Operators on weighted Bergman spaces.

Oscar Blasco*

Abstract

Let $\rho : (0,1] \to \mathbb{R}^+$ be a weight function and let X be a complex Banach space. We denote by $A_{1,\rho}(\mathbb{D})$ the space of analytic functions in the disc $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|\rho(1-|z|)dA(z) < \infty$ and by $Bloch_\rho(X)$ the space of analytic functions in the disc $\mathbb{D}$ with values in X such that $\sup_{|z|<1} \frac{1-|z|}{\rho(1-|z|)} \|F'(z)\| < \infty$. We prove that, under certain assumptions on the weight, the space of bounded operators $L(A_{1,\rho}(\mathbb{D}), X)$ is isomorphic to $Bloch_\rho(X)$ and some applications of this result are presented. Several properties of generalized vector-valued Bloch functions are also considered.

1 Introduction and Preliminaries.

Weighted Bergman spaces appeared, denoted by B^p, in [?] when looking at the Banach enveloppe of the Hardy spaces H^p for $0 < p < 1$, although they had been implicitely used in the work of Hardy and Littlewood (see [?], or [?] page 87) who established that for $0 < p < 1$

$$\int_0^1 (1-r)^{1/p-2} M_1(F,r)dr \le C||F||_{H^p} \tag{1}$$

for any $F \in H^p$, where $M_q(F,r) = (\int_0^{2\pi} |F(re^{it})|^q \frac{dt}{2\pi})^{1/q}$, $0 < q < \infty$ and $||F||_{H^p} = \sup_{0<r<1} M_p(F,r)$.

This inequality was a crutial point in proving the duality $(H^p)^* = \Lambda_\alpha$ for $0 < p < 1$ and $\alpha = 1/p - 1$ between the Hardy classes H^p and the Lipschitz

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classes Λ_α (see [**?**]). In that paper they denoted by B^p the space of analytic functions in the disc such that

$$||F||_{B^p} = \int_0^1 (1-r)^{1/p-2} M_1(F,r) < \infty.$$

It is known that for $0 < p < 1$, $F \in B^p$ if and only if

$$\int_0^1 (1-r)^{1/p-1} M_1(F',r)dr < \infty.$$

Making $p = 1$ in the last formula one can denote by B^1 the space of analytic functions such that

$$||F||_{B^1} = |F(0)| + \int_0^1 M_1(F',r)dr < \infty.$$

This space was later shown to be the predual of the Bloch space (see [**?**]).

Of course, using polar coordinates, one can realize the spaces as particular weighted Bergman spaces, or weighted Besov spaces, given by analytic functions in the unit disk such that

$$\int_{\mathbb{D}} |F(z)|(1-|z|)^{1/p-2} dA(z) < \infty, \tag{2}$$

or

$$\int_{\mathbb{D}} |F'(z)|(1-|z|)^{1/p-1} dA(z) < \infty, \tag{3}$$

where $dA(z)$ stands for the normalized Lebesgue measure. The reader is referred to [**?**] for a proof of the duality between $A_1(\mathbb{D})$ and *Bloch*.

There are some natural conditions on a weight function $\rho : (0,1] \to \mathbb{R}^+$ which allow to extend several results that hold for $\rho(t) = t^\alpha$ to a more general context.

Definition 1.1 *Let* $0 \leq p, q < \infty$ *and let* $\rho : (0,1] \to \mathbb{R}^+$ *be a continuous function. It is said to be a* (b_q)*-weight if there exists a constant* $C > 0$ *such that*

$$\int_s^1 \frac{\rho(t)}{t^{1+q}} dt \leq C \frac{\rho(s)}{s^q}, 0 < s < 1. \tag{4}$$

It is said to be a (d_p)-weight (or to satisfy Dini condition of order p) if there exists a constant $C > 0$ such that

$$\int_0^s t^p \rho(t) \frac{dt}{t} \leq C s^p \rho(s), 0 < s < 1. \tag{5}$$

These notions turned out to be relevant for different purposes (see [**?**, **?**, **?**, **?**, **?**]). We refer the reader to those papers for examples and properties of these classes of weights.

In this paper we consider the weighted Bergman spaces $A_{1,\rho}(\mathbb{D})$. The just mentioned B^p-spaces correspond to the case $\rho(t) = t^{1/p-2}$ for $0 < p < 1$.

Definition 1.2 *Let $\rho : (0, 1] \to \mathbb{R}^+$ be an integrable function. An analytic function f in the unit disc $\mathbb{D}$ is said to belong to $A_{1,\rho}(\mathbb{D})$ if*

$$\|f\|_{A_{1,\rho}} = \int_{\mathbb{D}} |f(z)|\rho(1 - |z|)dA(z) < \infty.$$

Remark 1.1 *(1) If $\rho \in L^1((0, 1])$ then $H^\infty(\mathbb{D}) \subset A_{1,\rho}(\mathbb{D})$.*

(2) $A_{1,\rho}(\mathbb{D})$ is a closed subspace of the space $L^1(\mathbb{D}, \rho(1 - |z|)dA(z))$.

(3) The polynomials are dense in $A_{1,\rho}(\mathbb{D})$.

Let us denote by P_α and P^*_α, $\alpha > -1$, the operators

$$P_\alpha(f)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{(1 - \bar{w}z)^{2+\alpha}} f(w)dA(w)$$

$$P^*_\alpha(f)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{|1 - \bar{w}z|^{2+\alpha}} f(w)dA(w)$$

for $f \in L^1(\mathbb{D}, (1 - |w|^2)^\alpha dA(w)$

It is well known (see [**?**]) that P^*_α in bounded on $L^p(\mathbb{D})$ and that P_α is a projection on the Bergman spaces $A^p(\mathbb{D})$ if and only if $p > 1 + \alpha$.

The Bergman projection corresponds to $\alpha = 0$ and it will be denoted by P. Since P is not bounded on $L^1(\mathbb{D})$ we now study its boundedness in $L^1(\mathbb{D}, \rho(1 - |z|)dA)$.

Proposition 1.3 *Let $\alpha \geq 0$ and $\rho : (0, 1] \to \mathbb{R}^+$ be a continuous function. If ρ is a (d_1) and (b_α)-weight then P^*_α is a bounded on $L^1(\mathbb{D}, \rho(1 - |z|)dA)$.*

In particular, P_α defines a projection from $L^1(\mathbb{D}, \rho(1-|z|)dA)$ onto $A_{1,\rho}(\mathbb{D})$.

Proof. Let f belong to $L^1(\mathbb{D}, \rho(1-|z|)dA)$. We have

$$\begin{aligned}
\|P_\alpha^* f\|_{A_{1,\rho}(\mathbb{D})} &= \int_{\mathbb{D}} \rho(1-|z|)|P_\alpha^* f(z)|dA(z) \\
&= C\int_{\mathbb{D}} \rho(1-|z|)(\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\bar{w}|^{2+\alpha}}|f(w)|dA(w))dA(z) \\
&\le C\int_{\mathbb{D}} |f(w)|(1-|w|^2)^\alpha(\int_{\mathbb{D}} \frac{\rho(1-|z|)}{|1-z\bar{w}|^{2+\alpha}}dA(z))dA(w) \\
&\approx C\int_{\mathbb{D}} (1-|w|^2)^\alpha|f(w)|(\int_0^1 \frac{\rho(1-r)}{(1-r|w|)^{1+\alpha}}dr)dA(w) \\
&\approx C\int_{\mathbb{D}} |f(w)|(1-|w|^2)^\alpha(\int_0^1 \frac{\rho(t)}{t+(1-|w|)^{1+\alpha}}dt)dA(w) \\
&\approx C\int_{\mathbb{D}} \frac{|f(w)|}{1-|w|}(\int_0^{1-|w|} \rho(t)dt)dA(w) \\
&+ C\int_{\mathbb{D}} |f(w)|(1-|w|)^\alpha(\int_{1-|w|}^1 \frac{\rho(t)}{t^{1+\alpha}}dt)dA(w) \\
&\le C\int_{\mathbb{D}} |f(w)|\rho(1-|w|)dA(w).
\end{aligned}$$

□

Proposition 1.4 *Let $\rho : (0,1] \to \mathbb{R}^+$ be non-increasing, $\rho(1) > 0$ and (d_1)-weight. Set $\rho_1(t) = t\rho(t)$. Then $f \in A_{1,\rho}(\mathbb{D})$ if and only if $f' \in A_{1,\rho_1}(\mathbb{D})$.*

Moreover $\|f'\|_{A_{1,\rho_1}(\mathbb{D})} + |f(0)| \approx \|f\|_{A_{1,\rho}(\mathbb{D})}$.

PROOF.- Use that $M_1(f', r^2) \le C\frac{M_1(f,r)}{1-r}$ to get

$$\begin{aligned}
\|f'\|_{A_{1,\rho_1}(\mathbb{D})} &= \int_0^1 (1-r^2)\rho(1-r^2)M_1(f', r^2)rdr \\
&\le C\int_0^1 \rho(1-r^2)M_1(f,r)dr \le C\|f\|_{A_{1,\rho}(\mathbb{D})}.
\end{aligned}$$

On the other hand

$$|f(0)| \le \int_{\mathbb{D}} |f(w)|dA(w) \le \rho(1)^{-1}\int_{\mathbb{D}} |f(w)|\rho(1-|w|)dA(w).$$

To prove the other inequality, we use $M_1(f,r) \le \int_0^r M_1(f', s)ds + |f(0)|$.

Hence

$$\begin{aligned}
\int_0^1 \rho(1-r)M_1(f,r)dr &\leq \int_0^1 \rho(1-r)(\int_0^r M_1(f',s)ds)dr + |f(0)|\int_0^1 \rho(t)dt \\
&\leq \int_0^1 (\int_s^1 \rho(1-r)dr)M_1(f',s)ds) + |f(0)|\int_0^1 \rho(t)dt \\
&\leq \int_0^1 (\int_0^{1-s} \rho(u)du)M_1(f',s)ds) + |f(0)|\int_0^1 \rho(t)dt \\
&\leq C\int_0^1 (1-s)\rho(1-s)M_1(f',s)ds + |f(0)|\int_0^1 \rho(t)dt \\
&\leq C(\|f'\|_{A_{1,\rho}(\mathbb{D})} + |f(0)|).
\end{aligned}$$

$\square$

Let us now introduce the generalized Bloch classes, extending the notion of Bloch functions:

$$Bloch(X) = \{F : \mathbb{D} \to X \text{ analytic } : \sup_{|z|<1}(1-|z|^2)||F'(z)|| < \infty\}.$$

The reader is referred to [**?**, **?**, **?**, **?**, **?**, **?**, **?**] for different results concerning vector-valued Bloch functions.

Definition 1.5 *Let $\rho : (0,1] \to \mathbb{R}^+$ be a continuous function such that $\frac{\rho(t)}{t}$ is non-increasing and $\rho(1) > 0$ and let X be a complex Banach space. An analytic function from the disc $\mathbb{D}$ into X, $F(z) = \sum_{n=0}^{\infty} x_n z^n$ where $x_n \in X$, is said to belong to $Bloch_\rho(X)$ if there exists a constant $C > 0$ such that*

$$||F'(z)|| \leq C\frac{\rho(1-|z|)}{1-|z|}, \qquad z \in \mathbb{D}.$$

It is easy to see that $Bloch_\rho(X)$ becomes a Banach space under the norm

$$\|F\|_{Bloch_\rho(X)} = ||F(0)|| + \sup_{|z|<1}\{\frac{1-|z|}{\rho(1-|z|)}||F'(z)||\}$$

Definition 1.6 *Let $\rho : (0,1] \to \mathbb{R}^+$ be a continuous function such that $\frac{\rho(t)}{t}$ is non-increasing, $\rho(1) > 0$ and $\lim_{t\to 0^+} \frac{\rho(t)}{t} = \infty$ and let X be a complex Banach space. The little Bloch space $bloch_\rho(X)$ is the closed subspace of $Bloch_\rho(X)$ given by those functions for which*

$$\lim_{|z|\to 1} \frac{1-|z|}{\rho(1-|z|)}||F'(z)|| = 0.$$

Remark 1.2 *(1) There is no loss of generality in assuming $\frac{\rho(t)}{t}$ non-increasing, because the function $\tilde{\rho}$ defined by*

$$\frac{\tilde{\rho}(1-t)}{1-t} = \sup\{M_\infty(F', t) : \|F\|_{Bloch_\rho(X)} \leq 1\}$$

is non-increasing and $Bloch_\rho(X) = Bloch_{\tilde{\rho}}(X)$.

(2) The assumptions $\frac{\rho(t)}{t} \geq \rho(1) > 0$ and $\lim_{t\to 0} \frac{\rho(t)}{t} = \infty$ are needed to have the vector-valued polynomials in the spaces $Bloch_\rho(X)$ and $bloch_\rho(X)$ respectively.

(3) If $\frac{\rho(t)}{t}$ non-increasing then ρ is (b_q)-weight for $q > 1$.
Indeed,

$$\int_s^1 \frac{\rho(t)}{t^{1+q}} dt \leq C \frac{\rho(s)}{s} \int_s^1 \frac{dt}{t^q} \leq C \frac{\rho(s)}{s^q}$$

(4) If ρ is (b_1)-weight and $\frac{\rho(t)}{t} \geq C > 0$ then $\lim_{t\to 0} \frac{\rho(t)}{t} = \infty$.
Indeed,

$$Clog(\frac{1}{s}) = C \int_s^1 \frac{dt}{t} dt \leq \int_s^1 \frac{\rho(t)}{t^2} dt \leq C' \frac{\rho(s)}{s}.$$

In [**?**] it was shown that the boundedness of operators between weighted Bergman spaces and a general Banach space X can be characterized by the fact that a fixed associated vector-valued function belongs to certain Lipschizt space. In the papers [**?**] and [**?**] similar results were extended to weighted spaces $B_p(\rho)$ for $0 < p \leq 1$ and certain generalized Lipschizt classes for weights introduced by Janson (see [**?**]) . Some applications of these results to multipliers, Carleson measures and composition operators were achieved.

In this paper we present an independent proof of some of those results, where we shall addecuate the duality between the Bergman space $A_1(\mathbb{D})$ and Bloch (see [**?**]) to our vector-valued and weighted situation.

Let us now give a natural correspondence between operators and vector-valued analytic functions, that will allow us to identify the bounded operators from $A_{1,\rho}(\mathbb{D})$ into X with $Bloch_\rho(X)$ with equivalent norms. This idea has been used by the author several times with slight modifications (see [**?**], [**?**], [**?**] or [**?**]).

Given an analytic function $F(z) = \sum_{n=0}^\infty x_n z^n$ where $x_n \in X$ we can define a linear operator T_F which acts on polynomials as follows:

$$T_F(\sum_{k=0}^n \alpha_k z^k) = \sum_{k=0}^n \frac{\alpha_k x_k}{k+1}. \qquad (6)$$

Conversely, given a linear operator T defined on some space of analytic functions on the unit disc containing the polynomials and with range in a Banach space X one can define the vector-valued analytic function F_T given by

$$F_T(z) = \sum_{n=0}^{\infty} (n+1) T(u_n) z^n. \tag{7}$$

where $u_n(z) = z^n$.

Now we are ready to state the main theorem of the paper:

Theorem 1.7 *Let $\rho : (0,1] \to \mathbb{R}^+$ be a continuous function such that $\frac{\rho(t)}{t}$ is non-increasing and $\rho(1) > 0$ and let X be a complex Banach space. Assume ρ is a (d_1) and (b_1) weight.*

(i) If $F(z) = \sum_{n=0}^{\infty} T_n z^n \in Bloch(X)$ then T_F extends to a bounded operator in $L(A_{1,\rho}(\mathbb{D}), X)$.

(ii) If T extends to a bounded operator in $L(A_{1,\rho}(\mathbb{D}), X)$ then F_T belongs to $Bloch_\rho(X)$.

In particular, $Bloch_\rho(X) = L(A_{1,\rho}(\mathbb{D}), X)$ with equivalent norms.

It is known that $Bloch_\rho(\mathbb{C})$ for $\rho(t) = t^\alpha$, $0 < \alpha < 1$ and $X = \mathbb{C}$ coincides with the Lipschizt class defined in terms of the modulus of continuity Λ_α (see Theorem 5.1 in [**?**]). The proof works also in the vector-valued case, and then we can say that the space $Bloch_\rho(X)$ for $\rho(t) = t^\alpha$ and $0 < \alpha < 1$ coincides with

$$\Lambda_\alpha(X) = \{f \in C_X(\mathbb{T}) : w(f,t) = \sup_{s \in \mathbb{T}} ||f(e^{i(t+s)} - f(e^{is})|| = O(t^\alpha)\}. \tag{8}$$

Corollary 1.8 *Let $1/2 < p < 1, \alpha = 1/p - 2$ and let X be a Banach space. Then the following are equivalent.*

(i) $T : H_p \to X$ is bounded.

(ii) $F_T \in \Lambda_\alpha(X)$.

PROOF.- (i) $\Longrightarrow$ (ii). Using that $F'_T(z) = \sum_{n=1}^{\infty} (n+1) n T(u_n) z^{n-1}$. Hence if $G_z(w) = \sum_{n=1}^{\infty} (n+1) n w^n z^{n-1} = \frac{2w}{(1-wz)^3}$, one has $F'_T(z) = T(G_z)$.

Hence one has to estimate $||G_z||_{H^p}$. Observe now that

$$||G_z||_{H^p} \leq (\int_0^{2\pi} \frac{dt}{|1 - ze^{it}|^{3p}})^{1/p} \leq C \frac{1}{(1-|z|)^{3-1/p}} = C \frac{1}{(1-|z|)^{1-\alpha}}.$$

(ii) $\Longrightarrow$ (i) Since $\rho(t) = t^{1/p-2}$ satisfies the assumptions in Theorem **??**, one gets T is bounded from $A_{1,\rho}(\mathbb{D})$ into X. Now, by (**??**), $H^p \subset A_{1,\rho}(\mathbb{D})$ and then $T = T_{F_T}$ is also bounded from H^p. $\square$

The reader is referred to [**?**, **?**, **?**] for applications to multipliers, Carleson measures and composition operators of similar nature.

2 Proof of the main theorem.

We need some lemmas before starting the proof.

Lemma 2.1 *Let X be complex Banach spaces, $F(z) = \sum_{n=0}^{\infty} x_n z^n$ for $x_n \in X$ and $g(z) = \sum_{n=0}^{m} \alpha_n z^n$ for $\alpha_n \in \mathbb{C}$. Then*
(i) $T_F(g) = \int_{\mathbb{D}} F(w)g(\bar{w})dA(w)$.
(ii) $\int_{\mathbb{D}}(1-|w|^2)F'(w)g_1(\bar{w})dA(w) = \sum_{n=1}^{m} \frac{\alpha_n x_n}{n+1} = T_F(g) - F(0)g(0)$ where $g_1(z) = \sum_{n=1}^{m} x_n z^{n-1} = \frac{g(z)-g(0)}{z}$.

PROOF.-

$$(i) \int_{\mathbb{D}} F(w)g(\bar{w})dA(w) = \sum_{n,k\geq 0} \int_{\mathbb{D}} x_n \alpha_k w^n \bar{w}^k dA(w) = \sum_{n=0}^{m} \int_{\mathbb{D}} x_n \alpha_n |w|^{2n} dA(w) = \sum_{n=0}^{m} \frac{\alpha_n x_n}{n+1}.$$

$$\begin{aligned}(ii) \int_{\mathbb{D}}(1-|w|^2)F'(w)g_1(\bar{w})dA(w) &= \sum_{n\geq 1, k\geq 1} \int_{\mathbb{D}}(1-|w|^2)n x_n \alpha_k w^{n-1}\bar{w}^{k-1}dA(w) \\ &= \sum_{n=1}^{m} n\alpha_n x_n \int_{\mathbb{D}}(1-|w|^2)|w|^{2n-2}dA(w) \\ &= \sum_{n=1}^{m} \frac{\alpha_n x_n}{n+1}.\end{aligned}$$

$\square$

Lemma 2.2 *Let $\rho : (0,1] \to \mathbb{R}^+$ belong to $L^1((0,1])$ and $\rho(t) \geq Ct$ for $0 < t < 1$. Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ and $f_1(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1} = \frac{f(z)-f(0)}{z}$. Then $f \in A_{1,\rho}(\mathbb{D})$ if and only if $f_1 \in A_{1,\rho}(\mathbb{D})$.*
Morerover $\|f_1\|_{A_{1,\rho}(\mathbb{D})}) + |f(0)| \approx \|f\|_{A_{1,\rho}(\mathbb{D})}$

PROOF.- Using $|\alpha_n|r^n \leq M_1(f,r)$ we obtain for $n \geq 0$

$$\begin{aligned} ||f||_{A_{1,\rho}} &\geq \int_0^1 |\alpha_n|r^n\rho(1-r)dr \\ &\geq C\int_0^1 |\alpha_n|r^n(1-r)dr \\ &= C\frac{|\alpha_n|}{(n+1)^2} \end{aligned}$$

Hence

$$|\alpha_n| \leq C(n+1)^2||f||_{A_{1,\rho}}. \tag{9}$$

Assume first that $f_1 \in A_{1,\rho}(\mathbb{D})$. Since $f(z) = f(0) + zf_1(z)$ we have

$$\|f\|_{A_{1,\rho}(\mathbb{D})} \leq |f(0)|\int_0^1 \rho(t)dt + \|f_1\|_{A_{1,\rho}(\mathbb{D})}.$$

Conversely, assume that $f \in A_{1,\rho}(\mathbb{D})$. By (**??**)

$$\begin{aligned} \int_{\mathbb{D}} |f_1(z)|\rho(1-|z|)dA(z) &\leq C(\sum_{n=0}^{\infty}\frac{(n+1)^2}{2^n})\|f\|_{A_{1,\rho}(\mathbb{D})} \\ &+ 2\int_{1/2}^1 (|f(0)| + M_1(f,r))\rho(1-r)dr. \end{aligned}$$

This gives $\|f_1\|_{A_{1,\rho}(\mathbb{D})} \leq C\|f\|_{A_{1,\rho}(\mathbb{D})}$. □

Proof of Theorem **??**

(i)$\Longrightarrow$ (ii). From (ii) in Proposition **??** we have

$$T_F(g) - F(0)g(0) = \int_{\mathbb{D}}(1-|z|^2)F'(z)g_1(\bar{z})dA(z)$$

for each polynomial g. Hence

$$\begin{aligned} ||T_F(g)|| &\leq ||F(0)||\,|g(0)| + \int_{\mathbb{D}}(1-|z|^2)||F'(z)||\,|g_1(\bar{z})|dA(z) \\ &\leq ||F(0)||\,|g(0)| + \sup_{|z|<1}\frac{1-|z|}{\rho(1-|z|)}||F'(z)||\int_{\mathbb{D}}\rho(1-|z|)|g_1(\bar{z})|dA(z) \\ &\leq ||F||_{Bloch_\rho(X)}(|g(0)| + ||g_1||_{A_{1,\rho}}). \end{aligned}$$

Now we apply Lemma **??** to get $||T_F(g)|| \leq C||F||_{Bloch_\rho(X)}|g||_{A_{1,\rho}}$. Hence we can extend now to $A_{1,\rho}$ using the density of polynomials.

(ii)$\Longrightarrow$ (i). Assume T extends to a bounded operator from $A_{1,\rho}(\mathbb{D})$ into X. We write $u_n(z) = z^n$ and $K_z(w) = \frac{1}{(1-wz)^2}$ for the Bergman kernel.

Clearly, for $n \in \mathbb{N}$,

$$||u_n||_{A_{1,\rho}} = \int_0^1 r^n \rho(1-r)dr \leq C.$$

This shows that $K_z = \sum_{n=0}^\infty (n+1)u_n z^n$ is an absolutely convergent series in $A_{1,\rho}(\mathbb{D})$. Therefore

$$F_T(z) = \sum_{n=0}^\infty (n+1)T(u_n)z^n = T(K_z).$$

Same argument gives

$$F_T'(z) = \sum_{n=1}^\infty (n+1)nT(u_n)z^{n-1} = T(\sum_{n=1}^\infty (n+1)nu_n z^{n-1}).$$

Write $G_z(w) = \sum_{n=1}^\infty (n+1)nw^n z^{n-1} = \frac{2w}{(1-wz)^3}$.

Hence

$$||F_T'(z)|| \leq ||T||.||G_z||_{A_{1,\rho}}.$$

Let us now estimate $||G_z||_{A_{1,\rho}}$

$$\begin{aligned}
||G_z||_{A_{1,\rho}} &\leq C\int_0^1 \rho(1-r)(\int_0^{2\pi} \frac{d\theta}{|1-rze^{i\theta}|^3})dr \\
&\leq C\int_0^1 \frac{\rho(1-r)}{(1-r|z|)^2}dr \\
&= \int_0^{|z|} \frac{\rho(1-r)}{(1-r|z|)^2}dr + \int_{|z|}^1 \frac{\rho(1-r)}{(1-r|z|)^2}dr \\
&\leq \int_0^{|z|} \frac{\rho(1-r)}{(1-r)^2}dr + \frac{1}{(1-|z|)^2}\int_{|z|}^1 \rho(1-r)dr \\
&\leq \int_{1-|z|}^1 \frac{\rho(t)}{t^2}dt + \frac{1}{(1-|z|)^2}\int_0^{1-|z|} \rho(r)dr
\end{aligned}$$

Using now the (d_1) and (b_1) assumptions on ρ one gets

$$||F'_T(z)|| \leq ||T||.||G_z||_{A_{1,\rho}} \leq C\frac{\rho(1-|z|)}{1-|z|}.$$

□

3 Vector-valued generalized Bloch spaces.

Let us now indicate how to construct examples of functions in the generalized Bloch classes in the vector-valued case using Theorem **??**. See [**?**, **?**, **?**, **?**] for more examples.

Proposition 3.1 *Let ρ be a (d_1) and (b_1)-weight on $(0,1]$ and set $\rho_n = \int_0^1 r^n \rho(1-r)dr$. Then*

$$F(z) = (\rho_n z^n)_n = \sum_{n=0}^{\infty} \rho_n e_n z^n \in Bloch_\rho(\ell^1) \tag{10}$$

where $\{e_n\}$ stands for the canonical basis of ℓ^1.

PROOF.- It suffices to see that $F(z) = T(K_z)$ for a bounded operator $T \in L(A_{1,\rho}(\mathbb{D}), \ell^1)$.

Consider $T(g) = (\frac{\alpha_n \rho_n}{n+1})_{n\geq 0}$ for $g(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in A_{1,\rho}(\mathbb{D})$.

Note that Hardy inequality (see [**?**]) gives

$$\sum_{n=0}^{\infty} \frac{|\alpha_n| r^n}{n+1} \leq CM_1(g,r),$$

and then

$$\sum_{n=0}^{\infty} \frac{|\alpha_n|\rho_n}{n+1} = \int_0^1 \rho(1-r) \sum_{n=0}^{\infty} \frac{|\alpha_n| r^n}{n+1} dr \leq C\|g\|_{A_{1,\rho}}.$$

This shows that T is bounded from $A_{1,\rho}(\mathbb{D})$ into ℓ^1. Note now that

$$T(K_z) = (\frac{(n+1)z^n \rho_n}{n+1})_{n\geq 0} = \sum_{n=0}^{\infty} \rho_n e_n z^n.$$

□

Proposition 3.2 *Let $\gamma_n \geq 0$ such that $\gamma_0 > 0$ and*

$$\gamma_n \leq C\frac{1}{n+1}\sum_{k=0}^{n}\gamma_k \qquad n \geq 0.$$

Then

$$\sum_{n=0}^{\infty}\frac{\gamma_n|\alpha_n|}{n+1} \leq C\|g\|_{A_{1,\rho}(\mathbb{D})} \tag{11}$$

where $g(z) = \sum_{n=0}^{\infty}\alpha_n z^n$ and $\rho(1-t) = \sum_{n=0}^{\infty}\gamma_n t^n$.

PROOF.- Observe first that

$$\frac{\rho(1-t)}{1-t} = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}\gamma_k)t^n \geq C^{-1}\sum_{n=0}^{\infty}(n+1)\gamma_n t^n \geq C^{-1}\gamma_0.$$

Define $T(g) = (\frac{\gamma_n\alpha_n}{n+1})_{n\geq 0}$ for $g(z) = \sum_{n=0}^{\infty}\alpha_n z^n$. We have to show that T is bounded from $A_{1,\rho}(\mathbb{D})$ into ℓ^1. It suffices to show that $F(z) = (\gamma_n z^n)_n = \sum_{n=1}^{\infty}\gamma_n e_n z^n \in Bloch_\rho(\ell^1)$ where $\{e_n\}$ stands for the canonical basis of ℓ^1. (The reader should observe that the assumptions (b_1) and (d_1) on the weight are only used in the other implication.)

Note that $F'(z) = \sum_{n=1}^{\infty}n\gamma_n e_n z^{n-1}$ and $\|F'(z)\| = (\sum_{n=1}^{\infty}n\gamma_n|z|^{n-1})$. Hence

$$\|F'(z)\| \leq C(\sum_{n=1}^{\infty}(\sum_{k=1}^{n}\gamma_k)|z|^n) \leq C\frac{\rho(1-|z|)}{1-|z|}.$$

□

We now introduce certain related spaces which can be used to produce more examples.

Definition 3.3 *Let X ba a Banach space and let $\sigma : (0,1] \to \mathbb{R}^+$ be a continuous function, we define by $L^\sigma(X)$ (respect. $A^\sigma(X)$) the space of measurable (respect. analytic) functions in the unit disc $\mathbb{D}$ such that there exists $C > 0$ for which*

$$||F(z)|| \leq C\sigma(1-|z|)$$

for almost all $z \in \mathbb{D}$ with respect to the normalized Lebesgue measure $dA(z)$ (respect. for all $z \in \mathbb{D}$).

It becomes a Banach space under the norm $\|F\|_{L^\sigma(X)}$ (respect. $\|F\|_{A^\sigma(X)}$) given by $\sup_{z\in\mathbb{D}}\frac{1}{\sigma(1-|z|)}||F(z)||$.

Remark 3.1 *$A^\sigma(X) = H^\infty(X)$ for $\sigma(t) = 1$, $A^\sigma(X) = A^\alpha(X)$ (see [?]) for $\sigma(t) = t^{-\alpha}$ for $\alpha > 0$ and $A^\sigma(X) = A^0(X)$ for $\sigma(t) = log(\frac{1}{t})$ (see [?]). We use the notation $L^0(X)$ for the space $L^\sigma(X)$ for $\sigma(t) = log(\frac{1}{t})$ (see [?]).*

Proposition 3.4 *Let X be Banach space and let $\sigma : (0,1] \to \mathbb{R}^+$ be non-increasing and denote $\sigma_1(t) = \int_t^1 \frac{\sigma(s)}{s} ds$ for $0 < t < 1$. Then*

(i) $A^\sigma(X) \subset Bloch_\sigma(X)$.

(ii) $Bloch_\sigma(X) \subset A^{\sigma_1}(X)$.

If $\rho(t) = t\sigma(t)$ is (b_1)-weight then $A^\sigma(X) = Bloch_\sigma(X)$. In particular $A^\alpha(X) = Bloch_\rho(X)$ for $\rho(t) = t^{-\alpha}$.

Proof. (i) Since

$$F'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\xi)}{(z-\xi)^2} d\xi$$

for $\Gamma = \{re^{it} : t \in [0, 2\pi)\}$ where $r = \lambda |z|$ for some $1 < \lambda < \frac{1}{|z|}$, we get

$$\begin{aligned} ||F'(z)|| &\leq \frac{1}{2\pi} \int_\Gamma \frac{||F(\xi)||}{|z-\xi|^2} |d\xi| \\ &\leq C \int_\Gamma \frac{\sigma(1-|\xi|)}{|z-\xi|^2} |d\xi| \\ &\leq C\sigma(1-r) \int_0^{2\pi} \frac{1}{|z-re^{it}|^2} r dt \\ &\leq C\sigma(1-\lambda|z|) \int_0^{2\pi} \frac{1}{r|1-\frac{|z|}{r}e^{-it}|^2} dt \\ &\leq C\frac{\sigma(1-\lambda|z|)}{(r-|z|)} \\ &\leq C\frac{\sigma(1-|z|)}{(\lambda-1)|z|}. \end{aligned}$$

Hence, taking limits as λ goes to $1/|z|$ one gets

$$||F'(z)|| \leq C\frac{\sigma(1-|z|)}{1-|z|}.$$

(ii) Let $F \in Bloch_\sigma(X)$. Simply observe that $F(z) - F(0) = \int_0^1 zF'(rz)zdr$ and then

$$||F(z) - F(0)|| \leq \int_0^1 |z|||F'(rz)||dr \leq C|z| \int_0^1 \frac{\sigma(1-r|z|)}{1-r|z|} dr \leq C \int_{1-|z|}^1 \frac{\sigma(s)}{s} ds.$$

This shows that $||F(z)|| \leq ||F(0)|| + C\sigma_1(1-|z|)$.

Note that ρ in (b_1) then $\sigma_1 \leq C\sigma$ what easily gives the final observation. $\square$

Let us now give general properties of the spaces.

Proposition 3.5 *Let $\rho : (0,1] \to \mathbb{R}^+$ be a continuous function such that $\frac{\rho(t)}{t}$ is non-increasing and $\rho(1) > 0$. Let X be a complex Banach space and $F : \mathbb{D} \to X$ be analytic. Then*

$$\|F\|_{Bloch_\rho(X)} = \lim_{r\to 1} \|F_r\|_{Bloch_\rho(X)},$$

where $F_r(z) = F(rz)$ for $0 < r < 1$.

Proof. Note that $F_r(0) = F(0)$ for all r. Observe that

$$\frac{1-|z|}{\rho(1-|z|)}||F_r'(z)|| = r\frac{1-|z|}{\rho(1-|z|)}||F'(rz)|| = r\frac{(1-|z|)^{1+\alpha}}{\rho(1-|z|)(1-|z|)^\alpha}||F'(rz)||$$

Due to the fact that $\frac{t}{\rho(t)}$ is non-decreasing, we have

$$\frac{1-|z|}{\rho(1-|z|)}||F_r'(z)|| \leq \frac{1-|rz|}{\rho(1-|rz|)}||F'(rz)||$$

what implies that $\|F_r\|_{Bloch_\rho(X)} \leq \|F\|_{Bloch_\rho(X)}$ for all $0 < r < 1$.

Now, given $\varepsilon > 0$ take $z_0 \in \mathbb{D}$ such that $\frac{1-|z_0|}{\rho(1-|z_0|)}||F'(z_0)|| > \|F\|_{Bloch_\rho(X)} - \varepsilon/2$ and take r_0 verifying that

$$r\frac{1-|z_0|}{\rho(1-|z_0|)}||F'(rz_0)|| > \frac{1-|z_0|}{\rho(1-|z_0|)}||F'(z_0)|| - \varepsilon/2$$

for any $r > r_0$. Hence

$$\|F_r\|_{Bloch_\rho(X)} > \|f\|_{Bloch_\rho(X)} - \varepsilon.$$

$\square$

Theorem 3.6 *Let $\rho : (0,1] \to \mathbb{R}^+$ be a continuous function such that $\frac{\rho(t)}{t}$ is non-increasing, $\rho(1) > 0$ and $\lim_{t\to 0^+} \frac{\rho(t)}{t} = \infty$. Let X be a complex Banach space and $F \in Bloch_\rho(X)$. The following are equivalent:*

(i) $F \in bloch_\rho(X)$.

(ii) $\lim_{r\to 1} \|F - F_r\|_{Bloch_\rho(X)} = 0$.

(iii) F belongs to the closure of the X-valued polynomials.

Proof. (i)$\Rightarrow$ (ii) Assume that $\lim_{s\to 1} \frac{1-s}{\rho(1-s)} M_\infty(F', s) = 0$. Note that for all $0 < s < 1$ we have

$$\begin{aligned} \sup_{|z|<1} \frac{1-|z|}{\rho(1-|z|)} \|F'(z) - rF'(rz)\| &\leq 2 \sup_{|z|>s} \frac{1-|z|}{\rho(1-|z|)} M_\infty(F', |z|) \\ &+ C \sup_{|z|\leq s} \|F'(z) - F'_r(z)\|. \end{aligned}$$

Hence, given $\varepsilon > 0$ choose $s_0 < 1$ such that $\sup_{|z|>s_0} \frac{1-|z|}{\rho(1-|z|)} M_\infty(F', |z|) < \frac{\varepsilon}{4}$ and then use that F'_r converges uniformly on compact sets to get $r_0 < 1$ such that $\sup_{|z|\leq s_0} |F'(z) - F'_r(z)| < \frac{\varepsilon}{2}$ for $r > r_0$. Then $\|F - F_r\|_{Bloch_\rho} < \varepsilon$ for $r > r_0$.

(ii)$\Rightarrow$ (iii) Assume now that for each $\varepsilon > 0$ there exists $r_0 < 1$ such that $\|F - F_{r_0}\|_{Bloch_\rho} < \varepsilon/2$. Now we can take a Taylor polynomial of F_{r_0} $P_N = P_N(F_{r_0})$ such that $\|F'_{r_0} - P'_N\|_{H_\infty} < \varepsilon/2A$ where $A = sup_{0<r<1} \frac{1-r}{\rho(1-r)}$. Therefore

$$\|F - P_N(F_{r_0})\|_{Bloch_\rho} \leq \|F - F_{r_0}\|_{Bloch_\rho} + (\sup_{0<r<1} \frac{1-r}{\rho(1-r)}) \|F'_{r_0} - P'_N\|_{H_\infty} < \varepsilon.$$

(iii)$\Rightarrow$ (i) Let P be an X-valued polynomial. Using that

$$\frac{1-r}{\rho(1-r)} M_\infty(P', r) \leq \frac{1-r}{\rho(1-r)} \max_{|z|\leq 1} \|P'(z)\|$$

one has that $P \in bloch_\rho(X)$. The result follows because $bloch_\rho(X)$ is closed in $Bloch_\rho(X)$. $\square$

Proposition 3.7 *Let X be a complex Banach space and let $\rho : (0,1] \to \mathbb{R}^+$ be a continuous function such that $\frac{\rho(t)}{t}$ is non-increasing and $\rho(1) > 0$. Assume that ρ is a weight in (d_1) and (b_1).*

If $F \in bloch_\rho(X)$ then T_F is a compact operator from $A_{1,\rho}(\mathbb{D})$ into X.

PROOF.- Using (4) in Remark **??** and Theorem **??** we have a sequence of polynomials P_n with values in X which approaches F in $bloch_\rho(X)$. Note that the associated operators T_{P_n} are finite rank operators. Due to Theorem **??** one gets that T_{P_n} converges to T in norm. Therefore T is compact. $\square$

Remark 3.2 *The converse of Proposition* **??** *is not true.*

Take $F \in Bloch_\rho(\mathbb{C}) \setminus bloch_\rho(\mathbb{C})$ and $T = T_F$ the corresponding operator for $X = \mathbb{C}$. Now T is compact but $F_T = F \notin bloch_\rho(\mathbb{C})$.

We observe now that the Bergman projection is also well defined for X-valued integrable functions f in $L^1(\mathbb{D}, dA, X)$:

$$P(f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^2} dA(w).$$

Theorem 3.8 *Let X be a Banach space and let σ be a (d_1) and (b_1)-weight in $(0,1]$. Then the Bergman projection P defines a bounded operator from $L^\sigma(X)$ onto $Bloch_\sigma(X)$.*

Proof. Let f belong to $L^\sigma(X)$. Since σ is (d_1)-weight, in particular $\sigma \in L^1((0,1])$, and therefore $f \in L^1(\mathbb{D}, dA, X)$.

Since $(Pf)'(z) = \int_{\mathbb{D}} \frac{2\bar{w}}{(1-\bar{w}z)^3} f(w) dA(w)$ we have

$$\begin{aligned}
\|(Pf)'(z)\| &\leq \|f\|_{L^\sigma} \int_{\mathbb{D}} \frac{2\sigma(1-|w|)}{|1-z\bar{w}|^3} dA(w) \\
&\leq 2\|f\|_{L^\sigma} \int_0^1 \sigma(1-r) \Big(\int_0^{2\pi} \frac{1}{|1-\bar{z}re^{it}|^3} dt \Big) dr \\
&\leq C\|f\|_{L^\sigma} \int_0^1 \frac{\sigma(1-r)}{(1-|z|r)^2} dr \\
&\approx C\|f\|_{L^\sigma} \int_0^1 \frac{\sigma(1-r)}{((1-|z|)+(1-r))^2} dr \\
&\approx C\|f\|_{L^\sigma} \Big(\int_0^{1-|z|} \frac{\sigma(r)}{(1-|z|)^2} dr + \int_{1-|z|}^1 \frac{\sigma(r)}{r^2} dr \Big) \\
&\leq C\|f\|_{L^\sigma} \Big(\frac{1}{(1-|z|)^2} \int_0^{1-|z|} \sigma(t) dt + \frac{\sigma(1-|z|)}{1-|z|} \Big) \\
&\leq C\|f\|_{L^\sigma} \frac{\sigma(1-|z|)}{1-|z|}
\end{aligned}$$

Let us prove the surjectivity. Let $f \in Bloch_\sigma(X)$ with $f(0) = f'(0) = 0$.

If $f(z) = \sum_{n=2}^\infty x_n z^n$, let g be given by

$$g(z) = \frac{(1-|z|^2)f'(z)}{\bar{z}}.$$

We have that $g \in L^\sigma(X)$ since $f \in Bloch_\sigma(X)$ and $f'(0) = 0$. Now write $Pg(z) = \sum_{n=0}^\infty y_n z^n$ and take $n \geq 1$

$$\begin{aligned} y_n &= (n+1)\int_{\mathbb{D}} g(z)\bar{z}^n dA(z) = (n+1)\int_{\mathbb{D}}(1-|z|^2)f'(z)\bar{z}^{n-1}dA(z) \\ &= (n+1)\int_{\mathbb{D}} f'(z)\bar{z}^{n-1}dA(z) - (n+1)\int_{\mathbb{D}} zf'(z)\bar{z}^n dm(z) = x_n. \end{aligned}$$

Also $y_0 = 0$. That is $Pg = f$.

The general case follows by writting $f(z) = f(0) + f'(0)z + f_1(z)$ where f_1 is as above. So if $Pg_1 = f_1$ then $P(f(0) + f'(0)z + g_1) = f$. □

Corollary 3.9 *The Bergman projection maps $L^0(X)$ to $A^0(X)$.*

Proof. Note that $\rho(t) = log(\frac{e}{t})$ is (d_1) and (b_1)-weight. Indeed,

$$\int_0^s log(\frac{e}{t})dt = s(log(\frac{e}{s}) + 1) \leq Cslog(\frac{e}{s}).$$

$$\int_s^1 \frac{log(\frac{e}{t})}{t^2}dt \leq \log(\frac{e}{s})\int_s^1 \frac{dt}{t^2} \leq C\frac{log(\frac{e}{s})}{s}.$$

□

Remark 3.3 *Denote by R the adjoint operator of P_2^*, that is*

$$Rf(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1-\bar{z}w|^4}dA(w),$$

which corresponds to the Berezin transform. The reader is referred to the recent paper [?] for the results on the Berezin transform of functions on L^σ.

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Departamento de Matemáticas
Universidad de Valencia
46100 Burjassot
Valencia
Spain
oblasco@uv.es