OPERATORS ON WEIGHTED BERGMAN SPACES (0<p≤1) AND APPLICATIONS

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ABSTRACT: We describe the boundedness of a linear operator from

$$B_p(\rho) = \{f : D \to \mathbb{C} \text{ analytic} : \left(\int_D \frac{\rho(1-|z|)}{(1-|z|)} |f(z)|^p dA(z) \right)^{1/p} < \infty\} \ ,$$

for $0 < p \leq 1$ under some conditions on the weight function ρ, into a general Banach space X by means of the growth conditions at the boundary of certain fractional derivatives of a single X-valued analytic function. This, in particular, allows us to characterize the dual of $B_p(\rho)$ for $0 < p < 1$ and to give a formulation of generalized Carleson measures in terms of the inclusion $B_1(\rho) \subset L^1(D, \mu)$. We then apply the result to the study of multipliers, Hankel operators and composition operators acting on $B_p(\rho)$ spaces.

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INTRODUCTION

During the last decade a big effort has been made to understand operators acting on Bergman and weighted Bergman spaces (see [A], [Z]). Different techniques have been developed for the study of different types of operators (see [AFP], [J2] for Hankel operators, [MS] for composition operators, [W] for multipliers, ...).

The aim of this paper is to deal with operators acting on weighted Bergman spaces in the case $0 < p \leq 1$ and for rather general weight functions. We shall show that in this case the boundedness of an operator into a general Banach space depends only upon the behaviour of a single vector valued analytic function. This will allow us to study Hankel operators, composition operators and multipliers acting on weighted Bergman spaces when $0 < p \leq 1$ from a unified and simple technique.

The vector valued function which represents a bounded operator is obtained by the action of the operator on the reproducing kernel. This has been previously used by N. Kalton (see [K1], [K2]) to characterize operators acting on H^p $(0 < p < 1)$ and related spaces into general q-Banach spaces and by the author (see [B]) to represent general operators acting on certain spaces of vector valued analytic functions.

We shall be concerned with weighted Bergman classes defined by weight functions of the type introduced by S. Janson (see [J1]) which will allow us to include the known cases and to cover new ones under the same scope.

Let ρ be a nondecreasing function on $(0,1)$ with $\rho(0^+) = 0$ and such that $\frac{\rho(t)}{t} \in L^1((0,1))$. ρ is said to be a *Dini-weight* if $\int_0^s \frac{\rho(t)}{t} dt \leq C\rho(s)$. For $0 < q < \infty$, ρ is said to be a b_q*-weight*, $\rho \in b_q$, if $\int_s^1 \frac{\rho(t)}{t^{q+1}} dt \leq C \frac{\rho(s)}{s^q}$.

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We say that an analytic function f on the unit disc belongs to $B_p(\rho)$, $0 < p \leq 1$, if

$$\|f\|_{p,\rho} = \left(\int_D \frac{\rho(1-|z|)}{(1-|z|)} |f(z)|^p dA(z) \right)^{1/p} < \infty.$$

For certain weights the spaces $B_p(\rho)$ have been extensively studied in the literature. They can be regarded as extensions of the classical Bergman spaces ($\rho(t) = t$). Although the condition appearing in the case $p = 1$ and $\rho(t) = t^{1/q-1}$ for $q < 1$ goes back to the work of Hardy and Littlewood (see [HL] Theorem 3.1.), the corresponding space was first studied as a Banach space by P. Duren, B.W. Romberg and A.L. Shield in [DRS] (denoted by B_q). Later, J. Shapiro [S] considered weighted Bergman spaces corresponding to the cases $\rho(t) = t^{(\alpha+1)}$, denoted by A^p_α, and also, T.M. Flett (see [F1], [F2]) studied similar cases even having mixed norms in their definition. Let us finally mention the paper by A.L. Shields and D.L. Williams [SW] where the case $p = 1$ and general pairs of weights were considered.

Let $(X, \| \ \|)$ be a Banach space, $\mathcal{P}$ denote the vector space of all polynomials and $u_n(z) = z^n$. For any linear map $T : \mathcal{P} \to X$ and $\beta > 0$ we define

$$F^\beta(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta + n + 1)}{\Gamma(\beta) n!} x_n z^n \ , \qquad x_n = T(u_n)$$

With this notation we are ready to state our main result.

Theorem 2.1. Let $0 < p \leq 1$, $0 < q < \infty$ and $\beta = \frac{q+1}{p} - 1$. Let ρ be a Dini-weight such that $\rho \in b_q$. The following are equivalent

(i) $T : B_p(\rho) \to X$ is continuous

(ii) F^β is a X-valued analytic functions satisfying

$$\|F^\beta(z)\|^p = O\left(\frac{\rho(1-|z|)}{(1-|z|)^q}\right) \qquad (|z| \to 1)$$

This theorem will allow us to obtain several known results that were proved by different methods, and also to produce lots of applications when considering particular operators acting on weighted Bergman spaces, whenever $0 < p \leq 1$. It can also be used to obtain applications to operators acting on H_p spaces for $0 < p < 1$. This is due to the fact (see [DRS]) that $B_1(\rho)$ for $\rho(t) = t^{1/p-1}$ is the "Banach envelope" of H^p.

The paper is divided into seven sections. Section 1 is devoted to the definitions and first properties of the spaces and weights that will be used in the following sections. The proof of the main result (Theorem 2.1) will be provided in Section 2. In Section 3 we apply the result to obtain the dual space of certain weighted Bergman spaces. In Section 4 we relate Theorem 2.1. to the notion of Carleson measures, proving a characterization of generalized Carleson measures in terms of the inclusion of $B_1(\rho)$ into $L^1(D,\mu)$. We deal with multipliers from weighted Bergman spaces in Section 5, using Theorem 2.1. to describe convolution, pointwise and sequence multipliers from $B_p(\rho)$ into different spaces. In Section 6 we consider Hankel operators, finding the condition on the symbol for the operator to map $B_1(\rho)$ into H^1. Section 7 is devoted to composition operators. We characterize the analytic funtions ϕ for which the operator C_ϕ maps $B_p(D)$ into $B_1(D)$ and $B_1(\rho)$ into H^1.

Throughout the paper $(X, \|.\|)$ will be a Banach space,and H^s and $B_s(D)$ denote the classical Hardy and Bergman spaces on the unit disc. A *weight function* ρ means a nondecreasing function on $(0,1)$ with $\rho(0^+) = 0$ such that $\frac{\rho(t)}{t} \in L^1((0,1))$, $M_p(f,r)$ stands for $\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}$ and C will be a numerical constant not necessarily the same in each instance.

PRELIMINAIRES

Definition 1.1.- Let $0 < p \leq 1$ and ρ a weight function. An analytic function f on the unit disc is said to belong to $B_p(\rho)$ if

$$\|f\|_{p,\rho} = \left(\int_D \frac{\rho(1-|z|)}{(1-|z|)}|f(z)|^p dA(z)\right)^{1/p} < \infty \qquad (1.1)$$

where $dA(z)$ stands for the area measure on D.

Let us first collect some properties of these spaces.

Given a holomorphic function on D and any value $0 < p \leq 1$, if we write $\sigma(s) = \int_0^s \frac{\rho(t)}{t} dt$, then

$$\sigma(1-|z|)M_p^p(f,|z|) \leq \int_{|z|}^1 \frac{\rho(1-r)}{1-r} M_p^p(f,r)dr$$

This implies that for functions in $B_p(\rho)$ one has

$$M_p(f,r) \leq \frac{\|f\|_{p,\rho}}{\sigma^{\frac{1}{p}}(1-r)} \tag{1.2}$$

Moreover

$$M_p(f,r) = o\left(\frac{1}{\sigma^{\frac{1}{p}}(1-r)}\right) \qquad (r \to 1) \tag{1.2'}$$

On the other hand it is well known (see [D], page 36) that

$$|f(z)| \leq C \frac{M_p(f,|z|)}{(1-|z|)^{1/p}}$$

This together with (1.2′) allows us to say that for $f \in B_p(\rho)$ we have

$$|f(z)| = o((1-|z|)^{-1/p}\sigma(1-|z|)^{-1/p}).$$

From this estimate and standard techniques (see [DRS], Theorem 3 for a special case) one can prove the following result.

Proposition 1.1.- Let $0 < p \leq 1$, ρ be a weight function.

(i) $B_p(\rho)$ is a p-Banach space, i.e. $\|f+g\|_{p,\rho}^p \leq \|f\|_{p,\rho}^p + \|g\|_{p,\rho}^p$.

(ii) The polynomials are dense in $B_p(\rho)$.

Definition 1.2.- ρ is said to be a Dini-weight if

$$\int_0^s \frac{\rho(t)}{t} dt \leq C\rho(s) \tag{1.3}$$

Given $0 < q < \infty$, ρ is said to be a b_q-weight, $\rho \in b_q$, if

$$\int_s^1 \frac{\rho(t)}{t^{q+1}} dt \le C \frac{\rho(s)}{s^q} \tag{1.4}$$

Remark 1.1.- Let us assume $\rho \in b_q$. Since ρ is nondecreasing then we always have $\frac{\rho(s)}{s^q} \le C \int_s^1 \frac{\rho(t)}{t^{q+1}} dt$. Therefore $\frac{\rho(t)}{t^q} \le C \frac{\rho(s)}{s^q}$ if $s \le t$ and $\rho(s) = q(\int_0^s t^{q-1} dt) \frac{\rho(s)}{s^q} \le C \int_0^s \frac{\rho(t)}{t} dt$.

In other words, if $\rho \in b_q$ for some $q > 0$, the Dini condition is equivalent to

$$\int_0^s \frac{\rho(t)}{t} dt = O(\rho(s)) \qquad (s \to 0) \tag{1.3'}$$

A very interesting example of a weight function is given in the following

Proposition 1.2.- Let $\alpha > 0,\ \beta \ge 0$ and $\rho(t) = t^\alpha (log\frac{e}{t})^\beta$.

Then ρ is a Dini-weight and $\rho \in b_q$ for $\alpha < q$.

Proof: Making the change of variable $t = su$ we get

$$\int_0^s t^{\alpha-1} (log\frac{e}{t})^\beta dt = s^\alpha \int_0^1 u^{\alpha-1} (log\frac{e}{s} + log\frac{1}{u})^\beta du$$

The Dini condition is easily obtained from $(a+b)^\beta \le C_\beta(\ a^\beta + b^\beta)$.

For $\alpha < q$ we have

$$\int_s^1 t^{\alpha-q-1} (log\frac{e}{t})^\beta dt \le (log\frac{e}{s})^\beta \int_s^1 t^{\alpha-q-1} dt \le C\ \frac{\rho(s)}{s^q}. \qquad ///$$

Let us now present some useful estimates on weights in the following lemma. I would like to point out that (1.5) below was already shown in [BS] and we include its proof (although quite elementary) for the sake of completeness.

Lemma 1.1.- Let ρ be a Dini weight such that $\rho \in b_q$. Then

$$\int_0^1 \frac{\rho(1-r)}{(1-r)(1-rs)^q} dr \le C \frac{\rho(1-s)}{(1-s)^q} \tag{1.5}$$

$$\int_0^1 \frac{\rho(1-r)}{(1-r)}\ s^n dr = O(\rho(\frac{1}{n})) \qquad (n\to\infty) \tag{1.6}$$

Proof:

$$\int_0^1 \frac{\rho(1-r)}{(1-r)(1-rs)^q}dr = \int_0^s \frac{\rho(1-r)}{(1-r)(1-rs)^q}dr + \int_s^1 \frac{\rho(1-r)}{(1-r)(1-rs)^q}dr$$

$$\leq \int_0^s \frac{\rho(1-r)}{(1-r)^{q+1}}dr + \frac{1}{(1-s)^q}\int_s^1 \frac{\rho(1-r)}{(1-r)}dr$$

$$\leq \int_{1-s}^1 \frac{\rho(t)}{t^{q+1}}dt + \frac{1}{(1-s)^q}\int_0^{1-s} \frac{\rho(t)}{t}dt \leq C\ \frac{\rho(1-s)}{(1-s)^q}$$

which gives (1.5)

Let us now prove (1.6). From Remark 1.1. we can write

$$C\rho(\frac{1}{n}) \leq (1-\frac{1}{n})^n \int_0^{\frac{1}{n}} \frac{\rho(t)}{t}dt \leq \int_0^{\frac{1}{n}} \frac{\rho(t)}{t}(1-t)^n dt \leq \int_0^1 \frac{\rho(1-r)}{(1-r)}\ s^n dr$$

On the other hand

$$\int_0^1 \frac{\rho(1-r)}{(1-r)}\ s^n dr = \int_0^{1-\frac{1}{n}} \frac{\rho(1-r)}{(1-r)}\ s^n dr + \int_{1-\frac{1}{n}}^1 \frac{\rho(1-r)}{(1-r)}\ s^n dr$$

$$\leq \int_{\frac{1}{n}}^1 \frac{\rho(t)}{t}\ (1-t)^n dt + \int_0^{\frac{1}{n}} \frac{\rho(t)}{t}dt \leq \int_{\frac{1}{n}}^1 \frac{\rho(t)}{t}\ (1-t)^n dt + C\ \rho(\frac{1}{n})$$

It remains to show $\int_{\frac{1}{n}}^1 \frac{\rho(t)}{t}\ (1-t)^n dt \leq C\ \rho(\frac{1}{n})$. Using $\frac{\rho(t)}{t^q} \leq C\frac{\rho(s)}{s^q}$ for $s \leq t$ we have

$$\int_{\frac{1}{n}}^1 \frac{\rho(t)}{t}\ (1-t)^n dt = \int_{\frac{1}{n}}^1 \frac{\rho(t)}{t^q}\ t^{q-1}(1-t)^n dt$$

$$\leq C\ \rho(\frac{1}{n})n^q \int_0^1 t^{q-1}(1-t)^n dt = C\ \rho(\frac{1}{n})n^q B(q,n+1) \leq C\ \rho(\frac{1}{n})$$

where $B(q,n+1) = \frac{\Gamma(q)n!}{\Gamma(n+q+1)} = O(n^{-q})$. ///

Definition 1.3.- Let G be an X-valued function, analytic on D and continuous on the unit circle and let ρ be a weight funtion. We say that G satisfies a generalized Lipschitz condition, denoted by $G \in \Lambda_\rho(X)$, if

$$sup_{0<\theta<1}\|G(e^{2\pi i(\theta+t)}) - G(e^{2\pi i\theta})\| = O(\rho(|t|)), \qquad (t\to 0) \tag{1.7}$$

We shall denote by $\Lambda_\alpha(X)$ the case $\rho(t) = t^\alpha$.

Similar ideas to those used by Hardy and Littlewood (see [D], page 74) allow us to prove next result. We omit the details of this vector valued situation (see [BlS],[BS] for particular cases).

Lemma 1.2.- Let ρ be a Dini weight such that $\rho \in b_1$. Let G be a vector valued analytic function on D such that $G \in C(\mathbb{T}, X)$. The following are equivalent

(i) $G \in \Lambda_\rho(X)$

(ii) $\|G'(z)\| = O(\frac{\rho(1-|z|)}{1-|z|}) \qquad (|z| \to 1)$.

THE MAIN THEOREM

The proof of the theorem is based on several lemmas. The first one shows that for $0 < p < 1$, $B_p(\rho)$ is continuosly included in $B_1(\rho_p)$ for certain weight functions ρ_p.

Lemma 2.1.- Let $0 < p < 1$, ρ be a Dini weight such that $\rho \in b_q$ for some $q > 0$ and $f \in B_p(\rho)$. Then

$$\int_D \frac{\rho^{1/p}(1-|z|)}{(1-|z|)^{2-1/p}} |f(z)| dA(z) \leq C \ \|f\|_{p,\rho} \tag{2.1}$$

Proof: We first recall the following well known inequality (see [D], page 84)

$$M_1(f,r) \leq C \frac{M_p(f,r)}{(1-r)^{\frac{1}{p}-1}} \tag{2.2}$$

Hence

$$\int_0^1 \frac{\rho(1-r)}{(1-r)^p} M_1^p(f,r) dr \leq C \ \|f\|_{p,\rho}^p \tag{2.3}$$

Now using (1.3′), (1.2) and (2.2), one easily obtains

$$\frac{\rho^{1/p-1}(1-r)}{(1-r)^{2-p-1/p}} M_1^{1-p}(f,r) \leq \|f\|_{p,\rho}^{1-p} \tag{2.4}$$

Combining the inequalities (2.3) and (2.4) we get

$$\int_D \frac{\rho^{1/p}(1-|z|)}{(1-|z|)^{2-1/p}} |f(z)| dA(z) = \int_0^1 \frac{\rho(1-s)}{(1-s)^p} \ M_1^p(f,s) \frac{\rho^{1/p-1}(1-s)}{(1-s)^{2-p-1/p}} \ M_1^{1-p}(f,s) s \ ds$$

$$\leq C\ \|f\|_{p,\rho}^{1-p}\ \int_0^1 \frac{\rho(1-r)}{(1-r)^p} M_1^p(f,r)dr \leq C\ \|f\|_{p,\rho}. \qquad ///$$

If we denote by A^p_α the space $B_p(\rho)$ for $\rho(t) = t^{\alpha+1}$ we obviously have the following corollary.

Corollary 2.1.- ([S], Theorem 3) Let $0 < p < 1$ and $\alpha > -1$.

Then $A^p_\alpha \subset A^1_\sigma$ for $\sigma = \frac{\alpha+2}{p} - 2$

Remark 2.1.- Let me point out that Shapiro's proof of the previous result was based on fractional integration and non trivial results by [HL], while our proof covers more cases and is much more elementary.

Let $\mathcal{P}$ denote the vector space of all analytic polynomials and set $u_n(z) = z^n$. Given any linear map $T : \mathcal{P} \to X$, we can define the following formal power series with values in X

$$F(z) = \sum_{n=0}^{\infty} x_n z^n \ , \qquad x_n = T(u_n)$$

For any value $\beta > 0$, let us consider the fractional derivative

$$F^\beta(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} x_n z^n \ , \qquad x_n = T(u_n)$$

Clearly for $\beta = m$, m a positive integer, we have

$$F^m(z) = \frac{1}{(m-1)!}(z^m F(z))^{(m)}$$

where $g^{(m)}$ stands for the mth-derivative.

Let us consider

$$K_z^\beta(w) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} w^n z^n = \frac{1}{(1-zw)^{\beta+1}}$$

We shall look at this as a vector-valued function. The next lemma gives an estimate of its norm in $B_p(\rho)$.

Lemma 2.2.- Let $0 < p \leq 1,\ 0 < q < \infty$ and $\beta = \frac{q+1}{p} - 1$. Let ρ be a Dini-weight and $\rho \in b_q$. Then

$$\|K_z^\beta\|_{p,\rho}^p = O(\frac{\rho(1-|z|)}{(1-|z|)^q}) \tag{2.5}$$

Proof: The following estimate is easy and well know (see [D], page 65): For $|a| < 1$ and $q > 1$

$$\int_{|\xi|=1} \frac{d\xi}{|1-a\xi|^q} = O(\frac{1}{(1-|a|)^{q-1}}) \qquad (|a| \to 1)$$

This shows that

$$M_p(K_z^\beta, r) = O(\frac{1}{(1-r|z|)^{\beta+1-\frac{1}{p}}})$$

Therefore

$$\|K_z^\beta\|_{p,\rho}^p = \int_0^1 \frac{\rho(1-r)}{(1-r)} M_p^p(K_z^\beta, r)dr \leq C \int_0^1 \frac{\rho(1-r)}{(1-r)(1-r|z|)^{p(\beta+1)-1}} dr$$

Now we get (2.5) from (1.5) in Lemma 1.1. ///

Theorem 2.1.- Let $0 < p \leq 1,\ 0 < q < \infty$ and $\beta = \frac{q+1}{p} - 1$. Let ρ be a Dini-weight such that $\rho \in b_q$ and X a Banach space. The following are equivalent

(i) $T : B_p(\rho) \to X$ is continuous

(ii) F^β is a X-valued analytic functions satisfying

$$\|F^\beta(z)\|^p = O(\frac{\rho(1-|z|)}{(1-|z|)^q}) \qquad (|z| \to 1).$$

Proof: Let us observe first that since $B_p(\rho)$ is a p-Banach space and $\frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} = O(n^\beta)$ then

$$\| \sum_{n=N}^{M} \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} z^n u_n\|_{p,\rho}^p \leq C \sum_{n=N}^{M} n^{\beta p}|z|^{np}\|u_n\|_{p,\rho}^p$$

Since $\|u_n\|_{p,\rho} \leq C$ then the following series converges in $B_p(\rho)$ for $|z| < 1$

$$K_z^\beta = \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} z^n u_n$$

Now the continuity of T implies that $F^\beta(z) = T(K_z^\beta)$ and then lemma 2.2 gives (ii).

Let us now do the converse. From Proposition 1.1. we only have to prove that $\|T(\phi)\| \leq C\|\phi\|_{p,\rho}$ for all polynomials $\phi \in \mathcal{P}$.

Let us take $\phi(z) = \sum_{n=0}^m \alpha_n z^n$. Since $x_n = T(u_n)$ then $T(\phi) = \sum_{n=0}^m \alpha_n x_n$. We shall use the following equality

$$\frac{\Gamma(\beta)n!}{\Gamma(\beta+n+1)} = \int_0^1 (1-r)^{\beta-1} r^n dr = 2\int_0^1 (1-r^2)^{\beta-1} r^{2n+1} dr$$

Using this we can write

$$T(\phi) = \sum_{n=0}^m \alpha_n \ x_n = 2\int_0^1 (1-r^2)^{\beta-1} \sum_{n=0}^m \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} \alpha_n \ x_n r^{2n+1} dr$$

On the other hand

$$\sum_{n=0}^m \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} \alpha_n \ x_n r^{2n+1} = \frac{1}{2\pi}\int_0^{2\pi} \phi(re^{i\theta}) F^\beta(re^{-i\theta}) r d\theta.$$

Therefore

$$\|T(\phi)\| \leq C \int_0^1 \int_0^{2\pi} (1-r^2)^{\beta-1} |\phi(re^{i\theta}| \|F^\beta(re^{-i\theta})\| r dr d\theta$$

$$\leq C \int_D (1-|z|)^{\frac{q+1}{p}-2} \frac{\rho^{1/p}(1-|z|)}{(1-|z|)^{q/p}} |\phi(z)| dA(z)$$

Finally Lemma 2.1 implies $\|T(\phi)\| \leq C\|\phi\|_{p,\rho}$. ///

For special cases this theorem gives a representation of the operators in terms of vector valued Lipschitz functions. To see this let us mention the following elementary lemma.

Lemma 2.3.- Let $\rho \in b_1$, and F bea vector valued analytic function. The following are equivalent

(i) $\|F^1(z)\| = O(\frac{\rho(1-|z|)}{1-|z|})$

(ii) $\|F'(z)\| = O(\frac{\rho(1-|z|)}{1-|z|})$

Proof: Since $F^1(z) = zF'(z) + F(z)$, it suffices to show that both (i) and (ii) implies $\|F(z)\| = O(\frac{\rho(1-|z|)}{1-|z|})$.

Note that for $z = |z|e^{2\pi i\theta}$ we can recover $F(z)$ from F^1 or from F' as follows

$$F(z) = \int_0^{|z|} F'(se^{2\pi i\theta})ds + F(0) \qquad \text{or} \qquad F(z) = \int_0^1 F^1(rz)dr$$

Observe that $\rho \in b_1$ implies $C \leq \frac{\rho(t)}{t}$. This gives in the first case

$$\|F(z)\| \leq C\int_0^{|z|} \frac{\rho(1-s)}{1-s}ds + \|F(0)\| \leq C\int_{1-|z|}^1 \frac{\rho(t)}{t}dt + C$$

$$\leq C\int_{1-|z|}^1 \frac{\rho(t)}{t^2}dt + C\frac{\rho(1-|z|)}{1-|z|} \leq C\frac{\rho(1-|z|)}{1-|z|}.$$

For the second situation

$$\|F(z)\| \leq C\int_0^1 \frac{\rho(1-s|z|)}{1-s|z|}ds = C\int_{1-|z|}^1 \frac{\rho(t)}{t}dt \leq C\frac{\rho(1-|z|)}{1-|z|}. \qquad ///$$

Let us denote by $L(X,Y)$ the space of bounded linear operators. Combining Lemma 2.3, Lemma 1.3 and Theorem 2.1 we obtain the following corollaries.

Corollary 2.2.- Let ρ be a Dini weight such that $\rho \in b_1$.

$$L(B_1(\rho), X) = \Lambda_\rho(X).$$

Remark 2.2.- Since for $0 < p < 1$ and $\rho(t) = t^{1/p-1}$ it is known that $(H_p)^* = (B_1(\rho))^*$ (see [DRS]) we obviously have that for any Banach space X, $L(H_p, X) = L(B_1(\rho), X)$.$\blacksquare$ From this observation one recovers the following result.

Corollary 2.3.-([K_1] Theorem 5.1) Let $1/2 < p < 1$, $\alpha = \frac{1}{p} - 1$.

$$L(H_p, X) = \Lambda_\alpha(X).$$

Remark 2.3.- Note that Corollary 2.3 can be also obtained as a consequence of the duality $(H^p)^* = \Lambda_\alpha$, because $F \in \Lambda_\alpha(X)$ is equivalent to $\xi F \in \Lambda_\alpha$ for all $\xi \in X^*$ (where

$\xi F(z) =< \xi, F(z) >$). Nevertheless Kalton's proof covers the case of quasi-Banach spaces X, where this simple argument can not be applied.

DUALITY RESULTS

Note that putting $X = \mathbb{C}$ in Theorem 2.1 allows us to characterize the dual spaces of $B_p(\rho)$ for $0 < p \leq 1$, but in fact by composing with functionals in X^* one can also get Theorem 2.1. from the knowledge of the dual.

The dual of $B_1(\rho)$ was obtained for particular weight functions by different authors. The case $\rho(t) = t^\alpha$ $(\alpha > 0)$, denoted B^q $(q = \frac{1}{\alpha+1})$, was first proved by P. Duren, B.W. Romberg and A.L. Shields (see [DRS], Theorem 8) and later by T.M. Flett (see [F2], Theorem 2.2), denoted there by $H\Lambda(1, 1, \alpha)$. The limiting case $\alpha = 0$ was considered by J.M. Anderson, J. Clunie and C. Pommerenke (see [ACP], Theorem 2.3). We refer the reader to [SW] for duality results for general pairs of weights, and to [BlS] and [BS] for duality results in the case p=1 and weights of the type we are dealing with.

The case $p < 1$ and $\rho(t) = t^\alpha$ can be obtained from Shapiro's result on the Banach envelope of A_p^α (see [S], Theorem 3). Very recently a direct proof has been provided by M. Marzuq ([M]).

Using Theorem 2.1 (whose proof is inspired by ideas from the papers quoted above) we can get a unified approach to the previous duality results together with new cases. We would like to state the following new result, which covers all of the cases mentioned above.

Let us use the notation $B_p(\alpha, \beta)$ for the case $\rho(t) = t^\alpha (log\frac{e}{t})^\beta$.

Theorem 3.1 Let $0 < p \leq 1,\ 0 < \alpha,\ 0 \leq \beta$ and $m = [\frac{\alpha+1}{p}]$. Then

$$(B_p(\alpha, \beta))^* = \{f : D \to \mathbb{C} \text{ analytic} : |f^{(m}(z)| = O\left(\frac{log\frac{1}{(1-|z|)}^{\beta/p}}{(1-|z|)^{m+1-\frac{\alpha+1}{p}}}\right)\} \tag{3.1}$$

Proof: We know from Proposition 1.2 that $\rho(t) = t^\alpha (log(\frac{e}{t}))^\beta$ satisfies the Dini condition and that $\rho \in b_{\alpha+\varepsilon}$ for any $\varepsilon > 0$. Let us take $\varepsilon = ([\frac{\alpha+1}{p}] + 1)p - (\alpha + 1)$ and then

$\beta = m$. It is not hard to see that we can replace the estimate on $|f^{\beta}(z)|$ by the same on the mth-derivative $|f^{(m)}(z)|$. Hence we get (3.1) as a consequence of Theorem 2.1. under the pairing duality given by

$$< f, \phi >= \sum_{n=1}^{m} a_n \alpha_n \qquad \text{where } f(z) = \sum_{n=1}^{\infty} a_n z^n, \ \phi(z) = \sum_{n=1}^{m} \alpha_n z^n \qquad ///$$

CARLESON MEASURES

We refer the reader to [G], [D] or [Z] for the definitions and properties of Carleson measures and to [D2], [H] and Section 4 of [MS] for the extension of Carleson's theorem to the setting of Hardy and Bergman spaces. In this section we shall see that a similar theorem can be obtained in this more general context and for generalized Carleson measures.

Let us fix some notation. Given $\theta_0 \in [0, 2\pi)$ and $0 < h < 1$ we define

$$S(\theta_0, h) = \{z \in D : |z - e^{i\theta_0}| \leq 2h\}.$$

Let us also denote by $P_z(t) = \frac{1-r^2}{1+r^2-2rcos(\theta-t)} = \frac{1-|z|^2}{|e^{it}-z|^2}$ the Poisson kernel at $z = re^{i\theta}$.

Definition 4.1.- Let ρ be a weight function and μ a finite Borel measure on the disc. μ is said to be a ρ-Carleson measure if

$$\mu(S(\theta, h)) = O(\rho(h)) \qquad (0 \leq \theta < 2\pi, \ 0 < h < 1).$$

The case $\rho(t) = t^{\alpha}$ is called an α-Carleson measure. (We refer the reader to [MS] and [H] for results on α-Carleson measures in the same spirit as our main application in this section.)

We also define the "balayage" of μ to be the following harmonic function

$$P(\mu)(z) = \int_D P_z(w) d\mu(w).$$

Lemma 4.1.- Let μ be a Borel measure on the disc D and let $\beta > 0$ and $\rho \in b_q$ for some $q > 0$. The following are equivalent

(i) $\int_D \frac{d\mu(w)}{|1-\bar{z}w|^\beta} = O(\frac{\rho(1-|z|)}{(1-|z|)^q})$

(ii) $\mu(S(\theta,h)) = O(\rho(h)h^{\beta-q}) \qquad (0<\theta\le 2\pi,\ 0<h<1).$

Proof: Let us assume (i) and fix $0<\theta_0\le 2\pi,\ 0<h<1$. Let us take $z_0=(1-h)e^{i\theta_0}$.

If $w\in S(\theta_0,h)$ then clearly $|1-\bar{z}_0w|\le 3h$ therefore

$$\mu(S(\theta_0,h)) = \int_{S_0} d\mu(w) \le Ch^\beta \int_D \frac{d\mu(w)}{|1-\bar{z}_0w|^\beta} \le C.\rho(h)h^{\beta-q}$$

Conversely let us take $z_0=r_0e^{i\theta_0}$ with $r_0>3/4$, and consider

$$E_0 = S(\theta_0,(1-|z_0|))$$
$$E_n = S(\theta_0,2^n(1-|z_0|)) - S(\theta_0,2^{n-1}(1-|z_0|))$$

An elementary computation shows that for $n\in\mathbf{N}$

$$|1-\bar{z}_0w| \ge 2^{n-1}(1-|z_0|), \qquad w\in E_n$$

Using this estimate and taking $M\in\mathbf{N}$ such that $2^M(1-|z_0|)\ge 2$ we have

$$\int_D \frac{d\mu(w)}{|1-\bar{z}_0w|^\beta} = \sum_{n=0}^{M}\int_{E_n}\frac{d\mu(w)}{|1-\bar{z}_0w|^\beta}$$

$$\le \frac{\mu(S(\theta_0,(1-|z_0|))}{(1-|z_0|)^\beta} + \sum_{n=1}^{M}\frac{\mu(S(\theta_0,2^{n-1}(1-|z_0|))}{2^{(n-1)\beta}(1-|z_0|)^\beta} \le C\sum_{n=1}^{M}\frac{\rho(2^{n-1}(1-|z_0|))}{2^{(n-1)q}(1-|z_0|)^q}$$

$$\le C\sum_{n=1}^{M}\int_{2^{n-1}(1-|z_0|)}^{2^n(1-|z_0|)}\frac{\rho(t)}{t^{q+1}}dt \le C\int_{1-|z_0|}^{1}\frac{\rho(t)}{t^{q+1}}dt = O(\frac{\rho(1-|z_0|)}{(1-|z_0|)^q}) \qquad ///$$

Theorem 4.1.- Let μ be a finite Borel measure on the disc and ρ a Dini weight such that $\rho\in b_2$. The following are equivalent

(i) μ is a ρ-Carleson measure

(ii) $|P(\mu)(z)| = O(\frac{\rho(1-|z|)}{1-|z|})$

(iii) $B_1(\rho)\subset L_1(D,\mu)$ with continuity.

Proof: Lemma 4.1 gives (i) if and only if (ii).

The equivalence between (ii) and (iii) follows from Theorem 2.1. ///

Corollary 4.1.-([D2]) Let $1/3 < p < 1$, μ be a finite Borel measure on D and $\alpha = \frac{1}{p} - 1$. The following are equivalent

(i) $H_p(D) \subset L_1(D,\mu)$ with continuity.

(ii) μ is α-Carleson measure.

Proof: Use Remark 2.2. and observe that $\rho(t) = t^\alpha \in b_2$. ///

MULTIPLIERS

Throughout this section $0 < p \leq 1$ and $1 \leq s \leq \infty$, $\lambda = (\lambda_n)$ will denote a bounded sequence and $g_\lambda(z) = \sum_{n=0}^\infty \lambda_n z^n$.

Definition 5.1.-

$\lambda \in \mathcal{M}(B_p(\rho), H^s)$ (*convolution multiplier* from $B_p(\rho)$ into H^s) if

$$g_\lambda * f(z) = \sum_{n=0}^\infty \lambda_n a_n z^n \in H^s \qquad \text{for all } f(z) = \sum_{n=0}^\infty a_n z^n \in B_p(\rho) \tag{5.1}$$

Theorem 5.1.- Let $0 < p \leq 1 \leq s \leq \infty,\ 0 < q < \infty$ and $\beta = \frac{q+1}{p} - 1$. Let ρ be a Dini-weight such that $\rho \in b_q$. The following are equivalent

(i) $\lambda \in \mathcal{M}(B_p(\rho), H^s)$

(ii) $M_s(g_\lambda^\beta, r) = O\left(\frac{\rho(1-r)^{1/p}}{(1-r)^{q/p}}\right)$

Proof: Consider the operator given by $T(u_n) = \lambda_n u_n$. The associated vector valued function is

$$F(z)(w) = \sum_{n=0}^\infty \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!}\lambda_n (zw)^n = g_\lambda^\beta(zw)$$

Hence applying Theorem 2.1 we get the result. ///

Let us consider the Zygmund classes ([Zy])

$$\Lambda_*^s = \{g : D \to \mathbb{C} \text{ analytic } : M_s(g'', r) = O(\frac{1}{1-r})\}$$

Corollary 5.1.- Let $1 \leq s \leq \infty$. Then $\lambda \in \mathcal{M}(B_1(D), H^s) = \Lambda^s_*$.

Corollary 5.2.-([DS1], Theorem 4) Let $0 < p < 1 \leq s \leq \infty$ and let $\frac{1}{n+1} \leq p < \frac{1}{n}$. Then $\lambda \in \mathcal{M}(H^p, H^s)$ if and only if $M_q(g_\lambda^{(n+1}, r) = O\left(\frac{1}{(1-r)^{n+2-1/p}}\right)$.

Remark 5.1.- I would like to point out that this vector valued approach for the characterization of multipliers has already been used by the author in [B], and that one can use the argument above, this time using the representation given by N. Kalton (see [K1] Theorem 5.1) of operators from H^p spaces into r-Banach spaces (for $0 < p < r \leq 1$), to get the extension of the previous corollary to the case $0 < p < q \leq \infty$ that has been recently proved by M. Mateljevic and M. Pavlovic (see [MP], Theorem 2).

Definition 5.2.-

$\lambda \in \mathcal{PM}(B_p(\rho), B_s(D))$ (*pointwise multiplier* from $B_p(\rho)$ into $B_s(D)$) if

$$f.g_\lambda(z) = \sum_{n=0}^{\infty}(\sum_{j+l=n} \lambda_j a_l)z^n \in B^s \qquad \text{for all } f(z) = \sum_{n=0}^{\infty} a_n z^n \in B_p(\rho) \tag{5.2}$$

We refer the reader to [S] and [KS] for results concerning pointwise multipliers on Dirichlet type spaces and their connection with α-Carleson measures and ρ-Carleson measures.

Theorem 5.2.- Let $0 < p \leq 1 \leq s < \infty$, $0 < q < \infty$ and $\beta = \frac{q+1}{p} - 1$, let ρ be a Dini-weight such that $\rho \in b_q$ and let $\rho_{p,s}(t) = t^{s/p}\rho^{s/p}(t)$. The following are equivalent

(i) $\lambda \in \mathcal{PM}(B_p(\rho), B_s(D))$

(ii) $d\mu(z) = |g_\lambda(z)|^s dA(z)$ is a $\rho_{p,s}$-Carleson measure.

Proof: Condition (i) means that $B_p(\rho) \subset L^s(D, d\mu)$ for $d\mu(z) = |g_\lambda(z)|^s dA(z)$.

This, using Theorem 2.1, is equivalent to

$$\left(\int_D \frac{|g_\lambda(w)|^s dA(w)}{|1 - zw|^{(\beta+1)s}}\right)^{p/s} = O\left(\frac{\rho(1-|z|)}{(1-|z|)^q}\right)$$

Observe now that $\rho \in b_q$ implies $\rho^{s/p} \in b_{qs/p}$, which allows the use of Lemma 4.1 to obtain

$$\mu(S(\theta,h)) = O(\rho^{s/p}(h)h^{s/p}) \qquad (0 < \theta \leq 2\pi, 0 < h < 1). \qquad ///$$

Definition 5.3.-

$\lambda \in \mathcal{SM}(B_p(\rho), l^s)$ (*sequence multiplier* from $B_p(\rho)$ into l^s) if

$$(\lambda_n a_n) \in l^s \qquad \text{for all } f(z) = \sum_{n=0}^{\infty} a_n z^n \in B_p(\rho) \tag{5.3}$$

Lemma 5.1. Let ρ be a Dini weight and $\rho(t) \in b_q$ for some $q > 0$. Let $(\alpha_n) \geq 0$ and $\beta \geq q-1$. The following are equivalent

(i) $\quad \sum_{n=0}^{\infty} \alpha_n r^n \leq C\frac{\rho(1-r)}{(1-r)^\beta}$

(ii) $\quad N^{-\beta}\sum_{n=0}^{N} \alpha_n \leq C\rho(\frac{1}{N})$

Proof: Assume (i) and consider $r = 1 - 1/N$. Since $(1-1/N)^k \geq (1-1/N)^N \geq C$ for all $k \leq N$ we have

$$\sum_{n=1}^{N} \alpha_n \leq C\sum_{n=1}^{N} \alpha_n (1-\frac{1}{N})^k \leq C\rho(\frac{1}{N})N^\beta$$

For the converse implication, let us write

$$\frac{1}{(1-r)}(\sum_{n=0}^{\infty} \alpha_n r^n) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} \alpha_k)r^n \leq C\sum_{n=0}^{\infty} \rho(\frac{1}{n})n^\beta r^n$$

Now use (1.6) in Lemma 1.1 to get

$$\sum_{n=0}^{\infty} \rho(\frac{1}{n})n^\beta r^n \leq C\sum_{n=0}^{\infty}\int_0^1 \frac{\rho(1-s)}{(1-s)}n^\beta (sr)^n ds$$

$$= C\int_0^1 \frac{\rho(1-s)}{(1-s)}\sum_{n=0}^{\infty} n^\beta (sr)^n ds \leq C\int_0^1 \frac{\rho(1-s)}{(1-s)(1-rs)^{\beta+1}}ds$$

$$\leq C\frac{1}{(1-r)^{\beta+1-q}}\int_0^1 \frac{\rho(1-s)}{(1-s)(1-rs)^q}ds$$

Combining the previous estimates and using (1.5) in Lemma 1.1 we have

$$\frac{1}{(1-r)}(\sum_{n=0}^{\infty} \alpha_n r^n) \leq C\frac{\rho(1-r)}{(1-r)^{\beta+1}} \qquad ///$$

Lemma 5.2.- Let ρ be a Dini weight such that $\rho(t) \in b_q$ for some $q > 0$. Let $(\alpha_n) \geq 0$. The following are equivalent

(i) $\sum_{n=1}^{N} n^q \alpha_n \leq C\rho(\frac{1}{N}) N^q$

(ii) $\sum_{n=N}^{\infty} \alpha_n \leq C\rho(\frac{1}{N})$

(iii) $\sum_{n=2^k}^{2^{k+1}} \alpha_n \leq C\rho(2^{-k})$

Proof: (i) $\Rightarrow$ (ii). Let $s_n = \sum_{k=1}^{n} k^q \alpha_k = O(\rho(\frac{1}{n}) n^q)$.

$$\sum_{n=N}^{M} \alpha_n = \sum_{n=N}^{M} (s_n - s_{n-1}) n^{-q} = \sum_{n=N}^{M-1} s_n (n^{-q} - (n+1)^{-q}) + s_M M^{-q} - s_{N-1} N^{-q}$$

$$\leq C \sum_{n=N}^{M-1} s_n n^{-(q+1)} + s_M M^{-q} \leq C\rho(\frac{1}{N})\frac{1}{N} + C \sum_{n=N+1}^{\infty} \rho(\frac{1}{n})\frac{1}{n} + C\rho(\frac{1}{M})\frac{1}{M}$$

Observe that $\frac{1}{n}\rho(\frac{1}{n}) \leq C \int_{\frac{1}{n}}^{\frac{1}{n-1}} \frac{\rho(t)}{t} dt$. Therefore

$$\sum_{n=N}^{M} \alpha_n \leq C\rho(\frac{1}{N}) + C \int_0^{1/N} \frac{\rho(t)}{t} dt + C\rho(\frac{1}{M})\frac{1}{M}$$

Taking limits as $M \to \infty$ and using Dini condition we get $\sum_{n=N}^{\infty} \alpha_n \leq C\rho(\frac{1}{N})$

(ii) $\Rightarrow$ (i). Let $R_n = \sum_{k=n}^{\infty} \alpha_k = O(\rho(\frac{1}{n}))$.

$$\sum_{n=1}^{N} n^q \alpha_n = \sum_{n=1}^{N} (R_n - R_{n+1}) n^q = \sum_{n=2}^{N} R_n (n^q - (n-1)^q) - N^q R_{N+1}$$

$$\leq R_1 + C \sum_{n=2}^{N} n^q \frac{1}{n} \rho(\frac{1}{n}) \leq R_1 + C \sum_{n=2}^{N} \int_{\frac{1}{n}}^{\frac{1}{n-1}} \frac{\rho(t)}{t^{q+1}} dt \leq C + C \int_{\frac{1}{N}}^{1} \frac{\rho(t)}{t^{q+1}} dt \leq C\rho(\frac{1}{N}) N^q$$

(ii) $\Rightarrow$ (iii). Obvious

(iii) $\Rightarrow$ (ii).

$$\sum_{n=N}^{\infty} \alpha_n = \sum_{k=log_2 N}^{\infty} \sum_{n=2^k}^{2^{k+1}} \alpha_n \leq C \sum_{k=log_2 N}^{\infty} \rho(2^{-k})$$

$$\leq C \sum_{k=log_2 N}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \frac{\rho(t)}{t} dt = \int_0^{\frac{1}{n}} \frac{\rho(t)}{t} dt \leq C\rho(\frac{1}{N}) \qquad ///$$

Theorem 5.3.- Let $0 < p \leq 1 \leq s < \infty$. Let ρ be a Dini-weight such that $\rho \in b_q$ for some $q > 0$. The following are equivalent

(i) $\lambda \in \mathcal{SM}(B_p(\rho), l^s)$

(ii) $(\sum_{n=2^k}^{2^{k+1}} |\lambda_n|^s)^{p/s} = O(\rho(2^{-k})2^{-k(1-p)})$

Proof: From the closed graph theorem $\lambda_n \in \mathcal{SM}(B_p(\rho), l^q)$ is equivalent to the boundedness of the operator T_λ, defined by $T_\lambda(f) = (\lambda_n a_n)$ for $f(z) = \sum_{n=1}^\infty a_n z^n$.

Since $T_\lambda(u_n) = \lambda_n e_n$ (where e_n stands for the canonic basis of l^s), we have that $F^\beta(z) = \sum_{n=0}^\infty \lambda_n \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} z^n e_n$. Hence $F^\beta(z) = (c_n(\beta)\lambda_n z^n)_n$ where $c_n(\beta) = \frac{\Gamma(\beta+n+1)}{\Gamma(\beta)n!} = O(n^\beta)$. Therefore condition (ii) of Theorem 2.1. applied to our case can be rephrased as follows

$$\left(\sum_{n=1}^\infty |\lambda_n|^s n^{\beta s} |z|^{ns}\right)^{p/s} = O(\frac{\rho(1-|z|)}{(1-|z|)^q}) \ ,$$

or in other words, for any (β_n) in the unit ball of $(l^{s/p})^*$,

$$\sum_{n=1}^\infty |\lambda_n|^{1/p} n^{\beta/p} \beta_n |z|^{n/p} = O(\frac{\rho(1-|z|)}{(1-|z|)^q}).$$

Hence from Lemma 5.1 we have

$$\sum_{n=1}^N |\lambda_n|^{1/p} n^{\beta/p} \beta_n = O(\rho(\frac{1}{N})N^q)$$

Finally apply Lemma 5.2 with $\alpha_n = |\lambda_n|^{1/p} n^{\beta/p} \beta_n n^{-q}$ to get

$$\sum_{n=2^k}^{2^{k+1}} |\lambda_n|^{1/p} \beta_n n^{(\frac{\beta}{p}-q)} = O\left(\rho(2^{-k})\right)$$

Taking the supremum over all (β_n) in the unit ball of $(l^{s/p})^*$ we get

$$\left(\sum_{n=2^k}^{2^{k+1}} |\lambda_n|^s n^{\beta s - qs/p}\right)^{p/s} = O(\rho(2^{-k}))$$

which, since $\beta s - qs/p = (\frac{1}{p} - 1)s$, gives (ii). ///

Remark 5.2.- This result can be used to get multipliers from H_p $(0 < p < 1)$ and from $B_p(D)$ $(0 < p \leq 1)$ into l^s . It is easy to see that a duality argument allows us to also get multipliers from l^s into $\mathcal{B}$, where $\mathcal{B}$ stands for the class of Bloch functions (see [AS]). We refer the reader to [DS1], [JP] for sequence multipliers on H^p for $0 < p < 1$ and to [W] for results concerning multipliers on Bergman spaces.

Let us recall the notion of Köthe dual for spaces of sequences. Given a sequence Banach space, A say, we define the Köthe dual as the space

$$A^K = \{\lambda_n : \lambda_n a_n \in l^1 \text{ for all } a_n \in A\}$$

Identifying the spaces $B_p(\rho)$ with the sequence spaces given by the Taylor coefficients, Theorem 5.3. can be applied to get the Köthe dual of those spaces. As far as we know only the case p=1 (see [AS], [DS2]) in the next corollary was known.

Let us use the notation $l(p,q) = \{\lambda_n : \left(\sum_{n=0}^{\infty}(\sum_{k=2^n}^{2^{n+1}} |\lambda_k|^p)^{q/p}\right)^{1/q} < \infty\}$ (with the obvious modifications when either p or q are equal ∞).

Corollary 5.3.- Let $0 < p \leq 1$. Then

$$(B_p(D))^K = \{\lambda_n : n^{2/p-1}\lambda_n \in l(1,\infty)\}.$$

HANKEL OPERATORS

Hankel operators have traditionally been considered in the context of Hilbert space theory (see [P]), more specifically considering their action on the Hardy space H^2, but in the last decade a big effort has been made to understand their action on the Bergman class $B_2(D)$ and the weighted Bergman classes A_2^{α} (see [Z], [AFP] and [J2]). Also the action of Hankel operators on other spaces of analytic functions, not necessarily in the Hilbert context, has been considered by different authors (see [JPS], [BM] and [CS]). Our applications are addressed to this last situation.

Definition 6.1.- Let $b \in H^\infty(D)$. If P denotes the projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$, we define the Hankel operator with symbol b by

$$H_b(f) = P(\bar{b}f) \qquad f \in H^2(\mathbb{T}).$$

In other words, if $b \in H^\infty(D)$ and f is an analytic polynomial

$$H_b(f)(w) = \int_{|\xi|=1} \frac{f(\xi)\bar{b}(\xi)}{(1-\bar{\xi}w)} \frac{d\xi}{\xi}$$

Let us consider the funtion $K_z(\xi) = \sum_{n=0}^\infty u_n(\xi)z^n \qquad (|\xi| = 1)$

An elementary computation shows

$$H_b(K_z)(\xi) = \frac{\bar{b}(\xi) - \bar{b}(\bar{z})}{1 - \xi z} \tag{6.1}$$

Lemma 6.1.- Let ρ be a Dini weight such that $\rho \in b_1$. Let g be an analytic function in D with continuous extension at the boundary. The following are equivalent

(i) $\quad |g'(z)| = O(\frac{\rho(1-|z|)}{1-|z|}) \qquad (|z| \to 1)$

(ii) $\quad \int_{|\xi|=1} \frac{|g(\xi)-g(z)|}{|1-\bar{\xi}z|^2} d\xi = O(\frac{\rho(1-|z|)}{1-|z|}) \qquad (|z| \to 1).$

Proof: (ii) $\Rightarrow$ (i) Obvious from the Cauchy formula $g'(z) = \int_{|\xi|=1} \frac{g(\xi)-g(z)}{(1-\bar{\xi}z)^2}\bar{\xi}^2 d\xi$.

For the converse, take $\xi = e^{2\pi i\theta}$ and $z = |z|e^{2\pi i\theta_z}$. Let us first estimate

$$|g(\xi) - g(z)| \le |g(\xi) - g(|z|e^{2\pi i\theta})| + |g(|z|e^{2\pi i\theta}) - g(z)|$$

On the one hand, using the Dini condition, we have

$$|g(\xi) - g(|z|e^{2\pi i\theta})| \le \int_{|z|}^1 |g'(se^{2\pi i\theta})|ds \le C\int_{|z|}^1 \frac{\rho(1-s)}{1-s}ds \le C\rho(1-|z|)$$

On the other hand we use Lemma 1.2. with $X = \mathbb{C}$ and $G(e^{2\pi i\theta}) = g(|z|e^{2\pi i\theta})$ to get

$$|g(|z|e^{2\pi i(\theta_z+t)}) - g(z)| \le C\rho(|t|).$$

Therefore

$$\int_{|\xi|=1} \frac{|g(\xi)-g(z)|}{|1-\bar{\xi}z|^2}d\xi \le C\rho(1-|z|)\int_{|\xi|=1}\frac{d\xi}{|1-\bar{\xi}z|^2} + \int_{-1}^1 \frac{|g(|z|e^{2\pi i(\theta_z+t)}) - g(z)|}{|e^{2\pi it} - |z||^2}dt$$

$$\leq C\frac{\rho(1-|z|)}{1-|z|}+C\int_0^1\frac{\rho(t)}{(1-|z|)^2+2|z|sin^2(\pi t)}dt$$

Let us finally use the facts that ρ is nondecreasing and belongs to b_1 to estimate

$$\int_0^1\frac{\rho(t)}{(1-|z|)^2+2|z|sin^2(\pi t}dt\leq C\int_0^1\frac{\rho(t)}{(1-|z|)^2+Ct^2}dt$$

$$\leq C\frac{1}{(1-|z|)^2}\int_0^{1-|z|}\frac{\rho(t)}{1+(\frac{t}{(1-|z|)})^2}dt+C\int_{1-|z|}^1\frac{\rho(t)}{t^2}$$

$$\leq C\frac{1}{(1-|z|)}\int_0^1\frac{\rho((1-|z|)s)}{1+s^2}ds+C\frac{\rho(1-|z|)}{1-|z|}$$

$$\leq C\frac{\rho(1-|z|)}{1-|z|}(\int_0^1\frac{1}{1+s^2}ds+1)\leq C\frac{\rho(1-|z|)}{1-|z|}. \qquad ///$$

Theorem 6.1.- Let ρ be a Dini weight such that $\rho\in b_1$. Let $b\in H^\infty(D)$. The following are equivalent

(i) $H_b : B_1(\rho)\to H^1$ is bounded

(ii) $|b'(z)|=O\left(\frac{\rho(1-|z|)}{(1-|z|)log\frac{1}{1-|z|}}\right) \qquad (|z|\to 1)$

Proof: Denote $F(z)=H_b(K_z)$, and use (6.1) to write

$$F'(z)(\xi)=\frac{\xi(\bar{b}(\xi)-\bar{b}(\bar{z}))}{(1-\xi z)^2}-\frac{\bar{b}'(\bar{z})}{1-\xi z} \qquad (6.2)$$

Let us assume (i). Applying Corollary 2.2 we have

$$\|F'(z)\|_{H_1}=O\left(\frac{\rho(1-|z|)}{(1-|z|)}\right) \qquad (6.3)$$

Now $H_b(f)(0)=\int_{|\xi|=1}\bar{b}(\xi)f(\xi)\frac{d\xi}{\xi}$, so the boundedness of H_b implies

$$|\int_{|\xi|=1}\bar{b}(\xi)f(\xi)\frac{d\xi}{\xi}|\leq\|H_b(f)\|_{H_1}\leq C\|f\|_{B_1(\rho)}$$

This implies $b\in(B_1(\rho))^*$, which coincides with Λ_ρ according to Corollary 2.2.

Hence we can apply Lemma 6.1 to obtain

$$\int_{|\xi|=1}\frac{|b(\xi)-b(z)|}{|1-\bar{\xi}z|^2}d\xi=O(\frac{\rho(1-|z|)}{1-|z|}), \qquad (|z|\to 1) \qquad (6.4)$$

From (6.2) we have

$$|b'(z)|\int_{|\xi|=1}\frac{d\xi}{|1-\xi\bar{z}|}\leq \|F'(\bar{z})\|_{H_1}+\int_{|\xi|=1}\frac{|b(\xi)-b(z)|}{|1-\bar{\xi}z|^2}d\xi$$

Using $\int_{|\xi|=1}\frac{d\xi}{|1-\xi\bar{z}|}=O(log(\frac{1}{1-|z|}))$, (6.3) and (6.4) we get (ii).

Let us now assume (ii). From Theorem 2.1 we have to show (6.3). Using (6.2) again we have

$$\|F'(z)\|_{H_1}\leq |b'(\bar{z})|\int_{|\xi|=1}\frac{d\xi}{|1-\xi\bar{z}|}+\int_{|\xi|=1}\frac{|b(\xi)-b(z)|}{|1-\bar{\xi}z|^2}d\xi$$

Now the estimate (6.3) follows easily by using (ii) and Lemma 6.1. ///

Corollary 7.1.- ([BM], Theorem 4),([CS], Theorem 3) Let $1/2 < p < 1$ and let $b \in H^\infty$. Then

$H_b : H^p \to H^1$ if and only if $\qquad |b'(z)| = O(\frac{1}{(1-|z|)^{1/p-2}log\frac{1}{1-|z|}})$.

COMPOSITION OPERATORS

The reader is referred to Chapter 10 in [Z] for definitions and results on composition operators acting on Hardy and Bergman spaces.

Definition 7.1.- Let $\phi : D \to D$ be an analytic function. If $\mathcal{H}(D)$ denotes the space of analytic functions on the open unit disc, we can define the following linear map on $\mathcal{H}(D)$

$$C_\phi(f)(z) = f(\phi(z)).$$

C_ϕ is called the composition operator induced by ϕ.

One of the main tools in dealing with composition operators is the Littlewood subordination principle (see [D], [Z]): If $0 < p < \infty,\ 0 < r < 1$, $\psi : D \to D$ with $\psi(0) = 0$ and $f \in \mathcal{H}(D)$ then

$$\int_0^1 |f(\psi(re^{2\pi i\theta}))|^p d\theta \leq \int_0^1 |f(re^{2\pi i\theta})|^p d\theta$$

From this one easily gets the boundedness of C_ϕ on weighted Bergman spaces.

Proposition 7.1.- Let ρ be a weight function. Let $\phi : D \to D$ be analytic and $0 < p < \infty$, then

$$\int_D \frac{\rho(1-|z|)}{1-|z|}|f(\phi(z))|^p dA(z) \le \frac{1+|\phi(0)|}{1-|\phi(0)|}\int_D \frac{\rho(1-|z|)}{1-|z|}|f(z)|^p dA(z)$$

Proof: Let $a = \phi(0)$ and consider $\psi(z) = \phi_a(\phi(z))$ where $\phi_a(w) = \frac{w-a}{1-\bar{a}w}$. Since $\psi : D \to D$, $\psi(0) = 0$ and $\phi(z) = \phi_a(\psi(z))$ we can use Littlewood subordination principle to get

$$\int_0^1 |f(\phi(re^{2\pi i\theta}))|^p d\theta \le \int_0^1 |f(\phi_a(re^{2\pi i\theta}))|^p d\theta$$

Making the change of variable $re^{2\pi it} = \phi_a(re^{2\pi i\theta})$ one gets

$$\int_0^1 |f(\phi(re^{2\pi i\theta}))|^p d\theta \le (1-|a|^2)\int_0^1 \frac{|f(re^{2\pi it})|^p}{|1-\bar{a}re^{2\pi it}|^2}dt$$

Therefore

$$\int_0^1 |f(\phi(re^{2\pi i\theta}))|^p \le \frac{1+|\phi(0)|}{1-|\phi(0)|}\int_0^1 |f(re^{2\pi it})|^p dt$$

Hence multiplying by $\frac{\rho(1-r)}{1-r}$ and integrating one gets

$$\|C_\phi(f)\|_{B_p(\rho)} \le \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{1/p} \|f\|_{B_p(\rho)} \qquad ///$$

Remark 7.1.- We have included the proof, although it is very elementary, because the change of variable at the right moment can improve the estimate of the norm (see [Z, Theorem 10.3.2] where $\|C_\phi\|_{\mathcal{L}(B_p,B_p)}$ is estimated by $(\frac{1+|\phi(0)|}{1-|\phi(0)|})^{2/p})$.

We are mainly concerned with analyzing when Hankel operators improve the condition of integrability. To this purpose we need the following notion (see [MS]).

Given $\phi : D \to D$ analytic, let us consider the following image measure A_ϕ on D defined by

$$A_\phi(B) = \int_{\phi^{-1}(B)} dA(z) \quad \text{for any Borel set } B \subset D \tag{7.1}$$

Theorem 7.1.- Let $0 < p \leq 1$ and $\phi : D \to D$ analytic. The following are equivalent

(i) $C_\phi : B_p(D) \to B_1(D)$

(ii) A_ϕ is a $\frac{2}{p}$-Carleson measure.

Proof: Since $\rho(t) = t \in b_{1+\varepsilon}$ for any $\varepsilon > 0$, we can apply Theorem 2.1. to this case and get that (i) means

$$\int_D \frac{dA(w)}{|1 - \phi(w)z|^{\frac{2+\varepsilon}{p}}} = O(\frac{1}{(1 - |z|)^{\varepsilon/p}})$$

which, in terms of the image measure A_ϕ, says

$$\int_D \frac{dA_\phi(w)}{|1 - wz|^{\frac{2+\varepsilon}{p}}} = O(\frac{1}{(1 - |z|)^{\varepsilon/p}}) \tag{7.2}$$

A look at Lemma 4.1 shows that (7.2) is equivalent to the fact that A_ϕ is a $2/p$-Carleson measure. ///

Remark 7.1.- Observe that (7.2) for $\varepsilon = 2$ and $p = 1$ gives the following interesting characterization for the norm of C_ϕ as operator on $B_1(D)$

$$\|C_\phi\|_{\mathcal{L}(B_1, B_1)} \approx \sup_{|z|<1} \int_D |\phi_z'(w)|^2 dA_\phi(w) \tag{7.3}$$

where $\phi_z(w) = \frac{w-z}{1-\bar{z}w}$.

As before our results can also be used to describe composition operators acting on H_p spaces for $0 < p < 1$. Given $\phi : D \to D$ analytic, write ϕ^* for the "function" (defined a.e.) consisting of its boundary limits. Let us consider the following image measure m_ϕ on $\bar{D}$ defined by

$$m_\phi(B) = m((\phi^*)^{-1}(B)) \;\text{ for any Borel set } B \subset \bar{D} \tag{7.4}$$

(m stands for the normalized Lebesgue measure on the unit circle.)

Arguing as in the previous theorem it is easy to get the next result.

Theorem 7.2.- Let ρ be a Dini weight such that $\rho \in b_1$, and $\phi : D \to D$ analytic. $C_\phi : B_1(\rho) \to H^1$ if and only if m_ϕ is a $t\rho(t)$-Carleson measure.

Remark 7.2.- We refer the reader to [HJ] for results about composition operators which improve integrability. As it was pointed out there $C_\phi : H^p \to H^1$ is equivalent to $C_\phi : H^1 \to H^{1/p}$. Hence Theorem 7.2. applied to $\rho(t) = t^{1/p-1}$ together with the previous observation allows us to recover some results by Hunziker and Jarchow (see [HJ] Theorem 3.1.).

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REFERENCES

[ACP] J.M. Anderson, J. Clunie, Ch. Pommerenke: On bloch functions and normal functions. *J. Reine Angew. Math.* 270, (1974), 12-37.

[AS] J.M. Anderson, A.L. Shields: Coefficient multipliers on Bloch functions. *Trans. Amer. Math. Soc.* 224, (1976), 256-265.

[AFP] J.Arazy, S. Fisher, J. Peetre: Hankel operators on weighted Bergman spaces. *Amer. J. Math.* 110, (1988), 989-1054.

[A] S. Axler: Bergman spaces and their operators. *Surveys on some recent results on Operator Theory, Vol I* (J.B. Conway, B.B. Morrel, editors.) Pitman Research Notes in Math., 171, 1988, 1-50.

[B] O. Blasco: Spaces of vector valued analytic functions and applications. *Geometry of Banach spaces.* Proc. conference held in Strobl, Austria 1989 (P. Muller, W Schachermayer , editors). London Math. Soc. Lecture Note Ser. 158 Cambridge Univ. Press (1990), 33-48.

[BS] O. Blasco, G.S. de Souza: Spaces of analytic functions on the disc where the growth of $M_p(F, r)$ depends on a weight. *J. Math. Anal. and Appl.* 147, (1990), 580-598.

[BlS] S. Bloom, G.S. de Souza: Weighted Lipschitz spaces and their analytic characterization. *Preprint* .

[BM] A. Bonami, Y. Mesrar: Inegalités H^p pour les operatèurs de Hankel a symbol lipschitzien. *C.R. Acad. Sc. Paris* 303, (1986), 853-856.

[CS] J. Cima, D.A. Stegenga: Hankel operators on H^p. *Analysis at Urbana, Vol I* (Urbana, IL. 1986-87), London Math. Soc. Lecture Note Ser. 137. Cambrigde Univ. Press, Cambrigde, 1989, 133-150.

[D] P. Duren: *Theory of H_p-spaces.* Academic Press. New York. 1970.

[D2] P. Duren: Extension of a Theorem of Carleson. *Bull. Amer. Math. Soc.* 75, (1969), 143-146.

[DS1] P. Duren, A.L. Shields: Coefficient multipliers of H^p and B^p spaces. *Pacific J. Math.* 32, (1970), 69-78.

[DS2] P. Duren, A.L. Shields: Properties of H^p $(0 < p < 1)$ and its containing Banach space. *Trans. Amer. Math. Soc.* 141, (1969),255-262.

[DRS] P.L. Duren, B.W. Romberg, A.L. Shields: Linear functionals on H_p-spaces with $0 < p < 1$. *J. Reigne Angew. Math.* 238, (1969), 32-60.

[F1] T.M. Flett: The dual of an inequality of Hardy and Littlewood and some related inequalities. *J. Math. Anal. and Appl.* 38, (1972), 746-765.

[F2] T.M. Flett: Lipschitz spaces of functions on the circle and the disc. *J. Math. Anal. and Appl.* 39, (1972), 125-158.

[G] J.B. Garnett: *Bounded Analytic functions.* Academic Press, New York. 1981.

[HL] G.H. Hardy, J.E. Littlewood: Some properties of fractional integrals II. *Math. Z.* 34, (1932), 403-439.

[H] W.W. Hastings: A Carleson measure theorem for Bergman spaces. *Proc. Amer. Math. Soc.* 52, (1975), 237-241.

[HJ] H. Hunziker, H. Jarchow: Composition operators which improve integrability. *Math. Nach.* to appear.

[J1] S. Janson: Generalization on Lipschitz spaces and applications to Hardy spaces and bounded mean oscillation. *Duke Math. J.* 47, (1980), 959-982.

[J2] S. Janson: Hankel operators between weighted Bergman spaces. *Ark. Mat.* 26, (1988), 205-219.

[JPS] S. Janson, J. Peetre, S. Semmes: On the action of Hankel and Toeplizt operators on some function spaces.. *Duke J. Math.* 51, (1984), 937-958.

[JP] M. Jevtic, M. Pavlovic: On multipliers from H^p to l^q, $0 < q < p < 1$. *Arch. Math.* 56, (1991), 174-180.

[K1] N. Kalton: Analytic functions in non locally convex spaces and applications. *Studia Math.* 83, (1986), 275-303.

[K2] N. Kalton: Some applications of vector-valued analytic and harmonic functions. *Probability and Banach spaces* Proceedings, Zaragoza 1985. Lecture Notes in Math. 1221, (1986) , 114-140.

[KS] R. Kerman, E. Sawyer: Carleson measures and multipliers of Dirichlet- type spaces. *Trans. Amer. Math. Soc.* 309, (1988), 87-98.

[MS] B.D. MacCluer, J.H. Shapiro: Angular derivatives and compact composition operators on Hardy and Bergman spaces. *Can. J. Math.* 38, (1986), 878-906..

[M] M. Marzuq: Linear functionals on some weighted Bergman spaces. *Bull. Austral. Math. Soc.* 42, (1990), 417-425.

[MP] M. Mateljevic, M. Pavlovic: Multiliers of H^p and $BMOA$. *Pacific J. Math.* 146, (1990), 71-84.

[P] S. Power: Hankel operators on Hilbert space. Pitman research Notes in Math. 64, Pitman, London, 1981.

[S] J.H. Shapiro: Mackey topologies, reproducing kernels and diagonal maps on the Hardy and Bergman spaces. *Duke Math. J.* 43, (1976), 187-202.

[SW] A.L.Shields, D.L. Williams: Bounded projections, duality and multipliers in spaces of analytic functions. *Trans. Amer. Math. Soc.* 162, (1971), 287-302.

[St] D.A. Stegenga: Multipliers of the Dirichlet space. *Illinois J. Math.* 24, (1980), 113-139.

[W] P. Wojtaszcyk: Multipliers into Bergman spaces and Nevalinna class. *Canad. Math. Bull.* 33, (1990), 151-161.

[Z] K. Zhu: *Operator theory in function spaces.* Marcel Dekker, Inc., New York. 1990.

[Zy] A. Zygmund: *Trigonometric series.* Cambrigde Univ. Press, New York. 1959.