CARLESON'S COUNTEREXAMPLE AND A SCALE OF LORENTZ-BMO SPACES ON THE BITORUS

OSCAR BLASCO AND SANDRA POTT

Abstract. We introduce a scale of Lorentz-BMO spaces $BMO_{L^{p,q}}$ on the bidisk, and show that these spaces do not coincide for different values of p. Our main tool is a detailed analysis of Carleson's construction in [C].

1. Introduction and notation

L. Carleson showed in [C] in an ingenious geometric construction that for each $N \in \mathbb{N}$, there exists a finite collection Φ_N of dyadic rectangles in $[0,1]^2$ such that

1. the total area of all rectangles is 1, i. e. $\sum_{R\in\Phi_N} |R| = 1$
2. the rectangles "intersect heavily", $|\cup_{R\in\Phi_N} R| < C_1 \frac{1}{N}$
3. the rectangles are evenly distributed over the unit square in the sense that a localized version of (1) holds, i. e. for each dyadic rectangle R, we have $\sum_{R'\in\Phi_N, R'\subseteq R} |R'| \leq C_2|R|$.

Here, C_1 and C_2 are absolute constants independent from N.

This construction was originally devised to show that the naive generalization of the Carleson Embedding Theorem to two variables is not true. However, it contains much more information. R. Fefferman used the construction to show that the dual of the Hardy space $H^1(\mathbb{T}^2)$ does not coincide with the so-called rectangular BMO space in two variables [Fef]. C. Sadosky and the second author used the Carleson construction to give a new proof of the fact that the Carleson Embedding theorem also does not extend to operator-valued measures (first proved in [NTV]), and to show that Bonsall's Theorem does not hold for little Hankel operators on the bidisk [PS].

In this note, we introduce a scale of BMO-Lorentz spaces on the bitorus and distinguish the spaces in this scale by a detailed analysis of the Carleson counterexample.

This provides also a proof of the fact that for $1 \leq p_1 < p_2 < \infty$, the spaces $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect},\mathrm{p_1}}$ and $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect},\mathrm{p_2}}$ are different (for an alternative proof, see [BP]).

Throughout the paper $\mathcal{D}$ denotes the set of dyadic intervals in the unit circle $\mathbb{T}$. We write $\mathcal{R} = \mathcal{D}\times\mathcal{D}$ for the dyadic rectangles in the bitorus $\mathbb{T}^2$, $|I|$ for the length of I and $|R|$ for the area of R. $(h_I)_{I\in\mathcal{D}}$ stands for the Haar basis in $L^2(\mathbb{T})$ and $(h_R)_{R\in\mathcal{R}}$ for the product Haar basis of $L^2(\mathbb{T}^2)$.

Here $h_I(t) = \frac{1}{|R|^{1/2}}(\chi_{I^+}(t) - \chi_{I^-}(t))$ for each dyadic interval $I \in \mathcal{D}$, where I^- denotes the left half of I, and I^+ denotes the right half of I. For each dyadic rectangle $R = I\times J \in \mathcal{R}$, h_R is defined by $h_R(s,t) = h_I(s)h_J(t)$.

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For any $f \in L^2(\mathbb{T}^2)$, we use the notation $f_R = \langle f, h_R \rangle$ for the Haar coefficients of f, and $m_I f(s) = \frac{1}{|I|}\int_I f(t,s)dt$, $m_J f(s) = \frac{1}{|J|}\int_J f(t,s)ds$ and $m_R f(t) = \frac{1}{|R|}\int_R f(t,s)dtds$ for the averages in the first, second and both variables respectively. We will use "$\approx$" to denote equivalence of expressions.

Given a complex-valued measurable function $f \in L^2(\mathbb{T}^2)$, let μ_f denote the distribution function of f. Here, $\mu_f(\lambda) = |E_\lambda|$ for $\lambda > 0$, where $E_\lambda = \{w \in \mathbb{T}^2 : |f(w)| > \lambda\}$.

Furthermore, let $f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\}$ be the nonincreasing rearrangement of f, and let $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

Now, given a measurable set $\Omega \subseteq \mathbb{T}^2$ and $0 < p, q \le \infty$, the Lorentz space $L^{p,q}_\Omega = L^{p,q}(\Omega, \mu_\Omega)$, where $\mu_\Omega(A) = \frac{|A|}{|\Omega|}$ is the normalized Lebesgue measure, consists of those measurable functions f supported in Ω such that $\|f\|^*_{L^{p,q}_\Omega} < \infty$, where

$$\|f\|^*_{L^{p,q}_\Omega} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^1 t^{\frac{q}{p}} f^*(t)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < p < \infty,\ 0 < q < \infty, \\ \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & 0 < p \le \infty,\ q = \infty. \end{cases} \tag{1}$$

We write $L^{p,q}$ for the Lorentz space over $L^{p,q}_{\mathbb{T}^2}$.

The reader should be aware that $\|f\|^*_{L^{p,q}_\Omega}$ is in general not a norm on $L^{p,q}_\Omega$. Nevertheless, replacing f^* by f^{**} in (1) and writing $\|f\|_{L^{p,q}_\Omega} = \|f^{**}\|^*_{L^{p,q}_\Omega}$, one gets a norm on $L^{p,q}_\Omega$ for $1 < p \le \infty$, $1 \le q \le \infty$, which is equivalent to $\|f\|^*_{L^{p,q}_\Omega}$ (see e. g. [SW]). The space $L^{p,p}_\Omega$, for which we will write L^p_Ω, is then the ordinary L^p space $L^p(\Omega, \mu_\Omega)$.

We write $S[f]$ for the dyadic square function of an integrable function f, $S[f] = (\sum_{R\in\mathcal{R}} \frac{\chi_R}{|R|}|f_R|^2)^{1/2}$. It is well-known that $\|S[f]\|_p \approx \|f\|_p$ for $1 < p < \infty$. Using interpolation, one has also $\|S[f]\|_{L^{p,q}} \approx \|f\|_{L^{p,q}}$ for $1 < p, q < \infty$.

For each measurable set $\Omega \subseteq \mathbb{T}^2$, let P_Ω be the orthogonal projection on the subspace spanned by the Haar functions $h_{R'}$, $R' \in \mathcal{R}$, $R' \subseteq \Omega$. In particular, for each dyadic rectangle $R \in \mathcal{R}$ and for $f = \sum_{R'\in\mathcal{R}} h_{R'} f_{R'} \in L^2(\mathbb{T}^2)$, one has

$$P_R f = \sum_{R'\in\mathcal{R}, R'\subseteq R} h_{R'} f_{R'}.$$

It is easy to see that for $R = I \times J \in \mathcal{R}$,

$$P_R f = (f - m_I f - m_J f + m_{I\times J} f)\chi_{I\times J}. \tag{2}$$

For $1 \le p < \infty$, a function $\varphi \in L^2(\mathbb{T}^2)$ is said to belong to $BMO^d_{rect,p}$, if

$$\|\varphi\|_{rect,p} = \sup_{R\in\mathcal{R}} \|P_R \varphi\|_{L^p_R} < \infty.$$

In contrast to the one-dimensional situation, functions in these spaces are not necessarily in the so-called product BMO space $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$, the dual of the dyadic Hardy space $H^1_d(\mathbb{T}^2)$, $H^1_d = \{f \in L^1(\mathbb{T}^2) : S[f] \in L^1(\mathbb{T}^2)\}$.

For $p = 2$, a continuous version of this fact was shown in [Fef]. In [BP], this was shown for all $1 \le p < \infty$. (For an overview of the theory of BMO spaces in two variables and characterizations of the duals of $H^1_d(\mathbb{T}^2)$ and $H^1(\mathbb{T}^2)$ in terms of the projections P_Ω, see [Be], [Ch], [ChFef1], [ChFef2].)

It was also shown in [BP] that the spaces BMO_{rect,p_1} and BMO_{rect,p_2}, again in contrast to the one-dimensional situation, are different for different values of p_1, p_2 (for $p_1 = 2$ and $p_2 = 4$, this is contained in [Fef]).

Here we improve the results given in [BP] by considering the BMO spaces defined by Lorentz space norms. We also give a new proof for the inequality of the $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect,p}}$ spaces.

Now we introduce our scale of Lorentz-BMO spaces. Let $BMO_{L^{p,q}}$ be the space of all $\varphi \in L^2(\mathbb{T}^2)$ such that

$$\|\varphi\|_{BMO_{L^{p,q}}} = \sup_{R\in\mathcal{R}} \|P_R\varphi\|_{L^{p,q}_R} < \infty.$$

Certainly $BMO_{L^{p,q}} \subseteq L^{p,q}(\mathbb{T}^2)$, since $m_I(f) = m_I(P_{I\times\mathbb{T}}f)$ and $m_J(f) = m_I(P_{\mathbb{T}\times J}f)$.

Note that for $f \geq 0$ and $supp(f) \subseteq \Omega_1 \subseteq \Omega_2$,

$$\|f\|_{L^{p,q}_{\Omega_1}} = \left(\frac{|\Omega_2|}{|\Omega_1|}\right)^{1/p} \|f\|_{L^{p,q}_{\Omega_2}}. \tag{3}$$

It is well-known that for $1 \leq p_1 < p_2 \leq \infty$ and $1 \leq q_1, q_2 \leq \infty$, we have $L^{p_2,q_2} \subseteq L^{p_1,q_1}$, and the embedding is continuous.

Therefore for $f \geq 0$, $supp(f) \subseteq \Omega$ and $1 \leq p_1 < p_2 \leq \infty$, $1 \leq q_1, q_2 \leq \infty$,

$$\|f\|_{L^{p_1,q_1}} \leq C_{p_1,p_2,q_1,q_2}|\Omega|^{1/p_1-1/p_2}\|f\|_{L^{p_2,q_2}}. \tag{4}$$

Hence, we have for all $R \in \mathcal{R}$ and $1 \leq p_1 < p_2 \leq \infty$, $1 \leq q_1, q_2 \leq \infty$ that

$$\|P_Rf\|_{L^{p_1,q_1}_R} \leq \|P_Rf\|_{L^{p_2,q_2}_R}.$$

This shows that $BMO_{L^{p_2,q_2}} \subseteq BMO_{L^{p_1,q_1}}$.

The aim of this note is to show that if $2 \leq p_1 < p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$, then $BMO_{L^{p_2,q_2}} \neq BMO_{L^{p_1,q_1}}$ (see [Fef] and [BP] for the cases $p_1 = q_1 = 2$ and $p_2 = q_2 = 4$ and $p_1 = q_1 < p_2 = q_2$ respectively). Namely we prove the following:

Theorem 1.1. *Let $1 < p_1 < p_2 < \infty$, and $1 \leq q_1, q_2 \leq \infty$.*
Then $BMO_{L^{p_1,q_1}} \nsubseteq L^{p_2,q_2}(\mathbb{T}^2)$.
In particular, $BMO_{L^{p_1,q_1}} \neq BMO_{L^{p_2,q_2}}$.

As in the proof in [Fef], our proof here will be based on Carleson's counterexample in [C] and provide concrete counterexamples. We shall show that the functions arising from Carleson's counterexample have $BMO_{L^{p,q}}$-norm equivalent to $L^{p,q}$-norm which easily leads to the desired result.

We first need to look at Carleson's construction in some detail. In [C], for each $N \in \mathbb{N}$, a collection of dyadic rectangles $\Phi_N \subset \mathcal{R}$ is constructed such that

1. $\sum_{R\in\Phi_N} |R| = 1$
2. $|\cup_{R\in\Phi_N} R| < C_1\frac{1}{N}$
3. For each $R \in \mathcal{R}$, $\sum_{R'\in\Phi_N, R'\subseteq R} |R'| \leq C_2|R|$.

Here, C_1 and C_2 are absolute constants independent from N. Let, as in [C], $\phi_N = \sum_{R\in\Phi_N} h_R|R|^{1/2}$. Here comes our key result, which makes use of a "localization property" of ϕ_N.

Theorem 1.2. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Then there exists a constant A_p such that*

$$\|\phi_N\|_{L^{p,q}} \leq \|\phi_N\|_{BMO_{L^{p,q}}} \leq A_p \max_{1\leq k\leq N-1} \|\phi_k\|_{L^{p,q}}$$

for all $N \in \mathbb{N}$.

Corollary 1.3. *Let $1 < p < \infty$ and $1 \le q \le \infty$. There exists a constant A'_p such that*

$$\|\phi_N\|_{L^{p,q}} \le \|\phi_N\|_{BMO_{L^{p,q}}} \le A'_p \|\phi_N\|_{L^{p,q}}$$

for all $N \in \mathbb{N}$.

Proof. of Theorem 1.1. Let $\Omega_N = \cup_{R\in\Phi_N} R$. Since $p_1 < p_2$ and ϕ_N is supported on Ω_N, one has

$$\|\phi_N\|_{L^{p_1,q_1}} \le C_{p_1,p_2,q_1,q_2} |\Omega_N|^{1/p_1 - 1/p_2} \|\phi_N\|_{L^{p_2,q_2}}.$$

by (4). Therefore, Corollary 1.3 yields

$$\begin{aligned} \|\phi_N\|_{L^{p_2,q_2}} &\ge C^{-1}_{p_1,p_2,q_1,q_2} \|\phi_N\|_{L^{p_1,q_1}} |\Omega_N|^{1/p_2 - 1/p_1} \\ &\ge {A'_p}^{-1} C_{p_1,p_2,q_1,q_2} C_1^{1/p_2-1/p_1} N^{1/p_1-1/p_2} \|\phi_N\|_{BMO_{L^{p_1,p_2}}}. \end{aligned}$$

Since $\|\phi_N\|_{L^{p_2,q_2}} \le \|\phi_N\|_{BMO_{L^{p_2,q_2}}}$, it also follows from this inequality that $BMO_{L^{p_1,q_1}} \supsetneq BMO_{L^{p_2,q_2}}$. □

2. The Carleson construction

Before we can turn to the proofs of Theorem 1.2 and Corollary 1.3, we need some more details of the construction of Φ_N in [C]. (For a nice description of the Carleson Counterexample, see also [T].) Φ_N is obtained by the following process. We first identify $\mathbb{T}^2$ with the unit square $[0,1]^2$. Take a sufficiently fast decreasing $N+1$-tuple $(A_N, \ldots, A_0)$ (for our purposes, we want to assume that this is the tuple $2^{(2^N)}, 2^{(2^{N-1})}, \ldots, 2^{(2^0)}$). Now cut the unit square into A_N vertical rectangles with sides parallel to the axis, of sidelength $A_N^{-1} \times 1$. Discard every second of these rectangles, and denote the collection of the remaining rectangles by $\Phi^{(1)}_{N,y}$. Then cut the unit square into A_N horizontal rectangles with sides parallel to the axis, of sidelength $1 \times A_N^{-1}$. Discard every second of these rectangles, and denote the remaining collection by $\Phi^{(1)}_{N,x}$. The collection of the thus kept horizontal and vertical rectangles, $\Phi^{(1)}_{N,x} \cup \Phi^{(1)}_{N,y}$, is denoted by $\Phi^{(1)}_N$.

Now we repeat the process and slice each rectangle in $\Phi^{(1)}_N$ vertically and horizontally into A_{N-1} rectangles with sides parallel to the boundary and again discard every second of them to obtain the collection $\Phi^{(2)}_N$. This process is iterated, until we get $\Phi_N := \Phi^{(N+1)}_N$. Since the tuple $(A_N, \ldots, A_0)$ decreases very fast, each rectangle in Φ_N has a unique "history" in the sense that it is generated from the unit square by a unique sequence of vertical and horizontal slicings. In paricular, writing $\Phi_{N,x}$ for the collection of those $R \in \Phi_N$ which are generated from a rectangle in $\Phi^{(1)}_{N,x}$, and $\Phi_{N,y}$ for the collection of those $R \in \Phi_N$ which are generated from a rectangle in $\Phi^{(1)}_{N,y}$, we find that $\Phi_{N,x} \cap \Phi_{N,y} = \varnothing$ and of course $\Phi_{N,x} \cup \Phi_{N,y} = \Phi_N$.

Moreover, for $R \in \Phi_N$, we have that $R \in \Phi_{N,x}$ if and only if there exists $R' \in \Phi^{(1)}_{N,x}$ with $R \subseteq R'$. One direction of this equivalence is clear, since each $R \in \Phi_{N,x}$ is generated from some $R' \in \Phi^{(1)}_{N,x}$ and therefore contained in this R'. Conversely, if $R \in \Phi_{N,y}$, then its width in y-direction is greater or equal than $A^{-1}_{N-1} \cdots A^{-1}_0 > A^{-1}_N$. Therefore, R cannot be contained in any $R' \in \Phi^{(1)}_{N,x}$. A corresponding statement holds for $\Phi_{N,y}$.

Another property of the construction we shall frequently use is that for each $R' \in \Phi_N^{(1)}$, the collection $\{R \in \Phi_N : R \subseteq R'\}$ is up to translation and dilation equal to the collection Φ_{N-1}.

For each $R \in \Phi_N^{(1)}$, we write $\tau_R^{(N)}$ for the composition of the translation and dilation which transform R into the unit square, $\tau_R^{(N)}(R) = [0,1] \times [0,1]$. Given a dyadic rectangle $Q = I \times J \subseteq R \in \Phi_N^{(1)}$, we have that $\tau_R^{(N)}(Q)$ is a dyadic rectangle, and $|Q| = |R||\tau_R^{(N)}(Q)|$. With this notation, our statement above means that for each $R \in \Phi_N^{(1)}$,

$$\{\tau_R^{(N)}(Q) : Q \in \Phi_N, Q \subseteq R\} = \Phi_{N-1}, \tag{5}$$

and consequently

$$\phi_{N-1} \circ \tau_R^{(N)} = P_R \phi_N |_R. \tag{6}$$

3. Proof of 1.2 and 1.3

Lemma 3.1. *For $Q = I \times J \subseteq R \in \Phi_N^{(1)}$, we write $Q' = \tau_R^{(N)}(Q)$. Then*

$$\|S[P_Q \phi_N]\|_{L_Q^{p,q}} = \|S[P_{Q'} \phi_{N-1}]\|_{L_{Q'}^{p,q}}.$$

In particular

$$\|S[P_R \phi_N]\|_{L_R^{p,q}} = \|S[\phi_{N-1}]\|_{L^{p,q}}$$

for any $R \in \Phi_N^{(1)}$.

Proof. Observe that

$$|\{x \in Q : S[P_Q \phi_N](x) > \lambda\}| = |R||\{x \in Q' : S[P_{Q'} \phi_{N-1}](x) > \lambda\}|.$$

Therefore

$$\mu_Q(\{x \in Q : S[P_Q \phi_N](x) > \lambda\}) = \mu_{Q'}(\{x \in Q' : S[P_{Q'} \phi_{N-1}](x) > \lambda\}). \tag{7}$$

This gives the result. □

Lemma 3.2. *If $Q \in [0,1] \times \mathcal{D}$, say $Q = [0,1] \times J$, and $|J| > A_N^{-1}$, then*

$$\|S[P_Q \phi_{N,x}]\|_{L_Q^{p,q}} = 2^{-1/p} \|S[\phi_{N-1}]\|_{L^{p,q}} \tag{8}$$

and

$$\|S[P_Q \phi_{N,y}]\|_{L_Q^{p,q}} = 2^{-1/p} \|S[P_Q \phi_{N-1}]\|_{L_Q^{p,q}}. \tag{9}$$

In particular

$$\|S[\phi_{N,y}]\|_{L^{p,q}} = \|S[\phi_{N,x}]\|_{L^{p,q}} = 2^{-1/p} \|S[\phi_{N-1}]\|_{L^{p,q}}. \tag{10}$$

A corresponding statement holds for $Q \in \mathcal{D} \times [0,1]$, $Q = I \times [0,1]$ with $|I| > A_N^{-1}$.

Proof. We write $\phi_{N,x} = \sum_{R \in \Phi_{N,x}} h_R |R|^{1/2}$ and $\phi_{N,y} = \sum_{R \in \Phi_{N,y}} h_R |R|^{1/2}$. Note that $S[\phi_{N,x}] = (\sum_{R \in \Phi_{N,x}} \chi_R)^{1/2}$ and $S[\phi_{N,y}] = (\sum_{R \in \Phi_{N,y}} \chi_R)^{1/2}$ are supported on the union of the disjoint collection of rectangles $\Phi_{N,x}^{(1)}$ and $\Phi_{N,y}^{(1)}$, respectively.

In the first situation, we have

$$S^2[P_Q \phi_{N,x}] = \sum_{R \in \Phi_{N,x}^{(1)}, R \subseteq Q} S^2[P_R \phi_N],$$

where the $S[P_R \phi_N]$ are equimeasurable for different R and disjointly supported.

Then we have, for any $R \subseteq Q$, $R \in \Phi^{(1)}_{N,x}$,

$$|\{x \in Q : S^2[P_Q\phi_{N,x}](x) > \lambda\}| = \frac{|Q|}{2|R|}|\{x \in R : S^2[P_R\phi_N](x) > \lambda\}|.$$

Therefore by (7), for any $\lambda > 0$

$$\mu_Q(\{S[P_Q\phi_{N,x}] > \lambda\}) = \frac{1}{2}|\{S[\phi_{N-1}] > \lambda\}|.$$

This gives $S[P_Q\phi_{N,x}]^{**}(t) = S[\phi_{N-1}]^{**}(2t)$. Therefore we get (8).

For the second situation we have

$$S^2[P_Q\phi_{N,y}] = \sum_{R \in \Phi^{(1)}_{N,y}} S^2[P_{R\cap Q}\phi_N],$$

where $S^2[P_{R\cap Q}\phi_N]$ are equimeasurable for different R and disjointly supported.

Then we have, for any $R \in \Phi^{(1)}_{N,y}$,

$$|\{x \in Q : S^2[P_Q\phi_{N,y}](x) > \lambda\}| = \sum_{R \in \Phi^{(1)}_{N,y}} |\{x \in R \cap Q : S^2[P_{R\cap Q}\phi_N](x) > \lambda\}|.$$

Therefore, using (7) and observing that $\tau_R^{(N)}(Q \cap R) = Q$ for any $R \in \Phi^{(1)}_{N,y}$ we get

$$|\{S[P_Q\phi_{N,y}] > \lambda\}| = \left(\sum_{R \in \Phi^{(1)}_{N,y}} |R|\right) |\{S[P_Q\phi_{N-1}] > \lambda\}| = \frac{1}{2}|\{S[P_Q\phi_{N-1}] > \lambda\}|.$$

This gives (9). □

Now it is easy to prove Corollary 1.3.

Proof of 1.3. Given Theorem 1.2, it suffices to prove that there exists a constant D_p such that $\|S[\phi_{N-1}]\|_{L^{p,q}} \leq D_p\|S[\phi_N]\|_{L^{p,q}}$ for all $N \in \mathbb{N}$.

Since $S^2[\phi_N] = S^2[\phi_{N,x}] + S^2[\phi_{N,y}]$, we have

$$\begin{aligned} \|S[\phi_N]\|_{L^{p,q}} &= \|(S^2[\phi_{N,x}] + S^2[\phi_{N,y}])^{1/2}\|_{L^{p,q}} \\ &\geq \|S[\phi_{N,x}]\|_{L^{p,q}} \\ &= 2^{-1/p}\|S[\phi_{N-1}]\|_{L^{p,q}}. \end{aligned}$$

by Lemma 3.2.

□

Before we can prove the main technical result Theorem 1.2, we need to collect some more facts.

Lemma 3.3. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. There exists $C_p \geq (2^{1/p} - 1)^{-1}$ such that*

$$\|S[P_Q\phi_N]\|_{L^{p,q}_Q} \leq C_p \max_{1\leq k\leq N-1}\{\|S[\phi_k]\|_{L^{p,q}}\} \tag{11}$$

for any $Q \in [0,1] \times \mathcal{D}$.

A corresponding statement holds for $Q \in \mathcal{D} \times [0,1]$.

Proof. We shall prove this statement by induction. It is obvious for $N = 1$. Assume it holds true for $N - 1$.

We first consider the case $Q = [0,1] \times J$, where $|J| > A_N^{-1}$. Using the inequality

$$S[P_Q\phi_N] = \big(S^2[P_Q\phi_{N,x}] + S^2[P_Q\phi_{N,y}]\big)^{1/2} \le S[P_Q\phi_{N,x}] + S[P_Q\phi_{N,y}]$$

and Lemma 3.2, we obtain

$$\begin{aligned}
\|S[P_Q\phi_N]\|_{L_Q^{p,q}} &\le \|S[P_Q\phi_{N,x}] + S[P_Q\phi_{N,y}]\|_{L_Q^{p,q}} \\
&\le \|S[P_Q\phi_{N,x}]\|_{L_Q^{p,q}} + \|S[P_Q\phi_{N,y}]\|_{L_Q^{p,q}} \\
&= 2^{-1/p}(\|S[\phi_{N-1}]\|_{L^{p,q}} + \|S[P_Q\phi_{N-1}]\|_{L_Q^{p,q}})) \\
&\le 2^{-1/p}(C_p \max_{1\le k\le N-2}\{\|S[\phi_k]\|_{L^{p,q}}\} + \|S[\phi_{N-1}]\|_{L^{p,q}}) \\
&\le 2^{-1/p}(C_p+1) \max_{1\le k\le N-1}\{\|S[\phi_k]\|_{L^{p,q}}\}) \\
&\le C_p \max_{1\le k\le N-1}\{\|S[\phi_k]\|_{L^{p,q}}\}.
\end{aligned} \tag{12}$$

In case $Q = [0,1] \times J$ and $A_{N-1}^{-1} < |J| \le A_N^{-1}$, we either have $P_Q\phi_N = 0$, or $Q \subseteq R$ for some $R \in \Phi_{N,x}^{(1)}$ and $Q' = \tau_R^{(N)}(Q) = [0,1] \times J'$ with $|J'| > A_{N-1}^{-1}$. Now Lemma 3.1 and the previous case give

$$\|S[P_Q\phi_N]\|_{L_Q^{p,q}} = \|S[P_{Q'}\phi_{N-1}]\|_{L_{Q'}^{p,q}} \le C_p \max_{1\le k\le N-2}\{\|S[\phi_k]\|_{L^{p,q}}\}.$$

Similarly, we get the result for any $Q = [0,1] \times J$ with $|J| \le A_N^{-1}$ and $S[P_Q\phi_N] \ne 0$. □

Now we can proceed to prove our main technical result.
Proof of Theorem 1.2.
We will show that

$$\|S[P_Q\phi_N]\|_{L_Q^{p,q}} \le B_p \max_{1\le k\le N-1} \|S[\phi_k]\|_{L^{p,q}}. \tag{13}$$

for all $N \in \mathbb{N}$ and all $Q \in \mathcal{R}$. Now from Littlewood-Paley theory (albeit with a different value of the constant B_p) the result will follow.

We shall use induction again. The case $N = 1$ is trivial. Assume the statement holds true for $N - 1$.

First consider the case that Q is contained in some rectangle R in $\Phi_N^{(1)}$. Using Lemma 3.1, we obtain

$$\|S[P_Q\phi_N]\|_{L_Q^{p,q}} \le B_p \max_{1\le k\le N-2} \|S[\phi_k]\|_{L^{p,q}}.$$

Consider now the case that Q is not contained in any $R \in \Phi_N^{(1)}$.

Note that if $Q = I \times J$, with $|J| \le A_N^{-1}$, then $P_Q\phi_N = P_Q\phi_{N,x}$. This is due to the fact that $|J'| > A_N^{-1}$ for any $R' = I' \times J' \in \Phi_{N,y}^{(1)}$. So either $Q \subseteq R$ for some $R \in \Phi_{N,x}^{(1)}$, or $P_Q\phi_N = 0$. Similarly, one can deal with the case $Q = I \times J$, where $|I| \le A_N^{-1}$.

Hence it remains to consider the case that $Q = I \times J$, where $|I|, |J| > A_N^{-1}$. Let us write

$$S^2[P_Q\phi_N] = S^2[P_Q\phi_{N,x}] + S^2[P_Q\phi_{N,y}] = \sum_{R\in\Phi_{N,x}^{(1)}} S^2[P_{Q\cap R}\phi_N] + \sum_{R\in\Phi_{N,y}^{(1)}} S^2[P_{Q\cap R}\phi_N].$$

Observe now that

$$S^2[P_Q\phi_{N,x}] = \sum_{R\in\Phi^{(1)}_{N,x}, R\cap Q\neq\emptyset} S^2[P_{R\cap Q}\phi_N],$$

where the $S[P_{R\cap Q}\phi_N]$ are equimeasurable for different R, and disjointly supported. Then we have

$$|\{x\in Q : S^2[P_Q\phi_{N,x}](x)>\lambda\}| = \sum_{R\in\Phi^{(1)}_{N,x}, R\cap Q\neq\emptyset} |\{x\in R\cap Q : S^2[P_{R\cap Q}\phi_N](x)>\lambda\}|.$$

Hence, for any $R\in\Phi^{(1)}_{N,x}$ with $R\cap Q\neq\emptyset$, we can write

$$|\{x\in Q : S^2[P_Q\phi_{N,x}](x)>\lambda\}| = \frac{|J|}{2|R|}|\{x\in R\cap Q : S^2[P_{R\cap Q}\phi_N](x)>\lambda\}|.$$

This gives that for any $\lambda>0$,

$$\mu_Q(\{S[P_Q\phi_{N,x}]>\lambda\}) = \frac{|J||R\cap Q|}{2|R||Q|}\mu_{R\cap Q}(\{S[P_{R\cap Q}\phi_N]>\lambda\}) = \frac{1}{2}\mu_{R\cap Q}(\{S[P_{R\cap Q}\phi_N]>\lambda\}).$$

Hence $S[P_Q\phi_{N,x}]^{**}(t) = S[P_{R\cap Q}\phi_N]^{**}(2t)$, and consequently

$$\|S[P_Q\phi_{N,x}]\|_{L^{p,q}_Q} = 2^{-1/p}\|S[P_{Q\cap R}\phi_N]\|_{L^{p,q}_{R\cap Q}}.$$

Notice that $R\cap Q\subseteq R\in\Phi^{(1)}_{N,x}$, and that $\tau^{(N)}_R(R\cap Q) = I\times[0,1]$. Applying Lemmas 3.1 and 3.3, we have

$$\begin{aligned}(14)\quad \|S[P_Q\phi_{N,x}]\|_{L^{p,q}_Q} &= 2^{-1/p}\|S[P_{Q\cap R}\phi_N]\|_{L^{p,q}_{R\cap Q}}\\ &\leq 2^{-1/p}\|S[P_{I\times[0,1]}\phi_{N-1}]\|_{L^{p,q}_{I\times[0,1]}} \leq C_p 2^{-1/p}\max_{1\leq k\leq N-1}\|S[\phi_k]\|_{L^{p,q}}.\end{aligned}$$

A similar argument shows that

$$\|S[P_Q\phi_{N,y}]\|_{L^{p,q}_Q} \leq C_p 2^{-1/p}\max_{1\leq k\leq N-1}\|S[\phi_k]\|_{L^{p,q}}.$$

Finally, since $S[P_Q\phi_N]\leq S[P_Q\phi_{N,x}]+S[P_Q\phi_{N,y}]$, we get

$$\begin{aligned}\|S[P_Q\phi_N]\|_{L^{p,q}_Q} &\leq \|S[P_Q\phi_{N,x}]\|_{L^{p,q}_Q}+\|S[P_Q\phi_{N,y}]\|_{L^{p,q}_Q}\\ &\leq C_p 2^{1-1/p}\max_{1\leq k\leq N-1}\|S[\phi_k]\|_{L^{p,q}}.\end{aligned}$$

Letting $B_p = C_p 2^{1-1/p}$, we finish the proof. □

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Department of Mathematics, Universitat de Valencia, Burjassot 46100 (Valencia) Spain

E-mail address: `oscar.blasco@uv.es`

Department of Mathematics, University of York, York YO10 5DD, UK

E-mail address: `sp23@york.ac.uk`