ATOMIC DECOMPOSITION OF WEIGHTED BESOV SPACES

José Luis Ansorena and Oscar Blasco

Abstract. We find the atomic decomposition of functions in the weighted Besov spaces under certain factorization conditions on the weight.

Introduction.

After achieving the atomic decomposition of Hardy spaces (see [8,22, 33]), many of the function saces have been shown to admit similar decompositions. Let us mention the decomposition of B.M.O. (see [32, 25]), Bergman spaces (see [9, 23]), the predual of Bloch space (see [11]), Besov spaces (see [15, 4, 10]), Lipschitz spaces (see [18]), Triebel-Lizorkin spaces (see [16, 31]),...

They are obtained by quite different methods, but there is a unified and beautiful approach to get the decomposition for most of the spaces. This is the use of a formula due to A.P. Calderón (see [6, 7]). The reader is referred to the book by M. Frazier, B. Jawerth and G. Weiss [18] for a collection of spaces where the Calderón's formula produces the atomic decomposition and applications of it.

Atomic decompositions of weighted versions of different spaces have been also considered in the literature (see [27] for weighted Hardy spaces, [5] for Lipschitz spaces,...).

In this paper we shall be concerned with weighted Besov spaces $B^{p,q}_{\phi,w}$. We shall find some conditions on the weights to have atomic decomposition on the spaces.

We refer the reader to [19, 29, 18] for general notions and applications of atomic decomposition and to [1, 24, 30] for different formulations and properties of Besov classes.

The classes of weights where the results are achieved consist of those which factorize through powers of Dini and b_1 weights. Our arguments for the cases $1 < p, q < \infty$ will be based upon two main points: Calderón's formula and a quite simple Schur Lemma. To obtain the other extreme cases $p, q \in \{1, \infty\}$ we need some new results on the classes of weights which enable us to apply the same procedure as in the previous cases. The reader should be aware that the case $1 < q < \infty$ could have been shown by interpolation with the extreme cases, but a direct proof is presented in the paper.

1991 *Mathematics Subject Classification.* 42A45, 42B25, 42C15.

Key words and phrases. Besov spaces, atomic decomposition, weights.

The authors has been partially supported by the Spanish DGICYT, Proyectos PB89-0106 and PB92-0699

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Throughout the paper a weight $w : \mathbb{R}^+ \to \mathbb{R}^+$ will be a measurable function $w > 0$ a.e., $1 \leq p, q \leq \infty$ and p', q' stand for the conjugate exponents. $\mathcal{S}$ denotes the Schwartz class of test functions on $\mathbb{R}^n$, $\mathcal{S}'$ the space of tempered distributions, $\mathcal{S}_0$ the set of functions in $\mathcal{S}$ with mean zero and $\mathcal{S}'_0$ its topological dual. We denote by $\mathcal{D}$ the collection of dyadic cubes Q in $\mathbb{R}^n$, i.e. $Q = Q_{j,z} = \{x \in \mathbb{R}^n : 2^j z_i \leq x_i \leq 2^j(z_i+1)\}$ for $z \in \mathbb{Z}^n, j \in \mathbb{Z}$. As usual $|Q|, l(Q)$ stand for the volume and the side length of the cube Q respectively. We shall write $T(Q)$ for $Q \times (\frac{l(Q)}{2}, l(Q)] \in \mathbb{R}^{n+1}_+$ and cQ for a cube with the same center as Q but with side length equal $cl(Q)$.

Given a weight w, $\phi \in \mathcal{S}_0$ and $1 \leq p, q \leq \infty$ we shall denote by $B^{p,q}_{w,\phi}$ the space functions $f : \mathbb{R}^n \to \mathbb{C}$ with $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ such that

$$||f||_{B^{p,q}_{w,\phi}} = \left(\int_{\mathbb{R}^+} \frac{||\phi_t * f||_p^q}{w(t)^q} \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \qquad (1 \leq q < \infty)$$

or

$$||f||_{B^{p,\infty}_{w,\phi}} = \inf\{C > 0 : ||\phi_t * f||_p \leq Cw(t) \quad \text{a.e.} t > 0\} < \infty \qquad (q = \infty)$$

where $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$.

To state the results of the paper, let us first recall the following notions.

A weight w is said to satisfy Dini condition if there exists $C > 0$ such that

$$\int_0^s \frac{w(t)}{t} dt \leq Cw(s) \quad \text{a.e. } s > 0.$$

A weight w is said to be a b_1-weight if there exists $C > 0$ such that

$$\int_s^\infty \frac{w(t)}{t^2} dt \leq C\frac{w(s)}{s} \quad \text{a.e. } s > 0.$$

We shall denote by $\mathcal{W}_{0,1}$ the space of b_1-weights which satisfy Dini condition.

Let us also use the notation $\mathcal{A}_1$ for the class of functions $\phi \in \mathcal{S}_0$ such that

(a) $\int_0^\infty \left(\hat{\phi}(t\chi)\right)^2 \frac{dt}{t} = 1, (\chi \neq 0)$

(b) ϕ radial and real,

(c) $supp\ \phi \subseteq \{|x| \leq 1\} and$

(d) $\int_{\mathbb{R}^n} x_i \phi(x) dx = 0, (i = 1, ..., n)$.

We refer the reader to Section 1 for the notion of (A,p)-atom and the unexplained notations. The aim of the paper is to prove the following theorem.

Main Theorem. *Let $1 \leq p, q \leq \infty$, $\phi \in \mathcal{A}_1$ and w be a weight that can be factored as $w(t) = \lambda^{\frac{1}{q'}}(t)\mu^{\frac{-1}{q}}(t^{-1})$ where $\lambda, \mu \in \mathcal{W}_{0,1}$. Then if $w_Q = \left(\int_{l(Q)/2}^{l(Q)} w(t)^{q'} \frac{dt}{t}\right)^{\frac{1}{q'}}$ we have $f \in B^{p,q}_{w,\phi}$ if and only if there exist $A > 0$, $\{s_Q\}_{Q \in \mathcal{D}}$ and (A,p)-atoms $\{a_Q\}_{Q \in \mathcal{D}}$ such that $f = \sum_{Q \in \mathcal{D}} s_Q a_Q$ (convergence in $\mathcal{S}'_0$) and*

$$\left(\sum_{j=-\infty}^\infty \left(\sum_{l(Q)=2^j} \left|\frac{s_Q}{w_Q}\right|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

Moreover

$$||f||_{B^{p,q}_{w,\phi}} \approx inf \left\{ \left(\sum_{j=-\infty}^{\infty} \left(\sum_{l(Q)=2^j} |\frac{s_Q}{w_Q}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} : f = \sum_{Q\in\mathcal{D}} s_Q a_Q \right\}.$$

(with the obvious modifications for p and q equal ∞).

This can be understood as a generalization of the cases proved in [15, 4, 10] corresponding to $w(t) = t^\alpha$.

The paper is divided into three sections. Section 1 has a preliminary character and it is devoted to introduce the notation and the main Lemmas to be used later on. In Section 2 we get the atomic decomposition for the spaces in the case $1 \le p < \infty$ and $1 < q < \infty$ and we postpone the remaining cases to the last section.

§1. Preliminaries and Basic Lemmas.

Let us recall some notions on weights we shall need later.

Definition 1.1. *Let $\varepsilon, \delta \in \mathbb{R}$ and w be a weight. w is said to be a d_ε-weight if there exists $C > 0$ such that*

$$\int_0^s t^\varepsilon w(t) \frac{dt}{t} \le C s^\varepsilon w(s) \quad \text{a.e. } s > 0. \tag{1.1}$$

w is said to be a b_δ-weight if there exists $C > 0$ such that

$$\int_s^\infty \frac{w(t)}{t^\delta} \frac{dt}{t} \le C \frac{w(s)}{s^\delta} \quad \text{a.e. } s > 0. \tag{1.2}$$

If (d_ε)(respect. (b_δ)) denotes de class of d_ε-weights (respect. b_δ-weights) we write

$$\mathcal{W}_{\varepsilon,\delta} = (d_\varepsilon) \cap (b_\delta).$$

Remark 1.1. The main examples of such weights are given by

$$w_{\alpha,\beta}(t) = t^\alpha (1 + |\log t|)^\beta.$$

It is left to the reader to show that $w_{\alpha,\beta} \in \mathcal{W}_{\varepsilon,\delta}$ for any $\delta > \alpha$ and $\varepsilon > -\alpha$.

Let us collect some elementary properties to be used in the sequel.

$$w \in (d_\varepsilon) \Rightarrow w \in (d_{\varepsilon'}) \text{ for any } \varepsilon' > \varepsilon. \tag{1.3}$$

$$\text{Let } \overline{w}(t) = w(t^{-1}) \text{ then } w \in (b_\varepsilon) \Longleftrightarrow \overline{w} \in (d_\varepsilon). \tag{1.4}$$

$$w \in \mathcal{W}_{\varepsilon,\delta} \Rightarrow w(t) \ge C \min\left(t^{-\varepsilon}, t^\delta\right). \tag{1.5}$$

The next properties on weights belonging to $\mathcal{W}_{0,1}$ are needed for some results later on.

Lemma 1.1. *Let* $\varepsilon \geq 0$, $\delta \geq 0$ *and* $w \in \mathcal{W}_{\varepsilon,\delta}$. *Then*

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leq C \inf_{s/2\leq u<\infty} u^\varepsilon w(u). \tag{1.6}$$

$$\int_s^\infty \frac{w(t)}{t^\delta}\frac{dt}{t} \leq C \inf_{0<u\leq 2s} \frac{w(u)}{u^\delta} \tag{1.6$'$}$$

Proof. From (1.4) it is enough to prove (1.6). Let $u \geq s$. From (1.1)

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leq C\int_0^u t^\varepsilon w(t)\frac{dt}{t} \leq Cu^\varepsilon w(u).$$

If we integrate this inequality in $[s, 2s]$ against the weight $\dfrac{1}{u^{1+\varepsilon+\delta}}$

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t}\int_s^{2s}\frac{du}{u^{1+\varepsilon+\delta}} \leq C\int_s^{2s} w(u)\frac{du}{u^{1+\delta}} \leq C\int_s^\infty w(u)\frac{du}{u^{1+\delta}}.$$

Hence, if $s/2 \leq v \leq s$,

$$C'\frac{1}{s^{\varepsilon+\delta}}\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leq C\int_v^\infty w(u)\frac{du}{u^{1+\delta}} \leq C\frac{w(v)}{v^\delta}$$

and, finally,

$$\int_0^s t^\varepsilon w(t)\frac{dt}{t} \leq \frac{C}{C'}w(v)\frac{s^{\varepsilon+\delta}}{v^\delta} \leq \frac{2^{\varepsilon+\delta}C}{C'}w(v)v^\varepsilon.$$

Corollary 1.1. *Let* $w \in \mathcal{W}_{0,1}$. *Then for any* $s > 0$

$$\int_0^\infty min(\frac{s}{t}, 1)w(t)\frac{dt}{t} \leq C \inf_{s/2\leq u<s} w(u). \tag{1.7}$$

The next result was pointed out to the authors by F. Ruiz and J. Bastero, who showed us the proof we present here.

Lemma 1.2. *Let* $w \in (d_\varepsilon)$ *(respectively* $w \in (b_\delta)$*). Then there exists* $\rho > 0$ *such that* $w \in (d_{\varepsilon-\rho})$ *(respectively* $w \in (b_{\delta-\rho})$*).*

Proof. From (1.4) it is enough to consider $w \in (d_\varepsilon)$. Write $\lambda(t) = t^\varepsilon w(t)$. Clearly $\lambda \in (d_0)$. Let us define the operator

$$H(\lambda)(t) = \int_0^t \frac{\lambda(s)}{s}ds \quad (t > 0).$$

Since $H(\lambda) \leq C\lambda$ then $H^n(\lambda) \leq C^n\lambda$.

Applying Fubini's theorem and an easy induction one gets

$$H^n(\lambda)(t) = \frac{1}{(n-1)!}\int_0^t \frac{\lambda(s)}{s}\log^{n-1}\left(\frac{t}{s}\right) ds.$$

Take $\rho > 0$ such that $\rho C < 1$

$$\sum_{n=1}^{\infty} \rho^{n-1} H^n(\lambda) \leq \frac{C}{1-\rho C}\lambda.$$

Hence

$$\int_0^t \frac{\lambda(s)}{s}\left(\frac{t}{s}\right)^{\rho} ds \leq \frac{C}{1-\rho C}\lambda(t).$$

Finally this gives

$$\int_0^t s^{\varepsilon-\rho} w(s)\frac{ds}{s} \leq \frac{C}{1-\rho C} t^{\varepsilon-\rho} w(t). \quad \square$$

Definition 1.2. *Let $1 \leq p \leq \infty$ and $A > 0$. A differentiable funtion a_Q is called an (A,p)-smooth atom if*

$$\textit{supp } a_Q \subseteq 3Q \textit{ for some } Q \in \mathcal{D}. \tag{1.8}$$

$$\int a_Q(x)dx = \int x_i a_Q(x)dx = 0 \qquad (i = 1,2,...,n). \tag{1.9}$$

$$|a_Q(x)| \leq \frac{A}{l(Q)^{n/p}}, \qquad \left|\frac{\partial}{\partial x_i} a_Q(x)\right| \leq \frac{A}{l(Q)^{n/p+1}} \qquad (i = 1,2,...,n). \tag{1.10}$$

Let us now establish one of the main lemmas to be used later on. This result is closely related with Calderón reproducing formula, and gives a procedure to decompose functions in $L^1\left(\mathbb{R}^n, \dfrac{dx}{(1+|x|)^{n+1}}\right)$.

Lemma A. *(see [6, 18]). Let $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ and $\phi \in \mathcal{A}_1$. Given $1 \leq p \leq \infty, Q \in \mathcal{D}$, define*

$$s_Q(f) = |Q|^{\frac{-1}{p'}} \int\int_{T(Q)} |\phi_t * f(y)| dy \frac{dt}{t},$$

$$a_Q(f)(x) = \frac{1}{s_Q(f)} \int\int_{T(Q)} \phi_t(x-y)\phi_t * f(y) dy \frac{dt}{t}.$$

Then $a_Q(f)$ are (A,p)-smooth atoms for $A = 2^{n+1}\max\{|\phi(x)|, |\dfrac{\partial}{\partial x_i}\phi(x)|\ \ i = 1,\dots,n\}$ and

$$f = \sum_{Q\in\mathcal{D}} s_Q(f) a_Q(f) = \lim_{M\to\infty, N\to\infty} \sum_{k=-M}^{N} \sum_{l(Q)=2^k} s_Q(f) a_Q(f) \qquad \textit{in } \mathcal{S}'_0.$$

Lemma 1.3. *Let* $1 \le p \le \infty$, $\phi \in \mathcal{A}_1$, $f \in L^1\left(\mathbb{R}^n, \dfrac{dx}{(1+|x|)^{n+1}}\right)$. *If we write*

$$s_Q(f) = |Q|^{-1/p'} \iint_{T(Q)} |\phi_t * f(y)| dy \frac{dt}{t}$$

then for any $j \in \mathbb{Z}$

$$\left(\sum_{l(Q)=2^j} |s_Q(f)|^p \right)^{1/p} \le \int_{2^{j-1}}^{2^j} ||\phi_t * f||_p \frac{dt}{t} \tag{1.11}$$

Proof. Assume $p < \infty$ (the case $p = \infty$ is similar)

$$\begin{aligned}
\left(\sum_{l(Q)=2^j} |s_Q(f)|^p \right)^{\frac{1}{p}} &= \sup_{\sum \beta_Q^{p'}=1} \left| \sum_{l(Q)=2^j} \beta_Q s_Q(f) \right| \\
&= \sup_{\sum \beta_Q^{p'}=1} \left| \int_{2^{j-1}}^{2^j} \int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^j} \beta_Q |Q|^{-1/p'} \chi_Q \right) |\phi_t * f(y)| dy \frac{dt}{t} \right| \\
&\le \sup_{\sum \beta_Q^{p'}=1} \int_{2^{j-1}}^{2^j} \left\| \sum_{l(Q)=2^j} \beta_Q |Q|^{-1/p'} \chi_Q \right\|_{p'} ||\phi_t * f||_p \frac{dt}{t} \\
&\le \int_{2^{j-1}}^{2^j} ||\phi_t * f||_p \frac{dt}{t}. \quad \square
\end{aligned}$$

Lemma 1.4. *Let* $1 \le p \le \infty$, $A > 0$, $\{\alpha_Q\}_{Q \in \mathcal{D}} \subseteq \mathbb{C}$, *and* $\{a_Q\}_{Q \in \mathcal{D}}$ *verifying (1.8) and (1.10) for* A *and* p. *Then for* $t > 0$ *and* $j \in \mathbb{Z}$

$$\left\| \sum_{l(Q)=2^j} \alpha_Q a_Q \right\|_p \le C \left(\sum_{l(Q)=2^j} |\alpha_Q|^p \right)^{1/p}. \tag{1.12}$$

$$\left\| \sum_{l(Q)=2^j} \alpha_Q (\phi_t * a_Q) \right\|_p \le C \min\left(\frac{t}{2^j}, 1 \right) \left(\sum_{l(Q)=2^j} |\alpha_Q|^p \right)^{1/p}. \tag{1.13}$$

Proof.

First observe that $\sum_{l(Q)=2^j} s_Q a_Q$ has only a finite number of non-zero terms since there are a finite number of overlapping cubes of the form $\{3Q\}$. Hence

$$\left\| \sum_{l(Q)=2^j} \alpha_Q a_Q \right\|_p \le A \left\| \sum_{l(Q)=2^j} |\alpha_Q| |Q|^{-1/p} \chi_{3Q} \right\|_p \le C \left(\sum_{l(Q)=2^j} |\alpha_Q|^p \right)^{1/p}.$$

Use the previous estimate (1.12) and Young's inequality to get

$$\begin{aligned}\Big\|\sum_{l(Q)=2^j}\alpha_Q(\phi_t * a_Q)\Big\|_p &\le \|\phi_t\|_1\Big\|\sum_{l(Q)=2^j}\alpha_Q a_Q\Big\|_p\\ &\le \|\phi\|_1\left(\sum_{l(Q)=2^j}|\alpha_Q|^p\right)^{1/p}.\end{aligned}$$

On the other hand, assume $l(Q) = 2^j$ and $t < 2^j$. Note that $x \notin 5Q$ and $y \in 3Q$ implies $\left|\frac{x-y}{t}\right| > 1$ and so $\phi\left(\frac{x-y}{t}\right)a_Q(y) = 0$. Hence $supp\, \phi_t * a_Q \subseteq 5Q$. Moreover

$$|a_Q(y) - a_Q(x)| \le C \sup_{\xi\in[x,y]}|\triangledown a_Q(\xi)|\,|x-y| \le \frac{CA|x-y|}{l(Q)^{\frac{n}{p}+1}}.$$

Therefore, using that $\int \phi_t = 0$,

$$\begin{aligned}|\phi_t * a_Q(x)| &= \left|\int \phi_t(x-y)(a_Q(y)-a_Q(x))dy\right|\\ &\le \frac{CA}{l(Q)^{\frac{n}{p}+1}}\int|\phi_t(x-y)||x-y|dy\\ &= CA\frac{t}{l(Q)^{\frac{n}{p}+1}}\int|\phi(z)||z|dz.\end{aligned}$$

We have proved that $|\phi_t * a_Q(x)| \le C\frac{t}{l(Q)^{\frac{n}{p}+1}}\chi_{5Q}$. Hence

$$\begin{aligned}\Big\|\sum s_Q\phi_t * a_Q\Big\|_p &\le C\frac{t}{2^j}\Big\|\sum_{l(Q)=2^j}|\alpha_Q||Q|^{-1/p}\chi_{5Q}\Big\|_p\\ &\le C\frac{t}{2^j}\left(\sum_{l(Q)=2^j}|\alpha_Q|^p\right)^{1/p}. \quad\square\end{aligned}$$

Lemma 1.5. *Let $1 \le p \le \infty$, $\{\alpha_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$, $\{a_Q\}_{Q\in\mathcal{D}}$ $(A,p)-$smooth atoms. There exist $\{c_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$ with $c_Q \ne 0$ for finitely many $Q \in \mathcal{D}$ such that we have for $p < \infty$ and $p = \infty$ respectively*

$$\int_{\mathbb{R}^n}\frac{\left|\sum_{l(Q)=2^j}\alpha_Q(a_Q-c_Q)\right|}{(1+|x|)^{n+1}}dx \le C\min\left(1,2^{-j}\right)\left(\sum_{l(Q)=2^j}|\alpha_Q|^p\right)^{1/p} \tag{1.14}$$

$$\int_{\mathbb{R}^n}\frac{\left|\sum_{l(Q)=2^j}\alpha_Q(a_Q-c_Q)\right|}{(1+|x|)^{n+1}}dx \le C\min\left(1,\frac{|j|}{2^j}\right)\sup_{l(Q)=2^j}|\alpha_Q| \tag{1.14$'$}$$

Proof. Let

$$c_Q = \begin{cases} a_Q(0) & \text{if } l(Q) \geq 1 \text{ and } 3Q \cap B(0, l(Q)) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

For $j < 0$ we have $c_Q = 0$ and then we simply use the estimate

$$\left| \sum_{l(Q)=2^j} \alpha_Q (a_Q(x) - c_Q) \right| \leq A \sum_{l(Q)=2^j} |\alpha_Q| |Q|^{\frac{-1}{p}} \chi_{3Q}(x)$$

Hölder's inequality gives

$$\int_{\mathbb{R}^n} \frac{\left| \sum_{l(Q)=2^j} \alpha_Q (a_Q - c_Q) \right|}{(1+|x|)^{n+1}} dx \leq C \left(\sum_{l(Q)=2^j} |\alpha_Q|^p \right)^{1/p}.$$

For $j \geq 0$ we argue as follows:

Note that, for fixed j, there exist a finite number, independent of j, of dyadic cubes of length 2^j such that $3Q \cap B(0, l(Q)) \neq \emptyset$. Call such a family $\mathcal{F}_j$ and denote by $E_j = \cup_{Q \in \mathcal{F}_j} 3Q$.

In the same way as in the case $j < 0$ we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{\left| \sum_{Q \notin \mathcal{F}_j} \alpha_Q (a_Q - c_Q) \right|}{(1+|x|)^{n+1}} dx \\ &\quad \leq \left(\int \left| \sum_{Q \notin \mathcal{F}_j} \alpha_Q a_Q \right|^p \right)^{\frac{1}{p}} \left(\int_{|x| \geq 2^j} (1+|x|)^{-(n+1)p'} dx \right)^{\frac{1}{p'}} \\ &\quad \leq C 2^{-j(1+n/p)} \left(\sum_{Q \notin \mathcal{F}_j} |\alpha_Q|^p \right)^{1/p}. \end{aligned}$$

Para $Q \in \mathcal{F}_j$ y $x \notin E_j$ we use the simple estimate $|a_Q(x) - C_Q| \leq 2A2^{-jn/p}$.

$$\begin{aligned} \int_{\mathbb{R}^n \setminus E_j} \frac{\left| \sum_{Q \in \mathcal{F}_j} \alpha_Q (a_Q(x) - c_Q) \right|}{(1+|x|)^{n+1}} &\leq C 2^{-jn/p} \int_{|x| \geq 2^j} \frac{dx}{(1+|x|)^{(n+1)}} \sum_{Q \in \mathcal{F}_j} |\alpha_Q| \\ &\leq C 2^{-jn/p} \int_{2^j}^{\infty} t^{-2} dt \sum_{Q \in \mathcal{F}_j} |\alpha_Q| \\ &= C 2^{-j(1+n/p)} \sum_{Q \in \mathcal{F}_j} |\alpha_Q| \\ &\leq C 2^{-j(1+n/p)} \left(\sum_{Q \in \mathcal{F}_j} |\alpha_Q|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Finally, we observe that it follows from the mean value theorem and (1.10) that there exists $C' > 0$ such that

$$|a_Q(x) - c_Q| \leq C' \frac{|x|}{l(Q)^{\frac{n}{p}+1}} \quad (x \in E_j, Q \in \mathcal{F}_j).$$

It is clear that if $x \in E_j$ then there exist K, independent of j, such that $|x| \leq K2^j$. Therefore

$$\begin{aligned}
\int_{E_j} \frac{\left|\sum_{Q\in\mathcal{F}_j} \alpha_Q(a_Q - c_Q)\right|}{(1+|x|)^{n+1}} dx &\leq A2^{-\frac{jn}{p}-1} \int_{|x|\leq K2^j} \frac{|x|}{(1+|x|)^{n+1}} dx \sum_{Q\in\mathcal{F}_j} |\alpha_Q| \\
&\leq A2^{-\frac{jn}{p}-1} \left(\int_0^{K2^j} \frac{t^n}{(1+t)^{n+1}} dt \right) \sum_{Q\in\mathcal{F}_j} |\alpha_Q| \\
&\leq C2^{-\frac{jn}{p}-1} \left(1 + log(2^j)\right) \sum_{Q\in\mathcal{F}_j} |\alpha_Q| \\
&\leq C2^{-\frac{jn}{p}-1} |j| \sum_{Q\in\mathcal{F}_j} |\alpha_Q|. \\
&\leq C2^{-\frac{jn}{p}-1} |j| \left(\sum_{Q\in\mathcal{F}_j} |\alpha_Q|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Combining the previous estimates we have (1.14) and (1.14') $\square$

Observe that a net $\{\phi_i\}_{i\in\Lambda}$ converges to ϕ in $\mathcal{S}_0'$ if there exist $\{c_i\}_{i\in\Lambda} \subset \mathbb{C}$ such that $\phi_i - c_i$ converges to ϕ in S'. Using this it is elementary to show the following lemma.

Lemma 1.6. *Let $\{f_j\}_{j\in\mathbb{Z}}$ be measurable functions defined in $\mathbb{R}^n$ such that there exist $\{c_j\}_{j\in\mathbb{Z}}$ real numbers with*

$$\int_{\mathbb{R}^n} \frac{\sum_{j\in\mathbb{Z}} |f_j - c_j|}{1 + |x|^{n+1}} dx < \infty$$

Then $\sum_{j\in\mathbb{Z}} f_j$ converge in $\mathcal{S}_0'$ to some function $f \in L^1\left(\mathbb{R}^n, \dfrac{dx}{1+|x|^{n+1}}\right)$.

§2. Atomic Decompositions for $B^{p,q}_{\phi,w}$ for $1 \leq p < \infty$ and $1 < q < \infty$.

Let us first state a version of Schur Lemma that will be useful for our purposes and whose proof follows easily from Holder's inequality.

Lemma B. *Let $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and let $K : \Omega_1 \times \Omega_2 \to \mathbb{R}^+$ be a measurable function and write $T_K(f)$ for*

$$T_K(f)(w_2) = \int_{\Omega_1} K(w_1, w_2) f(w_1) d\mu_1(w_1).$$

If there exist $C>0$ and measurable functions $h_i:\Omega_i\to\mathbb{R}^+$ $(i=1,2)$ such that

$$\int_{\Omega_1} K(w_1,w_2)h_1^{q'}(w_1)d\mu_1(w_1)\le Ch_2^{q'}(w_2)\ \ \mu_2-\text{a.e.} \tag{2.1}$$

$$\int_{\Omega_2} K(w_1,w_2)h_2^{q}(w_2)d\mu_2(w_2)\le Ch_1^{q}(w_1)\ \ \mu_1-\text{a.e.} \tag{2.2}$$

Then T_K defines a bounded operator from $L^q(\Omega_1,\mu_1)$ into $L^q(\Omega_2,\mu_2)$.

Theorem 2.1. *Let $1\le p<\infty$, $1<q<\infty$, $\phi\in\mathcal{A}_1$, $w(t)=\lambda^{1/q'}(t)\mu^{-1/q}(t^{-1})$ where $\lambda,\mu\in\mathcal{W}_{0,1}$. Denoting $w_Q=\left(\int_{\frac{l(Q)}{2}}^{l(Q)} w^{q'}(t)\frac{dt}{t}\right)^{1/q'}$ we have $f\in B^{p,q}_{w,\phi}$ if and only if there exist $A>0$, $\{s_Q\}_{Q\in\mathcal{D}}\subseteq\mathbb{C}$, $\{a_Q\}_{Q\in\mathcal{D}}$ $(A,p)-$smooth atoms such that $f=\sum_{Q\in\mathcal{D}} s_Q a_Q$ in $\mathcal{S}_0'$ and*

$$\sum_{j=-\infty}^{j=\infty}\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{q/p}<\infty. \tag{2.3}$$

Moreover

$$\|f\|_{B^{p,q}_{w,\phi}}\approx\inf\left\{\left(\sum_{j=-\infty}^{j=\infty}\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{q/p}\right)^{1/q};\quad f=\sum_{Q\in\mathcal{D}} s_Q a_Q\right\}.$$

Proof. Assume $f\in B^{p,q}_{w,\phi}$ and write $w_j=\left(\int_{2^{j-1}}^{2^j} w^{q'}(t)\frac{dt}{t}\right)^{1/q'}$. The atomic decomposition is obtained from Lemma A. The only thing to prove is the estimate (2.3). Using (1.11) we have

$$\left(\sum_{l(Q)=2^j}\left(\frac{s_Q}{w_Q}\right)^p\right)^{1/p}\le\int_0^\infty\frac{w(t)}{w_j}\chi_{(2^{j-1},2^j)}(t)\frac{\|\phi_t*f\|_p}{w(t)}\frac{dt}{t}.$$

Hence from duality and Hölder's inequality

$$\begin{aligned}\left(\sum_{j=-\infty}^{\infty}\left(\sum_{l(Q)=2^j}\left(\frac{s_Q}{w_Q}\right)^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}&=\sup_{\sum\beta_j^{q'}=1}\int_0^\infty\left(\sum_{j=-\infty}^{\infty}\frac{\beta_j}{w_j}w(t)\chi_{(2^{j-1},2^j)}(t)\right)\frac{\|\phi_t*f\|_p}{w(t)}\frac{dt}{t}\\&\le\sup_{\sum\beta_j^{q'}=1}\left\|\sum_{j=-\infty}^{\infty}\beta_j\frac{w(t)}{w_j}\chi_{(2^{j-1},2^j)}(t)\right\|_{q'}\|f\|_{B^{p,q}_{w,\phi}}\\&=\|f\|_{B^{p,q}_{w,\phi}}.\end{aligned}$$

Conversely, let us assume that $f = \sum_{Q\in\mathcal{D}} s_Q a_Q$ where $\{s_Q\}_{Q\in\mathcal{D}}$ satisfies (2.3). We use now (1.14) in Lemma 1.5 and Lemma 1.6 to prove that $\sum_{Q\in\mathcal{D}} s_Q a_Q$ converges in $\mathcal{S}_0'$ to a function $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$. It suffices to prove that $w_j \min\left(1, 2^{-j}\right) \in l_{q'}(\mathbb{Z})$.

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} w_j^{q'} \min\left(1,2^{-j}\right)^{q'} \\
&\qquad \le C \sum_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^j} w^{q'}(t) \min\left(1, \frac{1}{t}\right)^{q'} \frac{dt}{t} \\
&\qquad \le C \int_0^\infty w^{q'}(t) \min\left(1, \frac{1}{t}\right)^{q'} \frac{dt}{t} \\
&\qquad \le C \left(\int_0^1 \lambda(t)\mu^{-q'/q}(t^{-1}) \frac{dt}{t} + \int_1^\infty \lambda(t)\mu^{-q'/q}(t^{-1}) \left(\frac{1}{t}\right)^{q'} \frac{dt}{t} \right).
\end{aligned}$$

Using (1.5) we have $\mu(s) \ge \min(1,s)$. Hence

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} w_j^{q'} \min\left(1,2^{-j}\right)^{q'} &\le C \left(\int_0^1 \lambda(t) \frac{dt}{t} + \int_1^\infty \lambda(t) t^{q'-1} t^{-q'} \frac{dt}{t} \right) \\
&\le C \left(\int_0^1 \lambda(t) \frac{dt}{t} + \int_1^\infty \lambda(t) \frac{dt}{t^2} \right) < \infty.
\end{aligned}$$

Since $\|\phi_t * f\|_p \le \sum_{j\in\mathbb{Z}} \|\sum_{l(Q)=2^j} s_Q(\phi_t * a_Q)\|_p$ we can use (1.13) in Lemma 1.4 to get

$$\frac{\|\phi_t * f\|_p}{w(t)} \le C \sum_{j=-\infty}^{\infty} \frac{w_j}{w(t)} \min\left(\frac{t}{2^j}, 1\right) \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q}\right)^p \right)^{1/p}.$$

Let us write $(\Omega_1, \Sigma_1, \mu_1) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), d\nu)$ where ν denote the counting measure and $(\Omega_2, \Sigma_2, \mu_2) = \left((0,+\infty), \mathcal{B}((0,+\infty)), \frac{dt}{t}\right)$ and consider the following kernel

$$K(j,t) = \frac{w_j}{w(t)} \min\left(\frac{t}{2^j}, 1\right).$$

Take

$$\alpha_j = \left(\int_{2^{j-1}}^{2^j} \lambda(t) \frac{dt}{t} \right)^{\frac{1}{qq'}} \left[\inf_{2^{-j} < t \le 2^{-j+1}} \mu(t) \right]^{1/qq'}$$

and take

$$g(t) = \lambda^{\frac{1}{qq'}}(t) \mu^{\frac{1}{qq'}}(t^{-1}).$$

Clearly we have

$$g^q(t) = w(t)\mu(t^{-1}), \quad g^{q'}(t)w(t) = \lambda(t). \tag{2.4}$$

On the other hand $w_j \le \left(\int_{2^{j-1}}^{2^j} \lambda(t)\frac{dt}{t}\right)^{\frac{1}{q'}} \left(\inf_{2^{-j}<t\le 2^{-j+1}} \mu(t)\right)^{\frac{-1}{q}}$. This implies

$$w_j \inf_{2^{-j}<t\le 2^{-j+1}} \mu(t) \le \alpha_j^q, \quad w_j\alpha_j^{q'} \le \int_{2^{j-1}}^{2^j} \lambda(t)\frac{dt}{t}. \tag{2.5}$$

Hence from (2.4), Corollary 1.1 and (2.5)

$$\begin{aligned}\int_0^\infty K(j,t)g^q(t)\frac{dt}{t} &= w_j \int_0^\infty \mu(t^{-1})\min\left(\frac{t}{2^j},1\right)\frac{dt}{t}\\ &= w_j\int_0^\infty \mu(s)\min\left(\frac{2^{-j}}{s},1\right)\frac{ds}{s}\\ &\le Cw_j \inf_{2^{-j}<t\le 2^{-j+1}}\mu(t) \le C\alpha_j^q.\end{aligned}$$

On the other hand, applying (2.4) and (2.5)

$$\begin{aligned}\sum_{j=-\infty}^{\infty} K(j,t)\alpha_j^{q'} &\le \frac{C}{w(t)}\sum_{j=-\infty}^{\infty}\min\left(\frac{t}{2^j},1\right)w_j\alpha_j^{q'}\\ &\le \frac{C}{w(t)}\sum_{j=-\infty}^{\infty}\min\left(\frac{t}{2^j},1\right)\int_{2^{j-1}}^{2^j}\lambda(s)\frac{ds}{s}\\ &\le \frac{C}{w(t)}\int_0^\infty \min\left(\frac{t}{s},1\right)\lambda(s)\frac{ds}{s}\\ &\le C\frac{\lambda(t)}{w(t)} = Cg(t)^{q'}.\end{aligned}$$

Hence, by Lemma B,

$$\begin{aligned}\left\|\frac{\|\phi_t * f\|_p}{w(t)}\right\|_{L^q(\frac{dt}{t})} &\le C\left\|T_K\left(\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{1/p}\right)\right\|_{L^q(\frac{dt}{t})}\\ &\le C\left\|\left(\sum_{l(Q)=2^j}\left(\frac{|s_Q|}{w_Q}\right)^p\right)^{1/p}\right\|_{l^q} < \infty. \quad \square\end{aligned}$$

§3. Atomic Decompositions for $B^{\infty,q}_{\phi,w}$, $B^{p,\infty}_{\phi,w}$ and $B^{p,1}_{\phi,w}$

Theorem 3.1. *Let $1 \le p < \infty$, $w \in \mathcal{W}_{0,1}$, $\phi \in \mathcal{A}_1$. Then $f \in B^{p,\infty}_{w,\phi}$ if and only if there exist $A > 0$, $\{s_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$, $\{a_Q\}_{Q\in\mathcal{D}}$ $(A,p)-$smooth atoms such that* $f = \sum_{Q\in\mathcal{D}} s_Q a_Q$

in $\mathcal{S}_0'$ *and*

$$\left(\sum_{l(Q)=2^j} |s_Q|^p\right)^{1/p} \leq C\int_{2^{j-1}}^{2^j} w(t)\frac{dt}{t}. \tag{3.1}$$

Moreover

$$\|f\|_{B^{p,\infty}_{w,\phi}} \approx \inf\left\{\sup_{j\in\mathbb{Z}} \frac{\left(\sum_{l(Q)=2^j} |s_Q|^p\right)^{1/p}}{\int_{2^{j-1}}^{2^j} w(t)\frac{dt}{t}}; \quad f=\sum_{Q\in\mathcal{D}} s_Q a_Q\right\}.$$

Proof. Assume $f \in B^{p,\infty}_{w,\phi}$. Apply Lemma A and Lemma 1.3 to obtain (3.1).

Conversely assume $\{s_Q\}_{Q\in\mathcal{D}}$ safisfies (3.1). Invoking Lemmas 1.5 and 1.6 we can see that $\sum_{Q\in\mathcal{D}} s_Q(a_Q - c_Q)$ converge in $\mathcal{S}_0'$ to a function $f \in L^1\left(\mathbb{R}^n, \dfrac{dx}{(1+|x|)^{n+1}}\right)$ as soon as we notice that $\min\left(1, 2^{-j}\right)\int_{2^{j-1}}^{2^j} w(t)\frac{dt}{t} \in l_1(\mathbb{Z})$.

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \min\left(1,2^{-j}\right)\int_{2^{j-1}}^{2^j} w(t)\frac{dt}{t} &\leq C\int_R w(t)\min\left(1,t^{-1}\right)\frac{dt}{t}\\ &\leq C\int_0^1 w(t)\frac{dt}{t} + \int_1^\infty w(t)\frac{dt}{t^2}.\end{aligned}$$

Note that $\phi_t * f = \sum_{j=-\infty}^{\infty}\sum_{l(Q)=2^j} s_Q(\phi_t * a_Q)$ in $\mathcal{S}_0'$. Hence

$$\|\phi_t * f\|_p \leq \sum_{j=-\infty}^{\infty}\left\|\sum_{l(Q)=2^j} s_Q(\phi_t * a_Q)\right\|_p. \tag{3.2}$$

Applying (3.2) and (1.13) in Lemma 1.4 we have

$$\begin{aligned}\|\phi_t * f\|_p &\leq C\sum_{j=-\infty}^{\infty}\min\left(\frac{t}{2^j},1\right)\int_{2^{j-1}}^{2^j} w(s)\frac{ds}{s}\\ &\leq C\sum_{j=-\infty}^{\infty}\int_{2^{j-1}}^{2^j}\min\left(\frac{t}{s},1\right) w(s)\frac{ds}{s}\\ &\leq C\left(\int_0^t w(s)\frac{ds}{s} + t\int_t^\infty w(s)\frac{ds}{s^2}\right)\\ &\leq Cw(t). \quad \square\end{aligned}$$

Theorem 3.2. *Let* $1 \le p < \infty$, $\phi \in \mathcal{A}_1$ w *a weight such that* $w_1(t) = w^{-1}(t^{-1}) \in \mathcal{W}_{0,1}$. *Let us denote* $w_j = \sup\{w(t); 2^{j-1} < t \le 2^j\}$ *and* $w_Q = w_{l(Q)}$. *Then* $f \in B^{p,1}_{w,\phi}$ *if and only if there exist* $A > 0$, $\{s_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$, $\{a_Q\}_{Q\in\mathcal{D}}$ $(A,p)-$*atoms such that* $f = \sum_{Q\in\mathcal{D}} s_Q a_Q$ *in* $\mathcal{S}_0'$ *and*

$$\sum_{j=-\infty}^{+\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} < \infty. \tag{3.3}$$

Moreover

$$\|f\|_{B^{p,1}_{w,\phi}} \approx \inf \left\{ \sum_{j=-\infty}^{+\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} : \ f = \sum_{Q\in\mathcal{D}} s_Q a_Q \right\}.$$

Proof. Assume $f \in B^{p,1}_{w,\phi}$. Use one more time Lemma A and (1.11) to have

$$\left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} \le \int_{2^{j-1}}^{2^j} \frac{\|\phi_t * f\|_p}{w(t)} \frac{dt}{t}.$$

Adding them up we get

$$\sum_{j=-\infty}^{+\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q} \right)^p \right)^{1/p} \le \|f\|_{B^{p,1}_{w,\phi}}.$$

Conversely take $\{s_Q\}_{Q\in\mathcal{D}}$ satisfying (3.3). As in Theorem 2.1 we shall prove that $\sum_{Q\in\mathcal{D}} s_Q a_Q$ converges in S_0' to a function $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$. Now it suffices to prove that $w_j \min\left(1, 2^{-j}\right) \in l_\infty(\mathbb{Z})$ which easily follows from (1.5).

As in Theorem 2.1 we apply (1.13) to get

$$\begin{aligned} \|\phi_t * f\|_p &\le \sum_{j=-\infty}^{\infty} \| \sum_{l(Q)=2^j} s_Q(\phi_t * a_Q)\|_p \\ &\le C \sum_{j=-\infty}^{\infty} \min\left(\frac{t}{2^{j-1}}, 1\right) \left(\sum_{l(Q)=2^j} |s_Q|^p \right)^{1/p}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^+} \frac{\|\phi_t * f\|_p}{w(t)} \frac{dt}{t} &\le C \sum_{j=-\infty}^{\infty} \left(\int_{\mathbb{R}^+} \frac{1}{w(t)} \min\left(\frac{t}{2^{j-1}}, 1\right) \frac{dt}{t} \right) \left(\sum_{l(Q)=2^j} |s_Q|^p \right)^{1/p} \\ &= C \sum_{j=-\infty}^{\infty} \left(\sum_{l(Q)=2^j} |s_Q|^p \right)^{1/p} \left(\int_0^{\infty} \min\left(\frac{2^{-j+1}}{s}, 1\right) w_1(s) \frac{ds}{s} \right) \end{aligned}$$

From Corollary 1.1 we can estimate

$$\left(\int_0^\infty \min\left(\frac{2^{-j+1}}{s},1\right) w_1(s)\frac{ds}{s}\right) \leq C \inf_{2^{-j}<t<2^{-j+1}} w_1(t) \leq C\frac{1}{w_j}$$

This shows

$$\int_{\mathbb{R}^+} \frac{\|\phi_t * f\|_p}{w(t)}\frac{dt}{t} \leq C \sum_{j=-\infty}^{\infty} \left(\sum_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q}\right)^p\right)^{1/p}. \quad \square$$

Analyzing the previous proofs one realize that the only difficulty to extend to the case $p=\infty$ comes from the failure of (1.14), which in this case can be replaced by (1.14'). This problem can be overcome by using Lemma 1.2.

Theorem 3.3. *Let $1 \leq q \leq \infty$, $\phi \in \mathcal{A}_1$, $w(t) = \lambda^{1/q'}(t)\mu^{-1/q}(t^{-1})$ where $\lambda, \mu \in \mathcal{W}_{0,1}$. Denoting $w_Q = \left(\int_{\frac{l(Q)}{2}}^{l(Q)} w^{q'}(t)\frac{dt}{t}\right)^{1/q'}$ for $q>1$ or $w_Q = \sup\{w(t); l(Q)/2 < t \leq l(Q)\}$ for $q=1$ we have $f \in B^{\infty,q}_{w,\phi}$ if and only if there exist $A>0$, $\{s_Q\}_{Q\in\mathcal{D}} \subseteq \mathbb{C}$, $\{a_Q\}_{Q\in\mathcal{D}}$ $(A,p)-$smooth atoms such that $f = \sum_{Q\in\mathcal{D}} s_Q a_Q$ in $\mathcal{S}'_0$ and*

$$\sum_{j=-\infty}^{+\infty} \sup_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q}\right)^q < \infty. \tag{2.8}$$

Moreover

$$\|f\|_{B^{\infty,q}_{w,\phi}} \approx \inf\left\{\left(\sum_{j=-\infty}^{+\infty} \sup_{l(Q)=2^j} \left(\frac{|s_Q|}{w_Q}\right)^q\right)^{1/q}; \quad f = \sum_{Q\in\mathcal{D}} s_Q a_Q\right\}.$$

(the obvious modifications for $q=\infty$).

Proof.

Denote $w_j = \left(\int_{2^{j-1}}^{2^j} w^{q'}(t)\frac{dt}{t}\right)^{\frac{1}{q'}}$ or $w_j = \sup\{w(t) : 2^{j-1} < t \leq 2^j\}$

Using Lemma 1.2 then $\mu, \lambda \in (b_\varepsilon)$ for some $0<\varepsilon<1$ and this is enough to show that $w_j \min\left(1, \frac{|j|}{2^j}\right) \in l^{q'}(\mathbb{Z})$.

Indeed, for $q=\infty$ we have $w(t) = \lambda(t)$ and $w_j \min\left(1, \frac{|j|}{2^j}\right) \leq C w_j \min\left(1, \frac{1}{2^{\varepsilon j}}\right)$ From this

$$\sum_{j\in\mathbb{Z}} w_j \min\left(1, \frac{|j|}{2^j}\right) \leq C\left(\int_0^1 \lambda(t)\frac{dt}{t} + \int_1^\infty \lambda(t)\frac{dt}{t^{\varepsilon+1}}\right) < \infty.$$

For $q=1$ we have $w(t) = \mu^{-1}(t^{-1})$ and using (1.5)

$$w_j \min\left(1, \frac{|j|}{2^j}\right) \leq C\frac{1}{\inf_{2^{-j}<t<2^{-j+1}} \mu(t)} \min\left(1, \frac{1}{2^{\varepsilon j}}\right) \leq C < \infty,$$

C independent of j.

For $1 < q < \infty$ take $0 < \alpha < 1$ such that $q'(1-\alpha) = 1-\varepsilon$ and use

$$w_j \min\left(1, \frac{|j|}{2^j}\right) \leq C w_j \min\left(1, \frac{1}{2^{\alpha j}}\right).$$

Therefore

$$\sum_{j\in\mathbb{Z}} w_j^{q'} (\min\left(1, \frac{|j|}{2^j}\right))^{q'} \leq C\left(\int_0^1 \lambda(t)\mu^{-q'/q}(t^{-1})\frac{dt}{t} + \int_1^\infty \lambda(t) t^{q'-1}\frac{dt}{t^{\alpha q'+1}}\right) < \infty.$$

Ackknowledgement. We would like to thank the referee for his careful reading of the paper and for fixing a little detail in the proof of lemma 1.5 that we had overlooked in the previous version.

References

[1] O. V. Besov, *On a family of function spaces in connection with embeddings and extensions*, Trudy Mat. Inst. Steklov **60** (1961), 42-81.

[2] O. Blasco, *Operators on weighted Bergman spaces and applications*, Duke Math. J. **66** (1992), 443-467.

[3] O. Blasco, G.S. de Souza, *Spaces of analytic functions on the disc where the growth of $M_p(F,r)$ depends on a weight*, J. Math. Anal. and Appl. **147** (1990), 580-598.

[4] H.-Q. Bui, *Representation theorems and atomic decompositions of Besov spaces*, Math. Nachr. **132** (1987), 301-311.

[5] S. Bloom, G.S. de Souza, *Atomic decomposition of generalized Lipschitz spaces*, Illinois J. Math. **33** (1989), 181-189.

[6] A. P. Calderón, *An atomic decomposition of distributions in parabolic H^p spaces*, Adv. in Math. **25** (1977), 216-225.

[7] A. P. Calderón A. Torchinsky, *Parabolic maximal functions associated with a distribution I*, Adv. in Math. **16** (1975), 1-64.

[8] R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 241-250.

[9] R. Coifman, R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque **77** (1980), 11-66.

[10] G. S. de Souza, *The atomic decomposition of Besov-Bergman-Lipschitz spaces*, Proc. Amer. Math. Soc. **94** (1985), 682-686.

[11] G. S. de Souza, G. Sampson, *A real characterizations of the predual of Bloch functions*, J. London Math. Soc. **27** (1983), 267-276.

[12] G. S. de Souza, R. O'Neill, G. Sampson, *Several characterizations of special atoms spaces and applications*, Rev. Mat. Iberoamericana **2** (1986), 333-355.

[13] T.M. Flett, *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. and Appl. **39** (1972), 125-158.

[14] T.M. Flett, *Temperatures, Bessel potentials and Lipschitz spaces*, Proc. London Math. Soc. **20** (1970), 749-768.

[15] M. Frazier, B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777-799.

[16] M. Frazier, B. Jawerth, *The ϕ-tansform and applications to distribution spaces*, Function spaces and Applications. Lecture Notes in Math., vol 1302 (1988), Springer Verlag, Berlin, New York, 223-246.

[17] M. Frazier, B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34-170.

[18] M. Frazier, B. Jawerth, G. Weiss, *Littlewood-Paley Theory and the study of function spaces*, C.B.M.S. n 79 Amer. Math. Soc., Providence, Rhode Island, 1991.

[19] J. García-Cuerva, J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math. Stud. 116, North-Holland, Amsterdam, New York, 1985.

[20] S. Janson, *Generalization on Lipschitz spaces and applications to Hardy spaces and bounded mean oscillation*, Duke Math. J. **47** (1980), 959-982.

[21] S. Janson, M. Taibleson, *I Teoremi di rappresentazione di Calderón*, Rend. Sem. Mat. Ist. Politecn. Torino **39** (1981), 27-35.

[22] R. Latter, *A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms*, Studia Math. **62** (1977), 92-101.

[23] R. Rochberg, *Decomposition theorems for Bergman spaces and their applications*, Operators and Function Theory (S. C. Power, ed) (1985), Reidel, Dordrecht, 225-278.

[24] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series., Durham, NC, 1976.

[25] R. Rochberg, S. Semmes, *A decomposition theorem for functions in BMO and applications*, Journal of Funct. Anal. **67** (1986), 228-263.

[26] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[27] J.O. Stromberg, A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Math. 1381, Springer Verlag, Berlin, 1989.

[28] M. Taibleson, *On the theory of Lipschitz spaces of distributions on euclidean n-space.*, I, II, III, J. Math. Mech. **13** (1964), 407-480; **65**, 821-840; **15** (1966), 973-981.

[29] A. Torchinsky, *Real variable methods in Harmonic Analysis*, Academic Press, Orlando, FL, 1986.

[30] H. Triebel, *Theory of function spaces*, Monographs in Math., vol. 78, Birkhäuser, Verlag, Basel, 1983.

[31] H. Triebel, *Atomic decompositions of F^{α}_{pq} spaces. Applications to Exotic Pseudodifferential and Fourier Integral operators*, Math. Nachr. **144** (1989), 189-222.

[32] A. Uchiyama, *A constructive proof of Fefferman-Stein decomposition of BMO($\mathbf{R}^n$)*, Acta Math. **148** (1982), 215-241.

[33] M. Wilson, *On the atomic decomposition for Hardy spaces*, Pacific J. Math. **116** (1985), 201-207.

Oscar Blasco. Departamento de Análisis Matemático, Universidad de Valencia, 46100 Burjassot (Valencia), Spain.

E-mail address: Blascod@mac.uv.es

José Luis Ansorena. Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain.

E-mail address: Blasco@cc.unizar.es