A remark on Carleson measures from H^p to $L^q(\mu)$ for $0 < p < q < \infty$

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1 Introduction.

These notes contain an extended version of the lecture I presented in October of 2003 in Málaga and they are part of the material in a joint paper with Hans Jarchow (see [2]).

We are going to work on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, its closure $\overline{\mathbb{D}}$ and the unit circle $\mathbb{T} = \partial\mathbb{D}$. In the sequel, m will be the Haar measure on $\mathbb{T}$ (Borel algebra), so that $dm \equiv dt/2\pi$ and $dA(z)$ the normalized area measure. Given a Borel set $B \subseteq \mathbb{T}$, we shall often write $|B|$ instead of $m(B)$. The Lebesgue spaces $L^p(m)$ will also be denoted $L^p(\mathbb{T})$, $0 < p \leq \infty$. The canonical norm (p-norm if $0 < p < 1$) on $L^p(\mathbb{T})$ is $\| \cdot \|_p$.

Let $\mathcal{I}$ be the collection of half-open intervals in $\mathbb{T}$ of the form $I = \{e^{it} : \theta_1 \leq t < \theta_2\}$ where $0 \leq \theta_1 < \theta_2 < 2\pi$. With each $0 \neq z \in D$, we associate the interval $I(z) \in \mathcal{I}$ such that $|I(z)| = 1 - |z|$ and $z/|z|$ is the center of $I(z)$. Let $S(z)$ be the half-open Carleson box over $I(z)$ which has z on its 'inner arc'; this inner arc and the boundary part 'to the right' are supposed to belong to $S(z)$. For convenience, let us also put $I(0) = \mathbb{T}$, $S(0) = \mathbb{D}$, and for any $I \in \mathcal{I}$ we write $S(I)$ the corresponding Carleson box $S(z_I)$ for $z_I = |z_I|\zeta_I$ where ζ_I is the center of I and $1 - |z_I| = |I|$.

We shall write $P_z(\zeta) = \frac{1-|z|^2}{|1-z\bar{\zeta}|^2}$ for the Poisson kernel. Clearly one has there exists $C > 0$ such that

$$\frac{\chi_{I(z)}}{1 - |z|^2} \leq C P_z. \qquad (1)$$

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Let $f : \mathbb{D} \to \mathbb{C}$ be analytic. For each $0 < r < 1$, $f_r : \overline{\mathbb{D}} \to \mathbb{C} : z \mapsto f(rz)$ is continuous, analytic on $\mathbb{D}$, and $M_p(f, r) := \|f_r\|_p < \infty$ for all $0 < p \leq \infty$. The classical *Hardy space* $H^p(\mathbb{D})$ consists of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that $\|f\|_{H^p} := \sup_{r<1} M_p(f, r)$ is finite. Again, we get a Banach space if $1 \leq p < \infty$, and a p- Banach space if $0 < p < 1$. The usual Banach space of bounded analytic functions will be denoted by $H^\infty(\mathbb{D})$. If $0 < q < p < \infty$, then $H^\infty(\mathbb{D}) \hookrightarrow H^p(\mathbb{D}) \hookrightarrow H^q(D)$ continuously with 'norm' one.

Recall that f is in $H^p(\mathbb{D})$, for $0 < p < \infty$,then $f^*(e^{it}) = lim_{r\to 1} f_r(e^{it})$ exists m-a.e. on $\mathbb{T}$ (Fatou's Theorem). Moreover, an element f^* of $L^p(\mathbb{T})$ is generated in this way, and $f \mapsto f^*$ defines an isometric embedding $H^p(\mathbb{D}) \to L^p(\mathbb{T})$. Its range is the closure $H^p(\mathbb{T})$ of the set of polynomials in $L^p(\mathbb{T})$. This leads to the identification of $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$, and to the use of H^p as a common symbol.

We shall be investigating Carleson measures on $\overline{\mathbb{D}}$, i.e. finite, positive Borel measures μ for which the formal identity $J_\mu : H^p \to L^q(\mu)$ exists, for given values of $0 < p < q < \infty$, as a bounded operator from the Hardy space $H^p(\mathbb{D})$ into the Lebesgue space $L^q(\mu)$.

A characterization of measures on $\mathbb{D}$ for which J_μ is bounded for $p < q$ was obtained by P. Duren, using a modification of the argument given by L. Carleson in the case $p = q$.

Theorem 1.1 *(see [5], page 163) Let μ be a finite measure on $\mathbb{D}$ and let $0 < p < q < \infty$. Then $J_\mu : H^p(\mathbb{D}) \to L^q(\mu)$ is bounded if and only if*

$$\mu_D(S(z)) \leq C \cdot |I(z)|^{q/p} \qquad \forall\, 0 \neq z \in \mathbb{D}.$$

Examples of measures where $J_\mu : H^p(\mathbb{D}) \to L^q(\mu)$ is bounded for $p < q$ had appeared, for instance, in the result due to Hardy-Littlewood [8].

Theorem 1.2 *(see [5], page 87): Let $0 < p < q < \infty$. Then*

$$\left(\int_{\mathbb{D}} (1 - |z|)^{\frac{q}{p}-2} |f(z)|^q dA(z)\right)^{1/q} \leq C\|f\|_p \tag{2}$$

for all $f \in H^p(\mathbb{D})$.

Another example for such a Carleson measures is given by the embedding from H^1 into the Bergman space B^2

$$\left(\int_{\mathbb{D}} |f(z)|^2 dA(z)\right)^{1/2} \leq C\|f\|_1. \tag{3}$$

Let us remark that (3) can also be shown as a consequence of Hardy inequality (see [5], page 48).

We shall present here an alternative proof of Duren's theorem, which does not use ideas in Carleson approach. We shall see that it is actually equivalent to the Hardy and Littlewood result in Theorem 1.2.

When studying Carleson measures is, sometimes, important to consider not only measures on $\mathbb{D}$ but in $\overline{\mathbb{D}}$. For instance measures concentrated on $\mathbb{T}$, or measures coming from composition operators. We shall denote $\mu_{\mathbb{D}}$ and $\mu_{\mathbb{T}}$ the induced measures on $\mathbb{D}$ and $\mathbb{T}$.

Let $0 < p, q < \infty$. A measure μ on $\overline{\mathbb{D}}$ is called a *(p,q) - Carleson measure* if $f \mapsto f$ defines a (linear, bounded) operator

$$J_\mu : H^p(\mathbb{D}) \longrightarrow L^q(\mu) \ .$$

If μ is a (p, q) - Carleson measure then $J_{\mu_{\mathbb{D}}} : H^p(\mathbb{D}) \to L^q(\mu_{\mathbb{D}}) : f \mapsto f$ and $J_{\mu_{\mathbb{T}}} : H^p(\mathbb{T}) \to L^q(\mu_{\mathbb{T}}) : f^* \mapsto f^*$ are well-defined operators.

We first observe that this notion only depends on the ratio p/q.

Lemma 1.3 *(see [2]) Let μ be a measure on $\overline{\mathbb{D}}$ and let $0 < p, s, q < \infty$. Then μ is (p, q)-Carleson measure if and only if μ is (sp, sq)-Carleson measure.*

This says that the case $p < q$ and be reduced to $p/q < 1$. Our main theorem then establishes the following characterization.

Theorem 1.4 *Let μ be a finite measure on $\overline{\mathbb{D}}$ and let $0 < p < 1$. Then the following statements are equivalent:*

(i) $J_\mu : H^p(\mathbb{D}) \to L^1(\mu)$ is bounded if and only if $\mu_{\mathbb{T}} = 0$ and

$$\mu_{\mathbb{D}}(S(z)) \ \leq \ C \cdot |I(z)|^{1/p} \qquad \forall\, 0 \neq z \in \mathbb{D}.$$

(ii) There exists $C > 0$ such that

$$\int_{\mathbb{D}} (1 - |z|)^{\frac{1}{p}-2} |f(z)| dA(z) \leq C \|f\|_p,$$

for all $f \in H^p(\mathbb{D})$.

Direct proofs of (i) and (ii) in Theorem 1.4 can be found in [5]. The proof of (i) follows the same steps as the one by $p = q$. The proof of (ii) uses factorization, but also it can be achieved by using interpolation (see [6]).

2 Proof of Theorem 1.4.

We shall make use of a characterization of Carleson measures in terms of the Poisson kernel. The following lemma is a modification of Lemma 3.3 in [7], see also [1] for a proof.

Lemma 2.1 *(see [2]) Let μ be a finite measure on $\overline{\mathbb{D}}$ and let $0 < \alpha < \beta$. Then*

$$max\{\mu_{\mathbb{D}}(S(z)), \mu_{\mathbb{T}}(I(z))\}, \leq C \cdot |I(z)|^{\alpha} \qquad \forall\, 0 \neq z \in \mathbb{D}$$

if and only if

$$\sup_{|z|<1} \int_{\overline{\mathbb{D}}} \frac{(1-|z|^q)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}} d\mu(w) < \infty.$$

Proof of the theorem.

(i)$\Rightarrow$ (ii) Consider $d\mu_{\mathbb{D}}(z) = (1-|z|)^{\frac{1}{p}-2} dA(z)$ and $d\mu_{\mathbb{T}} = 0$. Clearly

$$\mu_{\mathbb{D}}(S(z)) \approx (1-|z|) \int_{|z|}^{1} (1-r)^{\frac{1}{p}-2} dr = (1-|z|) \int_{0}^{1-|z|} s^{\frac{1}{p}-2} ds = \frac{p}{1-p}(1-|z|)^{1/p}.$$

Hence μ is a $(p,1)$-Carleson measure.

(ii)$\Rightarrow$ (i) Let μ be a $(p,1)$-Carleson measure. Take $z \in \mathbb{D}$ and $f(w) = \frac{1}{(1-\bar{z}w)^{2/p}}$. Hence $\|f\|_p = \frac{1}{(1-|z|^2)^{1/p}}$ and the assumption gives that

$$\int_{\overline{\mathbb{D}}} \frac{1}{|1-\bar{z}w|^{2/p}} d\mu(w) \leq C \frac{1}{(1-|z|^2)^{1/p}}.$$

Hence an application of Lemma 2.1 for $\alpha = 1/p$ and $\beta = 2/p$ shows that

$$max\{\mu_{\mathbb{D}}(S(z)), \mu_{\mathbb{T}}(I(z))\}, \leq C \cdot |I(z)|^{1/p} \qquad \forall\, 0 \neq z \in \mathbb{D}.$$

Let us see that $\mu_{\mathbb{T}} = 0$

Every open set $\Omega \subseteq \mathbb{T}$ is the union of countably many disjoint intervals $I(z)$ and $p < 1$, we may conclude that $\mu_{\mathbb{T}}(\Omega)^p \leq C \cdot |\Omega|$. By regularity of these measures, we even get $\mu_{\mathbb{T}}(B)^p \leq C \cdot |B|$ for all Borel sets $B \subseteq \mathbb{T}$. In particular, $\mu_{\mathbb{T}} \ll m$ and so $d\mu_{\mathbb{T}} = F\, dm$ for some $F \in L^1(m)$. From Lebesgue differenciation theorem one gets $F(\zeta) = \lim_{|I|\to 0, \zeta \in I} \frac{1}{|I|} \int_I F dm \leq \lim_{|I|\to 0, \zeta\in I} |I|^{1/p-1} = 0$ m-a.e. This gives $\mu_{\mathbb{T}} = 0$.

Conversely, by Lemma 2.1, we assume that

$$\sup_{|z|<1}\int_{\mathbb{D}}\frac{(1-|w|^2)^{2-1/p}}{|1-\bar{w}z|^2}d\mu(z)<\infty.$$

Writting $f(z)=\int_{\mathbb{D}}\frac{f(w)}{(1-\bar{w}z)^2}dA(w)$ one has

$$\begin{aligned}\int_{\overline{\mathbb{D}}}|f(z)|d\mu(z) &\leq \int_{\overline{\mathbb{D}}}(\int_{\mathbb{D}}\frac{|f(w)|}{|1-\bar{w}z|^2}dA(w))d\mu(z)\\ &= \int_{\mathbb{D}}(\int_{\overline{\mathbb{D}}}\frac{d\mu(z)}{|1-\bar{w}z|^2})|f(w)|dA(w)\\ &\leq C\int_{\mathbb{D}}|f(w)|(1-|w|)^{1/p-2}dA(w)\\ &\leq C\|f\|_p.\end{aligned}$$

□

3 Compactness of (p,q)-Carleson measures

We say that a measure μ on $\overline{\mathbb{D}}$ is a *compact (p,q) - Carleson measure* if the formal identity $J_\mu : H^p(\mathbb{D}) \to L^q(\mu)$ exists as a compact operator.

As the boundedness, the condition of compactness only depends on p/q.

Lemma 3.1 *(see[2]) Let $0<p,q,r<\infty$ be given and let μ be a measure on $\overline{\mathbb{D}}$. Then μ is a compact (p,q)-Carleson measure if and only if μ is a compact (ps,qs)-Carleson measure.*

Lemma 3.2 *(see [2]) Let μ be a finite measure on $\overline{\mathbb{D}}$ and let $0<\alpha<\beta$. Then*

$$\lim_{|I|\to 0}\frac{max\{\mu_{\mathbb{D}}(S(I)),\mu_{\mathbb{T}}(I)\}}{|I|^\alpha}=0$$

if and only if

$$\lim_{|z|\to 1}\int_{\overline{\mathbb{D}}}\frac{(1-|z|^q)^{\beta-\alpha}}{|1-\bar{w}z|^\beta}d\mu(w)=0.$$

We now present the proof of the formulation of compact embeddings.

Theorem 3.3 *Let $0 < p < q < \infty$ and μ a measure on $\overline{\mathbb{D}}$. Then μ is a compact (p,q)-Carleson measure if and only if $\mu_{\mathbb{T}} = 0$ and*

$$\lim_{|I|\to 0} \frac{\mu_{\mathbb{D}}(S(I))}{|I|^{q/p}} = 0.$$

PROOF. Take an increasing sequence r_n converging to 1. Put $f_n(w) = \frac{(1-r_n^2)^{1/p}}{(1-r_n w)^{2/p}}$. Hence $\|f_n\|_p = 1$ for all $n \in \mathbb{N}$. By assumption there exists a subsequence $f_{r_{n_k}}$ convergent in $L^q(\mu)$. Note that since the pointwise limit is zero then

$$\lim_{k\to\infty} \int_{\mathbb{D}} \frac{(1-r_{n_k}^2)^{q/p}}{|1-r_{n_k} w|^{2q/p}} d\mu(w) = 0.$$

The proof of the implication is completed by invoking Lemma 3.2.

Conversely, assume $\mu_{\mathbb{T}} = 0$ and $\lim_{|I|\to 0} \frac{\mu_{\mathbb{D}}(S(I))}{|I|^{q/p}} = 0$.

Let us show that $J_{\mu_{\mathbb{D}}} : H^{p/q}(\mathbb{D}) \to L^1(\mu_{\mathbb{D}})$ is compact (what is enough invoking Lemma 3.1).

Lemma 3.2 gives that for $\epsilon > 0$ there exists $\delta > 0$ such that, for $1-|z| < \delta$,

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{2-q/p}}{|1-\bar{w}z|^2} d\mu(z) < \epsilon.$$

Same argument as in Theorem 1.4 implies

$$\begin{aligned}
\int_{\overline{\mathbb{D}}} |f(z)| d\mu(z) &\leq \int_{\overline{\mathbb{D}}} \int_{\mathbb{D}} \frac{|f(w)|}{|1-\bar{w}z|^2} dA(w) d\mu(z) \\
&= \int_{\mathbb{D}} \Big(\int_{\overline{\mathbb{D}}} \frac{d\mu(z)}{|1-\bar{w}z|^2}\Big) |f(w)| dA(w) \\
&\leq C \int_{|w|\leq 1-\delta} |f(w)|(1-|w|)^{q/p-2} dA(w) \\
&+ C\epsilon \int_{|w|>1-\delta} |f(w)|(1-|w|)^{q/p-2} dA(w) \\
&\leq C \int_{|w|\leq 1-\delta} |f(w)|(1-|w|)^{q/p-2} dA(w) + C\epsilon \|f\|_p.
\end{aligned}$$

Let (f_n) be a bounded sequence in $H^p(\mathbb{D})$. Then (f_n) is relatively compact in $\mathcal{H}(\mathbb{D})$ and then there exists a subsequence convergent uniformly on compact sets. This and the previous estimates finish the proof. □

Corollary 3.4 *Suppose that $p \leq r < q$. Every (p,q) - Carleson measure is a compact (p,r) - Carleson measure.*

Let X and Y be quasi-Banach spaces with separating duals. Recall that an operator $u : X \to Y$ is *completely continuous* if $\lim_n \|ux_n\|_Y = 0$ holds for every weak null sequence (x_n) in X.

Theorem 3.5 *Let $0 < p < 1$ and μ on $\overline{\mathbb{D}}$ be $(p,1)$ - Carleson measure. Then $J_\mu : H^p(\mathbb{D}) \to L^1(\mu)$ is completely continuous.*

PROOF. Since $\mu_{\mathbb{T}} = 0$ then J_μ is now the formal identity $H^p(\mathbb{D}) \to L^1(\mu_{\mathbb{D}}) : f \mapsto f$ since $\mu = \mu_{\mathbb{D}}$. Let (f_n) be a weak null sequence in $H^p(\mathbb{D})$. By continuity of point evaluations, $\lim_n f_n(z) = 0 \;\forall\, z \in D$. Also, (f_n) is uniformly integrable in $L^1(\mu_{\mathbb{D}})$: given $\varepsilon > 0$ there is a $\delta > 0$ such that if $B \subseteq \mathbb{D}$ is any Borel set with $\mu(B) < \delta$ then $\int_B |f_n|\, d\mu < \varepsilon$ for all n. But $f_n \to 0$ pointwise, so that Egorov's Theorem provides us with a Borel set $B \subseteq \mathbb{D}$ such that $\mu(B) < \delta$ and $\lim_n f_n(z) = 0$ uniformly on $\mathbb{D} \setminus B$. Accordingly, there is an $n_\varepsilon \in \mathbb{N}$ such that $\int_{\mathbb{D}\setminus B} |f_n|\, d\mu < \varepsilon$ for $n \geq n_\varepsilon$. We conclude that $\|f_n\|_{L^1(\mu)} < 2\varepsilon$ for all $n \geq n_\varepsilon$: (f_n) is a null sequence in the Banach space $L^1(\mu)$. $\square$

Theorem 3.6 *Let $0 < p < 1$, and let μ be $(p,1)$- Carleson measure on $\overline{\mathbb{D}}$. Then $J_\mu : H^p(\mathbb{D}) \to L^1(\mu)$ is a weakly compact operator if and only if J_μ is compact.*

PROOF. Assume J_μ is weakly compact. The compactness follows essentially by repeating an argument from the proof of Theorem 3.5. Let (f_n) be a bounded sequence in $H^p(\mathbb{D})$. By Montel's Theorem, some subsequence of (f_n) converges locally uniformly to some $f \in \mathcal{H}(\mathbb{D})$. By Fatou's Lemma, f is in $H^p(\mathbb{D})$. Therefore it suffices to look at a bounded sequence (f_n) in $H^p(\mathbb{D})$ which converges to zero pointwise. By hypothesis and since $\mu_{\mathbb{T}} = 0$, (f_n) is uniformly integrable in $L^1(\mu) = L^1(\mu_{\mathbb{D}})$. But $f_n \to 0$ pointwise on $\mathbb{D}$. In combination with Egorov's Theorem this yields $\lim_n \|f_n\|_{L^1(\mu)} = 0$. $\square$

4 Applications

We shall use the previous results to analyze embedding between Hardy and weighted Bergman spaces. Let $\rho : (0,1] \to [0,\infty)$ be an integrable function. Let us denote by $A^p(\rho)$ the space of analytic functions in the unit disc such that

$$\int_{\mathbb{D}} |f(z)|^p \rho(1-|z|)dA(z) < \infty.$$

The case $\rho(t) = t^{\alpha p - 1}$ is usually denoted A^p_α. The reader is referred to [1] for some results on these spaces.

Theorem 4.1 *Let $0 < p < q < \infty$ and let $\rho : (0,1] \to [0,\infty)$ be an integrable function. Then*

(i) $H^p(\mathbb{D}) \subset A^q(\rho)$ if and only if

$$\int_0^s \rho(t)dt \leq Cs^{\frac{q-p}{p}}.$$

(ii) $H^p(\mathbb{D})$ is compactly contained into $A^q(\rho)$ if and only if

$$\lim_{s\to 0} s^{\frac{p-q}{p}}(\int_0^s \rho(t)dt) = 0.$$

In particular, $H^p \subset A^q_\alpha$ if and only if $\alpha \geq \frac{1}{p} - \frac{1}{q}$.

H^p is compactly contained into A^q_α if and only if $\alpha > \frac{1}{p} - \frac{1}{q}$.

PROOF. (i) Consider $d\mu(z) = \rho(1-|z|)dA(z)$. Using Theorem 1.1 we have that the condition for the embedding is that

$$\int_{S(I)} \rho(1-|z|)dA(z) = |I| \int_{1-|I|}^1 \rho(1-r)dr \leq C|I|^{q/p}.$$

(ii) Same argument but applying Theorem 3.3.

$\square$

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