Bol. Soc. Mat. Mexicana (3) Vol. 4, 1998

CONVOLUTION MULTIPLIERS ON WEIGHTED BESOV SPACES

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Abstract

We give a complete characterization of convolution multipliers between particular cases of weighted Besov classes and a description of multipliers between two weighted Besov spaces $B_w^{p,q}$ for all values of $0 < p, q \leq \infty$ in terms of multipliers between L^p spaces.

0. Introduction

The theory of Fourier multipliers on a Banach space or between two different Banach spaces defined on the torus or on Euclidean space has a long history. The first space to be studied was $L^p(\mathbb{R}^n)$, but the difficulties appearing for values of p different from $p = 1$ or $p = 2$ pushed analysts to the study of other spaces such as Hardy spaces, Bergman spaces, Besov spaces, L^p spaces with a general measure, and so on, where more information can be obtained.

Recently new results on multipliers on Banach spaces of analytic functions on the torus, such as Bergman and Besov spaces, have been achieved by several authors; see, for example, [A, B1, JJ, W]. The aim of this paper is to investigate the situation for Besov spaces defined on $\mathbb{R}^n$ and not only for potential weights but for more general ones.

The study of multipliers on Besov spaces defined on $\mathbb{R}^n$ goes back to Hardy-Littlewood, who gave a description of multipliers from a Besov space to itself in terms of multipliers on $L^p(\mathbb{R}^n)$ spaces. In [FS, P1, P2, SZ, T] the reader can find results that contain those of Hardy-Littlewood. The embeddings existing between Besov spaces and Hardy spaces make very useful the knowledge of the situation for Besov spaces to get new results in the setting of Hardy spaces (see [BaS, Jo, He]).

In this paper we get a complete description of certain cases using Herz spaces, extending previous results by R. Johnson ([Jo]) and also, by regarding Besov spaces as retracts of certain mixed norm spaces, we are able to give a complete characterisation of multipliers between Besov spaces in terms of

1991 *Mathematics Subject Classification*: 42A45, 42B25, 42C15.

Keywords and phrases: Besov spaces, weights, multipliers.

The authors have been partially supported by the Spanish DGICYT, Projects PB92-0699 and PB95-0261

those between $L^p(\mathbb{R}^n)$ spaces, extending previous cases achieved by J. Peetre ([P1]).

Let us now introduce the Besov spaces we shall be dealing with.

We will denote by $\mathcal{S}$ the space of test functions in $\mathbb{R}^n$ with the topology given by the family of seminorms

$$P_N(\phi) = \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |(D_\alpha \phi)(x)|$$

We shall write $\mathcal{S}_\infty$ for the closed subspace of functions of $\mathcal{S}$ with null moments of all orders and $\mathcal{S}'$ and $\mathcal{S}'_\infty$ stand for their topological duals.

Throughout the paper we shall work with functions defined on the Euclidean space $\mathbb{R}^n$. So, the symbol n will denote always the dimension of the space. Given $f: \mathbb{R}^n \to \mathbb{C}$ and $t > 0$, we will denote $f_t(x) = \frac{1}{t^n} f\left(\frac{x}{t}\right)$, $f^x(y) = f(x-y)$ and $\tilde{f}(y) = f(-y)$.

Given a sequence of positive real numbers $\{w(k)\}_{k\in\mathbb{Z}}$, $p \in (0, +\infty]$, $q \in (0, +\infty]$ and a test function $\phi \in \mathcal{S}$. We define

$$\|f\|_{B_w^{p,q}(\phi)} = \left(\sum_{k\in\mathbb{Z}} \left[\frac{\|\phi_{2^k} * f\|_p}{w(k)} \right]^q \right)^{1/q} ;$$

for $f \in \mathcal{S}'$ and $q < +\infty$ (with the obvious modification for $q = \infty$).

The function ϕ in the definition will be chosen among the functions $\phi \in \mathcal{S}$ whose Fourier transform have compact support contained in $\mathbb{R}^n \setminus \{0\}$ and such that for every $x \in \mathbb{R}^n \setminus \{0\}$ there exists $k \in \mathbb{Z}$ such that $\hat{\phi}(2^k x) \neq 0$. We shall denote the set of such functions by $\mathcal{B}$.

There are two conditions on the weight w in $\mathbb{Z}$ which appear to be natural for different purposes:

Condition (1): There exist $C, a > 0$ such that

$$C^{-1} 2^{-a|k|} \leq w(k) \leq C 2^{a|k|} \quad \forall\, k \in \mathbb{Z}.$$

Condition (2): There exist $C, a > 0$ such that

$$C^{-1} w(k+1) \leq w(k) \leq C w(k+1) \quad \forall\, k \in \mathbb{Z}.$$

It is shown in the first section that (1) gives the embedding $\mathcal{S}_\infty \subset B_w^{p,q}(\phi) \cap B_{w^{-1}}^{p,q}(\phi)$ and (2) is used to get that the Besov classes are independent of the choice of the function $\phi \in \mathcal{B}$.

Since, clearly (2) implies (1), we believe that assumption (2) is the right condition to work in the weighted situation.

The paper is divided into three sections. The first one has a preliminary character and it is mainly devoted to the proof that $\mathcal{S}_\infty$ is continuously contained in any Besov space (under assumption (1)) and to show the density of $\mathcal{S}_\infty$ in $B_w^{p,q}$ unless $p = \infty$ or $q = \infty$. We will use the notation $(B')_w^{p,q}$ for the closure of $\mathcal{S}_\infty$ in $B_w^{p,q}$.

In this section is also shown that, under condition (2), these spaces are independent of the particular function ϕ taken in the definition and hence they will be denoted by $B_w^{p,q}$. With certain conditions on the weight stronger than the ones used in this paper several characterisations of these spaces, similar to those known in the potential case, can be obtained. The interested reader is referred to [AB, B2, Bu] to get other formulations of weighted Besov classes.

Let p_0, p_1, q_0, $q_1 \in (0, +\infty]$ and let w_0, w_1 be weights in $\mathbb{Z}$. A Fourier multiplier (or simply a multiplier) from $B_{w_0}^{p_0,q_0}$ on $B_{w_1}^{p_1,q_1}$ will be a distribution $a \in \mathcal{S}'_\infty$ such that $a * f \in B_{w_1}^{p_1,q_1}$ if $f \in \mathcal{S}_\infty$ and there exists a constant $C > 0$ such that $\|a * f\|_{B_{w_1}^{p_1,q_1}} \leq C\|f\|_{B_{w_0}^{p_0,q_0}}$, or in other words, a bounded operator $M_a: (B')_{w_0}^{p_0,q_0} \to B_{w_1}^{p_1,q_1}$ such that $M_a(f) = a * f$ for all $f \in \mathcal{S}_\infty$. We will write $a \in \left(B_{w_0}^{p_0,q_0}, B_{w_1}^{p_1,q_1}\right)$ and $\|a\|_{(B_{w_0}^{p_0,q_0}, B_{w_1}^{p_1,q_1})}$ for the norm $\|M_a\|$ as an operator between the corresponding spaces.

Actually a multiplier is a convolution operator with a distribution a whose Fourier transform is a function, say m, defined on $\mathbb{R}^n$. To analyze the multiplier it is known to be useful to look at the behaviour of the function m according to the localization of m in annuli centered at the origin (see for example [BaS,Ho, P1]). This idea appears in this paper and also leads to the use of Herz spaces in the weighted situation.

In Section 2 Herz spaces are defined and elementary results on multipliers are given.

Last section is devoted to prove our main theorem, which extends the case $q_0 = q_1$ for potential weights due to J. Peetre (see [P1]), and to give some applications of it.

Denoting by $K_B = \{x : |x| \leq B\}$ and $L^p[B] = \{f \in L^p(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset K_B\}$ we can state the following theorem.

MAIN THEOREM. *Let p_0, p_1, q_0, $q_1 \in (0, +\infty]$ and let w_0 and w_1 be weights in $\mathbb{Z}$ which satisfy (2). Let $\eta \in \mathcal{B}$ and $B \in (0, +\infty)$ such that* $\operatorname{supp} \hat{\eta} \subseteq K_B$ *and $a \in \mathcal{S}'$.*

Then $a \in \left(B_{w_0}^{p_0,q_0}, B_{w_1}^{p_1,q_1}\right)$ if and only if $\eta_{2^k} * a \in (L^{p_0}([2^{-k}B], L^{p_1}(\mathbb{R}^n))$ for all $k \in \mathbb{Z}$, and

$$\frac{w_0(k)}{w_1(k)} \|\eta_{2^k} * a\|_{(L^{p_0}([2^{-k}B], L^{p_1}(\mathbb{R}^n))} \in \ell_s(\mathbb{Z}),$$

for $\dfrac{1}{s} = \max\left\{\dfrac{1}{q_1} - \dfrac{1}{q_0}, 0\right\}$.

If $p_0 \geq 1$ we can replace $L^p[2^{-k}B]$ by $L^p(\mathbb{R}^n)$ in the theorem. Then, B does not play any role.

The pairing between distributions and test functions and also the inner product in $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$. If we denote by $\mathcal{P}$ the space of polynomials, it is clear that $\mathcal{P} \subset \mathcal{S}'$, regarding a polynomial as the distribution given by means of integration against it. Furthermore, a distribution is null on $\mathcal{S}_\infty$ if and only if is a polynomial. Hence, taking into account Hahn-Banach theorem, we can identify $\mathcal{S}'_\infty$ with the quotient space $\mathcal{S}'/\mathcal{P}$.

Given a quasi-Banach space X, with quasi-norm $\|.\|_X$, and $\lambda \in (0, +\infty)$, we will denote λX for the same space, with quasi-norm $\lambda\|.\|_X$.

We shall use, for the sake of homogeneity, the notation $\ell_p = \ell_p^0$ for $p < +\infty$, and $\ell_\infty^0 = c_0$. Also we write c_{00} for the space of sequences with only a finite number of non zero terms. In similar way, we denote $L_0^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, if $p < +\infty$, and $L_0^\infty(\mathbb{R}^n) = C_0(\mathbb{R}^n)$. Finally, given a sequence of quasi-Banach spaces $\{X_k\}_{k\in\mathbb{Z}}$ we denote by $\ell_p(X_k, k \in \mathbb{Z})$ the space of sequences $\{x_k\}_{k\in\mathbb{Z}}$ such that $x_k \in X_k$ and $\left(\sum_{k\in\mathbb{Z}} \|x_k\|_{X_k}^p\right)^{1/p} < \infty$.

Throughout the paper, $p \in (0, +\infty]$ and p' and n_p are defined by means of $\frac{1}{p'} = \max\left\{1 - \frac{1}{p}, 0\right\}$ and $n_p = n \max\left\{\frac{1}{p} - 1, 0\right\}$ and as usual C will denote a positive constant that may change from line to line.

1. Some preliminaries on weighted Besov spaces

We shall follow J. Peetre's approach to Besov classes (see [P1]).

Definition (1.1). Let $w(k)_{k\in\mathbb{Z}}$ be a sequence of positive numbers; let $p \in (0, +\infty]$, $q \in (0, +\infty]$ and $\phi \in \mathcal{S}_\infty$. Given $f \in \mathcal{S}'$ we define, for $q < +\infty$,

$$\|f\|_{B_w^{p,q}(\phi)} = \left(\sum_{k\in\mathbb{Z}} \left[\frac{\|\phi_{2^k} * f\|_p}{w(k)}\right]^q\right)^{1/q};$$

and, for $q = +\infty$,

$$\|f\|_{B_w^{p,\infty}(\phi)} = \sup_{k\in\mathbb{Z}} \frac{\|\phi_{2^k} * f\|_p}{w(k)}.$$

Note that $\|f\|_{B_w^{p,q}(\phi)} = 0$ if and only if $f \in \mathcal{P}$. Therefore, we can consider the quasi-normed space (normed if $p, q \geq 1$)

$$B_w^{p,q}(\phi) = \left\{f \in \mathcal{S}'_\infty;\ \|f\|_{B_w^{p,q}(\phi)} < +\infty\right\}.$$

In this section we shall assume conditions on the defining function ϕ and the weight w to ensure that some facts which are valid in the potential case $w(k) = 2^{\alpha k}$ still hold in our setting.

Usually ϕ is assumed to be a test function such that $\hat{\phi}$ is radial, $0 \leq \hat{\phi}(x) \leq 1$ and with support in $\{x : \frac{1}{4} \leq |x| \leq 4\}$. We shall be a bit more generous and assume a bit less on the function ϕ.

We will denote by $\mathcal{B}$ the set of functions $\phi \in \mathcal{S}$ whose Fourier transform have compact support contained in $\mathbb{R}^n \setminus \{0\}$ and such that $2^{\mathbb{Z}}(\operatorname{supp}\hat{\phi}) = \mathbb{R}^n \setminus \{0\}$, i.e. for every $x \in \mathbb{R}^n \setminus \{0\}$ there exists $k \in \mathbb{Z}$ such that $\hat{\phi}(2^k x) \neq 0$.

For such functions we have the following fact:

LEMMA (1.2). *Let $\phi \in \mathcal{B}$. There exists $\psi \in \mathcal{B}$ such that*

$$\hat{\phi}(x)\hat{\psi}(x) \geq 0 \qquad \forall\, x \in \mathbb{R}^n;$$

$$\sum_{k\in\mathbb{Z}} \hat{\phi}(2^k x)\hat{\psi}(2^k x) = 1 \qquad \forall\, x \neq 0.$$

Proof. Take $\hat{\psi}(x) = \dfrac{\overline{\hat{\phi}}(x)}{\sum_{k\in\mathbb{Z}} \left|\hat{\phi}^2(2^k x)\right|}$ and observe that $\operatorname{supp}\hat{\psi} = \operatorname{supp}\hat{\phi}$.

These properties will be used (taking $\alpha = \phi * \psi$) in combination with the following discrete version of Calderón reproducing formula.

LEMMA A. *(see* [FJW]*). Let $\alpha \in \mathcal{S}_\infty$ such that $\hat{\alpha}(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\sum_{k\in\mathbb{Z}} \hat{\alpha}(2^k x) = 1$ for every $x \in \mathbb{R}^n \setminus \{0\}$. If $f \in \mathcal{S}'_\infty$ then $\sum_{k\in\mathbb{Z}} \alpha_{2^k} * f$ converges to f in $\mathcal{S}'_\infty$.*

Let us mention another useful lemma about test functions.

LEMMA B. *(see* [FJW]*). For each a, b, $M \in (0, +\infty)$, we have that for every $\phi \in \mathcal{S}_\infty$ and $\psi \in \mathcal{S}_\infty$ there exists $N \in \mathbb{N}$ and $C > 0$ (depending only on a, b and M) such that*

$$|\phi_t * \psi(x)| \leq C P_N(\phi) P_N(\psi) \left(1 + \frac{|x|}{t}\right)^{-M} \min\left\{t^a, t^{-b}\right\},$$

for all $t \in (0, +\infty)$ and $x \in \mathbb{R}^n$.

Let us start by looking for some conditions on $w = \{w(k)\}_{k\in\mathbb{Z}}$ to get $\mathcal{S}_\infty \subseteq B^{p,q}_w(\phi)$.

PROPOSITION (1.3). *Let p, $q \in (0, +\infty]$ and let $\phi \in \mathcal{B}$. Then $\mathcal{S}_\infty \subseteq B^{p,q}_w(\phi)$ (with continuity) if and only if there exist constants $C, \gamma > 0$ such that $w(k) \geq C2^{-|k|\gamma}$.*

Proof. Assume $\mathcal{S}_\infty \subseteq B^{p,q}_w(\phi)$. Then also $\mathcal{S}_\infty \subseteq B^{p,\infty}_w(\phi)$. Therefore there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$\|\phi_{2^k} * \psi\|_p \leq C w(k) P_N(\psi)$$

for any $\psi \in \mathcal{S}_\infty$ and any $k \in \mathbb{Z}$.

Take now $\psi = \phi_{2^k}$. Note that $\|\phi_{2^k} * \phi_{2^k}\|_p = 2^{kn(1/p-1)}\|\phi * \phi\|_p$, and

$$\begin{aligned} P_N(\phi_{2^k}) &= \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^N 2^{-k(|\alpha|+n)}\left|(D_\alpha\phi)(\frac{x}{2^k})\right| \\ &= \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n} 2^{-k(|\alpha|+n)}\sum_{m=0}^{N}\frac{N!}{m!(N-m)!}2^{2km}|x|^{2m}|(D_\alpha\phi)(x)| \\ &\le \sum_{m=0}^{N}\frac{N!}{m!(N-m)!}2^{2km}\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}\sup_{0\le m\le N} 2^{-k(|\alpha|+n)}|x|^{2m}|(D_\alpha\phi)(x)| \\ &\le CP_N(\phi)(1+2^{2k})^N\sup_{|\alpha|\le N}2^{-k(|\alpha|+n)}. \end{aligned}$$

This shows that

$$w^{-1}(k) \le C_\phi(1+2^{2k})^N \sup_{|\alpha|\le N} 2^{-k(|\alpha|+n/p)}.$$

This gives

$$w(k) \ge C_\phi 2^{-|k|\max\{2N, N+n/p\}}.$$

Conversely take $A > \max\{\gamma, \frac{n}{p}\}$ and apply Lemma B for $M > \frac{n}{p}$, $a \ge A - \frac{n}{p}$, $b \ge A + \frac{n}{p}$ to get

$$\|\phi_{2^k} * \psi\|_p \le CP_N(\phi)P_N(\psi)2^{-|k|A}.$$

Therefore

$$\frac{\|\phi_{2^k} * \psi\|_p}{w(k)} \le CP_N(\phi)P_N(\psi)2^{-|k|(A-\gamma)},$$

which belongs to ℓ_q for in any $q \in (0,\infty]$.

Since the duality for $1 < p, q < \infty$ would say $(B_w^{p,q}(\phi))^* = B_{w^{-1}}^{p',q'}(\phi)$ and the Proposition (1.3) should work for both cases we shall assume the following condition on the weight:

There exist constants $C_1, C_2, \gamma_1, \gamma_1$ such that

$$C_1 2^{-|k|\gamma_1} \le w(k) \le C_2 2^{|k|\gamma_2} \qquad (k \in \mathbb{Z});$$

or equivalently there exist C, γ such that

$$C^{-1}2^{-|k|\gamma} \le w(k) \le C2^{|k|\gamma} \qquad (k \in \mathbb{Z}). \tag{1}$$

Let us now introduce some notation related to band-limited functions. If X is a space of tempered distributions (i.e., a space that can be embedded in $\mathcal{S}'$) we define

$$X[A] = \left\{f \in \mathcal{S}';\ f \in X,\ \operatorname{supp}\hat{f} \subseteq K_A\right\},$$

where $K_A = \{x \in \mathbb{R}^n; |x| \le A\}$.

LEMMA (1.4). *Let $A \in (0, +\infty)$ and $p \in (0, +\infty]$. Then $\mathcal{S}[A]$ is dense in $L_0^p[A]$.*

Proof. Let us assume that $f \in L_0^p[B]$ for some $B < A$ and take $\phi \in \mathcal{S}[A]$ such that $\hat{\phi}(x) = 1$ if $|x| \leq B$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ be such that $\lim_{n\in\mathbb{N}} \|f_n - f\|_p = 0$. Then $(\phi * f_n)_{n\in\mathbb{N}} \subset \mathcal{S}[A]$ and $\lim_{n\in\mathbb{N}} \|\phi * f_n - f\|_p = 0$.

Now combine this with the following facts: $\lim_{\substack{t\to1\\t>1}} f_t = f$ in $L^p(\mathbb{R}^n)$ for any $f \in L_0^p(\mathbb{R}^n)$ and $f_t \in L_0^p\left[t^{-1}A\right]$ $(t^{-1}A < A)$ for $f \in L_0^p[A]$ to get the desired density.

LEMMA C. (see [P1], p. 234-237). *Let $p \in (0, 1]$, $q \in [p, +\infty]$ and $A \in (0, +\infty)$. There exists $C \in (0, +\infty)$ such that if $f \in L^p[A]$ and $g \in L^q[A]$ then*

$$\|g * f\|_q \leq CA^{n(1/p-1)}\|g\|_q\|f\|_p. \tag{1.5}$$

$$\|f\|_q \leq CA^{n(1/p-1/q)}\|f\|_p. \tag{1.6}$$

Remark (1.7). Let $0 < p \leq \infty$. Using (1.6) we have the following fact: if $f \in L^p[A]$ and $g \in L^{p'}[A]$ then

$$\left|\int_{\mathbb{R}^n} fg\, dm\right| \leq CA^{n_p}\|f\|_p\|g\|_{p'}. \tag{1.8}$$

It is also clear that

$$\left|\sum_{i\in\mathbb{Z}} a_i b_i\right| \leq \|(a_i)_{i\in\mathbb{Z}}\|_p \, \|(b_i)_{i\in\mathbb{Z}}\|_{p'} \, . \tag{1.9}$$

Definition (1.10). Given $p, q \in (0, +\infty]$, w a weight and $\phi \in \mathcal{B}$ with $\text{supp}(\hat{\phi}) \subset K_A$, we define the bounded operator $S_\phi: B_w^{p,q}(\phi) \to \ell_q(w^{-1}L^p[2^{-k}A])$ given by

$$[S_\phi(f)]_k = \phi_{2^k} * f\, .$$

Our next objective will be to construct operators which go the other way, and to show that the definition of the spaces does not depend on the choice of the function $\phi \in \mathcal{B}$, i.e. if $\phi, \psi \in \mathcal{B}$ there exist constants $C_1, C_2 > 0$ such that

$$C_1\|f\|_{B_w^{p,q}(\phi)} \leq \|f\|_{B_w^{p,q}(\psi)} \leq C_2\|f\|_{B_w^{p,q}(\phi)} \tag{a}$$

for $f \in \mathcal{S}_\infty$.

Recall that $\phi_s * f_t = (\phi_{\frac{s}{t}} * f)_t$. Hence $\|\phi_s * f_t\|_p = t^{n(1/p-1)}\|\phi_{\frac{s}{t}} * f\|_p$. Then (a) implies that for any $t > 0$ the dilation operator $f \to f_t$ is bounded on $B_w^{p,q}(\phi)$.

In particular for $t = 2$ this implies the existence of constants $C_1, C_2 > 0$ for which

$$C_1(\sum_{k\in\mathbb{Z}}(\frac{\|\phi_{2^k} * f\|_p}{w(k+1)})^q)^{1/q} \le (\sum_{k\in\mathbb{Z}}(\frac{\|\phi_{2^k} * f\|_p}{w(k)})^q)^{1/q} \le C_2(\sum_{k\in\mathbb{Z}}(\frac{\|\phi_{2^k} * f\|_p}{w(k+1)})^q)^{1/q}.$$

This leads to the following natural assumption on the weight w.

There exists a constant $C > 0$ such that

$$C^{-1}w(k+1) \le w(k) \le Cw(k+1), \qquad (k \in \mathbb{Z}). \tag{2}$$

Remark (1.11). Note that (2) implies

$$C^{-|k|}w(0) \le w(k) \le C^{|k|}w(0)$$

which corresponds to assumption (1) with $\gamma = \log_2(C)$.

PROPOSITION (1.12). *Let p, $q \in (0, +\infty]$, w be a sequence satisfying (2), ϕ, $\psi \in \mathcal{B}$ and $B \in (0, +\infty)$. Denote $X_p(k) = w^{-1}(k)L^p\left[2^{-k}B\right]$ for $0 < p < 1$ and $X_p(k) = w^{-1}(k)L^p\left(\mathbb{R}^n\right)$ for $1 \le p \le \infty$.*

The operator T_ϕ given by $T_\phi(f) = \sum_{k\in\mathbb{Z}} \phi_{2^k} * f_k$ is well defined and bounded from $\ell_q\left(X_p(k), k \in \mathbb{Z}\right)$ to $B_w^{p,q}(\psi)$.

Proof. First, we shall prove that the map $\alpha \mapsto \sum_{k\in\mathbb{Z}} \langle \phi_{2^k} * f_k, \alpha\rangle$ defines an element of $\mathcal{S}'_\infty$. Take $A \in (0, +\infty)$ such that $\operatorname{supp}\hat{\phi} \subseteq K_A$ and consider the Besov space $B_\lambda^{p',q'}$, with $\lambda(k) = \frac{1}{w(k)}2^{kn_p}$. From Proposition (1.3) we have that, if $\alpha \in \mathcal{S}_\infty$ then $\left(\tilde{\phi}_{2^k} * \alpha\right)_{k\in\mathbb{Z}} \in \ell_{q'}\left(2^{-kn_p}w(k)L^{p'}\left[2^{-k}A\right], k \in \mathbb{Z}\right)$ with norm bounded by a continuous seminorm on $\mathcal{S}$ evaluated at α. Hence, from (1.8) and (1.9), we get that the series $\sum_{k\in\mathbb{Z}} \langle \phi_{2^k} * f_k, \alpha\rangle = \sum_{k\in\mathbb{Z}} \langle f_k, \tilde{\phi}_{2^k} * \alpha\rangle$ converges and its sum is bounded by $C_\alpha\|f\|$, where C_α is a continuous seminorm on $\mathcal{S}$, depending on n, p, q, w and ϕ. Therefore, $\sum_{k\in\mathbb{Z}} \phi_{2^k} * f_k$ defines an element in $\mathcal{S}'_\infty$ that we will denote $T_\phi(f)$.

Now, we must prove that the above distribution defines an element in the Besov space $B_w^{p,q}(\psi)$. Take $f \in \ell_q\left(X_p(k), k \in \mathbb{Z}\right)$ and notice that $\psi_{2^l} * T_\phi(f) = \sum_{k\in\mathbb{Z}} \phi_{2^k} * \psi_{2^l} * f_k$.

First take $M \in \mathbb{N}$ such that $\psi_{2^l} * \phi_{2^k} = 0$ if $|l - k| > M$. It follows from assumption (2) that there is $C_1 \in (0, +\infty)$ such that $\frac{w(k)}{w(l)} \le C_1$ for all $l, k \in \mathbb{Z}$ such that $|k - l| \le M$.

Observe now that

$$(b)\qquad \|\psi_{2^l} * \phi_{2^k} * f_k\|_p \le C\|f_k\|_p, \qquad |k-l| \le M.$$

Indeed this is obvious from Young inequality if $p \ge 1$ and for $p < 1$ we apply the following argument: there is a constant a such that

$$\psi_{2^l}, \phi_{2^k}, f_k \in L^p[2^{-k}a], \text{ for all } \quad |k-l| \le M, k \in \mathbb{Z}.$$

This allows us to apply Lemma C and to get (b) by using that $\|\psi_{2^l} * \phi_{2^k}\|_p \le C2^{kn(1/p-1)}$.

Hence,

$$\begin{aligned}\frac{\|\psi_{2^l} * T_\phi(f)\|_p}{w(l)} &= \frac{\left\|\sum_{|k-l|\le M} \psi_{2^l} * \phi_{2^k} * f_k\right\|_p}{w(l)} \\ &\le C \sum_{|k-l|\le M} \frac{\|\psi_{2^l} * \phi_{2^k} * f_k\|_p}{w(l)} \\ &\le C \sum_{|k-l|\le M} \frac{\|f_k\|_p}{w(l)} \\ &\le C \sum_{|k-l|\le M} \frac{\|f_k\|_p}{w(k)} \\ &= C \sum_{\substack{i\in\mathbb{Z}\\ |i|\le M}} \frac{\|f_{l+i}\|_p}{w(l+i)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|T_\phi(f)\|_{B_w^{p,q}(\psi)} &\le C \left(\sum_{|i|\le M} \left(\sum_{i\in\mathbb{Z}} \frac{\|f_{l+i}\|_p}{w(l+i)} \right)^q \right)^{1/q} \\ &\le C \sum_{|i|\le M} \left(\sum_{l\in\mathbb{Z}} \left(\frac{\|f_{l+i}\|_p}{w(l+i)} \right)^q \right)^{1/q} \\ &\le C'\|f\|_{l^q(X_p)}.\end{aligned}$$

Note that, if $p \ge 1$, the inequality $\|\psi_{2^l} * \phi_{2^k} * f_k\|_p \le C\|f_k\|_p$ holds, even if $\hat{f}_k$ has no support condition.

Remark (1.13). If we take (ϕ, α) as in Lemma (1.2) and $\operatorname{supp}\hat{\phi} \subseteq K_B$, we have, with the above notation, that $T_\alpha \circ S_\phi(f) = f$ for all $f \in B_w^{p,q}$; that is to say, the Besov space $B_w^{p,q}$ is a retract of the space $\ell_q\left(w^{-1}(k)L^p\left[2^{-k}B\right], k \in \mathbb{Z}\right)$ or $\ell_q\left(w^{-1}L^p(\mathbb{R}^n)\right)$ for $p < 1$ or $p \ge 1$ respectively.

COROLLARY (1.14). *Let p, $q \in (0, +\infty]$, w a weight in $\mathbb{Z}$ satisfying (2) and ϕ, $\psi \in \mathcal{B}$.*

Then $B_w^{p,q}(\psi) = B_w^{p,q}(\phi)$ with equivalent quasi-norms.

Proof. Clearly, it suffices to prove an inclusion between the spaces. Take $\alpha \in \mathcal{B}$ such that (α, ϕ) are as in Lemma (1.2) and $B \in (0, +\infty)$ such that $\phi \in \mathcal{S}[B]$. If $f \in B_w^{p,q}(\phi)$ then $S_\phi(f) \in \ell_q\left(w^{-1}(k)L^p\left[2^{-k}B\right], k \in \mathbb{Z}\right)$. Now $T_\alpha\left(S_\phi(f)\right) \in B_w^{p,q}(\psi)$ from Proposition (1.12). But, using Lemma A, $T_\alpha\left(S_\phi(f)\right) = f$.

Remark (1.15). In what follows we choose the particular case $\Phi \in \mathcal{S}$ such that Φ is radial, $\operatorname{supp}\hat{\Phi} \subset \{\frac{1}{4} \le |x| \le 4\}$, $\hat{\Phi}(\xi) \le 1$ and $0 \le \hat{\Phi}(\xi) = 1$ for $\frac{1}{2} \le |x| \le 2$. We will use the notation $B_w^{p,q}$ for the space $B_w^{p,q}(\Phi)$ and $(B')_w^{p,q}$ for the closure of $\mathcal{S}_\infty$ in $B_w^{p,q}$

COROLLARY (1.16). *Let $p, q \in (0, +\infty]$, w a weight in $\mathbb{Z}$ which satisfies (2) and $\phi \in \mathcal{B}$. Then a function $f \in (B')_w^{p,q}$ if and only if $S_\phi(f) \in \ell_q^0\left(w^{-1}L_0^p(\mathbb{R}^n)\right)$.*

Proof. If $\alpha \in \mathcal{S}_\infty$ then $\phi_{2^k} * \alpha \in \mathcal{S} \subseteq L_0^p(\mathbb{R}^n)$. Hence, by Proposition (1.3), $S_\phi(\alpha) \in \ell_q^0\left(w^{-1}L_0^p(\mathbb{R}^n)\right)$. Since S_ϕ is continuous and $\ell_q^0\left(w^{-1}L_0^p(\mathbb{R}^n)\right)$ is closed we obtain half of the corollary.

Take $\psi \in \mathcal{B}$ such that (ϕ, ψ) are as in Lemma (1.2) and $B \in (0, +\infty)$ such that $\operatorname{supp}\hat{\phi} \subseteq K_B$. With the help of Lemma (1.4) we get that $\ell_q^0(w^{-1}(k)L_0^p\left[2^{-k}B\right], k \in \mathbb{Z})$ is the closure of $c_{00}(w^{-1}(k)\mathcal{S}\left[2^{-k}B\right], k \in \mathbb{Z})$ in $\ell_q(w^{-1}(k)L^p\left[2^{-k}B\right], k \in \mathbb{Z})$.

Note that $T_\psi(f) \in \mathcal{S}_\infty$ for all $f \in c_{00}(w^{-1}(k)\mathcal{S}[2^{-k}B], k \in \mathbb{Z})$ and then the continuity of T_ψ on $\ell_q(w^{-1}(k)L^p[2^{-k}B], k \in \mathbb{Z})$ gives that $T_\psi(f) \in (B')_w^{p,q}$ for all $f \in \ell_q^0(w^{-1}(k)L_0^p[2^{-k}B], k \in \mathbb{Z})$.

Now, it only remains to recall that $S_\phi(f) \in \ell_q(w^{-1}(k)L^p[2^{-k}B], k \in \mathbb{Z})$ for all $f \in B_w^{p,q}$ and that $T_\psi \circ S_\phi$ is the identity operator on $B_w^{p,q}$.

COROLLARY (1.17). *$\mathcal{S}_\infty$ is dense in $B_w^{p,q}$ if and only if $p, q < +\infty$.*

Proof. Corollary (1.16) gives directly that $\mathcal{S}_\infty$ is dense in $B_w^{p,q}$ for $p, q < +\infty$.

If $p = +\infty$, then take $a \in \mathbb{R}^n \setminus \{0\}$, $\phi \in \mathcal{B}$ and define $f(x) = e^{i\langle a,x\rangle}$. We get that $\phi_{2^k} * f(x) = \hat{\phi}(2^k a)e^{i\langle a,x\rangle}$. Hence, $\|\phi_{2^k} * f\|_\infty = |\hat{\phi}(2^k a)| \in c_{00}(\mathbb{Z})$. Therefore, $f \in B_w^{\infty,q}$; but $\phi_{2^k} * f \notin C_0(\mathbb{R}^n)$ if we take a such that $\hat{\phi}\left(2^k a\right) \neq 0$.

If $q = +\infty$ then take $\phi \in \mathcal{B}$ such that $\hat{\phi}(x) = 1$ if $1 \le |x| \le 2$, and $\hat{\phi}(x) = 0$ if $|x| \le \frac{1}{2}$ or $4 \le |x|$.

For each $l \in \mathbb{Z}$, choose $f_l \in \mathcal{S}'$ such that $\operatorname{supp}\hat{f_l} \subseteq \{x \in \mathbb{R}^n;\ 2^{-3l} \le |x| \le 2^{-3l+1}\}$ and $\|f_l\|_p = w(3l)$. Define $f \in \mathcal{S}'_\infty$ such that $f = \sum_{l \in \mathbb{Z}} f_l$.

Due to the fact that for each $k \in \mathbb{Z}$, there exists a unique $j \in \mathbb{Z}$ such that $|k - 3j| \le 1$, we have $\phi_{2^k} * f_l = 0$ if $l \neq j$ and also that $\phi_{2^{3j}} * f_j = f_j$. Therefore, taking into account (2) and Lemma C,

$$\|\phi_{2^k} * f\|_p = \|\phi_{2^k} * f_j\|_p \le C\|f_j\|_p = Cw(3j) \le Cw(k),$$

and we get $f \in B_w^{p,\infty}$.

On the other hand since $\|\phi_{2^{3j}} * f\|_p = w(3j)$ then $\left(\frac{\|\phi_{2^k} * f\|_p}{w(k)}\right)_{k\in\mathbb{Z}} \notin c_0(\mathbb{Z})$.

2. Some preliminary results on multipliers

Recall that $B_w^{p,q}$ stands for $B_w^{p,q}(\Phi)$ for the particular case $\Phi \in \mathcal{S}$ such that Φ is radial, $\operatorname{supp}\hat{\Phi} \subset \{\frac{1}{4} \le |x| \le 4\}$, $0 \le \hat{\Phi}(\xi) \le 1$ and $\hat{\Phi}(\xi) = 1$ for $\frac{1}{2} \le |x| \le 2$ and all weights that appear from now on will satisfy condition (2).

Let us start with some elementary embeddings among the spaces.

PROPOSITION (2.1). *Let* $0 < p_1 \le p_2 \le \infty$, $0 < q_1 \le q_2 \le \infty$ *and* $w(k) \le Cu(k)$ *for all* $k \in \mathbb{Z}$. *Then*

(i) $B_w^{p_1,q_1} \subset B_w^{p_1,q_2}$.
(ii) $B_u^{p_1,q_1} \subset B_w^{p_1,q_1}$.
(iii) $B_{w_1}^{p_1,q_1} \subset B_{w_2}^{p_2,q_1}$ *for* $w_1(k)2^{nk/p_2} = w_2(k)2^{nk/p_1}$.

Proof. (i) and (ii) are obvious.

(iii) Assume $p_1 \le 1$. From (1.6) in Lemma C follows that

$$\|\Phi_{2^k} * f\|_{p_2} \le C2^{-nk(1/p_1-1/p_2)}\|\Phi_{2^k} * f\|_{p_1}.$$

Assume now that $1 < p_1$. Using Young's inequality one can write, for $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{r} - 1$,

$$\begin{aligned}\|(\Phi * \Phi)_{2^k} * f\|_{p_2} &= \|\Phi_{2^k} * \Phi_{2^k} * f\|_{p_2}\\ &\le \|\Phi_{2^k}\|_r\|\Phi_{2^k} * f\|_{p_1}\\ &\le C2^{nk(1/p_2-1/p_1)}\|\Phi_{2^k} * f\|_{p_1}\\ &\le C\frac{w_2(k)}{w_1(k)}\|\Phi_{2^k} * f\|_{p_1}.\end{aligned}$$

The proof is finished by computing the corresponding ℓ_{q_1} norm and invoking Corollary (1.14) for functions Φ and $\Phi * \Phi$.

Let us know get some results on the Fourier transform of functions in these classes.

Using the notation $A_k = \{2^{-k-1} \le |y| \le 2^{-k}\}$ it follows from Plancherel's theorem that

$$(c) \qquad \left(\int_{A_k} |\hat{f}(y)|^2 dy\right)^{1/2} \le \|\Phi_{2^k} * f\|_2 \le C\left(\int_{\cup_{j=k-2}^{k+1} A_j} |\hat{f}(y)|^2 dy\right)^{1/2}$$

From this one gets the following fact

PROPOSITION (2.2). *A function* $f \in B_w^{2,2}$ *if and only if* $\hat{f} \in L^2(W)$, *where* $W(x) = \sum_{k\in\mathbb{Z}} w^{-1}(k)\chi_{A_k}$.

Let us give now a Bernstein type theorem in this situation (see [He, Jo, P1]).

PROPOSITION (2.3). *Let W be a weight function (i.e. a measurable function such that $0 < W(x) < \infty$ a.e.). Let $1 < p \leq 2$ and $w^{-1}(k) = (\int_{A_k} W^p(x)dx)^{1/p}$.*

If $f \in B_w^{p,1}$ then $\hat{f} \in L^1(W)$. Moreover

$$\int_{\mathbb{R}^n} |\hat{f}(y)|W(y)dy \leq \|f\|_{B_w^{p,1}}.$$

Proof. Applying Hölder's and Hausdorff-Young's inequalities one has

$$\int_{A_k} |\hat{f}(y)|W(y)dy \leq \left(\int_{A_k} |\hat{f}(y)|^{p'} dy\right)^{1/p'} w^{-1}(k) \leq \|\Phi_{2^k} * f\|_p w^{-1}(k).$$

Adding them up over $k \in \mathbb{Z}$ we get the result.

Both results lead to the consideration of weighted Herz spaces, where they can easily be understood.

Definition (2.4). Let $u(k)_{k\in\mathbb{Z}}$ be a sequence of positive numbers; let $p \in (0, +\infty]$, $q \in (0, +\infty]$. Given $f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\})$ we define, for $q < +\infty$,

$$\|f\|_{K^{p,q}(u)} = \left(\sum_{k\in\mathbb{Z}} (\|\chi_{A_k} f\|_p u(k))^q\right)^{1/q} ;$$

and, for $q = +\infty$,

$$\|f\|_{K^{p,\infty}(u)} = \sup_{k\in\mathbb{Z}} \|\chi_{A_k} f\|_p u(k).$$

Therefore, we can consider the quasi-normed space (normed if $p, q \geq 1$)

$$K^{p,q}(u) = \left\{f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\});\ \|f\|_{K^{p,q}(u)} < +\infty\right\}.$$

The case $u(k) = 2^{k\alpha}$ was first considered in [He]. The reader is referred to [F, Jo] for some results on multipliers involving those spaces, and to [BaS] where the weighted version was already used.

Elementary properties, whose proofs are left to the reader, are the following:

PROPOSITION (2.5). *Let $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ and $u_1(k) \leq Cu_2(k)$ for all $k \in \mathbb{Z}$. Then*

(i) $K^{p,p}(u) = L^p(U)$ where $U(x) = \sum_{k\in\mathbb{Z}}(u(k))^p \chi_{A_k}$.
(ii) $K^{p_1,q_1}(u) \subset K^{p_1,q_2}(u)$.
(iii) $K^{p,q}(u_2) \subset K^{p,q}(u_1)$.
(iv) $K^{p_2,q_1}(u) \subset K^{p_1,q_1}(v)$ for $u(k)2^{nk/p_2} = v(k)2^{nk/p_1}$.

With all this one has the following elementary extension of the previous results.

THEOREM (2.6). *Let $\{w(k)\}_{k\in\mathbb{Z}}$ and $u(k)=\frac{1}{w(k)}$ be weights and let $0<q\leq\infty$.*

(i) $f\in B_w^{2,q}$ if and only if $\hat{f}\in K^{2,q}(u)$.

(ii) If $0<p\leq 1$ and $f\in B_w^{p,q}$ then $\hat{f}\in K^{\infty,q}(u')$ where $u'(k)=u(k)2^{nk(1/p-1)}$. Moreover

$$\|\hat{f}\|_{K^{\infty,q}(u')}\leq C\|f\|_{B_w^{p,q}}\,.$$

(iii) If $1<p\leq 2$ and $f\in B_w^{p,q}$ then $\hat{f}\in K^{p',q}(u)$. Moreover

$$\|\hat{f}\|_{K^{p',q}(u)}\leq\|f\|_{B_w^{p,q}}\,.$$

(iv) If $1<p\leq 2$ and $\hat{f}\in K^{p,q}(u)$ then $f\in B^{p',q}(w)$. Moreover

$$\|f\|_{B_w^{p',q}}\leq C\|\hat{f}\|_{K^{p,q}(u)}\,.$$

Proof. (i) follows from (c).

(ii) The result is obvious for $p=1$ since the Fourier transform maps L^1 into L^∞ and the case $0<p<1$ follows from the embedding (iii) in Proposition (2.1).

(iii) Use the estimate $(\int_{A_k}|\hat{f}(x)|^{p'}dx)^{1/p'}\leq\|\Phi_{2^k}*f\|_p$.

(iv) Use the estimate $\|\Phi_{2^k}*f\|_{p'}\leq C(\int_{\bigcup_{j=k-2}^{k+1}A_j}|\hat{f}(x)|^p dx)^{1/p}$.

The following description of pointwise multipliers between Hertz spaces follows easily from Hölder's inequality (see [K] for a proof in the non-weighted and the sequence spaces $l(p,q)$ that can be reproduced in our situation).

PROPOSITION (2.7). *Let $0<p_1,q_1\leq\infty$, $0<p_2,q_2\leq\infty$, with $p_1\geq p_2$, and $u_1(k)$, $u_2(k)$ sequences of positive real numbers. Let $0<p,q\leq\infty$ such that $\frac{1}{p}=\frac{1}{p_2}-\frac{1}{p_1}$ and $\frac{1}{q}=\max\left\{\frac{1}{q_2}-\frac{1}{q_1},0\right\}$. Then*

$$K^{p,q}\left(\frac{u_2}{u_1}\right)=\{f\in L^p_{loc}(\mathbb{R}^n\setminus\{0\}):fg\in K^{p_2,q_2}(u_2)\text{ for all }g\in K^{p_1,q_1}(u_1)\}\,.$$

COROLLARY (2.8). *Let q_1, $q_2\in(0,+\infty]$, w_1, w_2 be sequences of positive numbers and $a\in\mathcal{S}'$.*

Then $a\in(B_{w_1}^{2,q_1},B_{w_2}^{2,q_2})$ if and only if $\hat{a}(x)\in K^{\infty,q}(u)$ where $u(k)=\frac{w_1(k)}{w_2(k)}$ and $\frac{1}{q}=\max\left\{\frac{1}{q_2}-\frac{1}{q_1},0\right\}$.

Proof. Combine (i) in Theorem (2.6) and Proposition (2.7).

COROLLARY (2.9). *Let* $1 \leq p_1 \leq 2 \leq p_2$. *Let* $q_1, q_2 \in (0, +\infty]$ *and* w_1, w_2 *weights in* $\mathbb{Z}$. *If s and t are given by* $\frac{1}{t} = \frac{1}{p_1} - \frac{1}{p_2}$ *and* $\frac{1}{s} = \max\left\{\frac{1}{q_2} - \frac{1}{q_1}, 0\right\}$, *and if* $\hat{a} \in K^{s,t}(\frac{w_1}{w_2})$, *then* $a \in \left(B^{p_1,q_1}_{w_1}, B^{p_2,q_2}_{w_2}\right)$.

Proof. Given $f \in B^{p_1,q_1}_{w_1}$ from (iii) in Theorem (2.6) we have $\hat{f} \in K^{p_1',q_1}(\frac{1}{w_1})$. Now Proposition (2.7) gives that $\hat{a}\hat{f} \in K^{p_2,q_2}(\frac{1}{w_2})$ and again (iv) in Theorem (2.6) gives the desired result.

3. The general case of multipliers

In this section we shall solve the problem posed by Peetre (see [P1]) of finding a formulation of a multiplier between different Besov classes in terms of a sequence of multipliers between Lebesgue spaces having a certain size condition. We shall do several steps to get the main result.

PROPOSITION (3.1). *Let* $p_0, p_1, q_0, q_1 \in (0, +\infty]$ *and* w_0 *and* w_1 *weights in* $\mathbb{Z}$. *Let* $\phi, \psi \in \mathcal{B}$ *and* $B \in (0, +\infty)$ *such that* $\operatorname{supp}\hat{\phi} \subseteq K_B$ *and* $a \in \mathcal{S}'$. *Then,* $a \in \left(B^{p_0,q_0}_{w_0}, B^{p_1,q_1}_{w_1}\right)$ *if and only if there exists a bounded operator*

$$N: \ell^0_{q_0}(w_0^{-1}(k) L^{p_0}_0[2^{-k}B], k \in \mathbb{Z}) \to \ell_{q_1}(w_1^{-1} L^{p_1}(\mathbb{R}^n))$$

given by

$$N(f) = \left(\sum_{k\in\mathbb{Z}} a * \psi_{2^l} * \phi_{2^k} * f_k\right)_{l\in\mathbb{Z}}$$

for all $f \in c_{00}(\mathcal{S}[2^{-k}B], k \in \mathbb{Z})$.

If $p_0 \geq 1$, *no condition on the support of the Fourier transform of the functions has to be assumed and we can replace* $L^{p_0}_0[2^{-k}B]$ *by* $L^{p_0}_0(\mathbb{R}^n)$ *and* $\mathcal{S}[2^{-k}B]$ *by* $\mathcal{S}(\mathbb{R}^n)$.

Proof. Assume a is a multiplier, then we have a bounded operator

$$M: (B')^{p_0,q_0}_{w_0} \longrightarrow B^{p_1,q_1}_{w_1}$$

such that $M(f) = a * f$ for $f \in \mathcal{S}_\infty$.

Take

$$T: \ell^0_{q_0}(w_0^{-1}(k) L^{p_0}_0[2^{-k}B], k \in \mathbb{Z}) \to (B')^{p_0,q_0}_{w_0}$$

given by $T(f) = T_\phi(f)$.

Observe that to see that the range of T is in $(B')^{p_0,q_0}_{w_0}$ we can use the same argument as in corollary (1.16).

Let

$$S: B^{p_1,q_1}_{w_1} \to \ell_{q_1}(w_1^{-1} L^{p_1}(\mathbb{R}^n))$$

given by $S(f) = S_\psi(f)$.

We define a bounded operator

$$N: \ell^0_{q_0}(w_0^{-1}(k)L^{p_0}_0[2^{-k}B],\, k \in \mathbb{Z}) \to \ell_{q_1}(w_1^{-1}L^{p_1}(\mathbb{R}^n))$$

by $N = S \circ M \circ T$.

If $f \in c_{00}(\mathcal{S}[2^{-k}B],\, k \in \mathbb{Z})$, $T_\phi(f) \in \mathcal{S}_\infty$. Hence

$$N(f) = \left(\sum_{k\in\mathbb{Z}} a * \psi_{2^l} * \phi_{2^k} * f_k\right)_{l\in\mathbb{Z}}$$

For the converse, it is easy to see, using a density argument, that, given N as in the statement, it also holds that

$$N(f) = \left(\sum_{k\in\mathbb{Z}} a * \psi_{2^l} * \phi_{2^k} * f_k\right)_{l\in\mathbb{Z}}$$

for all $f \in \ell^0_{q_0}(\mathcal{S}[2^{-k}B],\, k \in \mathbb{Z})$.

Let α, β be such that $\operatorname{supp}\hat{\alpha} \subseteq K_B$ and (α, ϕ) and (β, ψ) are taken as in Lemma (1.2).

Take $S = S_\alpha$ defined on $(B')^{p_0,q_0}_{w_0}$, which gives

$$S: (B')^{p_0,q_0}_{w_0} \to \ell^0_{q_0}(w_0^{-1}(k)L^{p_0}_0[2^{-k}B],\, k \in \mathbb{Z})$$

and

$$T: \ell_{q_1}(w_1^{-1}L^{p_1}[2^{-k}A],\, k \in \mathbb{Z}) \to B^{p_1,q_1}_{w_1}$$

defined by $T(f) = T_\beta(f)$

Note that $\operatorname{supp}\hat{\psi} \subseteq K_A$ for some $A > 0$, and then the range of N is contained in $\ell_{q_1}(w_1^{-1}L^{p_1}[2^{-k}A],\, k \in \mathbb{Z})$.

Construct, then, the operator $M = T \circ N \circ S$, which turns out to be bounded from $(B')^{p_0,q_0}_{w_0}$ on $B^{p_1,q_1}_{w_1}$.

Note now that from Calderón's formula (Lemma A), if $f \in \mathcal{S}$ then

$$[N \circ S(f)]_l = \sum_{k\in\mathbb{Z}} a * \psi_{2^l} * \phi_{2^k} * \alpha_{2^k} * f = a * \psi_{2^l} * f$$

and

$$M(f) = \sum_{l\in\mathbb{Z}} a * \beta_{2^l} * \psi_{2^l} * f = a * f\,.$$

The proof for $p_0 \geq 1$ is an step by step repetition, taking into account that in this case we need no support condition.

Now, we realize an improvement of Proposition (3.1), just to get a *line to line* operator, acting on mixed norm spaces. This operator will be simpler for our study.

LEMMA (3.2). *Given $\eta \in \mathcal{B}$. There exists $\phi, \gamma \in \mathcal{B}$ such that $\phi = \gamma * \eta$.*

Proof. Given an open set W such that $\overline{W} \subseteq \mathbb{R}^n \setminus \{0\}$ is compact and a compact set $K \subseteq W$ such that $2^{\mathbb{Z}}K = \mathbb{R}^n \setminus \{0\}$, we can construct $\phi \in \mathcal{S}$ with $\hat{\phi}(x) \neq 0$ if $x \in K$ and $\hat{\phi}(x) = 0$ if $x \notin W$.

On the other hand if $W = \{x : \hat{\eta}(x) \neq 0\}$ we can easily construct $K \subseteq W$ such that $2^{\mathbb{Z}}K = \mathbb{R}^n \setminus \{0\}$ then considering the above function ϕ we have that $\hat{\gamma} = \frac{\hat{\phi}}{\hat{\eta}} \in \mathcal{S}$ what gives $\phi = \gamma * \eta$ and also that ϕ and γ belong to $\mathcal{B}$.

PROPOSITION (3.3). *Let $p_0, p_1, q_0, q_1 \in (0, +\infty]$ and w_0 and w_1 weights in $\mathbb{Z}$ satisfying condition (2). Let $\eta \in \mathcal{B}$ and $B \in (0, +\infty)$ such that* $\operatorname{supp} \hat{\eta} \subseteq K_B$. *Let $a \in \mathcal{S}'$. Then $a \in \left(B_{w_0}^{p_0,q_0}, B_{w_1}^{p_1,q_1}\right)$ if and only if there exists a bounded operator*

$$L: \ell_{q_0}^0(w_0^{-1}(k)L_0^{p_0}[2^{-k}B], k \in \mathbb{Z}) \to \ell_{q_1}(w_1^{-1}L^{p_1}(\mathbb{R}^n))$$

such that

$$L(f) = \left(a * \eta_{2^k} * f_k\right)_{k \in \mathbb{Z}}$$

for all $f \in c_{00}(\mathcal{S}[2^{-k}B], k \in \mathbb{Z})$.

If $p_0 \geq 1$, we can replace $L_0^{p_0}[2^{-k}B]$ by $L_0^{p_0}(\mathbb{R}^n)$ and $\mathcal{S}[2^{-k}B]$ by $\mathcal{S}(\mathbb{R}^n)$.

Proof. We only deal with the case $p_0 \leq 1$, the other case being similar but simpler.

Suppose that we have a bounded operator $L: \ell_{q_0}^0(w_0^{-1}(k)L_0^{p_0}([2^{-k}B], k \in \mathbb{Z}) \to \ell_{q_1}(w_1^{-1}L^{p_1}(\mathbb{R}^n))$ as in the assumption. Actually, the range of this operator is contained in $\ell_{q_1}(w_1^{-1}L^{p_1}[2^{-k}B])$.

First apply Lemma (3.2) to get $\phi, \gamma \in \mathcal{B}$ such that $\phi = \eta * \gamma$.

Given $\psi \in \mathcal{B}$, take $M \in \mathbb{N}$ such that $\psi * \phi_{2^k} = 0$ if $|k| > M$. Consider $\gamma_i = \gamma_{2^i} * \psi$ what allows to say $\psi * \phi_{2^i} = \gamma_i * \eta_{2^i}$ and $\operatorname{supp} \hat{\gamma}_i \subseteq \operatorname{supp} \hat{\psi} \subseteq K_A$ for some $A \in (0, +\infty)$.

By means of (2), there exist $C_1, C_2 \in (0, +\infty)$ such that $C_1 \leq \dfrac{w_1(l+i)}{w_1(l)} \leq C_2$ if $l, i \in \mathbb{Z}$ and $|i| \leq M$. Observe then that for each $i \in \mathbb{Z}$ such that $|i| \leq M$, the operator

$$\Lambda_i: \ell_{q_1}(w_1^{-1}(k)L^{p_1}[2^{-k}B], k \in \mathbb{Z}) \to \ell_{q_1}(w_1^{-1}L^{p_1}(\mathbb{R}^n))$$

given by

$$\Lambda_i(f) = \left((\gamma_i)_{2^l} * f_{l+i}\right)_{l \in \mathbb{Z}}$$

is bounded.

Indeed from Lemma C (if $p_1 < 1$) or Young's inequality we have

$$\|(\gamma_i)_{2^l} * f_{l+i}\|_{p_1} \leq C 2^{(M-l)n(1/p_1-1)} 2^{ln(1/p_1-1)} \|\gamma_i\|_{\min\{p_1,1\}} \|f_{l+i}\|_{p_1}.$$

Therefore,

$$N = \sum_{|i| \leq M} \Lambda_i \circ L$$

is bounded from $\ell^0_{q_0}\left(w_0^{-1}(k)L^{p_0}([2^{-k}B], k \in \mathbb{Z}\right)$ to $\ell_{q_1}\left(w_1^{-1}L^{p_1}(\mathbb{R}^n)\right)$.

But, given $f \in c_{00}(\mathcal{S}[2^{-k}B], k \in \mathbb{Z})$,

$$\begin{aligned} N(f)_l &= \sum_{|i|\le M} a * (\gamma_i)_{2^l} * \eta_{2^{l+i}} * f_{l+i} \\ &= \sum_{|i|\le M} a * \psi_{2^l} * \phi_{2^{l+i}} * f_{l+i} \\ &= \sum_{k\in\mathbb{Z}} a * \psi_{2^l} * \phi_{2^k} * f_k. \end{aligned}$$

Hence a is Fourier multiplier between the spaces according to Proposition (3.1).

Let us prove the converse. It is rather simple to get $\psi \in \mathcal{B}$ such that $\hat{\psi}(x) \neq 0$ for all $x \in \operatorname{supp}\hat{\eta}$. Take $\phi \in \mathcal{B}$ such that $\hat{\phi} = \dfrac{\hat{\eta}}{\hat{\psi}}$. Then $\operatorname{supp}\hat{\phi} \subseteq K_B$ and $\phi * \psi = \eta$.

Related with ϕ and ψ we have a bounded operator

$$N\colon \ell^0_{q_0}(w_0^{-1}(k)L^{p_0}_0[2^{-k}B], k \in \mathbb{Z}) \to \ell_{q_1}(w_1^{-1}L^{p_1}(\mathbb{R}^n)),$$

given by

$$N(f)_l = \left(\sum_{k\in\mathbb{Z}} a * \psi_{2^l} * \phi_{2^k} * f_k\right).$$

Let again $M \in \mathbb{N}$ such that $\phi_{2^k} * \psi = 0$ if $|k| > M$. For $i \in \mathbb{Z}$ with $|i| \le M$ we denote

$$\mathcal{H}_i = \left\{f \in \ell^0_{q_0}(w_0^{-1}(k)L^{p_0}_0[2^{-k}B], k \in \mathbb{Z}); f_k = 0 \text{ if } k \not\equiv i \operatorname{mod}(2M+1)\right\}$$

$$\mathcal{G}_i = \left\{f \in \ell_{q_1}(w_1^{-1}(k)L^{p_1}(\mathbb{R}^n), k \in \mathbb{Z}); f_k = 0 \text{ if } k \not\equiv i \operatorname{mod}(2M+1)\right\}$$

From N we construct bounded operators $N_i\colon \mathcal{H}_i \to \mathcal{G}_i$ given by

$$N_i(f)_k = \begin{cases} 0 & \text{if } k \not\equiv i \operatorname{mod}(2M+1), \\ N(f)_k & \text{if } k \equiv i \operatorname{mod}(2M+1). \end{cases}$$

If $f \in c_{00}(\mathcal{S}[2^{-k}B], k \in \mathbb{Z}) \cap \mathcal{H}_i$ and $k \equiv i \operatorname{mod}(2M+1)$ we have that

$$N_i(f)_k = \sum_{\substack{l\in\mathbb{Z} \\ l\equiv i \operatorname{mod}(2M+1)}} a * \psi_{2^k} * \phi_{2^l} * f_l = a * \psi_{2^k} * \phi_{2^k} * f_k = a * \eta_{2^k} * f_k.$$

Notice that

$$\ell^0_{q_0}(w_0^{-1}(k)L^{p_0}_0[2^{-k}B], k \in \mathbb{Z}) = \bigoplus_{|i|\le M} \mathcal{H}_i$$

and

$$\ell_{q_1}(w_1^{-1}(k)L^{p_1}(\mathbb{R}^n),\, k \in \mathbb{Z}) = \bigoplus_{|i|\leq M} \mathcal{G}_i\,.$$

This fact allows us to define a bounded operator

$$L\colon \ell^0_{q_0}(w_0^{-1}(k)L_0^{p_0}[2^{-k}B],\, k \in \mathbb{Z}) \to \ell_{q_1}(w_1^{-1}(k)L^{p_1}(\mathbb{R}^n),\, k \in \mathbb{Z})$$

by means of

$$L(f) = \sum_{|i|\leq M} N_i(f_i)$$

for all $f_i \in \mathcal{H}_i$ and $f = \sum_{|i|\leq M} f_i$.

If $f \in c_{00}(\mathcal{S}[2^{-k}B],\, k \in \mathbb{Z})$, we have $f = \sum_{|i|\leq M} f_i$, with $f_i \in c_{00}(\mathcal{S}[2^{-k}B],\, k \in \mathbb{Z}) \cap \mathcal{H}_i$. Given $k \in \mathbb{Z}$, there exists a unique $j \in \mathbb{Z}$ with $k \equiv j \bmod (2M+1)$ and $|j| \leq M$. Hence $N_i(f_i)_k = 0$ for $i \neq j$ and $N_j(f_j)_k = a * \eta_{2^k} * f_k$. We obtain that

$$L(f)_k = a * \eta_{2^k} * f_k\,.$$

Now, we are ready to prove the main result of this paper, stated in Introduction as Main Theorem.

Proof of Main Theorem. Let us start with the sufficiency of the conditions. Let $c = \{c_k\}_{k\in\mathbb{Z}}$ defined by

$$c_k = \frac{w_0(k)}{w_1(k)} \|\eta_{2^k} * a\|_{(L^{p_0}([2^{-k}B],L^{p_1}(\mathbb{R}^n))}$$

and assume that $c \in \ell_s$. Let $\phi = \eta * \gamma$ as in Lemma (3.2). Then, from inequality (1.9) we get

$$\begin{aligned}\|a * f\|_{B^{p_1,q_1}_{w_1}(\phi)} &= \left(\sum_{k\in\mathbb{Z}} \left(\frac{\|\gamma_{2^k} * \eta_{2^k} * a * f\|_{p_1}}{w_1(k)}\right)^{q_1}\right)^{1/q_1} \\ &\leq \left(\sum_{k\in\mathbb{Z}} \left(c_k \frac{\|\gamma_{2^k} * f\|_{p_0}}{w_0(k)}\right)^{q_1}\right)^{1/q_1} \\ &\leq \|c\|_s \|f\|_{B^{p_0,q_0}_{w_0}(\gamma)}.\end{aligned}$$

For the converse we use Proposition (3.3). It is immediate to show that the operator defined by $f \to (a * \eta_{2^k} * f_k)_{k\in\mathbb{Z}}$ is bounded from the space $\ell^0_{q_0}(w_0^{-1}(k)$ $L_0^{p_0}[2^{-k}B],\, k \in \mathbb{Z})$ to the space $\ell_{q_1}(w_1^{-1}L^{p_1}(\mathbb{R}^n))$ if and only if for each $k \in \mathbb{Z}$ the operator L_k defined by

$$f \mapsto a * \eta_{2^k} * f$$

for $f \in \mathcal{S}[2^{-k}B]$, is bounded from $L_0^{p_0}[2^{-k}B]$ to $L^{p_1}(\mathbb{R}^n)$ together with the fact the operator defined by

$$\{s_k\}_{k\in\mathbb{Z}} \mapsto \left(\frac{w_0(k)}{w_1(k)}\|L_k\|s_k\right)_{k\in\mathbb{Z}}$$

is bounded from $\ell^0_{q_0}$ on ℓ_{q_1}, which obviously means $\dfrac{w_0(k)}{w_1(k)}\|L_k\| \in \ell_s(\mathbb{Z})$.

Obvious modifications give the case $p_0 \geq 1$.

To get some consequences of the Main Theorem we shall need to know some duality and interpolation properties of $L^p[A]$ spaces. The following result says that the dual space of $L^p[A]$ ($p < 1$) is very close to $L^\infty[A]$.

LEMMA (3.4). *(See* [P1], *p. 236). Let $p < 1$ and $A \in (0, +\infty)$. The operator $T: L^\infty[A] \to (L^p[A])'$ given by $\langle T(f), g\rangle = \int_{\mathbb{R}^n} fg\, dm$ is well defined and bounded. Moreover, for each $B < A$, there exists a bounded operator $S: (L^p[A])' \to L^\infty[A]$ such that $\langle T(S(f)), g\rangle = \langle f, g\rangle$ for $g \in L^p[B]$.*

We deal with real interpolation method (see [BL,BS]). Let $L^{p,r}$ denote the Lorentz space.

LEMMA (3.5). *Let $A \in (0, +\infty)$. Let p_0, $p_1 \in (0, +\infty]$ with $p_0 \neq p_1$. Let $\theta \in (0, 1)$, $r \in [1, +\infty]$ and $B < A$. If $\dfrac{1}{p} = (1-\theta)\dfrac{1}{p_0} + \theta\dfrac{1}{p_1}$ then $L^{p,r}[B] \subseteq \left(L^{p_0}[A], L^{p_1}[A]\right)_{\theta,r}$ (with continuity).*

Proof. It suffices to prove that

$$K\left(f, t, L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n)\right) \leq CK\left(f, t, L^{p_0}[A], L^{p_1}[A]\right)$$

for all $f \in L^{p_0}[B] + L^{p_1}[B]$ and for all $t > 0$. Assume $f \in L^{p_0}[B] + L^{p_1}[B]$ and write $f = g + h$, with $g \in L^{p_0}(\mathbb{R}^n)$ and $g \in L^{p_1}(\mathbb{R}^n)$. Take $\phi \in \mathcal{S}[A]$ such that $\hat{\phi}(x) = 1$ if $|x| \leq B$. Therefore we can also write $f = \phi * g + \phi * h$, with $\|\phi * g\|_{p_0} \leq C\|g\|_{p_0}$ and $\|\phi * h\|_{p_1} \leq C\|h\|_{p_1}$, which gives the desired decomposition.

PROPOSITION (3.6). *Let p_0, p_1, q_0, $q_1 \in (0, +\infty]$ and w_0 and w_1 weights in $\mathbb{Z}$ satisfying the condition (2). If $p_0 > p_1$, we have that*

$$\left(B^{p_0,q_0}_{w_0}, B^{p_1,q_1}_{w_1}\right) = \{0\}\,.$$

Proof. Let us take $\eta \in \mathcal{B}$ and $a \in \left(B^{p_0,q_0}_{w_0}, B^{p_1,q_1}_{w_1}\right)$. We define $a_k = \eta_{2^k} * a$. It's enough to prove that $a_k = 0$ for all $k \in \mathbb{Z}$.

If $p_0 \geq 1$, by means of Main Theorem, a_k is a multiplier from $L^{p_0}(\mathbb{R}^n)$ on $L^{p_1}(\mathbb{R}^n)$. It's well known that, in this case $a_k = 0$.

For $p_0 < 1$ we take B such that $\eta \in \mathcal{S}[B]$. Let $B < C < D < E < +\infty$. By means of Main Theorem we obtain that a_k is a multiplier from $L^{p_0}[2^{-k}E]$ on

$L^{p_1}[2^{-k}E]$. Duality (Lemma (3.4)) gives that a_k is multiplier from $L^\infty[2^{-k}D]$ on $L^\infty(\mathbb{R}^n)$. By interpolation (Lemma (3.5)) a_k is a multiplier from $L^2[2^{-k}C]$ on $L^{p,2}(\mathbb{R}^n)$ for some p with $1 < p < 2$. We take $\phi \in \mathcal{S}[C]$ such that $\hat{\phi}(x) = 1$ if $|x| \leq B$. ϕ_{2^k} is a multiplier from $L^2(\mathbb{R}^n)$ to $L^2[2^{-k}C]$ and $\phi_{2^k} * a_k = a_k$. Hence, a_k is a multiplier from $L^2(\mathbb{R}^n)$ to $L^{p,2}(\mathbb{R}^n)$, which gives that $\hat{a_k}$ is a pointwise multiplier from $L^2(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ from the Hausdorff-Young inequality. Consequently, $\hat{a_k} = 0$.

PROPOSITION (3.7). *Let $p_0, p_1, q_0, q_1 \in (0, +\infty]$ and w_0 and w_1 weights in $\mathbb{Z}$ satisfying the condition (2). We have that*

$$\left(B^{p_0,q_0}_{w_0}, B^{p_1,q_1}_{w_1}\right) \subseteq B^{p_1,s}_{\lambda},$$

for $\lambda(k) = \dfrac{w_1(k)}{w_0(k)} 2^{kn(1/p_0-1)}$ *and* $\dfrac{1}{s} = \max\left\{\dfrac{1}{q_1} - \dfrac{1}{q_0}, 0\right\}$.

Proof. Let $\phi, \eta \in \mathcal{B}$ with $\phi * \eta = \eta$. Let $B \in (0, +\infty)$ such that $\phi \in \mathcal{S}[B]$. Then $\eta \in \mathcal{S}[B]$. We have that $\phi_{2^k} \in L^{p_0}[2^{-k}B]$ and $\|\phi_{2^k}\|_{p_0} = C2^{nk(1/p_0-1)}$.

Let $a \in \left(B^{p_0,q_0}_{w_0}, B^{p_1,q_1}_{w_1}\right)$. From the Main Theorem, since $a * \eta_{2^k} = a * \phi_{2^k} * \eta_{2^k} \in L^{p_1}(\mathbb{R}^n)$ then

$$\|a * \eta_{2^k}\|_{p_1} \leq A(k) 2^{nk(1/p_0-1)}$$

where $A(k)$ denotes the norm of the operator from $L^{p_0}[2^{-k}B]$ to $L^{p_1}(\mathbb{R}^n)$.

Hence

$$\|a * \eta_{2^k}\|_{p_1} 2^{-nk(1/p_0-1)} \frac{w_0(k)}{w_1(k)} \leq A(k) \frac{w_0(k)}{w_1(k)}$$

and therefore $a \in B^{p_1,s}_{\lambda}$.

PROPOSITION (3.8). *Let $p_0, p_1, q_0, q_1 \in (0, +\infty]$. Let w_0, w_1 weights in $\mathbb{Z}$ satisfying the condition (2). If $p_0 \leq p_1$ and $p_0 \leq 1$ we have that*

$$\left(B^{p_0,q_0}_{w_0}, B^{p_1,q_1}_{w_1}\right) = B^{p_1,s}_{\lambda},$$

for $\lambda(k) = \dfrac{w_1(k)}{w_0(k)} 2^{kn_{p_0}}$ *and* $\dfrac{1}{s} = \max\left\{\dfrac{1}{q_1} - \dfrac{1}{q_0}, 0\right\}$.

Proof. Proposition (3.7) give us a half of the result. For the other half, take $\eta \in \mathcal{B}$ and $B \in (0, +\infty)$ such that $\eta \in \mathcal{S}[B]$. Let $a \in B^{p_1,s}_{\lambda}$. We have that $a * \eta_{2^k} \in L^{p_1}[2^{-k}B]$ for every $k \in \mathbb{Z}$. Then, using Lemma C, $f * a * \eta_{2^k} \in L^{p_1}[2^{-k}B]$ for any $f \in L^{p_0}[2^{-k}B]$ and

$$\|f * a * \eta_{2^k}\|_{p_1} \leq C 2^{-kn_{p_0}} \|a * \eta_{2^k}\|_{p_1} \|f\|_{p_0}.$$

Consequently, $a * \eta_{2^k} \in (L^{p_0}[2^{-k}B], L^{p_1}(\mathbb{R}^n))$ with norm bounded by $A(k) = C2^{-kn_{p_0}} \|a * \eta_{2^k}\|_{p_1}$. The result now follows from Main Theorem.

We would like to thank the referee for his carefull reading of the paper and his helpfull remarks.

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