WEIGHTED LIPSCHITZ SPACES DEFINED BY A BANACH SPACE.

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Abstract. We consider general weighted Lipschitz spaces defined by a Banach space. Under certain assumptions on the weight and the space, we find a Littlewood-Paley type formulation for such spaces. This allows us to give a formulation for the predual space as a generalized Besov space. We also prove that operators acting on certain weighted Besov spaces corresponds to vector valued functions in a natural way.

Introduction.

It is well known that the use of Calderón's reproducing formula allows to give a Littlewood-Paley formulation of many functions spaces (see [FJW]).

The aim of this paper is to show that this method can be used also in the general setting of weighted Lipschitz spaces and then to give some applications to duality results and to characterize the bounded operators acting on certain weighted Besov classes.

In [J], S. Janson considered the spaces $Lip(\rho, E)$ defined by distributions whose moduli of continuity in the Banach space E are dominated by a function ρ which grows arbitraryly slowly. He also considered spaces $B(\rho, E)$ defined using convolution with test functions ψ such that $\int \psi \neq 0$. These spaces give a unified approach to certain Besov and Lipschitz classes. We shall define some closely related Banach spaces, but for a bit more general weight functions and under slightly different assumptions on the space E. The main difference from the spaces in [J] comes from the fact that our test functions will be of mean zero.

Let us start by recalling some definitions on the weights and the Banach space that will be the setting for our results.

Definition 1. *A weight ω will be a measurable function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega > 0$ a.e.. We shall say that ω satisties Dini condition if*

$$\int_0^s \frac{\omega(t)}{t} dt \leq C\omega(s)$$

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and ω is called a b_1-weight if

$$\int_s^\infty \frac{\omega(t)}{t^2}dt \leq C\frac{\omega(s)}{s}.$$

Remarks 1. This type of weights have been used by different authors (see [J, B1, BS, BlS]) and they turned out to be the natural setting for certain weighted versions of classical results.

The main examples are $\omega(t) = t^\alpha(1 + |\log t|)^\beta$ for $0 < \alpha < 1$.

As usual $\mathcal{S}$ denote the Schwartz class of test functions on $\mathbb{R}^n$ and $\mathcal{S}'$ the space of tempered distributions. We denote by $\mathcal{S}_0$ the set of functions in $\mathcal{S}$ with mean zero, and $\mathcal{S'}_0$ its topological dual.

We prefer working with measurable functions rather than distributions. Our Banach spaces will be formed by measurable functions satistying $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}}dx < \infty$.

Definition 2. *Let $(E, \|.\|_E)$ be a Banach space included in $L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}})$. E is said bounded under translations space if*

(i) $\mathcal{S} \subset E \subset \mathcal{S}'$ (with continuity)

(ii) If $\tau_x(f)(y) = f(x+y)$ for $x \in \mathbb{R}^n$, then $x \to \tau_x(f)$ is an E-valued bounded measurable function, that is $\sup_{x\in\mathbb{R}^n} \|\tau_x(f)\|_E \leq C_0\|f\|_E$.

Remark 2. A bounded under translations space E is a Banach L^1-module, that is

if $f \in L^1, g \in E$ then $f * g \in E$ and $\|f * g\|_E \leq C_0\|f\|_1\|g\|_E$.

Indeed, given $f \in L^1$ and $g \in E$ then $y \to f(-y)\tau_y(g) \in L^1(\mathbb{R}^n, E)$. Therefore, since $f * g = \int_{\mathbb{R}^n} f(-y)\tau_y(g)dy$, we have

$$\|f * g\|_E \leq C_0\|f\|_1\|g\|_E \tag{0.1}$$

Definition 3. *Given a weight ω, a function $\phi \in \mathcal{S}_0$ and a bounded under traslations Banach space E we define*

$$\Lambda_\omega^E = \{f \in L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}) : \|\Delta_x f\|_E \leq C\omega(|x|), x \in \mathbb{R}^n\}$$

$$B_{w,\phi}^{E,\infty} = \{f \in L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}) : \|\phi_t * f\|_E \leq C\omega(t), t > 0\}$$

$$B_{\omega,\phi}^{E,1} = \{f \in L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}) : \phi_t * f \in L^1(\omega(t)\frac{dt}{t}, E)\}$$

where $\Delta_x f(y) = f(x+y) - f(y)$ and $\phi_t(x) = \frac{1}{t^n}\phi(\frac{x}{t})$.

They are complete spaces under the following seminorms

$$\|f\|_{\Lambda_\omega^E} = inf\{C > 0 : \|\Delta_x f\|_E \leq C\omega(|x|)\}$$

$$\|f\|_{B^{E,\infty}_{w,\phi}} = inf\{C > 0 : \|\phi_t * f\|_E \leq C\omega(t)\}$$

$$\|f\|_{B^{E,1}_{\omega,\phi}} = \int_0^\infty \|\phi_t * f\|_E \omega(t)\frac{dt}{t}$$

Remarks 3.

Note that the assumption $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}}dx < \infty$ gives sense to the convolution $\phi_t * f$.

Observe also that $\int_0^\infty \omega(t)\frac{dt}{t} < \infty$ implies that E embeds into $B^{E,1}_{\omega,\phi}$.

Note that $\int_{\mathbb{R}^n} \phi(x)dx = 0$ implies that constant functions have seminorm equal zero in $B^{E,\infty}_{w,\phi}$ and $B^{E,1}_{\omega,\phi}$.

In order to avoid more notation we use the same formulation for the Banach spaces consisting of equivalence classes coming from the kernel of the seminorms.

Let us now recall Calderón's formula and formulate the version to be used later on.

Let $\phi \in \mathcal{S}_0$ a real and radial function with $\int_0^\infty \hat{(}\phi(t\xi))^2 \frac{dt}{t} = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. If $\psi \in \mathcal{S}$ then for $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\hat{\psi}(\xi) = \int_0^\infty (\phi_t * \phi_t * \psi)\hat{(}\xi)\frac{dt}{t}$$

This shows that $\psi_{\varepsilon,\delta} = \int_\varepsilon^\delta \phi_t * \phi_t * \psi\frac{dt}{t}$ converges to ψ in $\mathcal{S}$.

Lemma A. *(see* [FJW]*) Let $\phi \in \mathcal{S}_0$ a real and radial function with $\int_0^\infty \hat{(}\phi(t\xi))^2 \frac{dt}{t} = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. Let f be a measurable function with $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}}dx < \infty$, $0 < \varepsilon < \delta$ and write*

$$f_{\varepsilon,\delta} = \int_\varepsilon^\delta \phi_t * \phi_t * f\frac{dt}{t}$$

then $\lim_{\varepsilon\to 0,\delta\to\infty} f_{\varepsilon,\delta} = f$ *in* $\mathcal{S}'_0$.

The theorems and their proofs

Let us mention some properties for test funtions that are used later on.

If $\psi \in \mathcal{S}$ then for any $\varepsilon > 0$, taking

$$C_1 = max\Big(max\{|\psi(y)| : |y| \leq 1\}, max\{|y|^{n+\varepsilon}|\psi(y)| : |y| \geq 1\}\Big)$$

one has

$$|\psi_t(x)| \leq C_1 min(\frac{1}{t^n}, \frac{t^\varepsilon}{|x|^{n+\varepsilon}}) \tag{1.1}$$

Since $\|\Delta_x(\psi_t)\|_1 = \|\Delta_{\frac{x}{t}}\psi\|_1$ then taking $C_2 = max\Big(2\|\psi\|_1, \int_{\mathbb{R}^n} \max_{|z-y|\leq 1} |\nabla\psi(z)|dy\Big)$, one has

$$\|\Delta_x(\psi_t)\|_1 \leq C_2 min(1, \frac{|x|}{t}) \tag{1.2}$$

Lemma 1. *Let $\phi \in \mathcal{S}_0$. Then*

$$\|\phi_t * \phi_s\|_1 \leq Cmin(\frac{s}{t}, \frac{t}{s}) \tag{1.3}$$

Proof. Let us assume $s \leq t$. From the mean zero assumption we write

$$\phi_t * \phi_s(y) = \int_{\mathbb{R}^n} \phi_s(-x)\Delta_x(\phi_t)(y)dx$$

From Fubini, (1.1) for $\varepsilon = 2$ and (1.2)

$$\begin{aligned}
\|\phi_t * \phi_s\|_1 &\leq \int_{\mathbb{R}^n} |\phi_s(-x)| \|\Delta_x(\phi_t)\|_1 dx \\
&\leq C \int_{\mathbb{R}^n} min(\frac{1}{s^n}, \frac{s^2}{|x|^{n+2}}) \\
&\leq C\Big(\int_{|x|<s} \frac{|x|}{ts^n} dx + \int_{s\leq|x|\leq t} \frac{s^2}{t|x|^{n+1}} dx + \int_{|x|>t} \frac{s^2}{|x|^{n+2}} dx\Big) \\
&\leq C\Big(\frac{1}{ts^n}\int_0^s u^n du + \frac{s^2}{t}\int_s^t \frac{du}{u^2} + s^2\int_t^\infty \frac{du}{u^3}\Big) \\
&\leq C\Big(\frac{s}{t} + \frac{s^2}{t}(\frac{1}{s} - \frac{1}{t}) + \frac{1}{2}(\frac{s}{t})^2\Big) \leq C\frac{s}{t}.
\end{aligned}$$

Theorem 1. *Let E be a Banach space bounded under traslations, ω a b_1-weight satisfying Dini condition and $\phi \in \mathcal{S}_0$ a real radial function with $\int_0^\infty (\hat{\phi}(t\xi))^2 \frac{dt}{t} = 1, (\xi \neq 0)$, then*

$$\Lambda_\omega^E = B_{w,\phi}^{E,\infty} \text{ (with equivalent norms)}$$

Proof. Let us take $f \in \Lambda_\omega^E$, since $\int_{\mathbb{R}^n} \phi(x)dx = 0$ then

$$\phi_t * f = \int_{\mathbb{R}^n} \phi_t(-x)\Delta_x f dx.$$

Using (1.1) for $\varepsilon = 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\phi_t(-x)| \|\Delta_x f\|_E dx &\leq C\big(\frac{1}{t^n}\int_{|x|<t} \omega(|x|)dx + t\int_{|x|>t} \omega(|x|)\frac{dx}{|x|^{n+1}}\big) \\
&\leq C\Big(\int_0^t (\frac{s}{t})^n \omega(s)\frac{ds}{s} + t\int_t^\infty \omega(s)\frac{ds}{s^2}\Big) \leq C\omega(t).
\end{aligned}$$

This shows that $F_t(x) = \phi_t(-x)\Delta_x f$ belongs to $L^1(\mathbb{R}^n, E)$ and moreover

$$\|\phi_t * f\|_E \leq C\omega(t).$$

Conversely, let us take $f \in B^{E,\infty}_{w,\phi}$. Given $0 < \varepsilon < \delta$ we have

$$\Delta_x(f_{\varepsilon,\delta}) = \int_\varepsilon^\delta (\Delta_x \phi_t) * \phi_t * f \frac{dt}{t}$$

For each $x \in \mathbb{R}^n$, let us denote F_x the E-valued function defined by

$$F_x(t) = (\Delta_x \phi_t) * \phi_t * f$$

Using (0.1) and (1.2) we have

$$\|F_x(t)\|_E \leq C \|\Delta_x \phi_t\|_1 \|\phi_t * f\|_E \leq C min(1, \frac{|x|}{t}) \omega(t).$$

Note that

$$\int_0^\infty min(1, \frac{|x|}{t}) \omega(t) \frac{dt}{t} \leq C \Big(\int_0^{|x|} \omega(t) \frac{dt}{t} + |x| \int_{|x|}^\infty \frac{\omega(t)}{t^2} dt \Big) \leq C \omega(|x|).$$

This allows to say that $F_x \in L^1((0,\infty), \frac{dt}{t}, E)$. This implies $\Delta_x(f_{\varepsilon,\delta})$ is a Cauchy net in E and hence convergent in E. On the other hand, from Lemma A, $f_{\varepsilon,\delta} \to f$ in $\mathcal{S}'_0$ what implies $\Delta_x(f_{\varepsilon,\delta}) \to \Delta_x(f)$ in $\mathcal{S}'$. Combining both facts we have $\Delta_x(f_{\varepsilon,\delta}) \to \Delta_x(f)$ in E.

Finally take limit as $\varepsilon \to 0$ and $\delta \to \infty$ in

$$\|\Delta_x(f_{\varepsilon,\delta})\|_E \leq C \int_0^\infty min(1, \frac{|x|}{t}) \omega(t) \frac{dt}{t} \leq C \omega(|x|)$$

to show that $f \in \Lambda^E_\omega$.

Remark 4. This produces a Littlewood-Paley formulation of the Lipschitz classes Λ_α and Λ^p_α for $0 < \alpha < 1$ (see [FJW]).

Lemma 2. *Let $\phi \in \mathcal{S}_0$ and ω a b_1-weight such that $\omega_1(t) = t\omega(t)$ satisfies Dini condition. If $t > 0$ and $f \in E$ then $\phi_t * f \in B^{E,1}_{\omega,\phi}$ and*

$$\|\phi_t * f\|_{B^{E,1}_{\omega,\phi}} \leq C \omega(t) \|f\|_E \tag{1.4}$$

Proof.

$$\begin{aligned}
\|\phi_t * f\|_{B^{E,1}_{\omega,\phi}} &= \int_0^\infty \|\phi_t * \phi_s * f\|_E \omega(s) \frac{ds}{s} \\
&\leq C \int_0^\infty \|\phi_t * \phi_s\|_1 \|f\|_E \omega(s) \frac{ds}{s} \\
&\leq C \int_0^\infty min(\frac{t}{s}, \frac{s}{t}) \omega(s) \frac{ds}{s} \\
&\leq C \|f\|_E \Big(\frac{1}{t} \int_0^t \omega_1(s) \frac{ds}{s} + t \int_t^\infty \frac{\omega(s)}{s^2} ds \Big) \leq C \|f\|_E \omega(t).
\end{aligned}$$

Theorem 2. *Let E, E^* be Banach spaces bounded under traslations such that $\mathcal{S}$ is dense in E. Let ω be a b_1-weight such that $\omega_1(t) = t\omega(t)$ satisfies Dini condition and $\omega \in L^1(\frac{dt}{t})$. Let $\phi \in \mathcal{S}_0$ be a real radial function with $\int_0^\infty \hat{\phi}^2(t\xi)\frac{dt}{t} = 1, (\xi \neq 0)$.*

Then the dual space of $B^{E,1}_{\omega,\phi}$ is isomorphic to $\Lambda^{E^*}_\omega$.

Proof. Let us take $f \in \Lambda^{E^*}_\omega$ and $g \in \mathcal{S}$.

From Calderón's formula

$$\int_{\mathbb{R}^n} f(x)g(x)dx = \int_{\mathbb{R}^n} (\int_0^\infty \phi_t * \phi_t * g(x)\frac{dt}{t})f(x)dx = \int_0^\infty \int_{\mathbb{R}^n} \phi_t * f(x)\phi_t * g(x)dx\frac{dt}{t}$$

This implies

$$\begin{aligned} |\int_{\mathbb{R}^n} f(x)g(x)dx| &\leq \int_0^\infty \|\phi_t * g\|_E \|\phi_t * f\|_{E^*}\frac{dt}{t} \\ &\leq \|f\|_{\Lambda^{E^*}_\omega} \int_0^\infty \|\phi_t * g\|_E \omega(t)\frac{dt}{t}. \end{aligned}$$

Denoting by $\Phi(g) = \int_{\mathbb{R}^n} f(x)g(x)dx$ we get

$$|\Phi(g)| \leq \|f\|_{\Lambda^{E^*}_\omega} \|g\|_{B^{E,1}_{\omega,\phi}}.$$

Hence Φ exends to an operator on $B^{E,1}_{\omega,\phi}$.

Conversely, let us take $\Phi \in \left(B^{E,1}_{\omega,\phi}\right)^*$. It is immediate that $|\Phi(g)| \leq C\|g\|_E$ for any $g \in \mathcal{S}$. Hence Φ can be extended to a functional in E^*.

Therefore there exists $f \in E^*$ so that

$$\Phi(g) = \int_{\mathbb{R}^n} f(x)g(x)dx \qquad (g \in B^{E,1}_{\omega,\phi})$$

Let us show that $f \in \Lambda^{E^*}_\omega$.

$$\begin{aligned} \|\phi_t * f\|_{E^*} &= sup\{|\int_{\mathbb{R}^n} (\phi_t * f)(x)g(x)dx| : \|g\|_E \leq 1\} \\ &= sup\{|\int_{\mathbb{R}^n} (\phi_t * g)(x)f(x)dx| : \|g\|_E \leq 1\} \\ &= sup\{|\Phi(\phi_t * g)| : \|g\|_E \leq 1\} \end{aligned}$$

Applying the boundedness of Φ and Lemma 2 one has

$$\|\phi_t * f\|_{E^*} \leq C\omega(t).$$

Remark 5. See [J] for a proof of a similar result for ϕ with $\int \phi \neq 0$. Duality results for $E = L^p$ where achieved in [F1, F2, T1, T2, T3]. The reader is referred to [BS] for a proof under same conditions on the weight but for spaces defined on the torus $\mathbb{T}$ and convolution with the derivative of the Poisson kernel.

Theorem 3. *Let ω be a b_1-weight such that $\omega_1(t) = t\omega(t)$ satisfies Dini condition and $\omega \in L^1(\frac{dt}{t})$. Let $\phi \in \mathcal{S}_0$ be a real radial function with $\int_0^\infty \hat{\phi}^2(t\xi)\frac{dt}{t} = 1, (\xi \neq 0)$. Given a Banach space $(X, \|.\|_X)$ and a linear map $T : \mathcal{S} \to X$, define $F : \mathbb{R}_+^{n+1} \to X$ by*

$$F(x,t) = T(\tau_x(\phi_t)).$$

The following are equivalent
(i) $T : B_{\omega,\phi}^{L^1,1} \to X$ is bounded.
(ii) $\sup_{x\in\mathbb{R}^n} \|F(x,t)\|_X \leq C\omega(t)$

Proof. Assume that T is bounded. From (1.3) it follows

$$\|\tau_x(\phi_t)\|_{B_{\omega,\phi}^{L^1,1}} \leq C\int_0^\infty \|\phi_t * \phi_s\|_1\omega(s)\frac{ds}{s} \leq C\omega(t).$$

Hence

$$\|F(x,t)\|_X \leq \|T\|\|\tau_x(\phi_t)\|_{B_{\omega,\phi}^{L^1,1}} \leq C\omega(t).$$

Conversely let us assume $\sup_{x\in\mathbb{R}^n} \|F(x,t)\|_X \leq C\omega(t)$ and take $g \in B_{\omega,\phi}^{L^1,1}$. Let $0 < \varepsilon < \varepsilon' < \delta' < \delta$

Since

$$\begin{aligned}
\|g_{\varepsilon,\delta} - g_{\varepsilon',\delta'}\|_{B_{\omega,\phi}^{L^1,1}} &= \int_0^\infty \| \int_\varepsilon^{\varepsilon'} \phi_s * \phi_t * \phi_t * g\frac{dt}{t} + \int_{\delta'}^{\delta} \phi_s * \phi_t * \phi_t * g\frac{dt}{t}\|_1\omega(s)\frac{ds}{s} \\
&\leq C\int_0^\infty \Big(\int_\varepsilon^{\varepsilon'} + \int_{\delta'}^{\delta}\Big)\|\phi_s * \phi_t * \phi_t * g\|_1\frac{dt}{t}\omega(s)\frac{ds}{s} \\
&\leq C\Big(\int_\varepsilon^{\varepsilon'} + \int_{\delta'}^{\delta}\Big)\|\phi_t * g\|_1\Big(\int_0^\infty \|\phi_s * \phi_t\|_1\omega(s)\frac{ds}{s}\Big)\frac{dt}{t} \\
&\leq C\Big(\int_\varepsilon^{\varepsilon'} + \int_{\delta'}^{\delta}\Big)\|\phi_t * g\|_1\Big(\int_0^\infty min(\frac{s}{t}, \frac{t}{s})\omega(s)\frac{ds}{s}\Big)\frac{dt}{t} \\
&\leq C\Big(\int_\varepsilon^{\varepsilon'} + \int_{\delta'}^{\delta}\Big)\|\phi_t * g\|_1\omega(t)\frac{dt}{t}.
\end{aligned}$$

This implies that

$$\lim_{\varepsilon\to 0,\delta\to\infty} \int_\varepsilon^\delta \int_{\mathbb{R}^n} \phi_t(x-y))\phi_t * g(y)dy\frac{dt}{t} = g \qquad (in \qquad B_{\omega,\phi}^{L^1,1})$$

Which show

$$\int_0^\infty \int_{\mathbb{R}^n} F(x,t)\phi_t * g(y)dy\frac{dt}{t} = T(g)$$

Hence

$$\|T(g)\|_X \leq \int_0^\infty \int_{\mathbb{R}^n} \|F(x,t)\|_X |\phi_t * g(y)| dy \frac{dt}{t}$$
$$\leq C \int_0^\infty \|\phi_t * g\|_1 \omega(t) \frac{dt}{t} \leq C\|g\|_{B^{E,1}_{\omega,\phi}}.$$

Remark 6. In [B1] a similar and more general result is established and there are several applications of it to Carleson measures, multipliers, composition operators. Applications of similar nature, with the analogue formulation in $\mathbb{R}^n$, can be obtained from Theorem 3.

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