The Bergman Projection on weighted spaces: L^1 and Herz spaces

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Abstract

In this paper we study conditions on radial weights w so that the Bergman projection is bounded on the Herz spaces $K_p^q(w)$.

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1 Introduction and preliminaries.

The purpose of this paper is to study spaces of analytic functions in the unit disc $\mathbb{D}$ provided with a norm of a weighted Herz space. More precisely we will consider the classical family of Bergman projections P_s, $s > -1$, and we give necessary and sufficient conditions on the weight making these projections continuous in the corresponding weighted Herz space. The continuity of the projections P_s has been studied by many authors in several settings like weighted L^p continuity or weighted mixed norms (see for example [2, 5, 8, 12, 15, 16, 18] and [1, 3, 4, 17, 19] for related literature on Bergman type spaces).

Throughout the paper $dm(z)$ is de normalized area measure on the disc, that is $dm(z) = \frac{1}{\pi} r dr d\theta$. For a weight w we understand a function such that $0 < w(z) < \infty$. If f is a function in $\mathbb{D}$ and $s \geq 0$, we will denote $f_s(z) = (1-|z|^2)^s f(z)$. We will write $r_n = 1-2^{-n}$, $I_n = \{r : r_n < r < r_{n+1}\}$ and $A_n = \{z \in \mathbb{D} : r_n < |z| < r_{n+1}\}$. We denote

$$||f||_{L^p(w)} = (\int_{\mathbb{D}} |f(z)|^p w(z) dm(z))^{1/p},$$

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and

$$M_p^p(f,r) = \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}.$$

We will write $w(A) = \int_A w(z)dm(z)$ for any measurable subset A of $\mathbb{D}$. Given a function ψ integrable on $[0,1)$ we denote be $M_n(\psi) = \int_0^1 \psi(r)r^n dr$ the moment of order n for $n \in \mathbb{N}$ or $n = 0$.

Define the spaces $K_q^{\alpha,p}$ consisting of all measurable functions f in $\mathbb{D}$ such that

$$\sum_{n=1}^{\infty} 2^{-\alpha q n} \left(\int_{A_n} |f(z)|^p dm(z) \right)^{q/p} < \infty.$$

These spaces are a variant of those introduced by C. Herz in [10]. In this paper we will consider a more general class of spaces, the weighted Herz spaces $K_q^p(w)$, $1 \le p, q \le \infty$, introduced by Lu and Yang in [14] (see also [13] for power weights). These spaces consist of all measurable functions f in the disc such that $\left(\|f\|_{L_w^p(A_n)} \right) \in \ell^q$. The norm in $K_q^p(w)$ is defined by

$$||f||_{K_q^p(w)} = || \left(\|f\|_{L_w^p(A_n)} \right) ||_{\ell^q}.$$

Example 1.1 **a)** *If* $f = \sum_{m=1}^{\infty} a_n \chi_{A_n}$ *then* $f \in K_q^p(w)$ *if and only if*

$$\sum_{m=1}^{\infty} |a_n|^q w(A_n)^{q/p} < \infty.$$

b) *Let* w *be a radial weight and* $f(z) = \phi(r)\psi(\theta)$ *for* $z = re^{i\theta}$ *where* ϕ, ψ *are measurable functions in* $[0,1)$ *and* $[0,2\pi)$ *respectively. Then*

$$||f||_{K_q^p(w)} = ||\psi||_{L^p([0,2\pi))} (\sum_{n=1}^{\infty} \int_{I_n} |\phi(r)|^p w(r) r dr)^{q/p})^{1/q}.$$

For $s > -1$ we consider the family of Bergman projections

$$P_s f(z) = \int_{\mathbb{D}} K_s(z,\xi) f(\xi)(1-|\xi|^2)^s dm(\xi),$$

where

$$K_s(z,\xi) = \frac{1}{(1-z\bar{\xi})^{2+s}} = \frac{1}{\Gamma(s+2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{n!}z^n\bar{\xi}^n.$$

Lemma 1.2 **a)** *If* $f(z) = \phi(r)\psi(\theta)$ *for* $z = re^{i\theta}$ *then*

$$P_s(f)(z) = \frac{2}{\Gamma(s+2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{n!}M_{n+1}(\phi_s)\hat{\psi}(n)z^n.$$

b) *Fix* $s > -1$ *then* $P_s(\chi_{A_n})(z) = c_{n,s} \sim 2^{-n(s+1)}$ *for all* $z \in \mathbb{D}$.

Proof. To prove (a), we use polar coordinates to get

$$\begin{aligned} P_s f(z) &= 2\int_0^1 \Big(\int_0^{2\pi} \frac{\psi(\theta)}{(1-re^{-i\theta}z)^{2+s}}\frac{d\theta}{2\pi}\Big)(1-r^2)^s\phi(r)rdr \\ &= \frac{2}{\Gamma(s+2)}\int_0^1 \Big(\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{n!}\hat{\psi}(n)r^n z^n\Big)(1-r^2)^s\phi(r)rdr \\ &= \frac{2}{\Gamma(s+2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{n!}M_{n+1}(\phi_s)\hat{\psi}(n)z^n. \end{aligned}$$

The proof of (b) easily follows from (a). ■

Define now the spaces $H_{pq}(w)$ of all functions f holomorphic on the disc $\mathbb{D}$ such that

$$\left(\int_0^1 M_p^q(f,r)w(r)rdr\right)^{1/q} < \infty.$$

For the weight $w(r) = (1-r^2)^{q\alpha-1}$ the spaces are sometimes denoted by $H(p,q,\alpha)$.

Using that $M_p(f,r)$ is increasing for holomorphic functions one gets the following

Proposition 1.3 *Let* w *be a weight such that* $w(A_n) \le Cw(A_{n+1})$, *for instance* $w(r) = (1-r^2)^\beta$ *or* $w = \sum a_n\chi_{A_i}$ *with* $a_n/a_{n+1} \le M$. *Then*

1. $f \in H_{pq}(w)$ *if and only if* $\sum_{n=1}^{\infty} M_p^q(f,r_n)w(A_n) < \infty$.

2. $f \in K_q^p(w) \cap Hol(D)$ *if and only if* $\sum_{n=1}^{\infty} M_p^q(f,r_n)w^{q/p}(A_n) < \infty$.

In particular, $K_q^p(w) \cap Hol(D) = H_{pq}(w^{q/p})$.

1.1 The class B_s^p

In [2] Bekolle introduced the class B_s^p of weight functions. Let $1 < p < \infty$, a radial weight $w = w(r)$ belongs to B_s^p if

$$\left(\int_{1-h}^{1} w(r)(1-r^2)^s r dr\right)\left(\int_{1-h}^{1} w(r)^{-p'/p}(1-r^2)^s r dr\right)^{p/p'} \leq Ch^{(s+1)p}. \tag{1}$$

Example 1.4 *a) If $w = \sum_1^\infty a_n \chi_{A_n}$, with $a_n > 0$ then $w \in B_s^p$ if and only if*

$$\left(\sum_{k=n}^{\infty} a_k 2^{-(s+1)k}\right)\left(\sum_{k=n}^{\infty} a_k^{-p'/p} 2^{-(s+1)k}\right)^{p/p'} \leq C 2^{-(s+1)np}$$

b) If $w(r) = (1-r^2)^{\alpha - s}$ then $w \in B_s^p$ if and only if

$$0 < \alpha + 1 < p(s+1). \tag{2}$$

In [2] it was proved that the B_s^p is precisely the class of weight functions making P_s a continuous projection, namely

Theorem 1.5 *Let $1 < p < \infty$. P_s is continuous in $L^p(w_s)$ if and only if $w \in B_s^p$.*

Notice in particular that P_s is continuous on $L^p((1-r^2)^\alpha)$ if and only if the inequality (2) holds. Also for $p = 1$ the weak type continuity result was achieved in [2] and the B_s^p condition was shown to be equivalent to the boundedness in $L^p(w_s)$ of P_s^* where

$$P_s^*(f)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{(1-|\xi|^2)^s f(\xi)}{|1-\bar{\xi}z|^{2+s}} dm(\xi).$$

■

2 Continuity on $L^1(w)$

If we write the condition B_s^p as the existence of a constant $C > 0$ such that for all $0 < h < 1$

$$\left\|w^{1/p}\right\|_{L^p([h,1),d\nu_{h,s})} ||w^{-1/p}||_{L^{p'}([h,1),d\nu_{h,s})} \leq C, \tag{3}$$

with

$$d\nu_{h,s} = \frac{(1-r^2)^s r dr}{(1-h)^{s+1}},$$

then the natural substitute of (3) for $p = 1$ is true, namely

Proposition 2.1 *Let $w = w(r)$ and let P_s be bounded on $L^1(w_s)$. Then*

a) $M_{n+1}(w_s)\left(\sup_{0<r<1} r^n w^{-1}(r)\right) \leq \frac{C}{(n+1)^{s+1}}$.

b) $||w||_{L^1([h,1),d\nu_{h,s})}||w^{-1}||_{L^\infty([h,1),d\nu_{h,s})} \leq C$.

Proof. Let $f_n(re^{i\theta}) = \phi(r)e^{in\theta}$ for $\phi \geq 0$. Then

$$P_s(f_n)(z) = 2\frac{\Gamma(n+s+2)}{\Gamma(s+2)n!}M_{n+1}(\phi_s)z^n,$$

and

$$||P_s(f_n)||_{L^1(w_s)} = 2\frac{\Gamma(n+s+2)}{\Gamma(s+2)n!}M_{n+1}(w_s)\left(\int_0^1 \phi(r)(1-r^2)^s r^{n+1} dr\right).$$

Therefore, using the boundedness of the operator P_s, one gets

$$2\frac{\Gamma(n+s+2)}{\Gamma(s+2)n!}M_{n+1}(w_s)\left(\int_0^1 \phi(r)w(r)w^{-1}(r)(1-r^2)^s r^{n+1} dr\right)$$

$$\leq C\int_0^1 \phi(r)w_s(r)rdr.$$

This, by duality, implies that for all $n \geq 0$

$$\sup_{0<r<1} r^n w^{-1}(r) \leq \frac{C_s n!}{\Gamma(n+s+2)M_n(w_s)} \leq \frac{C_s}{(n+1)^{s+1}M_n(w_s)},$$

since by the Stirling formula we have that

$$\frac{\Gamma(n+s+2)}{n!} \sim (n+1)^{s+1} .$$

Notice in particular that w_s is integrable in $\mathbb{D}$ and w^{-1} is bounded.

To see (b) observe that for each $0 < h < 1$ we can take $n \in \mathbb{N}$ such that $1 - \frac{1}{n+1} < h \leq 1 - \frac{1}{n}$ and that for $r > 1 - \frac{1}{n}$ we have $r^n \geq (1 - \frac{1}{n})^n \geq C$, provided $n \geq 2$.

Hence

$$||w||_{L^1([h,1),d\nu_{h,s})}||w^{-1}||_{L^\infty([h,1),d\nu_{h,s})} =$$

$$= \left(\frac{1}{(1-h)^{s+1}} \int_h^1 w(r)(1-r^2)^s r dr\right) \left(\sup_{h<r<1} w^{-1}(r)\right) \leq$$

$$\leq C \left((n+1)^{s+1} \int_{1-\frac{1}{n}}^1 w(r)(1-r^2)^s r dr\right) \left(\sup_{1-\frac{1}{n+1}<r<1} w^{-1}(r) r^n\right) \leq$$

$$\leq C(n+1)^{s+1} M_n(w) \sup_{0<r<1} w^{-1}(r) r^n \leq C.$$

■

Remark 2.2 *If P_s is bounded on $L^1(w_s)$ is then P_s is also bounded on $L^p(w_s)$ for all $1 < p < \infty$. Indeed, part (b) in Proposition 2.1 implies Bekolle's condition as in (3).*

Let us now get a neccesary condition for the boundedness of P_s on $L^1(w)$ for a general weight w.

Theorem 2.3 *Let w be a radial weight. If P_s is bounded on $L^1(w)$ then there exists a constant $C > 0$ so that*

$$\int_0^1 \frac{w(r)}{(1-rt)^{s+1}} rdr \leq C \frac{w(t)}{(1-t)^s} log(\frac{1}{1-t}),$$

and there exist $C_\alpha > 0$ for all $\alpha > 0$ such that

$$\int_0^1 \frac{w(r)}{(1-rt)^{s+\alpha+1}} dr \leq C \frac{w(t)}{(1-t)^{s+\alpha}}.$$

Proof. Let us assume P_s is bounded on $L^1(w)$ and take $f = \phi(r)\psi(\theta)$ where $\psi \in H^1(\mathbb{T}) = \{\psi \in L^1([0, 2\pi) : \hat{\psi}(n) = 0 \ n < 0\}$. Recall that Hardy inequality (see [7]) gives that for all $0 < r < 1$

$$\sum_{n=0}^{\infty} \frac{|\hat{\psi}(n)| r^n}{n+1} \leq C M_1(\psi, r).$$

Then

$$\begin{aligned} ||P_s f||_{L^1(w)} &= \int_0^1 w(r) M_1((P_s(f), r) rdr \\ &\geq C_s \int_0^1 w(r) \left(\sum_{n=0}^{\infty} \frac{\Gamma(n+s+2)}{(n+1)!} M_{n+1}(\phi_s) |\hat{\psi}(n)| r^n \right) rdr \\ &= C_s \int_0^1 G(t)(1-t^2)^s \phi(t) tdt, \end{aligned}$$

where

$$G(t) = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{\Gamma(n+s+2)}{(n+1)!} t^n r^n |\hat{\psi}(n)| \right) w(r) rdr.$$

Using the continuity of P_s we obtain by duality that

$$\sup_{0<t<1} (1-t^2)^s w^{-1}(t) G(t) \leq C \|\psi\|_1.$$

If for each $\alpha \geq 0$ and $0 < t < 1$ we let $\psi(z) = \frac{1}{(1-tz)^{\alpha+1}}$, we have that

$$\|\psi\|_1 \sim \begin{cases} \frac{1}{(1-t)^\alpha}, \ \alpha > 0 \\ \\ \log\left(\frac{1}{1-t}\right), \ \alpha = 0 \end{cases}$$

For this ψ we obtain

$$G(t) = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{\Gamma(n+s+2)}{(n+1)!} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} t^{2n} r^n \right) w(r) r dr.$$

Then from $\frac{\Gamma(n+\lambda)}{n!} \sim n^{\lambda-1}$ and the expansion

$$\frac{1}{(1-t)^\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)n!} t^n,$$

it follows that

$$G(t) \sim \int_0^1 \frac{w(r)}{(1-rt^2)^{s+\alpha+1}} r dr \sim \int_0^1 \frac{w(r)}{(1-rt)^{s+\alpha+1}} r dr,$$

and the proof is complete. ■

We finish this section by showing that $\int_0^1 \frac{w(r)}{(1-rt)^{s+1}} r dr \leq C \frac{w(t)}{(1-t)^s}$ implies the continuity of P_s on $L^1(w)$. Actually this will be equivalent to the boundedness of P_s^*.

Lemma 2.4 *Let w be weight. If P_s^* is bounded on $L^1(w)$ if and only if*

$$\int_{\mathbb{D}} \frac{w(z)}{|1-\overline{y}z|^{2+s}} dm(z) \leq C \frac{w(y)}{(1-|y|^2)^s} \ a.e.$$

Proof. For any positive function f one has

$$\int_{\mathbb{D}} P_s^*(f)(z) w(z) dm(z) = \int_{\mathbb{D}} f(y) \left(1-|y|^2\right)^s \left(\int_{\mathbb{D}} \frac{w(z)}{|1-\overline{y}z|^{2+s}} dm(z) \right) dm(w).$$

Then the lemma follows by duality. ■

Proposition 2.5 *Let w be a radial weight. The following are equivalent*

a) *P_s^* is bounded on $L^1(w)$,*

b) *There exists a constant $C > 0$ so that $\int_0^1 \frac{w(r)}{(1-rt)^{s+1}} rdr \le C \frac{w(t)}{(1-t)^s}$ a.e.*

c) $\int_0^t \frac{w(r)}{(1-r)^{s+1}} rdr \le C \frac{w(t)}{(1-t)^s}$ *a.e. and* $\frac{1}{(1-t)} \int_t^1 w(r) rdr \le Cw(t)$ *a.e.*

Proof. (a) is equivalent to (b) according to the previous lemma using that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{-i\theta}|^{2+s}} \sim \frac{C}{(1 - |z|^2 r^2)^{1+s}}.$$

To see that (b) is equivalent to (c) observe that

$$\begin{aligned} \int_0^1 \frac{w(r)}{(1-rt)^{s+1}} rdr &= \int_0^t \frac{w(r)}{(1-rt)^{s+1}} rdr + \int_t^1 \frac{w(r)}{(1-rt)^{s+1}} rdr \\ &\sim \int_0^t \frac{w(r)}{(1-r)^{s+1}} dr + \frac{1}{(1-t)^{s+1}} \int_t^1 w(r) rdr. \end{aligned}$$

■

Let us recall that a weight w is called a normal weight (see [8] or [18]) if there exist a and b, $0 < a < b$, such that

i) $\frac{w(r)}{(1-r)^a}$ is nonincreasing with $\lim_{r \to 1} \frac{w(r)}{(1-r)^a} = 0$ and

ii) $\frac{w(r)}{(1-r)^b}$ is nondecreasing with $\lim_{r \to 1} \frac{w(r)}{(1-r)^b} = \infty$.

We shall denote by $b(w) = \inf\{b : b \text{ satisfies (ii)}\}$.

Corollary 2.6 *Let w be a normal weight. If $s > b(w)$ then P_s^* is bounded on $L^1(w)$.*

Proof. Let us check that (c) in Proposition 2.5 is satisfied. Set $b = b(w)$.

$$\int_0^t \frac{w(r)}{(1-r)^{s+1}} rdr = \int_0^t \frac{w(r)}{(1-r)^b} \frac{(1-r)^b}{(1-r)^{s+1}} rdr \le C \frac{w(t)}{(1-t)^s}$$

and

$$\frac{1}{(1-t)} \int_t^1 w(r) rdr = \frac{1}{(1-t)} \int_t^1 \frac{w(r)}{(1-r)^a} (1-r)^a rdr \le Cw(t).$$

■

3 Necessary conditions for the boundedness on Herz spaces

Proposition 3.1 *Let $1 \leq p, q < \infty$, and assume the constant functions belong to $K_q^p(w)$, that is $\sum_{n=1}^{\infty} w(A_n)^{q/p} < \infty$. If P_s is bounded on $K_q^p(w)$ then the sequence $\left(2^{-n(s+1)} w^{-1/p}(A_n)\right) \in \ell_{q'}$.*

Proof. Fix N and take $f = \sum_{n=1}^{N} \frac{a_n}{w(A_n)^{1/p}} \chi_{A_n}$. From Lemma 1.2

$$P_s(f) = \sum_{n=1}^{N} \frac{a_n}{w(A_n)^{1/p}} c_{n,s}.$$

Hence

$$||P_s f(z)||_{K_q^p(w)} = |\sum_{n=1}^{N} \frac{a_n}{w(A_n)^{1/p}} c_{n,s}| (\sum_{n=1}^{\infty} w(A_n)^{q/p})^{1/q}$$

and

$$||f||_{K_q^p(w)} = (\sum_{m=1}^{\infty} |a_n|^q)^{1/q}.$$

Now the result follows by duality. ■

Corollary 3.2 *Let $\alpha > -1$. If P_s is bounded on $K_q^p((1-r^2)^\alpha)$ then $\alpha + 1 < (s+1)p$.*

Proof. The proof follows from Proposition 3.1 and the fact that $w(A_n) \sim 2^{-n(\alpha+1)}$ in this case.

■

Let us now give some more accurate neccesary conditions for the boundedness of P_s on $K_q^p(w_s)$

Proposition 3.3 *Let w be a radial weight. If $1 < p, q < \infty$ and P_s is bounded on $K_q^p(w_s)$, then there exists a constant C such that for all $n \in \mathbb{N}$*

$$||r^n||_{K^p_q(w_s)}||r^n||_{K^{p'}_{q'}\left(\left(w^{-p'/p}\right)_s\right)} \leq \frac{C}{(n+1)^{s+1}}. \tag{4}$$

Proof. Applying the boundedness to functions $f_n(z) = \phi(r)e^{in\theta}$, $\phi \geq 0$ and $n \in \mathbb{Z}$ we have that

$$P_s(f_n)(z) = 2\frac{\Gamma(n+s+2)}{n!}M_{n+1}(\phi_s)z^n,$$

hence

$$||P(f_n)||_{K^p_q(w_s)} = M_{n+1}(\phi_s)\frac{\Gamma(n+s+2)}{\Gamma(s+2)n!}||r^n||_{K^p_q(w_s)} \leq C||\phi||_{K^p_q(w_s)},$$

what implies that for all $n \geq 0$

$$\int_0^1 \phi(r)r^{n+1}(1-r^2)^s dr \leq \frac{C\ \Gamma(s+2)n!}{\Gamma(n+s+2)||r^n||_{K^p_q(w_s)}}||\phi||_{K^p_q(w_s)}. \tag{5}$$

Writing

$$\int_0^1 \phi(r)r^{n+1}(1-r^2)^s dr = \sum_{k=1}^{\infty}\int_{I_k}\phi(r)w^{1/p}(r)w^{-1/p}(r)r^n(1-r^2)^s r dr$$

and taking the supremum over all $||\phi||_{K^p_q(w_s)} \leq 1$ one gets from the duality in Herz spaces (see [9, Th. 2.1]) that

$$\begin{aligned}\left(\sum_{k=1}^{\infty}(\int_{I_k} w^{-p'/p}(r)r^{np'}(1-r^2)^s r dr)^{q'/p'}\right)^{1/q'} &\leq \frac{C\ \Gamma(s+2)n!}{\Gamma(n+s+2)||r^n||_{K^p_q(w)}}\\ &\leq \frac{C_s}{(n+1)^{s+1}\ ||r^n||_{K^p_q(w)}}.\end{aligned}$$

■

Corollary 3.4 *Let w be a radial weight. If $1 < p, q < \infty$ and P_s is bounded on $K^p_q(w_s)$ then there exists a constant C such that for all $n \in \mathbb{N}$*

$$||\chi_{[h,1)}||_{K^p_q(w_s)}||\chi_{[h,1)}||_{K^{p'}_{q'}\left(\left(w^{-p'/p}\right)_s\right)} \leq C(1-h)^{s+1}. \tag{6}$$

Proof. Notice that there exists a positive number C such that $\chi_{[1-\frac{1}{n},1)} \leq Cr^n$ for all $n \in \mathbb{N}$. Given $0 < h < 1$ take n such that $1 - \frac{1}{n} < h \leq 1 - \frac{1}{n+1}$, then the lattice structure of these spaces gives

$$||\chi_{[h,1)}||_{K^p_q(w_s)}||\chi_{[h,1)}||_{K^{p'}_{q'}((w^{-p'/p})_s)} \leq C(1-h)^{s+1}.$$

∎

Remark 3.5 **a)** *If $w(r) = v(r^2)$, then for $p = q = 2$, the condition (4) can be written as*

$$M_n((1-r^2)^s v)^{1/2} M_n((1-r^2)^s v^{-1})^{1/2} \leq \frac{C}{(n+1)^{s+1}}.$$

b) *If $p = q$, then inequality (6) is precisely Bekolle's condition (1).*

4 Sufficient conditions for the boundedness on Herz spaces

Let us start with some conditions on the weight to have $w \in B^p_t$ for $s - \varepsilon < t < s + \varepsilon$.

Lemma 4.1 *If there exists $\gamma > 1$ such that*

$$\int_0^1 \frac{w(r)^\gamma (1-r^2)^s}{(1-rt)^{s+1}} r dr \leq C w(t)^\gamma$$

then $(1-r^2)^{\pm\varepsilon} w \in B^p_s$ for all $0 < \varepsilon < \min\{\frac{(s+1)}{\gamma'}, (p/p')(s+1)\}$.

Proof. By Proposition 2.5 we have that P^*_s is continuous in $L^1((w^\gamma)_s)$. Then Proposition 2.1 implies that

$$(\int_{1-h}^1 w^\gamma(r)(1-r^2)^s r dr)(\sup_{1-h<r<1} w^{-\gamma}(r)) \leq C h^{s+1}.$$

Let $-\frac{(s+1)}{\gamma'} < \varepsilon < (p/p')(s+1)$, then

$$\left(\int_{1-h}^{1} w(r)(1-r^2)^{\varepsilon+s} r dr\right)\left(\int_{1-h}^{1} w(r)^{-p'/p}(1-r^2)^{-\varepsilon(p'/p)+s} r dr\right)^{p/p'}$$

$$\leq \left(\int_{1-h}^{1} w(r)^{\gamma}(1-r^2)^{s} r dr\right)^{1/\gamma}\left(\int_{1-h}^{1} (1-r^2)^{\gamma'\varepsilon+s} r dr\right)^{1/\gamma'}$$

$$\times \sup_{1-h<r<1} w^{-1}(r)\left(\int_{1-h}^{1} (1-r^2)^{-\varepsilon(p'/p)+s} r dr\right)^{p/p'}$$

$$\leq C\left(\sup_{1-h<r<1} w^{-\gamma}(r)(\int_{1-h}^{1} w(r)^{\gamma}(1-r^2)^{s} r dr)\right)^{1/\gamma} h^{\varepsilon+(s+1)/\gamma'} h^{-\varepsilon+(s+1)p/p'}$$

$$\leq C h^{(s+1)p}.$$

■

Theorem 4.2 *If there exists $\gamma > 1$ such that*

$$\int_0^1 \frac{w(r)^{\gamma}(1-r^2)^{s}}{(1-rt)^{s+1}} r dr \leq C w(t)^{\gamma},$$

then P_s is bounded on $K_q^p(w_s)$ for every $1 < p < \infty$ and $1 \leq q < \infty$.

Proof. By Lemma 4.1, there exists $\varepsilon > 0$ such that $(1-r^2)^{\pm\varepsilon} w \in B_s^p$, hence P_s is continuous in $L^p((1-r^2)^{\pm\varepsilon} w_s)$, that is, there exists $C > 0$ such that

$$\int_D |P_s f(z)|^p (1-r^2)^{\pm\varepsilon} w_s(z) dm(z) \leq C \int_D |f(z)|^p (1-r^2)^{\pm\varepsilon} w_s(z) dm(z).$$

In particular, given $n, m \in$, if $supp(f) \subset A_n$

$$\int_{A_m} |P_s f(z)|^p w_s(z) dm(z) \leq C 2^{\pm\varepsilon(m-n)} \int_D |f(z)|^p w_s(z) dm(z).$$

Let f be any function in $K_q^p(w)$. Write $f = \sum f_n$, with $f_n = f\chi_{A_n}$. Assume that the series is a sum with only a finite number of terms.

Then

$$\begin{aligned}\|P_s f\|_{L^p_{w_s}(A_m)} &\le C\sum_n \|P_s f_n\|_{L^p_{w_s}(A_m)} = C\sum_n 2^{\pm\frac{\varepsilon}{p}(m-n)}\,\|f_n\|_{L^p_{w_s}(A_n)}\\ &= C\sum_{n<m} 2^{\pm\frac{\varepsilon}{p}(m-n)}\,\|f_n\|_{L^p_{w_s}(A_n)} + C\sum_{n\ge m} 2^{\pm\frac{\varepsilon}{p}(m-n)}\,\|f_n\|_{L^p_{w_s}(A_n)}\\ &= I_1 + I_{2.}\end{aligned}$$

Consider the sequences $X = (x_n)$, and $Y = (y_n)$, with $x_n = 2^{-\varepsilon|n|/p}$ and

$$y_n = \begin{cases} \|f_n\|_{L^p_{w_s}(A_n)} & n \ge 0 \\ 0 & n < 0 \end{cases}$$

Then we have that

$$\|P_s f\|_{L^p_{w_s}(A_m)} \le C X * Y(m),\ m \in \mathbb{N}.$$

Finally from Young's inequality it follows that

$$\|P_s f\|_{K_{pq}(w_s)} \le C\,\|X\|_{\ell^1}\,||f||_{K^p_q(w_s)}.$$

■

We notice that the proof of Theorem 4.2 was based on the existence of a positive number ε such that $(1-r^2)^{\pm\varepsilon} w \in B^p_s$. With the same idea we have the following

Theorem 4.3 *If $w \in B^p_t$ then P_s is bounded on $K^p_q(w_s)$ for every $s > t$.*

Proof. An easy calculation shows that for $\varepsilon > 0$ we have

$$(1-r^2)^{-\varepsilon} w \in B^p_{t+\varepsilon}$$

$$(1-r^2)^{\varepsilon} w \in B^p_{t+\varepsilon p'/p}.$$

If we let $\varepsilon > 0$ small enough so that $\max(t + \varepsilon p'/p, t+\varepsilon) < s$, we obtain

$$(1-r^2)^{\pm\varepsilon} w \in B^p_s$$

since the class B^p_t increases in t. ■

Corollary 4.4 *Let $\alpha > -1$ and $w(r) = (1-r^2)^\alpha$. Then P_s is continuous in $K_q^p(w)$ if and only if $\alpha + 1 < p(s+1)$. In this case P_s maps $K_q^p(w)$ onto $H_{pq}(w^{q/p})$.*

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