CHARACTERIZATION OF WEIGHTED BESOV SPACES

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Abstract. We find conditions on the weight $w$ in order characterize functions in weighted Besov spaces $B_{w,\phi}^{p,q}$ in terms of differences $\Delta_x f$.

Introduction.

There are many ways to define Besov spaces (see [1, 19, 24]). It is well known that Besov spaces can be defined, for instance in terms of convolutions $f \ast \phi_t$ with different kinds of smooth functions $\phi$ and that they can be also described by means of differences $\Delta_x f$ (see [10, 11, 22]).

Our objective will be to find weights (which extend the case $t^\alpha$) where we can still get such a characterization of weighted Besov spaces and to give a general procedure which works not only in the classical case but also in the weighted one. Our arguments will be based upon two main points: The Calderón’s formula, a quite simple Schur Lemma.

We want to notice that this characterization can be used to get the atomic decomposition of the spaces.

The paper is divided into two sections. Section 1 has a preliminary character and it is devoted to introduce the notation and the main lemmas to be used later on. In Section 2 we prove the result about coincidence of seminorms in the spaces defined by differences and convolutions.

Throughout the paper a weight $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ will be a measurable function $w > 0$ a.e., $1 \leq p, q \leq \infty$ and $p', q'$ stand for the conjugate exponents. $\mathcal{S}$ denotes the Schwartz class of test functions on $\mathbb{R}^n$, $\mathcal{S}'$ the space of tempered distributions, $\mathcal{S}_0$ the set of functions in $\mathcal{S}$ with mean zero and $\mathcal{S}'_0$ its topological dual.

Given a weight $w$ and $1 \leq p, q \leq \infty$ we shall denote by $\Lambda_{w}^{p,q}$ the space of measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$||f||_{\Lambda_{w}^{p,q}} = \left( \int_{\mathbb{R}^n} \frac{||\Delta_x f||_p^q}{w(|x|)^q |x|^n} dx \right)^{\frac{1}{q}} < \infty$$

or

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\[ ||f||_{B^{p,q}_{w,\phi}} = \inf\{ C > 0 : ||\Delta_x f||_p \leq Cw(|x|) \text{ a.e. } x \in \mathbb{R}^n \} \quad (q = \infty) \]

where \( \Delta_x f(y) = f(x+y) - f(y) \).

Given a weight \( w, \phi \in \mathcal{S}_0 \) and \( 1 \leq p, q \leq \infty \) we shall denote by \( B^{p,q}_{w,\phi} \) the space functions \( f : \mathbb{R}^n \to \mathbb{C} \) with \( f \in L^1 \left( \mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}} \right) \) such that

\[ ||f||_{B^{p,q}_{w,\phi}} = \left( \int_{\mathbb{R}^n} \frac{||\phi_t * f||_p^q}{w(t)^q} \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \quad (1 \leq q < \infty) \]

or

\[ ||f||_{B^{p,\infty}_{w,\phi}} = \inf\{ C > 0 : ||\phi_t * f||_p \leq Cw(t) \text{ a.e. } t > 0 \} < \infty \quad (q = \infty) \]

where \( \phi_t(x) = \frac{1}{t^n} \phi \left( \frac{x}{t} \right) \).

To state the results of the paper, let us first recall the following notions.

A weight \( w \) is said to satisfy Dini condition if there exists \( C > 0 \) such that

\[ \int_{s}^{\infty} w(t) \frac{dt}{t} \leq Cw(s) \text{ a.e. } s > 0. \]

A weight \( w \) is said to be a \( b_1 \)-weight if there exists \( C > 0 \) such that

\[ \int_{s}^{\infty} \frac{w(t)}{t^2} \frac{dt}{t} \leq C \frac{w(s)}{s} \text{ a.e. } s > 0. \]

We shall denote by \( \mathcal{W}_{0,1} \) the space of \( b_1 \)-weights which satisfy Dini condition.

Let us also use the notation \( \mathcal{A} \) and \( \mathcal{A}_1 \) for the following classes

\[ \mathcal{A} = \{ \phi \in \mathcal{S}_0 : \int_0^{\infty} \left( \hat{\phi}(t\xi) \right)^2 \frac{dt}{t} = 1 \text{ for } \xi \in \mathbb{R}^n \setminus \{0\} \}. \]

\[ \mathcal{A}_1 = \{ \phi \in \mathcal{A} : \phi \text{ radial and real, supp } \phi \subseteq \{|x| \leq 1|, \int_{\mathbb{R}^n} x_i \phi(x) dx = 0, i = 1, ..., n \}. \]

Section 2 is devoted to prove the following theorem.

**Main Theorem.** Let \( 1 \leq p, q \leq \infty \), \( \phi \in \mathcal{A} \) and \( w \) be a weight that can be factorized as \( w(t) = \lambda^\frac{1}{p'}(t)\mu^\frac{1}{q'}(t^{-1}) \) where \( \lambda, \mu \in \mathcal{W}_{0,1} \). Then

\[ \Lambda^{p,q}_w = B^{p,q}_{w,\phi} \text{ (with equivalent seminorms)}. \]

For particular cases \( w(t) = t^\alpha \) the reader is referred to [10, 11, 14, 22] for similar results for special functions \( \phi \) and their applications. In our weighted situation some closely related results for the unit disc are included in [3] and [5].

The reader should be aware that the case \( 1 < q < \infty \) in Main Theorem could have been shown by interpolation with the extreme cases, but a direct proof is presented in the paper.
§1. Preliminaries.

Let us recall some notions on weights we shall need later.

**Definition 1.1.** Let $\varepsilon \geq 0$, $\delta \geq 0$ and $w$ be a weight. $w$ is said to be a $d_{\varepsilon}$-weight if there exists $C > 0$ such that

$$
(1.1) \quad \int_0^s t^\varepsilon w(t) \frac{dt}{t} \leq C s^\varepsilon w(s) \quad \text{a.e. } s > 0.
$$

$w$ is said to be a $b_\delta$-weight if there exists $C > 0$ such that

$$
(1.2) \quad \int_s^\infty \frac{w(t)}{t^\delta} \frac{dt}{t} \leq C \frac{w(s)}{s^\delta} \quad \text{a.e. } s > 0.
$$

If $(d_{\varepsilon})$ (respect. $(b_\delta)$) denotes de class of $d_{\varepsilon}$-weights (respect. $b_\delta$-weights) we write

$$
\mathcal{W}_{\varepsilon,\delta} = (d_{\varepsilon}) \cap (b_\delta).
$$

The following properties are elementary and left to the interested reader

$$
(1.3) \quad w \in (d_{\varepsilon}) \Rightarrow w \in (d_{\varepsilon'}) \text{ for any } \varepsilon' > \varepsilon.
$$

(1.3') \quad w \in (b_\delta) \Rightarrow w \in (b_{\delta'}) \text{ for any } \delta' > \delta.

$$
(1.4) \quad \text{Let } \overline{w}(t) = w(t^{-1}) \text{ then } w \in (b_{\varepsilon}) \iff \overline{w} \in (d_{\varepsilon}).
$$

(1.5) \quad w \in \mathcal{W}_{\varepsilon,\delta} \Rightarrow w(t) \geq C \min\left(t^{-\varepsilon}, t^\delta\right).

Let us now give some examples.

It is elementary to see that if $\alpha \in \mathbb{R}$ and $w_\alpha(t) = t^\alpha$ then $w_\alpha \in \mathcal{W}_{\varepsilon,\delta}$ for any $\delta > \alpha$ and $\varepsilon > -\alpha$.

Let us give a bit more general example. Let $\alpha, \beta \in \mathbb{R}$ and $w_{\alpha,\beta}(t) = t^\alpha (1 + |\log t|)^\beta$. Then $w_{\alpha,\beta} \in \mathcal{W}_{\varepsilon,\delta}$ for any $\delta > \alpha$ and $\varepsilon > -\alpha$.

Indeed, let us take $\delta > \alpha$. Then making the change of variable $t = su$ we have

$$
\int_s^\infty \frac{w_{\alpha,\beta}(t)}{t^{\delta+1}} dt = \int_s^\infty \frac{t^{\alpha-\delta}(1 + |\log t|)^\beta dt}{t^\delta} \leq s^{\alpha-\delta} \int_1^\infty u^{\alpha-\delta}(1 + |\log s| + \log u)^\beta \frac{du}{u}.
$$
For $\beta < 0$ then
\[
\int_{s}^{\infty} \frac{w_{\alpha,\beta}(t)}{t^{\delta+1}} dt \leq \frac{1}{\delta - \alpha} s^{\alpha - \delta} (1 + |\log s|)^{\beta} = \frac{C_{\beta} w_{\alpha,\beta}(s)}{s^{\delta}}.
\]
For $\beta > 0$, using $(a+b)^{\beta} \leq C_{\beta} (a^{\beta} + b^{\beta})$, we have
\[
\int_{s}^{\infty} \frac{w_{\alpha,\beta}(t)}{t^{\delta+1}} dt \leq C_{\beta} s^{\alpha - \delta} \left( (1 + |\log s|)^{\beta} \int_{1}^{\infty} u^{\alpha - \delta} du + \int_{1}^{\infty} u^{\alpha - \delta} (\log u)^{\beta} \frac{du}{u} \right)
\leq C(\alpha, \beta, \delta) \frac{w_{\alpha,\beta}(s)}{s^{\delta}}.
\]
Since $w_{\alpha,\beta}(t) = w_{-\alpha,\beta}(t^{-1})$ then also have $w_{\alpha,\beta}$ is a $d_{\varepsilon}$-weight for $\varepsilon > -\alpha$. □

Let us now establish the main lemma to be used later on. Observe that a net $\{\phi_{i}\}_{i \in \Lambda}$ converges to $\phi$ in $S_{0}'$ if there exist $\{c_{i}\}_{i \in \Lambda} \subset \mathbb{C}$ such that $\phi_{i} - c_{i}$ converges to $\phi$ in $S'$. One of the main facts in our approach, which follows ideas from [6] and [14], is the use of the Calderón reproducing formula. Let $\phi \in \mathcal{A}$ and $\psi \in S$ then for $\xi \in \mathbb{R}^{n} \setminus \{0\}$,
\[
\hat{\psi}(\xi) = \int_{0}^{\infty} (\phi_{t} * \phi_{t} * \psi)(\xi) \frac{dt}{t}.
\]
This shows that $\psi_{\varepsilon,\delta} = \int_{\varepsilon}^{\delta} \phi_{t} * \phi_{t} * \psi \frac{dt}{t}$ converges to $\psi$ in $S$.

**Lemma A.** (see Appendix [14]). Let $f \in L^{1} \left( \mathbb{R}^{n}, \frac{dx}{(1+|x|)^{n+\varepsilon}} \right)$ and $\phi \in \mathcal{A}$. For $0 < \varepsilon < \delta$ define
\[
f_{\varepsilon,\delta}(x) = \int_{\varepsilon}^{\delta} (\phi_{t} * \phi_{t} * f)(x) \frac{dt}{t}.
\]
Then $f_{\varepsilon,\delta}$ converges to $f$ in $S_{0}'$ as $\varepsilon \to 0$ and $\delta \to \infty$.

To finish this preliminary section let us state a version of Schur lemma that will be useful for our purposes and whose elementary proof we include here for the sake of completeness.

**Lemma B.** Let $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let $(\Omega_{1}, \Sigma_{1}, \mu_{1})$ and $(\Omega_{2}, \Sigma_{2}, \mu_{2})$ be two $\sigma$-finite measure spaces and let $K : \Omega_{1} \times \Omega_{2} \to \mathbb{R}^{+}$ be a measurable function and write $T_{K}(f)$ for
\[
T_{K}(f)(w_{2}) = \int_{\Omega_{1}} K(w_{1}, w_{2}) f(w_{1}) d\mu_{1}(w_{1}).
\]
If there exist $C > 0$ and measurable functions $h_{i} : \Omega_{i} \to \mathbb{R}^{+}$ $(i = 1, 2)$ such that
\[
\int_{\Omega_{1}} K(w_{1}, w_{2}) h_{1}^{q'}(w_{1}) d\mu_{1}(w_{1}) \leq C h_{2}^{q'}(w_{2}) \mu_{2} \text{ a.e.}
\]
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\[(1.10) \quad \int_{\Omega_2} K(w_1, w_2) h^q_2(w_2) d\mu_2(w_2) \leq Ch^q_1(w_1) \mu_1 \text{ a.e.} \]

Then \( T_K \) defines a bounded operator from \( L^q(\Omega_1, \mu_1) \) into \( L^q(\Omega_2, \mu_2) \).

**Proof.** From (1.9) and Hölder’s inequality we have

\[
|T_K(f)(w_2)| \leq Ch^q_2(w_2) \left( \int_{\Omega_1} K(w_1, w_2) h^{-q}_1(w_1)|f(w_1)|^q d\mu_1(w_1) \right)^{\frac{1}{q}}.
\]

Apply now (1.10) and Fubini’s theorem to get

\[
\|T_K(f)\|_q \leq C \left( \int_{\Omega_1} \left( \int_{\Omega_2} K(w_1, w_2) h^q_2(w_2) d\mu_2(w_2) \right) h^{-q}_1(w_1)|f(w_1)|^q d\mu_1(w_1) \right)^{\frac{1}{q}} \leq C^2 \|f\|_q.
\]

§2. CHARACTERIZATION OF BESOV SPACES

Let us first establish some general facts that can be used to relate properties about differences \( \Delta_x f \) and convolutions \( \phi_t \ast f \).

**Lemma 2.1.** Let \( 1 \leq p \leq \infty, \rho \geq 0 \) and \( \phi \in A \). Then there exists \( C > 0 \) such that if \( f \in L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+\rho}}) \) then we have:

\[(2.1) \quad \|\phi_t \ast f\|_p \leq C \int_{\mathbb{R}^n} \min \left( \left( \frac{|x|}{t} \right)^{\rho}, \left( \frac{t}{|x|} \right)^\rho \right) \|\Delta_x f\|_p \frac{dx}{|x|^n}.\]

\[(2.2) \quad \|\Delta_x f\|_p \leq C \int_0^\infty \min \left( 1, \frac{|x|}{t} \right) \|\phi_t \ast f\|_p \frac{dt}{t}.\]

**Proof.** Notice that, since \( \int_{\mathbb{R}^n} \phi(x)dx = 0 \), then

\[\phi_t \ast f(y) = \int_{\mathbb{R}^n} \phi_t(x) \Delta_x f(y)dx.\]

From Minkowski’s inequality one gets

\[(2.1') \quad \|\phi_t \ast f\|_p \leq \int_{\mathbb{R}^n} \left( \frac{|x|^n}{t^n} \right) \phi \left( \frac{-x}{t} \right) \|\Delta_x f\|_p \frac{dx}{|x|^n}.\]

Hence (2.1) follows from the trivial estimates
\[ |y|^{n+\rho}|\phi(y)| \leq C \quad \text{if} \quad |y| \geq 1. \]
\[ |\phi(y)| \leq C \quad \text{if} \quad |y| \leq 1. \]

To prove (2.2) observe first that for \(0 < \varepsilon < \delta\)

\[ \Delta_x f_{\varepsilon,\delta}(y) = \int_{\varepsilon}^{\delta} (\Delta_{-x} \phi_t) * \phi_t * f(y) \frac{dt}{t}. \]

Hence Minkowski’s inequality and Young’s inequality give

\[ ||\Delta_x f_{\varepsilon,\delta}||_p \leq \int_{\varepsilon}^{\delta} ||\Delta_{-x} \phi_t||_1 ||\phi_t * f||_p \frac{dt}{t}. \]

Note that

\[ ||\Delta_y \phi||_1 \leq 2 ||\phi||_1 \quad \text{if} \quad |y| \geq 1. \]
\[ ||\Delta_y \phi||_1 \leq |y| \int_{\mathbb{R}^n} \max_{|z-u|<1} |\nabla \phi(z)| du \quad \text{if} \quad |y| \leq 1. \]

Hence

\[ ||\Delta_{-x} \phi_t||_1 = ||\Delta_{-x} \phi||_1 \leq C \min\left(1, \frac{|x|}{t}\right). \]

Therefore, using the previous estimate (2.2') and Lemma A, a simple limiting argument shows (2.2). \(\square\)

Although for the purposes of this paper only a particular case of next lemma will be used we state a general version of it that we find interesting in its own right.

**Lemma 2.2.** Given \(0 \leq \varepsilon, \delta < \infty\), \(1 < q < \infty\), and \(w\) a weight, let us consider

\[ R_{\varepsilon,\delta}(s,t) = \frac{w(s)}{w(t)} \min\left(\left(s, \frac{\varepsilon}{t}\right), \left(t, \frac{\delta}{s}\right)\right). \]

If \(w(s) = \lambda^{\frac{1}{q'}}(s)\mu^{-\frac{1}{q'}}(s^{-1})\) for some pair of weights \(\lambda, \mu \in \mathcal{W}_{\varepsilon,\delta}\), then there exist \(C > 0\) and \(g : \mathbb{R}^+ \to \mathbb{R}^+\) measurable such that

\[ \int_0^\infty R_{\varepsilon,\delta}(s,t) g^q(s) \frac{ds}{s} \leq C g^q(t). \]

\[ \int_0^\infty R_{\varepsilon,\delta}(s,t) g^q(t) \frac{dt}{t} \leq C g^q(s). \]

**Proof.** Let us take \(g(t) = \lambda^{\frac{1}{q'}}(t)\mu^{-\frac{1}{q'}}(t^{-1})\). Then \(g^q(s) = \frac{\lambda(s)}{w(s)}\) and \(g^q(t) = w(t)\mu(t^{-1})\).
Therefore
\[
\int_0^\infty R_{\epsilon, \delta}(s, t)g^q'(s) \frac{ds}{s} = \frac{1}{w(t)} \int_0^\infty \lambda(s) \min \left( \left( \frac{s}{t} \right)^\epsilon, \left( \frac{t}{s} \right)^\delta \right) \frac{ds}{s} = \frac{1}{t^\epsilon w(t)} \int_0^t s^\epsilon \lambda(s) \frac{ds}{s} + \frac{t^\delta}{w(t)} \int_t^\infty \frac{\lambda(s)}{s^\delta} ds
\]
\[
\leq C \frac{\lambda(t)}{w(t)} = C g^q'(t).
\]

On the other hand
\[
\int_0^\infty R_{\epsilon, \delta}(s, t)g^q(t) \frac{dt}{t} = w(s) \int_0^\infty \mu(t^{-1}) \min \left( \left( \frac{s}{t} \right)^\epsilon, \left( \frac{t}{s} \right)^\delta \right) \frac{dt}{t} = \frac{w(s)}{s^\delta} \int_0^s t^\delta \mu(t^{-1}) \frac{dt}{t} + s^\epsilon w(s) \int_s^\infty \frac{\mu(t^{-1})}{t^\epsilon} \frac{dt}{t}
\]
\[
= \frac{w(s)}{s^\delta} \int_{s^{-1}}^\infty \frac{\mu(t)}{t^\delta} \frac{dt}{t} + s^\epsilon w(s) \int_0^{s^{-1}} \frac{t^\delta}{t^\epsilon} \mu(t) \frac{dt}{t}
\]
\[
\leq C \mu(s^{-1}) w(s) = C g^q(s). \quad \square
\]

Let us now state the following result in order to avoid repeating arguments in several of the remaining proofs.

**Lemma 2.3.** Let $1 \leq p \leq \infty$ and let $f$ be a measurable function.

If $\|\Delta_x f\|_p \in L^1 \left( \mathbb{R}^n, \frac{dx}{(1+|x|)^{n+\tau}} \right)$ then $f \in L^1 \left( \mathbb{R}^n, \frac{dx}{(1+|x|)^{n+\tau}} \right)$.

**Proof.** Choose $\Psi \in L^{p'}(\mathbb{R}^n, dx)$ with $\Psi > 0$ a.e. Then Hölder’s inequality and Fubini’s theorem give

\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|f(x+y) - f(y)|}{(1+|x|)^{n+1}} dx \right) \Psi(y) dy < \infty.
\]

Therefore

\[
\int_{\mathbb{R}^n} \frac{|f(x+y) - f(y)|}{(1+|x|)^{n+1}} dx < \infty \quad \text{for a.e. } y \in \mathbb{R}^n.
\]

Since $(1+|x|)^{-(n+1)} \in L^1(\mathbb{R}^n)$ then

\[
\int_{\mathbb{R}^n} \frac{|f(x+y)|}{(1+|x|)^{n+1}} dx < \infty \quad \text{for a.e. } y \in \mathbb{R}^n.
\]

Finally since there exists $C > 0$ such that $1 + |x+y| \geq C(1+|x|)$ for all $y \in \mathbb{R}^n$, then one has
\[
\int_{\mathbb{R}^n} \frac{|f(x)|}{(1 + |x|)^{n+1}} dx < \infty. \quad \square
\]

Let us now start with the case \( q = \infty \) in the Main Theorem which easily follows from Lemma 2.1.

**Theorem 2.1.** Let \( 1 \leq p \leq \infty, \phi \in \mathcal{A} \) and \( w \in \mathcal{W}_{0,1} \). Then

\[
\Lambda_w^{p,\infty} = B_w^{p,\infty},
\]

(with equivalent seminorms).

**Proof.** Assume \( f \in \Lambda_w^{p,\infty} \). Note that

\[
\int_{\mathbb{R}^n} \frac{||\Delta x f||_p}{(1 + |x|)^{n+1}} dx \leq C \int_{\mathbb{R}^n} \frac{w(|x|)}{(1 + |x|)^{n+1}} dx
\]

\[
\leq C \int_0^\infty w(t) t^{-n} \frac{dt}{(1 + t)^{n+1}}
\]

\[
\leq C \left( \int_0^1 w(t) \frac{dt}{t} + \int_1^\infty w(t) \frac{dt}{t^2} \right) < \infty
\]

what combined with Lemma 2.3 gives

\[
\int_{\mathbb{R}^n} \frac{|f(x)|}{(1 + |x|)^{n+1}} dx < \infty.
\]

Let us prove that \( ||\phi_t * f||_p \leq Cw(t) \). From (2.1) in Lemma 2.1 for \( \rho = 1 \) we have

\[
||\phi_t * f||_p \leq C \left( \frac{1}{t^n} \int_{|x| < t} ||\Delta x f||_p dx + t \int_{|x| > t} ||\Delta x f||_p \frac{dx}{|x|^{n+1}} \right)
\]

\[
\leq C \left( \frac{1}{t^n} \int_{|x| < t} w(|x|) dx + t \int_{|x| > t} w(|x|) \frac{dx}{|x|^{n+1}} \right)
\]

\[
\leq C \left( \int_0^t \frac{w(s)}{s^n} ds + t \int_t^\infty w(s) \frac{ds}{s^2} \right) \leq C w(t).
\]

Assume now \( f \in B_w^{p,\infty} \). Then from (2.2) we have

\[
||\Delta x f||_p \leq C \left( \int_0^{|x|} ||\phi * f||_p \frac{dt}{t} + |x| \int_{|x|}^\infty ||\phi * f||_p \frac{dt}{t^2} \right)
\]

\[
\leq C \left( \int_0^{|x|} w(t) \frac{dt}{t} + |x| \int_{|x|}^\infty \frac{w(t)}{t^2} dt \right) \leq C w(|x|). \quad \square
\]

We prove now the case \( q = 1 \) in the Main Theorem.
Theorem 2.2. Let \(1 \leq p \leq \infty\), \(\phi \in A\) and \(w\) such that \(\mu(t) = w^{-1}(t^{-1}) \in \mathcal{W}_{0,1}\). Then
\[
\Lambda_{w}^{p,1} = B_{w,\phi}^{p,1}
\] (with equivalent seminorms).

Proof. Assume \(f \in \Lambda_{w}^{p,1}\). Let us first prove that
\[
\int_{\mathbb{R}^n} \frac{|f(x)|}{(1 + |x|)^{n+1}} \, dx < \infty.
\]

From (1.5)
\[
\frac{1}{|x|^n w(|x|)} \geq C \frac{1}{|x|^n} \min\left(1, \frac{1}{|x|}\right) \geq C \frac{1}{|x|^n} \min\left(|x|^n, \frac{1}{|x|}\right) \geq \frac{C}{(1 + |x|)^{n+1}}.
\]

Hence
\[
\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{(1 + |x|)^{n+1}} \, dx \leq C \int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)} \frac{dx}{|x|^n} < \infty
\]
and we apply Lemma 2.3 again.

We shall now prove that \(||f||_{B_{w,\phi}^{p,1}} \leq C ||f||_{\Lambda_{w}^{p,1}}\).

Using (2.1) in Lemma 2.1 with \(\rho = 1\)
\[
\int_{0}^{\infty} \frac{||\phi_t \ast f||_p \, dt}{w(t)} \leq C \int_{0}^{\infty} \left[ \int_{\mathbb{R}^n} \min\left(\left(\frac{|x|}{t}\right)^n, \left(\frac{t}{|x|}\right)^n\right) \frac{||\Delta_x f||_p}{w(t)} \frac{dx}{|x|^n} \right] \frac{dt}{t}
\]
\[
= C \int_{\mathbb{R}^n} ||\Delta_x f||_p \left[ \int_{0}^{\infty} \min\left(\left(\frac{|x|}{t}\right)^n, \left(\frac{t}{|x|}\right)^n\right) \mu\left(\frac{t^{-1}}{t}\right) \frac{dt}{t} \right] \frac{dx}{|x|^n}
\]
\[
= C \int_{\mathbb{R}^n} ||\Delta_x f||_p \left[ \int_{|x|^{-1}}^{\infty} \frac{t \mu\left(\frac{t^{-1}}{t}\right) \, dt}{|x|^n} + \int_{|x|^{-1}}^{\infty} \frac{\mu\left(\frac{t^{-1}}{t}\right) \, dt}{t^n} \right] \frac{dx}{|x|^n}
\]
\[
\leq C \int_{\mathbb{R}^n} ||\Delta_x f||_p \mu\left(\frac{1}{t}\right) \frac{dt}{t^2} + \int_{0}^{\infty} \frac{\mu\left(\frac{t^{-1}}{t}\right) \, dt}{t^n} \frac{dx}{|x|^n}
\]
\[
= C \int_{\mathbb{R}^n} ||\Delta_x f||_p \frac{dx}{w(|x|)} \frac{dx}{|x|^n}.
\]

Take now \(f \in B_{w,\phi}^{p,1}\). From (2.2) in Lemma 2.1 and Fubini’s theorem.
\[
\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)} \frac{dx}{|x|^n} \leq C \int_0^\infty \|\phi_t * f\|_p \left[ \int_{\mathbb{R}^n} \mu(|x|^{-1}) \min \left( 1, \frac{|x|}{t} \right) \frac{dx}{|x|^n} \right] \frac{dt}{t} \\
= C \int_0^\infty \|\phi_t * f\|_p \left[ \int_0^\infty \mu(s) \min \left( 1, \frac{1}{st} \right) \frac{ds}{s} \right] \frac{dt}{t} \\
= C \int_0^\infty \|\phi_t * f\|_p \left[ \int_0^{t^{-1}} \frac{\mu(s)}{s} ds + \frac{1}{t} \int_{t^{-1}}^\infty \frac{\mu(s)}{s^2} ds \right] \frac{dt}{t} \\
\leq C \int_0^\infty \|\phi_t * f\|_p \mu^{-1}(t^{-1}) \frac{dt}{t} \\
= C \int_0^\infty \frac{\|\phi_t * f\|_p}{w(t)} \frac{dt}{t}. \quad \square
\]

**Theorem 1.3.** Let \(1 \leq p \leq \infty, 1 < q < \infty, \phi \in A\) and \(w\) a weight such that

\[
w(t) = \lambda^\frac{1}{q'}(t) \mu^\frac{-1}{q'}(t^{-1})
\]

for some pair of weights \(\lambda, \mu \in \mathcal{W}_{0,1}\). Then

\[\Lambda_{w}^{p,q} = B_{w,\phi}^{p,q}\] (with equivalent seminorms).

**Proof.** Assume \(f \in \Lambda_{w}^{p,q}\). Let us show first that

\[
\int_{\mathbb{R}^n} \frac{|f(x)|}{(1 + |x|)^{n+1}} dx < \infty.
\]

Let us denote

\[
\Phi(x) = \frac{w(|x|)|x|^n}{(1 + |x|)^{n+1}}.
\]

We shall see that under the assumptions \(\lambda, \mu \in \mathcal{W}_{0,1}\) one has that \(\Phi \in L^{q'}(\mathbb{R}^n, \frac{dx}{|x|^n})\).

Indeed

\[
\int_0^\infty \Phi^q'(t) \frac{dt}{t} = \int_0^\infty \lambda(t) \mu^{-q'/q} (t^{-1}) \frac{t^{nq'}}{(1 + t)^{q'(n+1)}} \frac{dt}{t}.
\]

Using (1.5) we have \(\mu(s) \geq C \min(1, s)\). Therefore

\[
\int_0^\infty \Phi^q'(t) \frac{dt}{t} \leq C \int_0^\infty \lambda(t) \max \left( 1, t^{(q'-1)} \right) \frac{t^{nq'}}{(1 + t)^{q'q + q'}} \frac{dt}{t}
\]

\[
\leq C \left( \int_0^1 \lambda(t) \frac{dt}{t} + \int_1^\infty \lambda(t) \frac{dt}{t} \right) < \infty.
\]

From Hölder’s inequality one has
\[
\int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)^{n+1}} \, dx = \int_{\mathbb{R}^n} \frac{||\Delta_x f||_p}{w(|x|)} \Phi(x) \frac{dx}{|x|^n} < \infty
\]

and we apply Lemma 2.3.

Let us now prove

\[
||f||_{L^p_w} \leq C ||f||_{\Lambda^{p,q}_w}.
\]

From (2.1) in Lemma 2.1 with \(\rho = 1\)

\[
||\phi_t \ast f||_p \leq C \int_{\mathbb{R}^n} K(x,t) \frac{||\Delta_x f||_p}{w(|x|)} \frac{dx}{|x|^n}
\]

where

\[
K(x,t) = \frac{w(|x|)}{w(t)} \min \left(1, \frac{t}{|x|}\right).
\]

Take

\[
(\Omega_1, \Sigma_1, \mu_1) = \left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \frac{dx}{|x|^n}\right)
\]

and

\[
(\Omega_2, \Sigma_2, \mu_2) = \left((0, \infty), \mathcal{B}((0, \infty)), \frac{dt}{t}\right).
\]

Since \(K(x,t) = R_{0,1}(|x|, t)\) we can apply Lemma 2.2 with \(\varepsilon = 0\) and \(\delta = 1\) to get a measurable function \(g\) satisfying (2.3) and (2.4).

Now write \(h_1(x) = g(|x|)\) and \(h_2(t) = g(t)\). Obviously, using polar coordinates, (2.3) and (2.4) give (1.3) and (1.4) in Lemma B, what shows that \(T_K\) is bounded from \(L^q(\mathbb{R}^n, \frac{dx}{|x|^n})\) into \(L^q((0, \infty), dt)\). Therefore

\[
||f||_{L^p_w} \leq C \left|\left| T_K \left( \frac{||\Delta_x f||_p}{w(|x|)} \right) \right|\right|_{L^q((0, \infty), \frac{dt}{t})}
\]

\[
\leq C \left|\left| \frac{||\Delta_x f||_p}{w(|x|)} \right|\right|_{L^q(\mathbb{R}^n, \frac{dt}{t})}
\]

\[
\leq C ||f||_{\Lambda^{p,q}_w}.
\]

(ii) Let us take \(f \in B^{p,q}_{w,\phi}\). From (2.2) in Lemma 2.1

\[
\frac{||\Delta_x f||_p}{w(|x|)} \leq C \int_0^\infty R(x,t) ||\phi_t \ast f||_p \frac{dt}{t}.
\]

where

\[
R(x,t) = \frac{w(t)}{w(|x|)} \min \left(1, \frac{|x|}{t}\right).
\]
Now take
\[(\Omega_1, \Sigma_1, \mu_1) = \left( (0, \infty), \mathcal{B}((0, \infty)), \frac{dx}{|x|^n} \right)\]
and
\[(\Omega_2, \Sigma_2, \mu_2) = \left( \mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \frac{dx}{|x|^n} \right).\]

Combine now again Lemma 2.2 and Lemma B to get the boundedness of \(T_R\) from \(L^q\left((0, \infty), \frac{dt}{t}\right)\) into \(L^q\left(\mathbb{R}^n, \frac{dx}{|x|^n}\right)\). Therefore

\[\|f\|_{\Lambda^{p,q}_{w}} \leq C \left| T_R \left( \frac{\|\phi_t \ast f\|_p}{w(t)} \right) \right|_{L^q\left(\frac{dt}{t}\right)} \leq C \left| \frac{\|\phi_t \ast f\|_p}{w(t)} \right|_{L^q\left(\frac{dt}{t}\right)} \leq C \|f\|_{B^{p,q}_{w,\phi}}. \quad \square\]

**Remark.** Note that in the previous theorem one of the embedding could have been proved under weaker assumptions. In fact, if \(\lambda, \mu \in \mathcal{W}_{n,1}\) then \(\Lambda^{p,q}_{w} \subseteq B^{p,q}_{w,\phi}\).

**References**


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