On functions of integrable mean oscillation.

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Abstract

Given $f \in L^1(\mathbb{T})$ we denote by $w_{mo}(f)$ and $w_{ho}(f)$ the moduli of mean and harmonic oscillation given by

$$w_{mo}(f)(t) = \sup_{0<|I|\leq t} \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi}$$

where $I \subseteq \mathbb{T}$ is an interval, $|I|$ stands for the normalized length of I and $m_I(f) = \frac{1}{\mid I \mid} \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}$ and

$$w_{ho}(f)(t) = \sup_{1-t\leq|z|<1} \int_{\mathbb{T}} \mid f(e^{i\theta}) - P(f)(z) \mid P_z(e^{i\theta}) \frac{d\theta}{2\pi}$$

where and $P_z(e^{i\theta})$ and $P(f)$ stand for the Poisson kernel and the Poisson integral of f respectively. .

It is shown that for each $1 \leq p < \infty$ there exists $C_p > 0$ such that

$$\int_0^1 [w_{mo}(f)(t)]^p \frac{dt}{t} \leq \int_0^1 [w_{ho}(f)(t)]^p \frac{dt}{t} \leq C_p \int_0^1 [w_{mo}(f)(t)]^p \frac{dt}{t}.$$

1 Introduction.

Let us denote by Δ the open unit disc $\{z \in \mathbb{C} : \mid z \mid< 1\}$ and by $\mathbb{T}$ the unit circle. Throughout the paper $I \subseteq \mathbb{T}$ is an interval, $|I|$ stands for the normalized length of I and $m_I(f) = \frac{1}{\mid I \mid} \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}$. Given $z \in \Delta \setminus \{0\}$, we denote by I_z the open interval in $\mathbb{T}$ centered at $\frac{z}{\mid z \mid}$ and $\mid I_z \mid= 1- \mid z \mid$.

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and for an interval $I \subset \mathbb{T}$ and $\lambda \leq \mid I \mid^{-1}$ we write λI for the interval with the same center as I but length $\lambda \mid I \mid$.

We write $P(f)(z) = \int_{\mathbb{T}} f(e^{i\theta}) P_z(e^{i\theta}) \frac{d\theta}{2\pi}$ for $\mid z \mid < 1$ and $P_z(e^{i\theta}) = \Re(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}) = \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}$ stands for the Poisson kernel. We denote by $T_t f(e^{i\theta}) = f(e^{i(\theta - t)})$ the translation operator and by $P_r(f)(e^{i\theta}) = P(f)(re^{i\theta})$ for $0 < r < 1$.

A function f is said to have bounded mean oscillation, in short $f \in BMO$ if

$$\|f\|_* = \sup_{I \subseteq \mathbb{T}} \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} < \infty.$$

We write $\|f\|_{BMO} = \mid \hat{f}(0) \mid + \|f\|_*$.

If $f \in L^1(\mathbb{T})$ *and* $U = P[f]$ then we say that $f \in BMOH$ if

$$\|f\|_{**} = \sup_{z \in \Delta} \int_{\mathbb{T}} \mid f(e^{i\theta}) - P(f)(z) \mid P_z(e^{i\theta}) \frac{d\theta}{2\pi} < \infty.$$

We write $\|f\|_{BMOH} = \mid P(f)(0) \mid + \|f\|_{**}$.

It is not difficult to prove (see [6]) that $f \in BMO$ if and only if $f \in BMOH$ with equivalent norms.

Let $f \in L^1(\mathbb{T})$ and $0 < t \leq 1$. We define the modulus of mean oscillation of f at the point t as

$$w_{mo}(f)(t) = \sup_{0 < |I| \leq t} \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi}.$$

Similarly we define the modulus of harmonic oscillation of f at the point t as

$$w_{ho}(f)(t) = \sup_{1-t \leq |z| < 1} \int_{\mathbb{T}} \mid f(e^{i\theta}) - P(f)(z) \mid P_z(e^{i\theta}) \frac{d\theta}{2\pi}.$$

With this notation out of the way we have that $f \in BMO$ if and only if $w_{mo}(f)(1) < \infty$ or if and only if $w_{ho}(f)(1) < \infty$.

A function f is said to have vanishing mean oscillation, in short $f \in VMO$, if

$$\lim_{|I| \to 0} \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} = 0.$$

This is a closed subspace of BMO, which can be characterized in many ways (see [7],[16] or [20]).

Theorem 1.1 *Let $f \in BMO$. The following statements are equivalent :*
(i) $f \in VMO$.
(ii) $\lim_{t\to 0^+} \|T_t f - f\|_{BMO} = 0$.
(iii) $\lim_{r\to 1} \|P_r(f) - f\|_{BMO} = 0$.
(iv) f *belongs to the closure of* $C(\mathbb{T})$ *in* BMO.
(v) $\lim_{t\to 0^+} w_{mo}(f)(t) = 0$.
(vi) $\lim_{t\to 0^+} w_{ho}(f)(t) = 0$.

It is also well-known that, using John-Nirenberg' lemma (see [6]), if $1 \leq p < \infty$ $f \in BMO$ if and only if

$$\sup_{I\subseteq\mathbb{T}} \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid^p \frac{d\theta}{2\pi} \ < \ \infty$$

or, equivalently,

$$\sup_{|z|<1} \int_{\mathbb{T}} \mid f(e^{i\theta}) - P(f)(z) \mid^p \ P_z(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Spaces of functions where $w_{mo}(f)(t) = O(\rho(t))$ for a fixed function ρ with certain properties have been considered by different authors, usually denoted by $BMO(\rho)$ (see [9], [18], [19]).

Our aim will be to analyze spaces where we do not know the function ρ but we do know its behaviour at the origin in terms of certain integrability conditions. Namely for $1 \leq p < \infty$, we will denote by $MO^p(\mathbb{T})$ and $HO^p(\mathbb{T})$ the spaces of integrable functions such that $\int_0^1 [w_{mo}(f)(t)]^p \frac{dt}{t} \ < \ \infty$ and $\int_0^1 [w_{ho}(f)(t)]^p \frac{dt}{t} \ < \ \infty$ respectively.

Our main result establishes that $MO^p(\mathbb{T}) = HO^p(\mathbb{T})$ with equivalent norms.

The paper is divided into three sections. The first one contains the definitions and properties of both modulus. The second one is devoted to introduce $MO^p(\mathbb{T})$ and prove some of its properties. Finally we introduce $HO^p(\mathbb{T})$ and show that coincides with $MO^p(\mathbb{T})$.

2 Mean and harmonic oscillation.

Definition 2.1 *Let* $f \in L^1(\mathbb{T})$, $I \subset \mathbb{T}$ *an interval and* $t \in (0,1]$. *We define the modulus of mean oscillation of* f *at the point* t *as*

$$w_{mo}(f)(t) = \sup_{0<|I|\leq t} \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi}.$$

Remark 2.1 *If* $0 < t \leq s \leq 1$ *then*

$$w_{mo}(f)(t) \leq w_{mo}(f)(s) \leq \max\{w_{mo}(f)(t), \frac{2\|f\|_1}{t}\}. \qquad (1)$$

This follows from the following estimate

$$\sup_{t<|I|\leq s} \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} \leq \frac{2}{t}||f||_1.$$

In particular, $f \in BMO$ *if and only if* $w_{mo}(f)(t) < \infty$ *for some (or for all)* $t < 1$.

Let us now prove the following useful lemma.

Lemma 2.2 *Let* $f \in L^1(\mathbb{T})$. *If* $\{I_n\}$ *be a sequence of intervals such that* $\lim_{n\to\infty} I_n = I$ *for some interval* I *with* $|I| > 0$ *then*

$$\lim_{n\to\infty} \frac{1}{\mid I_n \mid} \int_{I_n} \mid f(e^{i\theta}) - m_{I_n}(f) \mid \frac{d\theta}{2\pi} = \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi}.$$

PROOF. Let us first estimate

$$\begin{aligned} & \frac{1}{\mid I_n \mid} \int_{I_n} \mid f(e^{i\theta}) - m_{I_n}(f) \mid \frac{d\theta}{2\pi} - \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} \\ \leq & \frac{1}{\mid I_n \mid} \int_{I_n} \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} | + |m_{I_n}(f) - m_I(f)| \\ - & \frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi}. \end{aligned}$$

Notice that $\nu(A) = \int_A f(e^{i\theta}) \frac{d\theta}{2\pi}$ and $\nu_I(A) = \int_A \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi}$ are a complex and a finite measure respectively. Hence $\lim_{n\to\infty} \nu(I_n) = \nu(I)$, $\lim_{n\to\infty} \nu_I(I_n) = \nu_I(I)$ and $\lim_{n\to\infty} \mid I_n \mid = \mid I \mid$. Therefore the result follows passing to the limit. □

Proposition 2.3 *If* $f \in BMO$ *then* $w_{mo}(f)$ *is increasing and continuous in* $(0, 1]$.

PROOF. Obviously the modulus is increasing.

Let $0 < t_0 \leq 1$ and let us prove that it is left continuous at t_0. Given $\epsilon > 0$ we find $I_{t_0} \subset \mathbb{T}$ such that $0 <\mid I_{t_0} \mid\leq t_0$ and

$$w_{mo}(f)(t_0) \leq \frac{1}{\mid I_{t_0} \mid} \int_{I_{t_0}} \mid f(e^{i\theta}) - m_{I_{t_0}}(f) \mid \frac{d\theta}{2\pi} + \frac{\epsilon}{2}.$$

Let (t_n) be a sequence such that $t_n \leq t_0$ for all $n \in \mathbb{N}$ and converges to t_0.

If $\mid I_{t_0} \mid= t_0$, we can find $I_n \subset I_{t_0}$ such that $\lim_{n\to\infty} I_n = I_{t_0}$. Hence

$$w_{mo}(f)(t_0) - w_{mo}(f)(t_n) \leq$$
$$\leq \frac{1}{\mid I_{t_0} \mid} \int_{I_{t_0}} \mid f(e^{i\theta}) - m_{I_{t_0}}(f) \mid \frac{d\theta}{2\pi} - \frac{1}{\mid I_n \mid} \int_{I_n} \mid f(e^{i\theta}) - m_{I_n}(f) \mid \frac{d\theta}{2\pi} + \frac{\epsilon}{2}.$$

Now use Lemma 2.2 to get $\lim_{n\to\infty} w_{mo}(f)(t_0) - w_{mo}(f)(t_n) = 0$.

If $\mid I_{t_0} \mid< t_0$ there exists n_0 such that $\mid I_{t_0} \mid\leq t_n$ for $n \geq n_0$. Hence $w_{mo}(f)(t_0) - w_{mo}(f)(t_n) < \frac{\epsilon}{2}$ for $n \geq n_0$.

To see that it is right continuous at t_0, we shall argue as follows: Let (t_n) be a sequence such that $t_n \geq t_0$ for all $n \in \mathbb{N}$ and converges to t_0. We shall find a subsequence (t_{n_k}) such that $\lim_{k\to\infty} w_{mo}(f)(t_{n_k}) = w_{mo}(f)(t_0)$.

Given $\epsilon > 0$ we find $I_n \subset \mathbb{T}$ such that $0 <\mid I_n \mid\leq t_n$ and

$$w_{mo}(f)(t_n) \leq \frac{1}{\mid I_n \mid} \int_{I_n} \mid f(e^{i\theta}) - m_{I_n}(f) \mid \frac{d\theta}{2\pi} + \epsilon.$$

Let $\mathcal{F} = \{n \in \mathbb{N} :\mid I_n \mid> t_0\}$. If $\mathcal{F}$ is finite then $\mid I_n \mid\leq t_0$ for $n \geq n_0$ and

$$w_{mo}(f)(t_n) - w_{mo}(f)(t_0) < \epsilon \text{ for } n \geq n_0.$$

Without lost of generality we assume $\mid I_n \mid> t_0$ for all $n \in \mathbb{N}$.

Call $I_0 = \liminf I_n$. It is easy to see that I_0 is an interval and $|I_0| = t_0$. Take a subsequence n_k such that (I_{n_k}) converges to I_0. We have

$$w_{mo}(f)(t_{n_k}) - w_{mo}(f)(t_0) \leq w_{mo}(f)(t_{n_k}) - \frac{1}{\mid I_0 \mid} \int_{I_0} \mid f(e^{i\theta}) - m_{I_0}(f) \mid \frac{d\theta}{2\pi} \leq$$

$$\leq \frac{1}{\mid I_{n_k} \mid} \int_{I_{n_k}} \mid f(e^{i\theta}) - m_{I_{n_k}}(f) \mid \frac{d\theta}{2\pi} - \frac{1}{\mid I_0 \mid} \int_{I_0} \mid f(e^{i\theta}) - m_{I_0}(f) \mid \frac{d\theta}{2\pi} + \epsilon.$$

Applying Lemma 2.2 the proof is completed. □

Remark 2.2 *Let* $f \in BMO$ *and take* $a(f) = \lim_{t\to 0^+} w_{mo}(f)(t)$*. Hence* $f \in VMO$ *if and only if* $a(f) = 0$*.*

Remark 2.3 *One can define other moduli as*

$$w'_{mo}(f)(t) = \sup_{|I|\le t}\left(\frac{1}{|I|^2}\int_I\int_I |f(e^{i\theta}) - f(e^{i\varphi})| \frac{d\theta}{2\pi}\frac{d\varphi}{2\pi}\right)$$

or

$$\widetilde{w}_{mo}(f)(t) = \sup_{|I|\le t}\left(\inf_c\left(\frac{1}{|I|}\int_I |f(e^{i\theta}) - c| \frac{d\theta}{2\pi}\right)\right)$$

Clearly one gets

$$w_{mo}(f)(t) \le w'_{mo}(f)(t) \le 2w_{mo}(f)(t) \tag{2}$$

and

$$\widetilde{w}_{mo}(f)(t) \le w_{mo}(f)(t) \le 2\widetilde{w}_{om}(f)(t). \tag{3}$$

Remark 2.4 *If* $w_\infty(f)(t) = \sup_{\theta,\varphi\in I, |I|\le t} |f(e^{i\theta}) - f(e^{i\varphi})|$ *then (2) shows that*

$$w_{mo}(f)(t) \le w_\infty(f)(t). \tag{4}$$

In particular if $f \in C(\mathbb{T})$ *then* $f \in VMO$*.*

Definition 2.4 *Let* $f \in L^1(\mathbb{T})$ *and* $0 < t \le 1$*, we define the harmonic oscillation of* f *at the point* t *as*

$$w_{ho}(f)(t) \;=\; \sup_{1-t\le|z|<1}\int_{\mathbb{T}} |f(e^{i\theta}) - P(f)(z)| \; P_z(\theta)\frac{d\theta}{2\pi}.$$

Of course we can also define

$$\widetilde{w}_{ho}(f)(t) = \sup_{1-t\le|z|<1}\inf_c\int_{\mathbb{T}} |f(e^{i\theta}) - c| \; P_z(\theta)\frac{d\theta}{2\pi}.$$

Easily one gets

$$\widetilde{w}_{ho}(f)(t) \le w_{ho}(f)(t) \le 2\widetilde{w}_{ho}(f)(t). \tag{5}$$

Let us collect several known facts to be used later on.

Lemma 2.5 *There exist constants* $0 < C, C_1, C_2, C_3 < \infty$ *such that*

(i) $1- \mid z \mid \leq \mid e^{i\theta} - z \mid \leq C\ (1- \mid z \mid), \quad e^{i\theta} \in I_z, \quad$ *and* $z \in \Delta$.

(ii) $C_1 \frac{1}{|I_z|} \leq P_z(\theta) \leq C_2 \frac{1}{|I_z|}\ , e^{i\theta} \in I_z$ *and* $z \in \Delta$.

(iii) $\frac{1}{4^k |I_z|} \leq P_z(\theta) \leq C_3 \frac{1}{4^k |I_z|}\ ,\ e^{i\theta} \in 2^k I_z \setminus 2^{k-1} I_z\ ,\ k \in \{1, 2, \cdots, N+1\}$ *where* $N = [\log_2 \frac{1}{|I_z|}]$ *and* $z \in \Delta$.

PROOF. All the statements follow from the following estimates

$$(1 - |z|) \leq |e^{i\theta} - z| \leq |e^{i\theta} - \frac{z}{|z|}| + (1 - |z|)$$

and

$$|e^{i\theta} - \frac{z}{|z|}| \leq |e^{i\theta} - z| + (1 - |z|).$$

$\square$

Proposition 2.6 *If* $f \in L^1(\mathbb{T})$ *and* $0 < t \leq 1$ *then* $w_{mo}(f)(t) \leq w_{ho}(f)(t)$.

PROOF. Let us take $I \subseteq \mathbb{T}$ interval such that $\mid I \mid \leq t$. Consider $z \in \Delta$ for which $I = I_z$. From $\mid I_z \mid = 1- \mid z \mid \leq t$ we have $1 - t \leq \mid z \mid < 1$.

Using (ii) in Lemma 2.5 we have

$$\begin{aligned}
\frac{1}{\mid I \mid} \int_I \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} &\leq \frac{1}{\mid I_z \mid} \int_{I_z} \mid f(e^{i\theta}) - P(f)(z) \mid \frac{d\theta}{2\pi} \\
&+ |m_I(f) - P(f)(z)| \\
&\leq \frac{2}{\mid I_z \mid} \int_{I_z} \mid f(e^{i\theta}) - P(f)(z) \mid \frac{d\theta}{2\pi} \\
&\leq C \left(\int_{-\pi}^{\pi} \mid f(e^{i\theta}) - P(f)(z) \mid P_z(\theta) \frac{d\theta}{2\pi} \right) \\
&\leq C\ w_{ho}(f)(t)
\end{aligned}$$

Now taking supremum over all intervals we get $w_{mo}(f)(t)\ \leq\ C\ w_{ho}(f)(t)$.$\square$

3 Integrable mean oscillation.

Definition 3.1 *Let* $1 \leq p < \infty$. *A function* $f \in L^1(\mathbb{T})$ *is said to have modulus of mean oscillation p-integrable, in short* $f \in MO^p(\mathbb{T})$, *if*

$$\int_0^1 [w_{mo}(f)(t)]^p \frac{dt}{t} < \infty.$$

It is elementary to see that defining

$$\|f\|_{MO^p} = \|f\|_{L^1(\mathbb{T})} + \Big(\int_0^1 [w_{mo}(f)(t)]^p \frac{dt}{t}\Big)^{1/p}$$

one gets a normed space.

Remark 3.1 *Since* $w_{mo}(f)$ *is increasing then*

$$(\log 2)\sum_{k=1}^{\infty} w_{mo}(f)^p(\frac{1}{2^k}) \leq \int_0^1 [w_{mo}(t)(t)]^p \frac{dt}{t} \leq (\log 2)\sum_{k=0}^{\infty} w_{mo}(f)^p(\frac{1}{2^k}).$$

Remark 3.2 *From Remark 3.1 we have that*

$$MO^p(\mathbb{T}) \subseteq MO^q(\mathbb{T}) \qquad (1 \leq p \leq q < \infty).$$

Remark 3.3 *If* $Lip_\alpha(\mathbb{T})$ *stand for functions* f *such that*

$$w_\infty(f)(t) = \sup_{\theta,\varphi \in I, |I| \leq t} |f(e^{i\theta}) - f(e^{i\varphi})| \leq Ct^\alpha$$

then Remark 2.4 implies that $Lip_\alpha(\mathbb{T}) \subset MO^1(\mathbb{T})$ *for any* $0 < \alpha$.

Lemma 3.2 *If* $f \in MO^p(\mathbb{T})$ *and* $0 < s < 1$ *then* $w_{mo}(f)(s) \leq \|f\|_{MO^p} \left(\log \frac{1}{s}\right)^{-\frac{1}{p}}$.

PROOF. Let $0 < s < 1$. Since $w_{mo}(f)(t)$ is increasing then

$$w_{mo}(f)^p(s) \log\frac{1}{s} \leq \int_s^1 w_{mo}(f)^p(t)\frac{dt}{t} \leq \|f\|_{MO^p}.$$

□

Corollary 3.3 *$MO^p(\mathbb{T}) \subset VMO$. Moreover $\|f\|_{BMO} \le C\,\|f\|_{MO^p}$.*

PROOF. Clearly Lemma 3.2 gives that $\lim_{s\to 0^+} w_{mo}(f)(s) = 0$ for $f \in MO^p(\mathbb{T})$.

Use (1) and Lemma 3.2 to get $s \in (0,1)$

$$w_{mo}(f)(1) \le \|f\|_{MO^p} \max\{\frac{2}{s}, (\log\frac{1}{s})^{-1/p}\}.$$

Take s_p the solution of the equation $s^p = 2^p \log(1/s)$ and $C = 2/s_p$. □

Theorem 3.4 *$(MO^p(\mathbb{T}), \|.\|_{MO^p})$ is a Banach space.*

PROOF. We only show the completeness. Let $\{f_n\}$ be a Cauchy sequence in $MO^p(\mathbb{T})$. In particular, there exists $f \in BMO$ such that $\{f_n\}$ is converging to f.

Let $|I| \le t$, $0 < t \le 1$. Using that $f_n \to f$ en $L^1(\mathbb{T})$ we get that $m_I(f_n) \to m_I(f)$ and that there exists (m_{n_k}) such that $f_{m_k} \to f$ a.e.

Now

$$\begin{aligned}
&\frac{1}{|I|}\int_I |f_n(e^{i\theta}) - f(e^{i\theta}) - m_I(f_n - f)| \frac{d\theta}{2\pi}\\
= &\frac{1}{|I|}\int_I |f_n(e^{i\theta}) - \lim_k f_{m_k}(e^{i\theta}) - \lim_k\ m_I(f_n - f_{m_k})| \frac{d\theta}{2\pi}\\
= &\frac{1}{|I|}\int_I \lim_k |f_n(e^{i\theta}) - f_{m_k}(e^{i\theta}) - m_I(f_n - f_{m_k})| \frac{d\theta}{2\pi}\\
\le &\liminf_k \frac{1}{|I|}\int_I |f_n(e^{i\theta}) - f_{m_k}(e^{i\theta}) - m_I(f_n - f_{m_k})| \frac{d\theta}{2\pi}\\
\le &\liminf_k w_{mo}(f_n - f_{m_k})(t).
\end{aligned}$$

Therefore

$$w_{mo}(f_n - f)(t) \le \liminf_k w_{mo}(f_n - f_{m_k})(t).$$

Hence

$$\begin{aligned}
\int_0^1 [w_{mo}(f_n - f)(t)\frac{dt}{t}]^p &\le \int_0^1 \liminf_k [w_{mo}(f_n - f_{m_k})(t)]^p \frac{dt}{t}\\
&\le \liminf_k \int_0^1 [w_{mo}(f_n - f_{m_k})(t)]^p \frac{dt}{t}
\end{aligned}$$

Finally, using that f_n is a Cauchy sequence we get $\lim_{n\to\infty} \|f_n - f\|_{MO^p} = 0$ and that $f \in MO^p$. $\square$

Let us show that the spaces share some properties of BMO.

Proposition 3.5 *Si* $f \in MO^p(\mathbb{T}) \Rightarrow | f | \in MO^p(\mathbb{T})$.

PROOF. Let $t \in (0,1)$ and $I \subset \mathbb{T}$ with $| I | \leq t$. Then

$$\begin{aligned}
\frac{1}{| I |}\int_I | \, | f(e^{i\theta}) | - m_I(| f |) \, | \frac{d\theta}{2\pi} &\leq \frac{1}{| I |}\int_I | \, | f(e^{i\theta}) | - | m_I(f) | \, | \frac{d\theta}{2\pi} \\
&+ \; |m_I(|f|) - |m_I(f)|| \\
&\leq \frac{2}{| I |}\int_I | \, | f(e^{i\theta}) | - | m_I(f) | \, | \frac{d\theta}{2\pi} \\
&\leq \frac{2}{| I |}\int_I | f(e^{i\theta}) - m_I(f) | \frac{d\theta}{2\pi}
\end{aligned}$$

This shows that $w_{mo}(| f |)(t) \leq 2 \; w_{mo}(f)(t)$ and the proof is completed. $\square$

Theorem 3.6 *Let* $f \in MO^p(\mathbb{T})$*. Then* $\lim_{s\to 0^+} \|T_s f - f\|_{MO^p} = 0$.

PROOF. Since $f \in VMO$ we know that $\lim_{s\to 0^+} \|T_s f - f\|_{BMO} = 0$.

Note that $w_{mo}(T_s f - f)(t) \leq \|T_s f - f\|_{BMO}$ for all $0 < t \leq 1$.

On the other hand

$$\begin{aligned}
w_{mo}(T_s f - f)(t) &= \sup_{|I|\leq t} \frac{1}{| I |}\int_I \left|(T_s f - f)(e^{i\theta}) - m_I(T_s f - f)\right| \frac{d\theta}{2\pi} \\
&\leq \sup_{|I|\leq t} \frac{1}{| I |}\int_I \left|T_s f(e^{i\theta}) - m_I(T_s f)\right| \frac{d\theta}{2\pi} \\
&+ \sup_{|I|\leq t} \frac{1}{| I |}\int_I \left|f(e^{i\theta}) - m_I(f)\right| \frac{d\theta}{2\pi} \\
&= 2 \; w_{mo}(f)(t)
\end{aligned}$$

The Lebesgue dominated convergence theorem gives $\lim\limits_{s\to 0^+} \|T_s f - f\|_{MO^p} = 0$.
$\square$

4 Integrable harmonic oscillation.

Definition 4.1 *Let $1 \leq p < \infty$. A function $f \in L^1(\mathbb{T})$ is said to have modulus of harmonic oscillation p-integrable, in short $f \in HO^p(\mathbb{T})$, if*

$$\int_0^1 [w_{ho}(f)(t)]^p \frac{dt}{t} < \infty.$$

As above, defining

$$\|f\|_{HO^p} \; = \; \|f\|_{L^1(\mathbb{T})} \; + \; \Big(\int_0^1 [w_{ho}(f)(t)]^p \frac{dt}{t} \Big)^{1/p}$$

one gets a normed space.

Theorem 4.2 *Let $1 \leq p < \infty$. Then $MO^p(\mathbb{T}) = HO^p(\mathbb{T})$ with equivalent norms.*

PROOF. $HO^p(\mathbb{T}) \subseteq MO^p(\mathbb{T})$ follows from Proposition 2.6.

We first recall the following elementary estimate: If I, J are intervals in $\mathbb{T}$ such that $I \subset J$ then

$$| \, m_J(f) - m_I(f) \, | \leq \; \frac{| \, J \, |}{| \, I \, |} \; w_{mo}(f)(| \, J \, |). \tag{6}$$

Assume now that $f \in MO^p(\mathbb{T})$. Let us show that $f \in HO^p(\mathbb{T})$ and $\|f\|_{HO^p} \; \leq \; C \; \|f\|_{MO^p}$.

Take $t \in (0,1]$ and $z \in \Delta$ with $1 - t \; \leq | \; z \; | < 1$. Consider now the interval $I = I_z$, which gives $| \; I_z \; | = 1 - | \; z \; | \leq t$. Let $N = [\log_2 \frac{1}{t}]$ and $I_k \;\; (k = 0, 1, \cdots, N+1)$ be defined by $I_k = 2^k I_z$.

Using (iii) Lemma 2.5 we have

$$
\begin{aligned}
& \int_{\mathbb{T}} \mid f(e^{i\theta}) - P(f)(z) \mid P_z(\theta) \frac{d\theta}{2\pi} \\
\leq & \int_{\mathbb{T}} \mid f(e^{i\theta}) - m_I(f) \mid P_z(\theta) \frac{d\theta}{2\pi} \\
+ & \mid \int_{\mathbb{T}} (f(e^{i\theta}) - m_I(f)) P_z((e^{i\theta}) \frac{d\theta}{2\pi} \mid \\
\leq & \ 2 \int_{\mathbb{T}} \mid f(e^{i\theta}) - m_I(f) \mid P_z(\theta) \frac{d\theta}{2\pi} \\
\leq & \ C \ (\int_{I} \mid f(e^{i\theta}) - m_I(f) \mid P_z(\theta) \frac{d\theta}{2\pi} \\
+ & \sum_{k=1}^{N+1} \int_{I_k \setminus I_{k-1}} \mid f(e^{i\theta}) - m_I(f) \mid P_z(\theta) \frac{d\theta}{2\pi} \ ) \\
\leq & \ C \ (\frac{1}{\mid I \mid} \int_{I} \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} \\
+ & \sum_{k=1}^{N+1} \frac{1}{4^k \mid I \mid} \int_{I_k \setminus I_{k-1}} \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} \ ) \\
\leq & \ C \left(w_{mo}(f)(t) \ + \ \sum_{k=1}^{N+1} \frac{1}{2^k \mid I_k \mid} \int_{I_k} \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} \right).
\end{aligned}
$$

On the other hand from (6)

$$
\begin{aligned}
&\frac{1}{\mid I_k \mid}\int_{I_k} \mid f(e^{i\theta}) - m_I(f) \mid \frac{d\theta}{2\pi} \leq \\
\leq\ & \frac{1}{\mid I_k \mid}\int_{I_k} \mid f(e^{i\theta}) + (\sum_{j=1}^{k} m_{I_j}(f) - m_{I_{j-1}}(f)) - m_{I_k}(f) \mid \frac{d\theta}{2\pi} \\
\leq\ & \frac{1}{\mid I_k \mid}\int_{I_k} \mid f(e^{i\theta}) - m_{I_k}(f) \mid \frac{d\theta}{2\pi} \ + \ \sum_{j=1}^{k} \mid m_{I_j}(f) - m_{I_{j-1}}(f) \mid \\
\leq\ & \frac{1}{\mid I_k \mid}\int_{I_k} \mid f(e^{i\theta}) - m_{I_k}(f) \mid \frac{d\theta}{2\pi} \ + \ \sum_{j=1}^{k} \frac{\mid I_j \mid}{\mid I_{j-1} \mid} w_{mo}(f)(\mid I_j \mid) \\
\leq\ & w_{mo}(f)(\mid I_k \mid) \ + \ \sum_{j=1}^{k} 2\ w_{mo}(f)(\mid I_j \mid) \\
\leq\ & (1+2k)w_{mo}(f)(\mid I_k \mid).
\end{aligned}
$$

Combining both estimates one gets

$$
\begin{aligned}
&\int_{\mathbb{T}} \mid f(e^{i\theta}) - P(f)(z) \mid P_z(\theta)\frac{d\theta}{2\pi} \\
\leq\ & C\Big(w_{mo}(f)(t) + \sum_{k=1}^{N+1} \frac{1+2k}{2^k}\ w_{mo}(f)(\mid I_k \mid)\Big).
\end{aligned}
$$

Taking supremum over $1-t \leq\mid z \mid< 1$ and using $\mid I_k \mid= 2^k \mid I \mid\leq 2^k t$ we obtain

$$
w_{ho}(f)(t) \ \leq\ C\ \Big(\ w_{mo}(f)(t) \ + \ \sum_{k=1}^{N_t+1} \frac{1+2k}{2^k} w_{mo}(f)(2^k t)\Big)
$$

where $N = N_t = [\log_2 \frac{1}{t}]$.

For $1 \leq p$ we apply Hölder's inequality to obtain

$$
w_{ho}(f)^p(t) \quad \leq \quad C\Big(\ [w_{mo}(t)(t)]^p \ + \ \sum_{k=1}^{N_t+1} \frac{(1+2k)^p}{2^k}[w_{mo}(f)(2^k t)]^p\Big).
$$

Now integrating, and taking into account that $1 \leq k \leq N_t+1 = [\log_2 \frac{1}{t}]+1$ is equivalent to $0 < t \leq 2^{-k}$, we get

$$\begin{aligned}
\int_0^1 [w_{ho}(f)(t)]^p \frac{dt}{t} &\leq C \int_0^1 [w_{mo}(t)(t)]^p \frac{dt}{t} + C \int_0^1 \sum_{k=1}^{N_t+1} \frac{(1+2k)^p}{2^k} [w_{mo}(f)(2^k t)]^p \frac{dt}{t} \\
&\leq C \, \|f\|^p_{MO^p} + C \sum_{k=1}^{\infty} \frac{(1+2k)^p}{2^k} \int_0^{2^{-k}} [w_{mo}(f)(2^k t)]^p \frac{dt}{t} \\
&\leq C \, \|f\|^p_{MO^p} + C \sum_{k=1}^{\infty} \frac{(1+2k)^p}{2^k} \int_0^{1} [w_{mo}(f)(t)]^p \frac{dt}{t} \\
&\leq C \, \|f\|^p_{MO^p}.
\end{aligned}$$

Putting together all the estimates we have the result. □

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