_{\ell,r} + \sum_{r=r(\ell)+1}^{\infty} \gamma_{\ell,r} f_{\ell,r},$$

satisfy

$$\frac{1}{8C} \sum_{\ell=1}^{N} \xi_\ell f'_\ell \in \mathfrak{F}$$

where $r(\ell)$ *is chosen so that* $m_{r(\ell)} < \ell \leq m_{r(\ell)+1}$.

Proof. Write $I_N = [1, N] \cap \mathbb{N}$ and set

$$g'_r = \sum_{\ell \in [1,m_r] \cap I_N} \xi_\ell \gamma_{\ell,r} f_{\ell,r} \quad \text{and} \quad \nu_r = \sum_{\ell \in [1,m_r] \cap I_N} \xi_\ell \gamma_{\ell,r}.$$

Then by Lemma 3.1 we have that $g'_r \in 2\nu_r \mathcal{F}_r$.

Fix $r \in \mathbb{N}$ and let $s \leq r$. We consider the functions $(f_{\ell,s})_{\ell \in (m_r, m_{r+1}] \cap I_N} \subseteq \mathcal{F}_s$. Then, by Lemma 3.2 we know that there exists $A_{s,r} \subseteq \mathbb{N}$ with $card(A_{s,r}) = k_r$ such that

$$\chi_{A^c_{s,r}} \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} f_{\ell,s} \in 2C w_s \mathcal{F}_{r+1}$$

where $w_s = \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s}$. Set $B_r = \cup_{s=1}^r A_{s,r}$, and note that $card(B_r) \leq rk_r$. Since $r \leq m_r$, Lemma 3.1 gives that the function

$$g''_r = \chi_{B^c_r} \sum_{s \leq r} \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} f_{\ell,s} \leq \sum_{s \leq r} \chi_{A^c_{s,r}} \sum_{\ell \in (m_r, m_{r+1}] \cap I_N} \xi_\ell \gamma_{\ell,s} f_{\ell,s}$$

belongs to $4C\delta_r \mathcal{F}_{r+1}$, where $\delta_r = \sum_{s \leq r} w_s$. Therefore, applying Lemma 3.1 again we see that the function

$$g = \sum_{r=1}^{\infty} g'_r + g''_r$$

belongs to $8C\mathfrak{F}$. Observe also that $\sum_{r=1}^{\infty} \nu_r + \delta_r = 1$.

Now we are going to define functions f'_ℓ such that $\sum_{\ell \leq N} \xi_\ell f'_\ell = g$. Let us fix $\ell \in \{m_1, \ldots, N\}$, then there exist a unique r such that $m_r < \ell \leq m_{r+1}$. We denote by $r(\ell)$ this unique r and define the function

$$f'_\ell = \chi_{B^c_{r(\ell)}} \sum_{r=1}^{r(\ell)} \gamma_{\ell,r} f_{\ell,r} + \sum_{r=r(\ell)+1}^{\infty} \gamma_{\ell,r} f_{\ell,r}.$$

For $\ell \in \{1, \dots, m_1\}$, we define (corresponding to $r(\ell)=0$) the function $f'_\ell = \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r}$. Thus f'_ℓ can also be expressed as

$$f'_\ell = \sum_{r=1}^{\infty} \gamma_{\ell,r} f_{\ell,r} h_{\ell,r}$$

where $h_{\ell,r} = 1$ if $\ell \le m_r$ and $h_{\ell,r} = \chi_{B^c_{r(\ell)}}$ if $m_r < \ell$. The same proof as in Lemma 2.4 gives that $\sum_{\ell=1}^{N} \xi_\ell f'_\ell = g \in 8C\mathfrak{F}$. $\square$

We need also some general lemmas.

Lemma 3.4. *Let $\mathcal{F}$ be a generating family and let $1 < q < \infty$. Assume that $(x_\ell)_{\ell \le N}$ is a finite collection of elements in $X_q(\mathcal{F})$ and $B \subseteq \mathbb{N}$. Then,*

$$\sum_{\ell=1}^{N} \|x_\ell \chi_B\| \le card(B) \sup_{|\epsilon_\ell|=1} \Big\| \sum_{\ell=1}^{N} \epsilon_\ell x_\ell \Big\|.$$

Proof. Set $c = \sup_{f \in \mathcal{F}} \|f\|_{\ell_\infty}$. Since $c^{1/q'} \|x\|_{\ell_\infty} \le \|x\| \le c^{1/q'} \|x\|_{\ell_1}$, we have

$$\begin{aligned} \sum_{\ell=1}^{N} \|x_\ell \chi_B\| &\le \sum_{\ell=1}^{N} \sum_{i \in B} |x_\ell(i)| C^{1/q'} = \sum_{i \in B} \sum_{\ell=1}^{N} |x_\ell(i)| C^{1/q'} \\ &\le card(B) \sup_{|\epsilon_\ell|=1} \Big\| \sum_{\ell=1}^{N} \epsilon_\ell x_\ell \Big\|_{\ell_\infty} C^{1/q'} \le card(B) \sup_{|\epsilon_\ell|=1} \Big\| \sum_{\ell=1}^{N} \epsilon_\ell x_\ell \Big\| \end{aligned}$$

which yields the result. $\square$

Lemma 3.5. *Let $\mathcal{F}$ be a generating family, $\xi_\ell \in \mathbb{R}^+$, $\ell = 1, \dots, N$, and let $(f_\ell)_{\ell \le N}$ be a finite collection of functions in $\mathcal{F}$ such that $\sum_{\ell \le N} \xi_\ell f_\ell \in \mathcal{F}$.*

a) *If $1 < q < \infty$, then*

$$\sum_{\ell=1}^{N} \langle |x_\ell|, \sqrt[q']{\xi_\ell f_\ell} \rangle \le \Big\| \sum_{\ell=1}^{N} |x_\ell| \Big\|.$$

b) *If $2 \le q < \infty$, then*

$$\sum_{\ell=1}^{N} \langle |x_\ell|, \sqrt[q']{\xi_\ell f_\ell} \rangle \le \sqrt{2} \sup_{|\epsilon_\ell|=1} \Big\| \sum_{\ell=1}^{N} \epsilon_\ell x_\ell \Big\|.$$

Proof. Since $\sum_{\ell\le N}\xi_\ell f_\ell\in\mathcal{F}$, by Hölder's inequality we get

$$\sum_{\ell=1}^{N}\langle|x_\ell|,\sqrt[q']{\xi_\ell f_\ell}\rangle\le\Big\langle\Big(\sum_{\ell=1}^{N}|x_\ell|^q\Big)^{\frac{1}{q}},\sqrt[q']{\sum_{\ell=1}^{N}\xi_\ell f_\ell}\Big\rangle\le\Big\|\Big(\sum_{\ell=1}^{N}|x_\ell|^q\Big)^{\frac{1}{q}}\Big\|$$

. If $1<q<\infty$ then

$$\Big\|\Big(\sum_{\ell=1}^{N}|x_\ell|^q\Big)^{\frac{1}{q}}\Big\|\le\Big\|\sum_{\ell=1}^{N}|x_\ell|\Big\|.$$

Hence a) is true. If $q\ge 2$ by Kintchine's inequality (see [2, 1.10]) we have that there exists a constant $B_1=\sqrt{2}$ such that for all $i\in\mathbb{N}$

$$\Big(\sum_{\ell=1}^{N}|x_\ell(i)|^q\Big)^{\frac{1}{q}}\le\Big(\sum_{\ell=1}^{N}|x_\ell(i)|^2\Big)^{\frac{1}{2}}\le B_1\int_0^1\Big|\sum_{\ell=1}^{N}r_\ell(t)x_\ell(i)\Big|dt.$$

Therefore,

$$\sum_{\ell=1}^{N}\langle|x_\ell|,\sqrt[q']{\xi_\ell f_\ell}\rangle\le\sqrt{2}\int_0^1\Big\|\sum_{\ell=1}^{N}r_\ell(t)x_\ell(i)\Big\|dt\le\sqrt{2}\sup_{t\in[0,1]}\Big\|\sum_{\ell=1}^{N}r_\ell(t)x_\ell\Big\|.$$

From this we get b) and the proof is complete. □

Lemma 3.6. *Let $\mathcal{F}$ be a generating family and let $1<q<\infty$. Suppose that $(\eta_r)_{r=1}^{\infty}$ is an increasing sequence of real numbers and that $\{x_1,\dots,x_N\}$ is a finite collection of elements in $X_q(\mathcal{F})$ such that the sequence $(\|x_\ell\|)_{\ell\le N}$ is decreasing. Let $(C_r)_{r\ge1}$ be subsets of $\mathbb{N}$. Consider, for $r\ge1$, the subsets of $\mathbb{N}$*

$$H_r=\{\ell:1\le\ell\le N,\ m_r<\ell\le m_{r+1}\ \textit{and}\ \|x_\ell\|\le\eta_r\|x_\ell\chi_{C_r}\|\}$$

and let $H=\cup_{r\ge1}H_r$. Then,

$$\sum_{\ell\in H}\|x_\ell\|^q\le\Big(\sum_{\ell=1}^{N}\|x_\ell\|^q\Big)^{\frac{1}{q'}}\sup_{|\epsilon_\ell|=1}\Big\|\sum_{\ell=1}^{N}\epsilon_\ell x_\ell\Big\|\Big(\sum_{r=1}^{\infty}\frac{\eta_r card(C_r)}{\sqrt[q']{m_r}}\Big).$$

Proof. We assume that $\sup_{|\epsilon_\ell|=1}\|\sum_{\ell=1}^{N}\epsilon_\ell x_\ell\|=1$. By Lemma 3.4 and the definition of H_r we know that

$$\sum_{\ell\in H_r}\|x_\ell\|\le\eta_r\sum_{\ell\in H_r}\|x_\ell\chi_{C_r}\|\le\eta_r card(C_r).$$

Thus

$$\sum_{\ell\in H_r}\|x_\ell\|^q\le(\max_{\ell\in H_r}\|x_\ell\|^{q-1})\Big(\sum_{\ell\in H_r}\|x_\ell\|\Big)\le(\max_{\ell\in H_r}\|x_\ell\|^{q-1})\eta_r card(C_r).$$

On the other hand, since $(\|x_\ell\|)_{\ell\le N}$ is decreasing we get

$$\|x_\ell\|^q \le \frac{\sum_{\ell=1}^N \|x_\ell\|^q}{\ell} \le \frac{\sum_{\ell=1}^N \|x_\ell\|^q}{m_r}$$

if $\ell \in H_r$ and so $\|x_\ell\|^{q-1} \le \frac{(\sum_{\ell=1}^N \|x_\ell\|^q)^{\frac{1}{q'}}}{\sqrt[q']{m_r}}$. Whence we conclude that

$$\sum_{\ell\in H} \|x_\ell\|^q \le \Big(\sum_{\ell=1}^N \|x_\ell\|^q\Big)^{\frac{1}{q'}} \Big(\sum_{r=1}^\infty \frac{\eta_r card(C_r)}{\sqrt[q']{m_r}}\Big).$$

□

We are now ready to study the q-Orlicz property and a lower q-estimate of the space $X_q(\mathfrak{F})$.

Theorem 3.7. *Let $(\eta_r)_{r=1}^\infty$ be an increasing sequence of real numbers with $\eta_r \ge 2$. Assume that the sequence $(\alpha_s)_{s=0}^\infty$ satisfies $(*)$ and that the sequences $(\eta_r)_{r=1}^\infty$, $(k_r)_{r=1}^\infty$ and $(m_r)_{r=1}^\infty$ satisfy*

$$\sum_{r=1}^\infty \frac{r\eta_r k_r}{\sqrt[q']{m_r}} < \infty. \tag{3}$$

Then if $1 < q < \infty$ the space $X_q(\mathfrak{F})$ satisfies a lower q-estimate. Furthermore if $2 \le q < \infty$ the space $X_q(\mathfrak{F})$ has the q-Orlicz property.

Proof. Let $N \in \mathbb{N}$ and let $(x_\ell)_{\ell\le N}$ a collection of elements in $X_q(\mathfrak{F})$. We assume that the sequence $(\|x_\ell\|)_{\ell\le N}$ is decreasing. We set $S^q = \sum_{\ell=1}^N \|x_\ell\|^q$ and $\xi_\ell = \frac{\|x_\ell\|^q}{S^q}$. Hence $\sum_{\ell=1}^N \xi_\ell = 1$.

By definition of the norm in $X_q(\mathfrak{F})$, for each ℓ there exists a function $g_\ell \in \mathfrak{F}$ such that

$$\|x_\ell\| \le \frac{4}{3}\langle |x_\ell|, {g_\ell}^{1/q'}\rangle. \tag{4}$$

If we apply Theorem 3.3 to the functions g_ℓ and the numbers $\xi_\ell = \frac{\|x_\ell\|^q}{S^q}$, then we can find functions f'_ℓ so that $\sum_{\ell=1}^N \xi_\ell f'_\ell \in 8C\mathfrak{F}$ and subsets $B_r \subseteq \mathbb{N}$ with $card(B_r) \le rk_r$.

In order to estimate S^q we split it as

$$S^q = \sum_{\ell=1}^N \|x_\ell\|^q = \sum_{\ell=1}^{m_1} \|x_\ell\|^q + \sum_{\ell\in H} \|x_\ell\|^q + \sum_{\ell\notin H\cup\{1,\dots,m_1\}} \|x_\ell\|^q$$

where $H = \cup_{r\geq 1} H_r$ and

$$H_r = \{\ell : 1 \leq \ell \leq N,\ m_r < \ell \leq m_{r+1} \text{ and } \|x_\ell\| \leq \eta_r \|x_\ell \chi_{B_r}\|\}.$$

If $\ell \in H$ then by Lemma 3.6 we have

$$\sum_{\ell\in H} \|x_\ell\|^q \leq S^{q/q'} \sup_{|\epsilon_\ell|=1} \Big\|\sum_{\ell=1}^N \epsilon_\ell x_\ell\Big\| \Big(\sum_{r=1}^\infty \frac{\eta_r r k_r}{\sqrt[q']{m_r}}\Big) \leq T S^{q-1} \sup_{|\epsilon_\ell|=1} \Big\|\sum_{\ell=1}^N \epsilon_\ell x_\ell\Big\|,$$

where $T := \sum_{r=1}^\infty \frac{r\eta_r k_r}{\sqrt[q']{m_r}}$. On the other hand, if $\ell \in \{1, \ldots, m_1\}$ then $g_\ell \leq f'_\ell$ and hence

$$\sum_{\ell=1}^{m_1} \|x_\ell\|^q \leq \frac{4}{3} \sum_{\ell=1}^{m_1} \|x_\ell\|^{q-1} \langle |x_\ell|,\ \sqrt[q']{g_\ell}\rangle \leq \frac{4}{3} \sum_{\ell=1}^{N} \|x_\ell\|^{q-1} \langle |x_\ell|,\ \sqrt[q']{f'_\ell}\rangle.$$

Finally if we assume that $\ell \notin H \cup \{1, \ldots, m_1\}$, then there exists a number $r(\ell) \geq 1$ such that $m_{r(\ell)} < \ell \leq m_{r(\ell)+1}$ and by the definition of H_r we have for $\eta_r \geq 2$

$$\|x_\ell \chi_{B_{r(\ell)}}\| \leq \frac{\|x_\ell\|}{\eta_{r(\ell)}} \leq \frac{\|x_\ell\|}{2}$$

. Whence by (4) we have

$$\begin{aligned} \frac{1}{4}\|x_\ell\| &= \frac{3}{4}\|x_\ell\| - \frac{1}{2}\|x_\ell\| \leq \langle |x_\ell|,\ \sqrt[q']{g_\ell}\rangle - \|x_\ell \chi_{B_{r(\ell)}}\| \\ &\leq \langle |x_\ell|,\ \sqrt[q']{g_\ell}\rangle - \langle |x_\ell \chi_{B_{r(\ell)}}|,\ \sqrt[q']{g_\ell}\rangle \leq \langle |x_\ell \chi_{B^c_{r(\ell)}}|,\ \sqrt[q']{g_\ell}\rangle \\ &= \langle |x_\ell|,\ \sqrt[q']{g_\ell \chi_{B^c_{r(\ell)}}}\rangle \leq \langle |x_\ell|,\ \sqrt[q']{f'_\ell}\rangle \end{aligned}$$

where we have used the fact that $f'_\ell(i) \geq g_\ell \chi_{B^c_{r(\ell)}}(i)$ if $i \in B^c_{r(\ell)}$ and $f'_\ell(i) = \sum_{r=r(\ell)+1}^\infty \gamma_{\ell,r} f_{\ell,r} \geq 0$ if $i \in B_{r(\ell)}$. It follows from these relations that

$$\begin{aligned} \sum_{\ell \notin H \cup \{1,\ldots,m_1\}} \|x_\ell\|^q &\leq 4 \sum_{\ell \notin H \bigcup \{1,\ldots,m_1\}} \|x_\ell\|^{q-1} \langle |x_\ell|,\ \sqrt[q']{f'_\ell}\rangle \\ &\leq 4 \sum_{\ell=1}^N \|x_\ell\|^{q-1} \langle |x_\ell|,\ \sqrt[q']{f'_\ell}\rangle. \end{aligned}$$

Thus

$$\begin{aligned}\sum_{\ell=1}^{m_1}\|x_\ell\|^q+\sum_{\ell\notin H\cup\{1,\dots,m_1\}}\|x_\ell\|^q &\le (\frac{4}{3}+4)\sum_{\ell=1}^{N}\|x_\ell\|^{q-1}\langle|x_\ell|,\ \sqrt[q']{f'_\ell}\rangle\\ &= \frac{16}{3}\sum_{\ell=1}^{N}S^{q-1}\sqrt[q']{\xi_\ell}\,\langle|x_\ell|,\ \sqrt[q']{f'_\ell}\rangle\\ &= \frac{16}{3}S^{q-1}\sum_{\ell=1}^{N}\langle|x_\ell|,\ \sqrt[q']{\xi_\ell f'_\ell}\rangle.\end{aligned}$$

Assume that $1<q<\infty$. Then, by (a) in Lemma 3.5, we get

$$S^q\le\frac{16\sqrt[q']{8C}}{3}S^{q-1}\Big\|\sum_{\ell=1}^{N}|x_\ell|\Big\|+TS^{q-1}\sup_{|\epsilon_\ell|=1}\Big\|\sum_{\ell=1}^{N}\epsilon_\ell x_\ell\Big\|.$$

Therefore

$$\Big(\sum_{\ell=1}^{N}\|x_\ell\|^q\Big)^{\frac{1}{q}}\le\Big(\frac{16\sqrt[q']{8C}}{3}+T\Big)\Big\|\sum_{\ell=1}^{N}|x_\ell|\Big\|$$

and the space $X_q(\mathcal{F})$ satisfies a lower q-estimate.

If $2\le q<\infty$ by (b) in Lemma 3.5 we have

$$S^q\le\Big(\frac{16\sqrt[q']{8C}}{3}\sqrt{2}+T\Big)S^{q-1}\sup_{|\epsilon_\ell|=1}\Big\|\sum_{\ell=1}^{N}\epsilon_\ell x_\ell\Big\|$$

and hence the space $X_q(\mathfrak{F})$ has the q-Orlicz property. □

4. q-Concavity

In this section we show that the space $X_q(\mathfrak{F})$ is not q-concave if the family $\mathfrak{F}$ satisfies some further conditions. In order to do this we need to introduce another increasing sequence of natural numbers $(n_s)_{s=1}^{\infty}$ with $n_1=1$.

Again we need some lemmas.

Lemma 4.1. *Let s, $r\in\mathbb{N}$ with $r\le s$. Let $(n_s)_{s=1}^{\infty}$ be an increasing sequence of natural numbers, $n_1=1$, such that $n_s\le k_{s+1}$ for every $s\ge 1$, and assume that the sequence $(\alpha_s)_{s=0}^{\infty}$ satisfies $(*)$. Then for every function $f\in\mathcal{F}_r$ there exists a pair of functions f_1 and f_2 such* *that* $f = f_1 + f_2$ *with*

$$card(supp f_1) \leq 2m_r^s k_s \quad and \quad \sum_{i=1}^{n_s} f_2(i) \leq \alpha_{s+1}\Big(\frac{n_s}{k_{s+1}} + \frac{C}{2^s}\Big).$$

Proof. Since $f \in \mathcal{G}_r$ we can assume that

$$f = \sum_{\ell=0}^{\infty} 2^{-\ell} \sum_{j \leq m_r^\ell} \zeta_{j,\ell} h_{j,\ell}$$

where $h_{j,\ell} \in \mathcal{H}$, $\zeta_{j,\ell} \in \mathbb{R}^+$ and $\sum_{j\leq m_r^\ell} \zeta_{j,\ell} = 1$ for all ℓ. We know that for each $h_{j,\ell} \in \mathcal{H}$ we can find $h'_{j,\ell}$ and $h''_{j,\ell}$ such that $h_{j,\ell} = h'_{j,\ell} + h''_{j,\ell}$, with $card(supp\ h'_{j,\ell}) = k_s$ and $\|h''_{j,\ell}\|_{\ell_\infty} \leq \frac{\alpha_{s+1}}{k_{s+1}}$. Therefore we can decompose f as $f = f_1 + f_2$ where

$$f_1 = \sum_{\ell=0}^{s} 2^{-\ell} \sum_{j \leq m_r^\ell} \zeta_{j,\ell} h'_{j,\ell}$$

and

$$f_2 = \sum_{\ell=0}^{s} 2^{-\ell} \sum_{j \leq m_r^\ell} \zeta_{j,\ell} h''_{j,\ell} + \sum_{\ell=s+1}^{\infty} 2^{-\ell} \sum_{j \leq m_r^\ell} \zeta_{j,\ell} h_{j,\ell}.$$

Now, the support of f_1 has at most $2k_s m_r^s$ points. Indeed, since $m_1 \geq 2$ and $(m_s)_{s=1}^\infty$ is a strictly increasing sequence we have that

$$\sum_{\ell=0}^{s} m_r^\ell \leq \Big(m_r^s \sum_{\ell=0}^{\infty} (\frac{1}{m_r})^\ell\Big) = m_r^s \frac{1}{1 - \frac{1}{m_r}} \leq \frac{m_r^s}{1 - \frac{1}{2}} = 2m_r^s.$$

Therefore

$$card(supp\, f_1) \leq k_s \sum_{\ell=0}^{s} m_r^\ell \leq 2k_s m_r^s.$$

On the other hand, by $\sum_{i=1}^{n_s} h''_{j,\ell}(i) \leq n_s \frac{\alpha_{s+1}}{k_{s+1}}$, $n_s \leq k_{s+1}$ and (1) in Proposition 2.1,

$$\sum_{i=1}^{n_s} f_2(i) \leq n_s \frac{\alpha_{s+1}}{k_{s+1}} \sum_{\ell=0}^{s} 2^{-\ell} + \sum_{\ell=s+1}^{\infty} 2^{-\ell} \Big(\sum_{j=2}^{s+1} \alpha_j\Big)$$

. Finally, by $(*)$ we get

$$\sum_{i=1}^{n_s} f_2(i) \leq \alpha_{s+1} \frac{n_s}{k_{s+1}} + C\alpha_{s+1} 2^{-s}$$

and conclude the proof of the lemma. □

As a consequence we have:

Lemma 4.2. *Let s, $r \in \mathbb{N}$ with $r \leq s$, and let $(n_s)_{s=1}^{\infty}$ be an increasing sequence of natural numbers with $n_1 = 1$, such that $n_s \leq k_{s+1}$ for every $s \geq 1$. Finally assume that the sequence $(\alpha_s)_{s=1}^{\infty}$ satisfies $(*)$. If $(f_r)_{r=1}^{s}$ are functions in $\mathcal{F}_r$ and $\gamma_r \in \mathbb{R}^+$ so that $\sum_{r\geq 1} \gamma_r = 1$, then there exist f' and f'' functions of $\mathfrak{F}$ so that*

$$\sum_{r=1}^{s} \gamma_r f_r = f' + f''$$

with

$$card(suppf') \leq 2k_s\Big(\sum_{r=1}^{s} m_r^s\Big) \quad and \quad \sum_{i=1}^{n_s} f''(i) \leq \alpha_{s+1}\Big(\frac{n_s}{k_{s+1}} + \frac{C}{2^s}\Big).$$

The new assumption on the sequence $(\alpha_s)_{s=0}^{\infty}$ that will be needed is the following:

$(**)$ There exists a constant $K \geq 0$ such that $\frac{\alpha_{s+1}}{\alpha_s} \leq K$ for all $s \geq 2$.

Proposition 4.3. *Let $(n_s)_{s=1}^{\infty}$ be a 2-lacunary sequence of natural numbers, i.e. $2n_s \leq n_{s+1}$, $n_1 = 1$, such that $k_s \leq n_s \leq k_{s+1}$ and assume that the sequence $(\alpha_s)_{s=0}^{\infty}$ satisfies $(**)$. Let $\tau > 0$ be a fixed integer, $1 < q < \infty$ and let x and y be the vectors belonging to $X_q(\mathfrak{F})$ defined by*

$$x = \sum_{s=2}^{\tau} \frac{1}{\sqrt[q']{\alpha_s}\sqrt[q]{k_s}} \chi_{[k_{s-1},k_s)} \quad and \quad y = \sum_{s=2}^{\tau} \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \chi_{[n_{s-1},n_s)}.$$

Then, there exists a finite number of permutations of the set $\mathbb{N}$, $\{\sigma_1, \ldots, \sigma_N\}$, such that if we set $x_j = x\sigma_j$ then

$$\frac{1}{N}\sum_{j=1}^{N} x_j^q(i) \leq 2(2K^{q-1}+1)y^q(i), \quad for\ all\ i \in \mathbb{N}. \tag{5}$$

Proof. Let N$= n_\tau - n_{\tau-1}$ and let $\sigma \in \Pi(\mathbb{N})$ be defined as

$$\begin{cases} \sigma(n_s - 1) = n_{s-1}, & s \geq 2, \\ \sigma(i) = i + 1, & \text{otherwise.} \end{cases}$$

We take $x_j = x\sigma^j$, $j = 1, \dots, N$. Then for $i \in [n_{s-1}, n_s)$, $s \geq 2$, we have

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N} x_j^q(i) &\leq \frac{1}{N}\Big(\sum_{n_{s-1}\leq j<n_s} x^q(j)\Big)\Big(E\Big[\frac{N}{n_s - n_{s-1}}\Big] + 1\Big)\\ &\leq \frac{2}{n_s - n_{s-1}}\Big(\sum_{n_{s-1}\leq j<k_s} x^q(j) + \sum_{k_s\leq j<n_s} x^q(j)\Big)\\ &= 2\frac{\frac{1}{\alpha_s^{q-1}k_s}(k_s - n_{s-1}) + \frac{1}{\alpha_{s+1}^{q-1}k_{s+1}}(n_s - k_s)}{n_s - n_{s-1}}.\end{aligned}$$

Let $s \geq 2$ and $i \in [n_{s-1}, n_s)$. Since $k_s \leq n_s \leq k_{s+1}$, $n_s \geq 1$, $n_s - n_{s-1} \geq \frac{1}{2}n_s$ and $(\alpha_s)_{s=0}^{\infty}$ satisfies $(**)$ we conclude that

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N} x_j^q(i) &\leq 2\Big(\frac{k_s}{\alpha_s^{q-1}k_s}\frac{1}{(n_s - n_{s-1})} + \frac{(n_s - n_{s-1})}{\alpha_{s+1}^{q-1}k_{s+1}}\frac{1}{(n_s - n_{s-1})}\Big)\\ &\leq 2\Big(\frac{K^{q-1}}{\alpha_{s+1}^{q-1}(n_s - n_{s-1})} + \frac{1}{\alpha_{s+1}^{q-1}k_{s+1}}\Big)\\ &\leq 2\Big(\frac{2K^{q-1}}{\alpha_{s+1}^{q-1}n_s} + \frac{1}{\alpha_{s+1}^{q-1}n_s}\Big) = 2(2K^{q-1}+1)y^q(i).\end{aligned}$$

□

The main theorem of this section is the following:

Theorem 4.4. *Let $1 < q < \infty$ and let $(n_s)_{s=1}^{\infty}$ be a sequence of natural numbers with $n_1 = 1$. Assume that the sequence $(\alpha_s)_{s=0}^{\infty}$ satisfies $(*)$ and $(**)$, and that the sequences $(n_s)_{s=1}^{\infty}$ and $(k_s)_{s=1}^{\infty}$ are 2-lacunary and satisfy that $k_s \leq n_s \leq k_{s+1}$ for all $s \geq 1$. Assume further that the sequences $(k_s)_{s=1}^{\infty}$, $(n_s)_{s=1}^{\infty}$ and $(m_r)_{r=1}^{\infty}$ satisfy*

$$\sum_{s=1}^{\infty}\sqrt[q']{\frac{n_s}{k_{s+1}}} < \infty \quad \text{and} \quad \sum_{s=1}^{\infty}\sqrt[q]{\frac{k_s(\sum_{r=1}^{s} m_r^s)}{n_s}} < \infty.$$

Then, the space $X_q(\mathfrak{F})$ fails to be q-concave.

Proof. Let $\tau > 0$ be a fixed integer and let x, y and x_j, $j = 1, \dots, N$, be the vectors defined in Proposition 4.3. We know that $X_q(\mathfrak{F})$ is a rearrangement invariant space, $h \in \mathfrak{F}$ and $(k_s)_{s=1}^{\infty}$ is a lacunary sequenc.

Therefore $\|x_j\| = \|x\|$ for all $j = 1, \dots, N$ and

$$\|x\| \geq \langle |x|, \sqrt[q']{h} \rangle = \sum_{s=2}^{\tau} \frac{(k_s - k_{s-1}) \sqrt[q']{\alpha_s}}{\sqrt[q']{\alpha_s} \sqrt[q]{k_s} \sqrt[q']{k_s}} = \sum_{s=2}^{\tau} \frac{(k_s - k_{s-1})}{k_s} \geq \frac{1}{2}(\tau - 1).$$

Thus,

$$\sum_{j=1}^{N} \|x_j\|^q = N\|x\|^q \geq \frac{N}{2^q}(\tau - 1)^q.$$

In order to show that

$$\Big(\sum_{j=1}^{N} \|x_j\|^q\Big)^{\frac{1}{q}} \Big/ \Big\| \Big(\sum_{j=1}^{N} |x_j|^q\Big)^{\frac{1}{q}} \Big\|$$

is arbitrarily large we are going to find an upper bound for the denominator in the last expression. By Proposition 4.3 we know that $\frac{1}{N} \sum_{j \leq N} x_j^q(i) \leq 2(2K^{q-1} + 1) y^q(i)$ for all $i \in \mathbb{N}$, and hence it is enough to estimate $\|y\|$.

Let $f \in \mathfrak{F}$ and assume that $f \leq \sum_{r \geq 1} \gamma_r f_r$ with $f_r \in \mathcal{F}_r$, $\gamma_r \geq 0$ and $\sum_{r \geq 1} \gamma_r = 1$. Then

$$\langle |y|, \sqrt[q']{f} \rangle = \sum_{i=1}^{\infty} |y(i)| \sqrt[q']{f(i)} \leq \sum_{s=2}^{\tau} I(s) + II(s) + III(s)$$

where for $s \geq 2$

$$I(s) = \frac{1}{\sqrt[q']{\alpha_{s+1}} \sqrt[q]{n_s}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q']{\sum_{r=1}^{s} \gamma_r f_r(i)},$$

$$II(s) = \frac{1}{\sqrt[q']{\alpha_{s+1}} \sqrt[q]{n_s}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q']{\gamma_{s+1} f_{s+1}(i)},$$

$$III(s) = \frac{1}{\sqrt[q']{\alpha_{s+1}} \sqrt[q]{n_s}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q']{\sum_{r \geq s+2} \gamma_r f_r(i)}.$$

We shall first estimate $II(s)$. We observe that Hölder's inequality and (1) in Proposition 2.2 give us

$$II(s) \leq \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \sum_{i=1}^{n_s} \sqrt[q']{\gamma_{s+1} f_{s+1}(i)} \leq \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \sqrt[q]{n_s \gamma_{s+1}^{q/q'}} \sqrt[q']{\sum_{i=1}^{n_s} f_{s+1}(i)}$$

$$\leq \frac{\gamma_{s+1}^{1/q'}}{\sqrt[q']{\alpha_{s+1}}} \sqrt[q']{\sum_{i=1}^{n_s} f_{s+1}(i)} \leq \frac{\gamma_{s+1}^{1/q'}}{\sqrt[q']{\alpha_{s+1}}} \sqrt[q']{\sum_{\ell=1}^{s+1} \alpha_\ell}.$$

And by $(*)$ we have

$$II(s) \leq \sqrt[q']{\gamma_{s+1}} \sqrt[q']{\frac{C\alpha_{s+1}}{\alpha_{s+1}}} = \sqrt[q']{\gamma_{s+1}} \sqrt[q']{C}.$$

Thus, again, using Hölder's inequality, we have

$$\sum_{s=2}^{\tau} II(s) \leq \sqrt[q']{C} \sum_{s=2}^{\tau} \sqrt[q']{\gamma_{s+1}} \leq \sqrt[q']{C} \sqrt[q]{\tau - 1} \sqrt[q']{\sum_{s=2}^{\tau} \gamma_{s+1}} \leq \sqrt[q']{C} \sqrt[q]{\tau - 1}.$$

To bound $III(s)$, we observe that by Hölder's inequality

$$III(s) \leq \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \sqrt[q]{n_s} \sqrt[q']{\sum_{i=1}^{n_s} \sum_{r \geq s+2} \gamma_r f_r(i)} \leq \sqrt[q']{\frac{n_s}{k_{s+1}}}$$

where in the last step we used that $\|f_r\|_{\ell_\infty} \leq \frac{\alpha_{r-1}}{k_{r-1}} \leq \frac{\alpha_{s+1}}{k_{s+1}}$ for $r \geq s+2$.

Finally, we shall estimate $I(s)$. Let us fix $s \geq 2$. By Lemma 4.2 we can find functions f' and f'' such that $\sum_{r=1}^{s} \gamma_r f_r = f' + f''$ with

$$card(suppf') \leq 2k_s \Big(\sum_{r=1}^{s} m_r^s\Big) \quad \text{and} \quad \sum_{i=1}^{n_s} f''(i) \leq \alpha_{s+1} \Big(\frac{n_s}{k_{s+1}} + \frac{C}{2^s}\Big).$$

This allows us to split $I(s)$ as $I(s) \leq IV(s) + V(s)$ for all $s \geq 2$ where

$$IV(s) = \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q']{f'(i)}$$

and

$$V(s) = \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \sum_{n_{s-1} \leq i < n_s} \sqrt[q']{f''(i)}.$$

By Hölder's inequality,

$$\begin{aligned} IV(s) &\leq \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \sum_{i=1}^{n_s} \sqrt[q']{f'(i)\chi_{suppf'}(i)} \\ &\leq \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} \Big(\sum_{i=1}^{n_s} \chi_{suppf'}(i)\Big)^{\frac{1}{q}} \Big(\sum_{i=1}^{n_s} f'(i)\Big)^{\frac{1}{q'}} \\ &\leq \frac{1}{\sqrt[q']{\alpha_{s+1}}\sqrt[q]{n_s}} (card(suppf'))^{\frac{1}{q}} \Big(\sum_{i=1}^{n_s} f'(i)\Big)^{\frac{1}{q'}}. \end{aligned}$$

Since $card(suppf') \leq 2k_s(\sum_{r=1}^{s} m_r^s)$, $(*)$, (1) in Proposition 2.2 yields

$$IV(s) \leq \sqrt[q]{\frac{2k_s(\sum_{r=1}^{s} m_r^s)}{n_s}} \sqrt[q']{\frac{\sum_{\ell=1}^{s+1}\alpha_\ell}{\alpha_{s+1}}} = \sqrt[q]{2}\,\sqrt[q']{C}\sqrt[q]{\frac{k_s(\sum_{r=1}^{s} m_r^s)}{n_s}}.$$

On the other hand, Hölder's inequality and the fact that $\sum_{i=1}^{n_s} f''(i) \leq \alpha_{s+1}(\frac{n_s}{k_{s+1}} + \frac{C}{2^s})$ imply

$$V(s) \leq \frac{1}{\sqrt[q']{\alpha_{s+1}}} \sqrt[q']{\sum_{i=1}^{n_s} f''(i)} \leq \sqrt[q']{\frac{n_s}{k_{s+1}} + C2^{-s}}.$$

It follows from these relations that

$$\begin{aligned} \langle |y|, \sqrt[q']{f}\rangle &\leq \sqrt[q']{C}\,\sqrt[q']{\tau-1} + 2\sum_{s=2}^{\tau} \sqrt[q']{\frac{n_s}{k_{s+1}}} + \sqrt[q']{C}\sum_{s=2}^{\tau}\frac{1}{2^{s/q'}} \\ &+ \sqrt[q']{C}\sqrt[q]{2}\sum_{s=2}^{\tau}\sqrt[q]{\frac{k_s(\sum_{r=1}^{s} m_r^s)}{n_s}}. \end{aligned}$$

Since we are assuming that $A = \sum_{s=1}^{\infty} \sqrt[q']{\frac{n_s}{k_{s+1}}}$ and $B = \sum_{s=1}^{\infty} \sqrt[q]{\frac{k_s(\sum_{r=1}^{s} m_r^s)}{n_s}}$ are finite we have

$$\langle |y|, \sqrt[q']{f}\rangle \leq \sqrt[q']{C}\sqrt[q]{\tau-1} + 2A + \frac{\sqrt[q']{C}}{2^{1/q'}-1} + \sqrt[q]{2}\,\sqrt[q']{C}B \leq \sqrt[q']{C}\sqrt[q]{\tau-1} + S$$

where S is a constant independent of τ. Putting this altogether we have

$$\frac{\left(\sum_{j=1}^{N} \|x_j\|^q\right)^{\frac{1}{q}}}{\left\|\left(\sum_{j=1}^{N} |x_j|^q\right)^{\frac{1}{q}}\right\|} \geq \frac{\frac{1}{2}\sqrt[q]{N}(\tau-1)}{\sqrt[q]{2(2K^{q-1}+1)}\sqrt[q]{N}(\sqrt[q']{C}\sqrt[q]{\tau-1}+S)}$$

$$= \frac{(\tau-1)}{2\sqrt[q]{2(2K^{q-1}+1)}(\sqrt[q']{C}\sqrt[q]{\tau-1}+S)}.$$

This expression goes to infinity as τ goes to infinity. □

Proof of the Theorem 1.1. Let $1 < q < \infty$ and take $(k_s)_{s=0}^{\infty}$ to be the sequence of natural numbers defined by

$$k_0 = k_1 = 1,$$

$$k_{s+1} = 3^{2s+2s^2(E[q']+1)}\, k_s^{1+s(E[q']+1)}$$

and the sequences

$$\alpha_s = 3^{2s},\ (\alpha_0 = 9), \qquad m_s = (3^{2s}k_s)^{E[q']+1}, \qquad \eta_s = 3^s, \qquad n_s = \frac{k_{s+1}}{3^s}$$

for all $s \geq 1$. These sequences satisfy the assumptions in Theorem 3.7 and in Theorem 4.4, hence $X_q(\mathfrak{F})$ satisfies a lower q-estimate and is not q-concave. □

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