Some classes of p-summing type operators.

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Abstract

Let X, Y be Banach spaces and denote by $\ell_p^w(X,Y)$, $\ell_p^s(X,Y)$ and $\ell_p(X,Y)$ the spaces of sequences of operators (T_n) from X into Y such that $\sup_{||x||=1, ||y^*||=1} \sum | < T_n(x), y^* > | < \infty$, $\sup_{||x||=1} \sum ||T_n(x)|| < \infty$ and $\sum ||T_n|| < \infty$, respectively. Given Banach spaces X, Y, Z and W, we introduce and study the classes of bounded linear operators $\Phi : \mathcal{L}(X,Y) \to \mathcal{L}(Z,W)$ such that $(T_n) \to (\Phi(T_n))$ maps $\ell_p^s(X,Y)$ into $\ell_p(Z,W)$, $\ell_p^s(X,Y)$ into $\ell_p^s(Z,W)$ and $\ell_p^w(X,Y)$ into $\ell_p^s(Z,W)$.

1 Introduction

Throughout this paper X, Y, Z, W will stand for Banach spaces, $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X into Y and X^* for the dual of X. As usual B_X denotes the closed unit ball of X.

For $1 \leq p < \infty$ we write $\ell_p(X)$ for the Banach space of all *absolutely p-summable* sequences $(x_n)_{n=1}^\infty$ in X, i.e., the space of sequences such that $\|(x_n)\|_{\ell_p(X)} = (\sum_{n=1}^\infty \|x_n\|_X^p)^{\frac{1}{p}} < \infty$ and $\ell_p^w(X)$ for the space of all *weakly p-summable* sequences in X, i.e., the space of sequences $(x_n)_{n=1}^\infty$ such that $(\langle x^*, x_n \rangle)_{n=1}^\infty \in \ell_p$ for every $x^* \in X^*$, which becomes a Banach space with respect to the norm $\|(x_n)\|_{\ell_p^w(X)} = \sup_{x^* \in B_{X^*}} (\sum_{n=1}^\infty |\langle x^*, x_n \rangle|^p)^{\frac{1}{p}}$.

Recall that the space of p-summing operators $\Pi_p(X,Y)$ consists in those bounded linear operators $T \in \mathcal{L}(X,Y)$ such that $\tilde{T}((x_n)_{n=1}^\infty) = (T(x_n))_{n=1}^\infty$ defines a bounded linear operator from $\ell_p^w(X)$ into $\ell_p(Y)$. The reader is

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referred to [2], [3], [7], [11], [12] or [14] for definitions and results about these classes and their applications in Banach space theory.

We shall simply recall here two related notions which are the main motivations for our future considerations.

The first one appears when analyzing operators acting on spaces of vector-valued continuous functions $T : C(\Omega, X) \to Y$, where Ω is a compact Haussdorf space. An operator is called p-dominated operator (see [4], III.19.3) if there exist a constant $C > 0$ and a probability measure μ on Ω such that

$$\|T(f)\|^p \leq C \int_\Omega \|f(t)\|^p d\mu(t)$$

for all $f \in C(\Omega, X)$. The connection between p-summing and p-dominated operators was first given by A. Pietsch, who showed that for finite dimensional Banach spaces X both notions are the same. For infinite dimensional Banach spaces C. Swartz (see [13]) showed that operators in $\Pi_1(C(\Omega, X), Y)$ are always 1-dominated but the space of 1-dominated operators from $C(\Omega, X)$ into Y coincides with $\Pi_1(C(\Omega, X), Y)$ if and only if X is finite dimensional.

Later S. Kislyakov introduced (see [6], Def. 1.1.5) the following notion in the setting of injective tensor products: Given $1 \leq p < \infty$ and three Banach spaces E, X and Y, an operator T from the injective tensor product $X \check{\otimes} E$ into Y is said to be (p, E)-summing if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $u_1, u_2, ..., u_N \in X \otimes E$ we have

$$\sum_{k=1}^{N} \|T(u_k)\|_Y^p \leq C \sup_{F \in B_{X^*}} \sum_{k=1}^{N} \|u_k(F)\|_E^p$$

where $u_k(F) = \sum_{j=1}^{n_k} \langle F, x_{j,k} \rangle e_{j,k}$ for $u_k = \sum_{j=1}^{n_k} x_{j,k} \otimes e_{j,k}$ for some $e_{j,k} \in E$ and $x_{j,k} \in X$. The space of such operators is denoted by $\Pi_p^E(X \check{\otimes} E, Y)$.

Theorem 1.1.6 of [6] gives an analogue to Pietsch's domination theorem for (p, X)-summing operators. This result actually enables us to get the p-dominated operators in a very similar fashion to the p-summing ones. Since $C(\Omega) \check{\otimes} X = C(\Omega, X)$ then the class of p-dominated operators actually coincides with $\Pi_p^X(C(\Omega) \check{\otimes} X, Y)$.

Given two Banach spaces X and Y we shall be denoting by $\ell_p(X,Y)$ and $\ell_p^w(X,Y)$ the spaces $\ell_p(\mathcal{L}(X,Y))$ and $\ell_p^w(\mathcal{L}(X,Y))$ respectively.

Recall that if (e_n) is a sequence in a Banach space E then

$$\|(e_n)\|_{\ell_p^w(E)} = \sup_{(\alpha_n)\in B_{\ell_{p'}}} \Big\|\sum_{n=1}^{\infty} \alpha_n e_n\Big\|_E.$$

Hence one easily gets that

$$\|(T_n)\|_{\ell_p^w(X,Y)} = \sup_{x\in B_X} \sup_{y^*\in B_{Y^*}} \Big(\sum_{n=1}^{\infty} |\langle y^*, T_n x\rangle|^p\Big)^{\frac{1}{p}}. \tag{1.1}$$

The fact that we have the strong operator topology at our disposal allows to consider the following intermediate space of sequences of operators. We shall use the notation $\ell_p^s(X,Y)$ for the space of strongly p-summable sequences of operators (T_n), that is $\sup_{x\in B_X}(\sum_{n=1}^{\infty}\|T_n x\|_Y^p)^{1/p} < \infty$.

Let $\Phi : \mathcal{L}(X,Y) \longrightarrow \mathcal{L}(Z,W)$ be a bounded operator. The correspondence

$$\tilde{\Phi} : (T_n)_{n=1}^{\infty} \longmapsto (\Phi(T_n))_{n=1}^{\infty}$$

always induces a linear bounded operator from $\ell_p(X,Y)$ to $\ell_p(Z,W)$, as well as from $\ell_p^w(X,Y)$ to $\ell_p^w(Z,W)$. Recall that Φ is p-summing if $\tilde{\Phi}$ maps $\ell_p^w(X,Y)$ to $\ell_p(Z,W)$.

In this paper we study several questions concerning the class of operators Φ such that this vector-valued extension $\tilde{\Phi}$ produces a bounded linear operator either from $\ell_p^s(X,Y)$ to $\ell_p(Z,W)$, from $\ell_p^s(X,Y)$ to $\ell_p^s(Z,W)$, or from $\ell_p^w(X,Y)$ to $\ell_p^s(Z,W)$.

In particular we study operators Φ from $\mathcal{L}(X,Y)$ into Z such that $\tilde{\Phi}$ defines a bounded linear operator from $\ell_p^s(X,Y)$ into $\ell_p(Z)$. These operators will be called (ℓ_p^s,ℓ_p)-summing and denoted by $\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$.

Since $X\check{\otimes}E$ is a subspace of $\mathcal{L}(X^*,E)$ we easily get that if $\Phi : \mathcal{L}(X^*,E) \to Y$ is (ℓ_p^s,ℓ_p)-summing then its restriction to $X\check{\otimes}E$ is (p,E)-summing. Also, the theory can be applied, among other things, to operators acting on weakly p-summable sequences $\ell_p^w(X) = \mathcal{L}(\ell_{p'},X)$ or Pettis-p-integrable functions which are subspaces of $\mathcal{L}(L^{p'}(\mu),X)$.

The paper is divided into five sections. In Section 2 the notions of (ℓ_p^s, ℓ_p)-summing, (ℓ_p^w, ℓ_p^s)-summing and (ℓ_p^s, ℓ_p^s)-summing operators are introduced and studied. It is shown that the classes of (ℓ_p^w, ℓ_p^s)-summing and (ℓ_p^s, ℓ_p^s)-summing operators can be easily described in terms of the classes Π_p or $\Pi_{(\ell_p^s,\ell_p)}$ (see Theorems 2.13 and 2.16 below).

In section 3 several characterizations of the class $\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$ are achieved and several examples are exhibited.

In Section 4 some results about the coincidence of classes are shown, in particular an analogue of Maurey's theorem is presented.

Finally we pose two open problems in Section 5.

2 Definitions and preliminaries

Definition 2.1 *Let $1 \leq p \leq \infty$ and X, Y be Banach spaces. A sequence of operators $(T_n)_{n=1}^\infty \subseteq \mathcal{L}(X,Y)$ is said to be* strongly p-summable in $\mathcal{L}(X,Y)$ *if the vector-valued sequence $(T_n x)_{n=1}^\infty$ belongs to $\ell_p(Y)$ for every $x \in X$.*

We shall use the notation $\ell_p^s(X,Y)$ for the space of all strongly p-summable sequences. The norm in $\ell_p^s(X,Y)$ is given by

$$\|(T_n)\|_{\ell_p^s(X,Y)} = \sup_{x \in B_X} \left(\sum_{n=1}^\infty \|T_n x\|_Y^p \right)^{\frac{1}{p}}.$$

It is rather easy to see that $(\ell_p^s(X,Y), \|\cdot\|_{\ell_p^s(X,Y)})$ is a Banach space for any $1 \leq p \leq \infty$.

Easy examples of sequences in $\ell_p^s(X,Y)$ are given in the next propositions, whose proofs are left to the reader.

Proposition 2.2 *Let X, Y be Banach spaces and $1 \leq r_1, r_2, p \leq \infty$ with $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p}$. If $(x_n^*)_{n=1}^\infty \in \ell_{r_1}^w(X^*)$ and $(y_n)_{n=1}^\infty \in \ell_{r_2}(Y)$, then $(x_n^* \otimes y_n)_{n=1}^\infty \in \ell_p^s(X,Y)$.*

Proposition 2.3 *Let X, Y and Z be Banach spaces and $1 \leq p \leq \infty$. If $(T_n)_{n=1}^\infty \in \ell_p^w(X,Y)$ and $S \in \Pi_p(Y,Z)$, then $(ST_n)_{n=1}^\infty \in \ell_p^s(X,Z)$.*

As mentioned in the introduction $\ell_p(X,Y)$ stands for $\ell_p(\mathcal{L}(X,Y))$ and $\ell_p^w(X,Y)$ for $\ell_p^w(\mathcal{L}(X,Y))$. Let us first notice the inclusions appearing between these spaces and the new one.

Note that $\ell_p^s(\mathbb{K},Y) = \ell_p(\mathbb{K},Y) = \ell_p(Y)$ and $\ell_p^s(X,\mathbb{K}) = \ell_p^w(X,\mathbb{K}) = \ell_p^w(X^*)$. Observe also that for every bounded sequence (T_n) in $\mathcal{L}(X,Y)$ we have

$$\sup_n \|T_n\| = \sup_{x\in B_X} \sup_n \|T_n x\|_Y = \sup_{x\in B_X} \sup_{y^*\in B_{Y^*}} \sup_n |\langle y^*, T_n x\rangle|.$$

Hence for $p=\infty$, $\ell_\infty(X,Y) = \ell_\infty^s(X,Y) = \ell_\infty^w(X,Y)$.

Proposition 2.4 *Let X, Y be Banach spaces and $1 \leq p < \infty$. Then*

$$\ell_p(X,Y) \subseteq \ell_p^s(X,Y) \subseteq \ell_p^w(X,Y).$$

Moreover, each of the inclusions is strict in general.

PROOF. We shall only show the last statement, since the inclusions are immediate by definitions.

Let $(e_n)_{n=1}^\infty$ be the usual basis in $\ell_{p'}$ (or c_0 for $p=1$) and consider $(e_n \otimes e_n)_{n=1}^\infty$ as operators from ℓ_p into ℓ_∞. Since $(e_n) \in \ell_p^w(\ell_{p'})$ and $(e_n) \in \ell_\infty(\ell_\infty)$, by Proposition 2.2 we get that $(e_n \otimes e_n) \in \ell_p^s(\ell_p, \ell_\infty)$. On the other hand, $\|e_n \otimes e_n\|_{\ell_p,\ell_\infty} = \|e_n\|_{\ell_{p'}} \|e_n\|_{\ell_\infty} = 1$ thus $(e_n \otimes e_n) \notin \ell_p(\ell_p, \ell_\infty)$.

Let us now find a weakly p-summable sequence which is not strongly p-summable. Simply choose $p < r < \infty$ and take s so that $1/r + 1/p' = 1/s$. A direct computation, using (1.1), shows that $(e_n \otimes e_n) \in \ell_p^w(\ell_r, \ell_s)$. However $(e_n \otimes e_n) \notin \ell_p^s(\ell_r, \ell_s)$ as it is shown by selecting any $\lambda = (\lambda_n)_{n=1}^\infty \in \ell_r \setminus \ell_p$, since $\|\langle e_n, \lambda\rangle e_n\|_{\ell_s} = |\lambda_n| \notin \ell_p$. $\square$

Observe that if $(T_n)_{n=1}^\infty$ in $\mathcal{L}(X,Y)$ is a strongly p-summable sequence, using the closed graph theorem, then one can associate a linear and bounded operator $S : X \longrightarrow \ell_p(Y)$ defined by $S(x) = (T_n x)_{n=1}^\infty$. In this way we get $\ell_p^s(X,Y)$ as a closed subspace of $\mathcal{L}(X, \ell_p(Y))$. Let us see that they are actually isometrically isomorphic.

Proposition 2.5 *Let X, Y be Banach spaces and $1 \leq p \leq \infty$. For each $n \in \mathbb{N}$, let $Q_n : \ell_p(Y) \to Y$ be the operator $Q_n((y_i)_{i=1}^\infty) = y_n$. The correspondence $T \longmapsto (Q_nT)_{n=1}^\infty$ is an isometric isomorphism from $\mathcal{L}(X, \ell_p(Y))$ onto $\ell_p^s(X, Y)$.*

PROOF. Given $T \in \mathcal{L}(X, \ell_p(Y))$, the sequence $T_n = Q_nT$, $n = 1, 2, \dots,$ belongs to $\ell_p^s(X, Y)$ and

$$\|(T_n)\|_{\ell_p^s(X,Y)} = \sup_{x \in B_X} \left(\sum_{n=1}^\infty \|Q_nTx\|_Y^p \right)^{\frac{1}{p}} = \sup_{x \in B_X} \|Tx\|_{\ell_p(Y)} = \|T\|_{X, \ell_p(Y)}.$$

Therefore, the correspondence $T \longmapsto (Q_nT)_{n=1}^\infty$ induces an isometry. To see the isomorphism let us take $(S_n) \in \ell_p^s(X, Y)$ and note that $S : X \longrightarrow \ell_p(Y)$ defined by $S(x) = (S_nx)_{n=1}^\infty$ gives a bounded operator and $Q_nS = S_n$. $\square$

As a consequence, we get that $\ell_p^w(X^*)$ is isometrically isomorphic to $\mathcal{L}(X, \ell_p)$ for any $1 \leq p \leq \infty$ and any Banach space X (see [5], Prop. 19.4.3). Let us also recall that, for every Banach space Z, $\ell_p^w(Z)$ is also isometrically isomorphic to $\mathcal{L}(\ell_{p'}, Z)$ if $1 < p < \infty$ and $\mathcal{L}(c_o, Z)$ for $p = 1$. In these last cases the isomorphims are given by associating to each operator T the sequence $x_n = T(e_n)$. Of course the connection between both results goes through the adjoint operator.

Combining both identifications we get that the strongly p-summable sequences in $\mathcal{L}(\ell_{q'}, X)$ are precisely the weakly q-summable sequences in $\ell_p(X)$.

Corollary 2.6 *Let X be a Banach space, $1 \leq p \leq \infty$ and $1 < q < \infty$. Then the map $\Psi : \ell_q^w(\ell_p(X)) \to \ell_p^s(\ell_{q'}, X)$, defined by $\Psi((x_k)_k) = (T_n)_n$ where T_n are defined by $T_n(e_k) = Q_n(x_k)$ for all $n, k \in \mathbb{N}$, is an isometric isomorphism.*

Similar result is true for $q = 1$ replacing ℓ_∞ by c_o.

The concept of strongly p-summing sequence can now be used to define new classes of p-summing type operators.

Definition 2.7 *Let $1 \leq p < \infty$. Let X, Y, Z and W be Banach spaces.*

An operator $\Phi : \mathcal{L}(X,Y) \longrightarrow Z$ is said to be (ℓ_p^s, ℓ_p)-summing *if there exists a constant $C>0$ such that*

$$(2.1) \qquad \Big(\sum_{i=1}^{n} \|\Phi(T_i)\|_Z^p\Big)^{\frac{1}{p}} \leq C \sup_{x\in B_X} \Big(\sum_{i=1}^{n} \|T_i x\|_Y^p\Big)^{\frac{1}{p}}$$

for any finite family of operators $T_1, \dots, T_n$ in $\mathcal{L}(X,Y)$.

An operator $\Phi : X \longrightarrow \mathcal{L}(Y,Z)$ is said to be (ℓ_p^w, ℓ_p^s)-summing *if there exist a constant $C>0$ such that*

$$(2.2) \qquad \sup_{y\in B_Y} \Big(\sum_{i=1}^{n} \|\Phi(x_i)(y)\|_Z^p\Big)^{\frac{1}{p}} \leq C \sup_{x^*\in B_{X^*}} \Big(\sum_{i=1}^{n} |\langle x^*, x_i\rangle|^p\Big)^{\frac{1}{p}}$$

for every choice of elements $x_1, \dots, x_n$ in X.

An operator $\Phi : \mathcal{L}(X,Y) \longrightarrow \mathcal{L}(Z,W)$ is said to be (ℓ_p^s, ℓ_p^s)-summing *if there exist a constant $C>0$ such that*

$$(2.3) \qquad \sup_{z\in B_Z} \Big(\sum_{i=1}^{n} \|\Phi(T_i)(z)\|_W^p\Big)^{\frac{1}{p}} \leq C \sup_{x\in B_X} \Big(\sum_{i=1}^{n} \|T_i x\|_Y^p\Big)^{\frac{1}{p}}.$$

for every choice of operators $T_1, \dots, T_n$ in $\mathcal{L}(X,Y)$.

The least constants in (2.1), (2.2) and (2.3) are denoted by $\pi_{(\ell_p^s,\ell_p)}(\Phi)$, $\pi_{(\ell_p^w,\ell_p^s)}(\Phi)$ and $\pi_{(\ell_p^s,\ell_p^s)}(\Phi)$ respectively.

We shall denote by $\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$ the space of all (ℓ_p^s,ℓ_p)-summing operators from $\mathcal{L}(X,Y)$ to Z, by $\Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$ the space of all (ℓ_p^w,ℓ_p^s)-summing operators from X to $\mathcal{L}(Y,Z)$ and by $\Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,W))$ the space of all strongly (ℓ_p^s,ℓ_p)-summing operators from $\mathcal{L}(X,Y)$ to $\mathcal{L}(Z,W)$.

$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$, $\Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$ and $\Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,W))$ become Banach spaces with the norms $\pi_{(\ell_p^s,\ell_p)}(\cdot)$, $\pi_{(\ell_p^w,\ell_p^s)}(\cdot)$ $\pi_{(\ell_p^s,\ell_p^s)}(\cdot)$, respectively.

The corresponding definitions for $p=\infty$ would simply lead to the space of bounded operators in all cases.

Alternative definition for (ℓ_p^s, ℓ_p)-summing operators is the following one:

Remark 2.8 Let $1 \leq p < \infty$ and $\Phi \in \mathcal{L}(\mathcal{L}(X,Y),Z)$. The following are equivalent:

i) Φ is (ℓ_p^s, ℓ_p)-summing.

ii) Φ maps sequences $(T_n) \in \ell_p^s(X,Y)$ into sequences $(\Phi(T_n)) \in \ell_p(Z)$.

iii) The linear operator $\tilde{\Phi} : \ell_p^s(X,Y) \longrightarrow \ell_p(Z)$ defined by $\tilde{\Phi}((T_n)_{n=1}^{\infty}) = (\Phi(T_n))_{n=1}^{\infty}$ is continuous.

Similar equivalences are true for (ℓ_p^w, ℓ_p^s)-summing operators and (ℓ_p^s, ℓ_p^s)-summing operators.

Remark 2.9 It is rather easy to see that, when some of the spaces is finite dimensional, the classes $\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$, $\Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$ reduce either to bounded operators or to p-summing operators and that the class $\Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,U))$ reduces to one of the previous cases.

Remark 2.10 Recall that $dimX = \infty$ implies $\Pi_p(X,X) \subsetneq \mathcal{L}(X,X)$ and observe that

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(\mathbb{K},X^*),Y) = \mathcal{L}(X^*,Y)$$

and

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,\mathbb{K}),Y) = \Pi_p(X^*,Y).$$

Therefore, for any infinite dimensional X, we have

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(\mathbb{K},X^*),Y) \neq \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,\mathbb{K}),Y), \text{ for some Banach space } Y,$$

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,\mathbb{K}),\mathcal{L}(X,\mathbb{K})) \subsetneq \Pi_{(\ell_p^w,\ell_p^s)}(\mathcal{L}(X,\mathbb{K}),\mathcal{L}(X,\mathbb{K})),$$

$$\Pi_{(\ell_p^w,\ell_p^s)}(\mathcal{L}(\mathbb{K},X),\mathcal{L}(\mathbb{K},X)) \subsetneq \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(\mathbb{K},X),\mathcal{L}(\mathbb{K},X)).$$

Remark 2.11 If A and B are spaces of operator then

$$\Pi_p(A,B) \subseteq \Pi_{(\ell_p^w,\ell_p^s)}(A,B) \cap \Pi_{(\ell_p^s,\ell_p)}(A,B),$$

$$\Pi_{(\ell_p^w,\ell_p^s)}(A,B) \cup \Pi_{(\ell_p^s,\ell_p)}(A,B) \subseteq \Pi_{(\ell_p^s,\ell_p^s)}(A,B).$$

It is not difficult to show that the inclusions are strict in general.

Remark 2.12 (i) For each $x \in X$, the evaluation map $e_x : \mathcal{L}(X,Y) \to Y$ given by $e_x(T) = Tx$ is (ℓ_1^s, ℓ_1)-summing.

(ii) For each $y \in Y$, the operator $\Phi_y : X^* \to \mathcal{L}(X,Y)$ given by $\Phi_y(x^*) = x^* \otimes y$ is (ℓ_1^w, ℓ_1^s)-summing.

(iii) For each $S \in \mathcal{L}(Y,Z)$ the operator $\Phi_S : \mathcal{L}(X,Y) \to \mathcal{L}(X,Z)$ given by $\Phi_S(T) = ST$ is (ℓ_1^s, ℓ_1^s)-summing. Moreover, if $S \in \Pi_p(Y,Z)$, $1 \leq p < \infty$, then Φ_S is (ℓ_p^w, ℓ_p^s)-summing.

We now show that actually $\Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y,Z))$ and $\mathcal{L}(Y, \Pi_p(X,Z))$ can be identified.

Theorem 2.13 *Let $\Phi : X \longrightarrow \mathcal{L}(Y,Z)$ be a bounded operator and let us define $\Phi^\# : Y \longrightarrow \mathcal{L}(X,Z)$ by $\Phi^\#(y)(x) := \Phi(x)(y)$, $x \in X$, $y \in Y$.*

The correspondence $\Phi \longmapsto \Phi^\#$ is an isometric isomorphism between $\Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y,Z))$ and $\mathcal{L}(Y, \Pi_p(X,Z))$.

PROOF. Take $\Phi \in \Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y,Z))$ and $y \in Y$, then

$$\left(\sum_{i=1}^n \|\Phi^\#(y)(x_i)\|_Z^p\right)^{\frac{1}{p}} \leq \|y\|_Y \Pi_{(\ell_p^w, \ell_p^s)}(\Phi) \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{\frac{1}{p}}$$

for every choice of elements $\{x_1, \ldots, x_n\}$ in X. Thus, $\Phi^\#(y) \in \Pi_p(X,Z)$ and $\pi_p(\Phi^\#(y)) \leq \|y\|_Y \pi_{(\ell_p^w, \ell_p^s)}(\Phi)$. Hence $\Phi^\# \in \mathcal{L}(Y, \Pi_p(X,Z))$ and $\|\Phi^\#\| \leq \pi_{(\ell_p^w, \ell_p^s)}(\Phi)$.

On the other hand, if $\Psi \in \mathcal{L}(Y, \Pi_p(X,Z))$ then

$$\begin{aligned}
\sup_{y \in B_Y} \left(\sum_{i=1}^n \|\Psi^\#(x_i)(y)\|_Z^p\right)^{\frac{1}{p}} &= \sup_{y \in B_Y} \left(\sum_{i=1}^n \|\Psi(y)(x_i)\|_Z^p\right)^{\frac{1}{p}} \\
&\leq \sup_{y \in B_Y} \{\pi_p(\Psi(y))\} \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{\frac{1}{p}} \\
&\leq \|\Psi\| \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{\frac{1}{p}}
\end{aligned}$$

for every finite sequence $\{x_1, \ldots, x_n\}$ in X. Hence $\Psi^\# \in \Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y,Z))$ and $\pi_{(\ell_p^w, \ell_p^s)}(\Psi^\#) \leq \|\Psi\|$. Since $(\Psi^\#)^\# = \Psi$ the proof is finished. $\square$

Therefore, the class $\Pi_{(\ell_p^w,\ell_p^s)}$ inherits some properties from those in Π_p. For example, Grothendieck theorem (see [3], Theorem 1.13) implies the following:

Corollary 2.14 *Let X be a Banach space. Every operator from ℓ_1 into $\mathcal{L}(X,\ell_2)$ is (ℓ_1^w,ℓ_1^s)-summing, i.e. $\Pi_{(\ell_1^w,\ell_1^s)}(\ell_1,\mathcal{L}(X,\ell_2)) = \mathcal{L}(\ell_1,\mathcal{L}(X,\ell_2))$.*

Corollary 2.15 *Let X be a Banach space, K be a compact Hausdorff space and (Ω,μ) be a σ-finite measure space.*

Every operator from $C(K)$ into $\mathcal{L}(X,L_1(\Omega,\mu))$ is (ℓ_2^w,ℓ_2^s)-summing, i.e. $\Pi_{(\ell_2^w,\ell_2^s)}(C(K),\mathcal{L}(X,L_1(\mu))) = \mathcal{L}(C(K),\mathcal{L}(X,L_1(\mu)))$.

Furthermore, arguing as in Theorem 2.13 we get also the following result whose proof is left to the interested reader.

Theorem 2.16 *The correspondence $\Phi \longmapsto \Phi^{\#}$ is an isometric isomorphism between $\Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,W))$ and $\mathcal{L}(Z,\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),W))$.*

We would like to point out also certain behaviour of these classes as operator ideals and some composition results. The proof of the following proposition is straightforward.

Proposition 2.17 *Let $1 \le p < \infty$ and let X, Y, Z, W, U, X_0 and Z_0 be Banach spaces.*

1) If $\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$ and $\Psi \in \mathcal{L}(Z,Z_0)$, then the composition $\Psi\Phi$ belongs to $\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z_0)$ with $\pi_{(\ell_p^s,\ell_p)}(\Psi\Phi) \le \|\Psi\|_{Z,Z_0} \cdot \pi_{(\ell_p^s,\ell_p)}(\Phi)$. That is, $\mathcal{L} \circ \Pi_{(\ell_p^s,\ell_p)} \subseteq \Pi_{(\ell_p^s,\ell_p)}$.

2) If $\Phi \in \mathcal{L}(X_0,X)$ and $\Psi \in \Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$, then the composition $\Psi\Phi$ belongs to $\Pi_{(\ell_p^w,\ell_p^s)}(X_0,\mathcal{L}(Y,Z))$ with $\pi_{(\ell_p^w,\ell_p^s)}(\Psi\Phi) \le \pi_{(\ell_p^w,\ell_p^s)}(\Psi)\cdot\|\Phi\|_{X_0,X}$. That is, $\Pi_{(\ell_p^w,\ell_p^s)} \circ \mathcal{L} \subseteq \Pi_{(\ell_p^w,\ell_p^s)}$.

3) If $\Phi \in \Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,W))$ and $\Psi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(Z,W),U)$, then $\Psi\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),U)$ with $\pi_{(\ell_p^s,\ell_p)}(\Psi\Phi) \le \pi_{(\ell_p^s,\ell_p)}(\Psi)\cdot\pi_{(\ell_p^s,\ell_p^s)}(\Phi)$. That is, $\Pi_{(\ell_p^s,\ell_p)}\circ \Pi_{(\ell_p^s,\ell_p^s)} \subseteq \Pi_{(\ell_p^s,\ell_p)}$.

4) If $\Phi \in \Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$ and $\Psi \in \Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(Y,Z),\mathcal{L}(W,U))$, then $\Psi\Phi \in \Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(W,U))$ with $\pi_{(\ell_p^w,\ell_p^s)}(\Psi\Phi) \leq \pi_{(\ell_p^w,\ell_p^s)}(\Psi)\cdot\pi_{(\ell_p^w,\ell_p^s)}(\Phi)$. That is, $\Pi_{(\ell_p^s,\ell_p^s)}\circ\Pi_{(\ell_p^w,\ell_p^s)} \subseteq \Pi_{(\ell_p^w,\ell_p^s)}$.

5) If $\Phi \in \Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$ and $\Psi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(Y,Z),W)$, then $\Psi\Phi \in \Pi_p(X,W)$ with $\pi_p(\Psi\Phi) \leq \pi_{(\ell_p^s,\ell_p)}(\Psi)\cdot\pi_{(\ell_p^w,\ell_p^s)}(\Phi)$. That is, $\Pi_{(\ell_p^s,\ell_p)}\circ\Pi_{(\ell_p^w,\ell_p^s)} \subseteq \Pi_p$.

The classical theorem of Pietsch stated that if Φ is q-summing and Ψ is p-summing then $\Psi\Phi$ is r-summing, with $\frac{1}{r} = min\{1, \frac{1}{p}+\frac{1}{q}\}$ (see [3], Theorem 2.22 or [5], Theorem 19.10.3). Next we establish that when Ψ is (ℓ_p^w,ℓ_p^s)-summing then $\Psi\Phi$ is (ℓ_r^w,ℓ_r^s)-summing. The proof follows along the lines of Theorem 2.22 in [3].

Proposition 2.18 *Let $\Phi \in \Pi_q(X,Y)$ and $\Psi \in \Pi_{(\ell_p^w,\ell_p^s)}(Y,\mathcal{L}(Z,W))$ with $1 \leq p,q < \infty$. Define $1 \leq r < \infty$ by $\frac{1}{r} = min\{1,\frac{1}{p}+\frac{1}{q}\}$. Then $\Psi\Phi$ is (ℓ_r^w,ℓ_r^s)-summing with $\pi_r^s(\Psi\Phi) \leq \pi_{(\ell_p^w,\ell_p^s)}(\Psi)\pi_q(\Phi)$. That is, $\Pi_{(\ell_p^w,\ell_p^s)}\circ\Pi_q \subseteq \Pi_{(\ell_r^w,\ell_r^s)}$.*

PROOF. We assume first that $\frac{1}{p}+\frac{1}{q} \leq 1$ then $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. Let $(x_n) \in \ell_r^w(X)$ be given. Applying Lemma 2.23 in [3] we get sequences $(\sigma_n) \in \ell_q$ and $(y_n) \in \ell_p^w(Y)$ such that $\|(\sigma_n)\|_{\ell_q} \leq \|(x_n)\|_{\ell_r^w(X)}^{r/q}$, $\|(y_n)\|_{\ell_p^w(Y)} \leq \pi_q(\Phi)\|(x_n)\|_{\ell_r^w(X)}^{r/p}$ and $\Phi(x_n) = \sigma_n y_n$ for all n. By Hölder's inequality, using that Ψ is (ℓ_p^w,ℓ_p^s)-summing and that $[\Psi\Phi(x_n)](z) = \sigma_n[\Psi(y_n)(z)]$ for every $z \in Z$, we get

$$\begin{aligned}
\sup_{z\in B_Z}\Big(\sum_{n=1}^k \|[\Psi\Phi(x_n)](z)\|_W^r\Big)^{\frac{1}{r}} &\leq \Big(\sum_{n=1}^k |\sigma_n|^q\Big)^{\frac{1}{q}} \sup_{z\in B_Z}\Big(\sum_{n=1}^k \|\Psi(y_n)(z)\|_W^p\Big)^{\frac{1}{p}} \\
&\leq \|(x_n)\|_{\ell_r^w(X)}^{r/q}\pi_{(\ell_p^w,\ell_p^s)}(\Psi)\sup_{y^*\in B_{Y^*}}\Big(\sum_{n=1}^k |\langle y^*,y_n\rangle|^p\Big)^{\frac{1}{p}} \\
&\leq \pi_{(\ell_p^w,\ell_p^s)}(\Psi)\cdot\pi_q(\Phi)\cdot\|(x_n)\|_{\ell_r^w(X)}.
\end{aligned}$$

Then $\Psi\Phi$ belongs to $\Pi_{(\ell_r^w,\ell_r^s)}(X,\mathcal{L}(Z,W))$.

If $\frac{1}{p}+\frac{1}{q} > 1$ and we assume that $1 < p \leq q$, then $\Phi \in \Pi_{p'}(X,Y)$ with $\pi_{p'}(\Phi) \leq \pi_q(\Phi)$ and applying the first part with p and p' we get $\Psi\Phi \in \Pi_{(\ell_1^w,\ell_1^s)}(X,\mathcal{L}(Z,W))$, which completes the proof. $\square$

3 (ℓ_p^s, ℓ_p)-summing operators

Let us now present several examples of such operators. The first one connects the notion of p-summing and (ℓ_p^s, ℓ_p)-summing operators.

Let $1 \le p < \infty$ and let $T : X \longrightarrow Y$ be a bounded operator. Denote $\tilde{T} : \mathcal{L}(\ell_{p'}, X) \longrightarrow \ell_p(Y)$ the map given by $\tilde{T}(S) = (TSe_n)_{n=1}^\infty$. Recall that T is p-summing if and only if $\tilde{T}$ is bounded and $\pi_p(T) = \|\tilde{T}\|$.

We first study when $\tilde{T}$ is (ℓ_p^s, ℓ_p)-summing operator. The answer is given in the following theorem.

Theorem 3.1 *Let X, Y be Banach spaces, $1 < p < \infty$ and $T \in \mathcal{L}(X, Y)$. Then $\tilde{T} \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(\ell_{p'}, X), \ell_p(Y))$ if and only if $\bar{T} \in \Pi_p(\ell_p(X), \ell_p(Y))$ where $\bar{T} : \ell_p(X) \to \ell_p(Y)$ is given by $\bar{T}((x_j)_{j=1}^\infty) = (T(x_j))_{j=1}^\infty$. Moreover $\pi_p(\bar{T}) = \pi_{(\ell_p^s, \ell_p)}(\tilde{T})$.*

Same result holds for $p = 1$ with the replacement of ℓ_∞ by c_0.

PROOF. Assume that $\tilde{T} \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(\ell_{p'}, X), \ell_p(Y))$. Given $(x_k)_k \in \ell_p^w(\ell_p(X))$ we define the operators $T_n : \ell_{p'} \to X$ such that $T_n(e_k) = Q_n(x_k)$ for all $n, k \in \mathbb{N}$. Since Corollary 2.6 gives that $(T_n) \in \ell_p^s(\ell_{p'}, X)$ then $(\tilde{T}(T_n))_n = ((TT_n(e_k))_k)_n \in \ell_p(\ell_p(Y))$. Now we have

$$\begin{aligned}
\sum_{k=1}^\infty \|\bar{T}(x_k)\|_{\ell_p(Y)}^p &= \sum_{n=1}^\infty \sum_{k=1}^\infty \|T(Q_n(x_k))\|_Y^p = \sum_{n=1}^\infty \|\tilde{T}(T_n)\|_{\ell_p(Y)}^p \\
&\le (\pi_{(\ell_p^s, \ell_p)}(\tilde{T}))^p \|(T_n)\|_{\ell_p^s(\ell_{p'}, X)}^p = (\pi_{(\ell_p^s, \ell_p)}(\tilde{T}))^p \|(x_k)\|_{\ell_p^w(\ell_p(X))}^p.
\end{aligned}$$

For the converse assume that $\bar{T}$ is p-summing and, as above, we write

$$\begin{aligned}
\sum_{n=1}^\infty \|\tilde{T}(T_n)\|_{\ell_p(Y)}^p &= \sum_{k=1}^\infty \|\bar{T}(x_k)\|_{\ell_p(Y)}^p \\
&\le (\pi_p(\bar{T}))^p \|(x_k)\|_{\ell_p^w(\ell_p(X))}^p = (\pi_p(\bar{T}))^p \|(T_n)\|_{\ell_p^s(\ell_{p'}, X)}^p.
\end{aligned}$$

□

Example 3.2 Let $1 < p < \infty$, X, Y be Banach spaces. Assume that $T \in \Pi_p(X^*, Y)$, then the operator

$$\begin{array}{rccc}
\Phi_T : & \mathcal{L}(X, \ell_p) & \longrightarrow & \ell_p(Y) \\
& u & \rightsquigarrow & (Tu^*(e_i))_{i=1}^\infty
\end{array}$$

is (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s,\ell_p)}(\Phi_T) = \pi_p(T)$.

PROOF. For any choice of $u_1, \dots, u_N \in \mathcal{L}(X, \ell_p)$, since $T : X^* \longrightarrow Y$ is p-summing we have

$$\begin{aligned}
\sum_{n=1}^{N} \|\Phi_T(u_n)\|^p_{\ell_p(Y)} &= \sup_m \Big\{ \sum_{i=1}^{m} \sum_{n=1}^{N} \|T u_n^*(e_i)\|^p_Y \Big\} \\
&\leq (\pi_p(T))^p \sup_m \sup_{x \in B_X} \Big\{ \sum_{i=1}^{m} \sum_{n=1}^{N} |\langle u_n^*(e_i), x\rangle|^p \Big\} \\
&= (\pi_p(T))^p \sup_{x \in B_X} \sup_m \Big\{ \sum_{i=1}^{m} \sum_{n=1}^{N} |\langle e_i, u_n x\rangle|^p \Big\} \\
&\leq (\pi_p(T))^p \sup_{x \in B_X} \Big\{ \sum_{n=1}^{N} \|u_n x\|^p_{\ell_p} \Big\}.
\end{aligned}$$

Therefore Φ_T is (ℓ_p^s, ℓ_p)-summing and $\pi_{(\ell_p^s,\ell_p)}(\Phi_T) \leq \pi_p(T)$. Therefore Φ_T is (ℓ_p^s, ℓ_p)-summing and $\pi_{(\ell_p^s,\ell_p)}(\Phi_T) \leq \pi_p(T)$. It is straightforward to get equality of norms since $\pi_{(\ell_p^s,\ell_p)}(\Phi_T) \geq \|\Phi_T\| = \pi_p(T)$. $\square$

Example 3.3 Let $1 \leq p < \infty$, let X be a reflexive Banach space, Y be a separable Banach space and $L_p(\mu, Y)$ denote the space of Bochner p-integrable functions. If μ is a finite Borel measure on the compact topological space $(B_X, \sigma(X, X^*))$ then the operator $\Theta_p : \mathcal{L}(X, Y) \longrightarrow L_p(\mu, Y)$ defined by $\Theta_p(T)(x) = T(x)$, $x \in B_X$, is (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s,\ell_p)}(\Theta_p) \leq \mu(B_X)^{\frac{1}{p}}$.

PROOF. We first note that $x \rightarrow T(x)$ is weakly continuous function on B_X and hence weakly measurable. Using the separability of Y we get that it is a bounded measurable function and then in $L_p(\mu, Y)$. This shows that the operator Θ_p is well defined and bounded.

If $T_1, \dots, T_n$ are in $\mathcal{L}(X, Y)$, then

$$\begin{aligned}
\sum_{i=1}^{n} \|\Theta_p(T_i)\|^p_{L_p(\mu,Y)} &= \sum_{i=1}^{n} \int_{B_X} \|T_i(x)\|^p_Y d\mu(x) \\
&\leq \mu(B_X) \sup_{x \in B_X} \Big\{ \sum_{i=1}^{n} \|T_i(x)\|^p_Y \Big\}.
\end{aligned}$$

Hence we have that Θ_p is (ℓ_p^s, ℓ_p)-summing and $\pi_{(\ell_p^s,\ell_p)}(\Theta_p) \leq \mu(B_X)^{\frac{1}{p}}$. $\square$

Example 3.4 Let (Ω, μ) be a σ-finite measure space, X, Y be Banach spaces, $1 \leq p < \infty$ and denote by $L_p(\mu, X)$ the space of Bochner p-integrable functions. For each $f \in L_p(\mu, X)$, the operator $\Phi_f : \mathcal{L}(X, Y) \longrightarrow L_p(\mu, Y)$ defined by $\Phi_f(T)(w) = T(f(w))$, $w \in \Omega$, is (ℓ_p^s, ℓ_p)-summing and $\pi_{(\ell_p^s,\ell_p)}(\Phi_f) \leq \|f\|_{L_p(\mu,X)}$.

PROOF. Observe first that the operator is well defined. Let $T_1, \ldots, T_n$ operators from X into Y. If $E = \{w \in \Omega : f(w) \neq 0\}$ then

$$\begin{aligned}
\sum_{i=1}^{n} \|\Phi_f(T_i)\|^p_{L_p(\mu,Y)} &= \sum_{i=1}^{n} \int_E \|T_i(f(w))\|_Y^p d\mu(w) \\
&= \sum_{i=1}^{n} \int_E \|T_i(\frac{f(w)}{\|f(w)\|})\|_Y^p \|f(w)\|^p d\mu(w) \\
&\leq \|f\|^p_{L_p(\mu,X)} \sup_{x \in B_X} \Big(\sum_{i=1}^{n} \|T_i x\|_Y^p\Big).
\end{aligned}$$

$\square$

Example 3.5 Let $1 \leq p < \infty$ and let X, Y be Banach spaces. Any sequence $(T_n)_{n=1}^{\infty} \in \ell_p(X, Y)$ induces an operator

$$\begin{array}{rccl}
\Delta_T : & \mathcal{L}(\ell_1, X) & \longrightarrow & \ell_p(Y) \\
& S & \rightsquigarrow & (T_n S(e_n))_{n=1}^{\infty}
\end{array}$$

which is (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s,\ell_p)}(\Delta_T) \leq \|(T_n)\|_{\ell_p(X,Y)}$.

PROOF. Take $S_1, \ldots, S_N \in \mathcal{L}(\ell_1, X)$, then

$$\sum_{k=1}^{N} \|\Delta_T(S_k)\|^p = \sum_{k=1}^{N} \sum_{n=1}^{\infty} \|T_n S_k(e_n)\|_Y^p \leq (\sum_{n=1}^{\infty} \|T_n\|^p) \sup_{a \in B_{\ell_1}} \Big(\sum_{k=1}^{N} \|S_k(a)\|_X^p\Big).$$

This gives that $\pi_{(\ell_p^s,\ell_p)}(\Delta_T) \leq \|(T_n)\|_{\ell_p(X,Y)}$. $\square$

To finish the section we give several equivalent formulations for the notion of (ℓ_p^s, ℓ_p)-summing operator.

Note that we can write the fact $(T_n)_{n\in\mathbb{N}} \in \ell_p^s(X,Y)$ using duality:

$$\sup_{x\in B_X}\Big(\sum_{i=1}^{n}\|T_i x\|_Y^p\Big)^{\frac{1}{p}} = \sup_{(y_i^*)\in B_{\ell_{p'}^n(Y^*)}}\Big\|\sum_{i=1}^{n} T_i^* y_i^*\Big\|_{X^*}.$$

Proposition 3.6 *Let $1 \le p < \infty$ and $\Phi \in \mathcal{L}(\mathcal{L}(X,Y),Z)$. The following statements are equivalent:*

i) $\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$.

ii) There exists a constant $C > 0$ such that

$$\Big(\sum_{i=1}^{n}\|\Phi(T_i)\|_Z^p\Big)^{\frac{1}{p}} \le C \sup_{(y_i^*)\in B_{\ell_{p'}^n(Y^*)}}\Big\|\sum_{i=1}^{n} T_i^* y_i^*\Big\|_{X^*} \tag{3.1}$$

for every $T_1,\dots,T_n$ in $\mathcal{L}(X,Y)$.

Moreover $\pi_{(\ell_p^s,\ell_p)}(\Phi) = inf\{C : C$ verifying (3.1)$\}$.

Proposition 3.7 *Let $1 \le p < \infty$ and $\Phi \in \mathcal{L}(\mathcal{L}(X,Y),Z)$. The following statements are equivalent:*

i) $\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$.

ii) For each $u \in \mathcal{L}(X,\ell_p(Y))$ the operator

$$\begin{array}{rccc} \Phi_u : & \mathcal{L}(\ell_p(Y),Y) & \longrightarrow & Z \\ & T & \rightsquigarrow & \Phi(Tu) \end{array}$$

is (ℓ_p^s,ℓ_p)-summing.

iii) There exist a constant c such that $\pi_{(\ell_p^s,\ell_p)}(\Phi_u) \le c\|u\|$ for each $n \in \mathbb{N}$ and each $u \in \mathcal{L}(X,\ell_p^n(Y))$.

Moreover

$$\begin{aligned} \pi_{(\ell_p^s,\ell_p)}(\Phi) &= \sup\{\pi_{(\ell_p^s,\ell_p)}(\Phi_u) : u \in \mathcal{L}(X,\ell_p(Y),||u||=1\} \\ &= \inf\{c : c \text{ verifies (iii)}\}. \end{aligned}$$

PROOF. (i)⇒(ii) Observe that $\Psi_u : \mathcal{L}(\ell_p(Y), Y) \to \mathcal{L}(X, Y)$, $\Psi_u(T) = Tu$, is strongly (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s, \ell_p^s)}(\Psi_u) \leq \|u\|$ and $\Phi_u = \Phi\Psi_u$. Then by (iii) in Proposition 2.17, Φ_u is (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s, \ell_p)}(\Phi_u) \leq \pi_{(\ell_p^s, \ell_p)}(\Phi) \cdot \|u\|$.

(ii)⇒(iii) It follows easily from the closed graph theorem.

(iii)⇒(i) Let $T_1, \dots, T_n$ be operators in $\mathcal{L}(X, Y)$ and let $S_n : X \longrightarrow \ell_p^n(Y)$ be defined by $S_n(x) = (T_i x)_{i=1}^n$. Note that $\|S_n\|_{X, \ell_p^n(Y)} = \|(T_i)_{i=1}^n\|_{\ell_p^s(X,Y)}$.

For $i = 1, \dots, n$ we denote by $Q_{i,n}$ the projections $Q_{i,n} : \ell_p^n(Y) \longrightarrow Y$ given by $Q_{i,n}(y_j)_{j=1}^n = y_i$. Hence for each $n \in \mathbb{N}$ and $i = 1, \dots, n$, $T_i = Q_{i,n} S_n$ and $\Phi(T_i) = \Phi(Q_{i,n} S_n) = \Phi_{S_n}(Q_{i,n})$. By hypothesis Φ_{S_n} is (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s, \ell_p)}(\Phi_{S_n}) \leq B\|S_n\|_{X, \ell_p^n(Y)}$, thus

$$\begin{aligned}
\Big(\sum_{i=1}^n \|\Phi(T_i)\|_Z^p\Big)^{\frac{1}{p}} &= \Big(\sum_{i=1}^n \|\Phi_{S_n}(Q_{i,n})\|_Z^p\Big)^{\frac{1}{p}} \\
&\leq B\|S_n\|_{X, \ell_p^n(Y)} \sup\Big\{\Big(\sum_{i=1}^n \|Q_{i,n}(\lambda)\|_Y^p\Big)^{\frac{1}{p}} : \lambda \in B_{\ell_p^n(Y)}\Big\} \\
&\leq B\|(T_i)_{i=1}^n\|_{\ell_p^s(X,Y)}.
\end{aligned}$$

Consequently, Φ is (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s, \ell_p)}(\Phi) \leq B$. □

4 Relations between the classes

As in the case of p-summing operators we have the following inclusions.

Proposition 4.1 *Let X, Y, Z and W be Banach spaces and $1 \leq p < q \leq \infty$. Then*

(i) $\Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y, Z)) \subset \Pi_{(\ell_q^w, \ell_q^s)}(X, \mathcal{L}(Y, Z))$.

(ii) $\Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), Z) \subset \Pi_{(\ell_q^s, \ell_q)}(\mathcal{L}(X, Y), Z)$.

(iii) $\Pi_{(\ell_p^s, \ell_p^s)}(\mathcal{L}(X, Y), \mathcal{L}(Z, W)) \subset \Pi_{(\ell_q^s, \ell_q^s)}(\mathcal{L}(X, Y), \mathcal{L}(Z, W))$.

PROOF. (i) follows from Theorem 2.13 and $\Pi_p(X, Z) \subset \Pi_q(X, Z)$.

To see (ii) we take $\Phi \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), Z)$ and $T_k \in \mathcal{L}(X, Y)$ for $k = 1, ..., n$. Let us write $\sum_{k=1}^n \|\Phi(T_k)\|^q = \sum_{k=1}^n \|\Phi(\beta_k T_k)\|^p$ where $\beta_k = \|\Phi(T_k)\|^{\frac{q-p}{p}}$.

Applying Hölder's inequality with conjugate indices $\frac{q}{p}$ and $\frac{q}{q-p}$ we get

$$\begin{aligned}
\sum_{k=1}^{n} \|\Phi(T_k)\|^q &\leq (\pi_{(\ell_p^s,\ell_p)}(\Phi))^p \sup_{x\in B_X} \Big(\sum_{k=1}^{n} \|T_k(x)\|^p \beta_k^p\Big) \\
&\leq (\pi_{(\ell_p^s,\ell_p)}(\Phi))^p \sup_{x\in B_X} \Big(\sum_{k=1}^{n} \|T_k(x)\|^q\Big)^{p/q} \Big(\sum_{k=1}^{n} \beta_k^{\frac{qp}{q-p}}\Big)^{\frac{(q-p)}{q}} \\
&= (\pi_{(\ell_p^s,\ell_p)}(\Phi))^p \sup_{x\in B_X} \Big(\sum_{k=1}^{n} \|T_k(x)\|^q\Big)^{p/q} \Big(\sum_{k=1}^{n} \|\Phi(T_k)\|^q\Big)^{\frac{(q-p)}{q}}.
\end{aligned}$$

This gives that

$$\Big(\sum_{k=1}^{n} \|\Phi(T_k)\|^q\Big)^{1/q} \leq \pi_{(\ell_p^s,\ell_p)}(\Phi) \sup_{x\in B_X} \Big(\sum_{k=1}^{n} \|T_k(x)\|^q\Big)^{1/q}.$$

(iii) now follows using Theorem 2.16 and (ii). □

Next we are going to see that, under some assumptions on the Banach spaces, these classes coincide, at least for certain values of p and q.

Let us recall that some classical result, due to B. Maurey (see [10] or [3], Theorem 11.13), states that if Y has cotype 2 and $2 < p < \infty$ then

$$\Pi_p(X,Y) = \Pi_2(X,Y).$$

Using Theorem 2.13 we get the following corollary.

Corollary 4.2 *Let X, Y, Z be Banach spaces and $2 < p < \infty$. Assume that Z has cotype 2, then $\Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z)) = \Pi_{(\ell_2^w,\ell_2^s)}(X,\mathcal{L}(Y,Z))$.*

It is natural to ask whether there are generalizations in the framework of (ℓ_p^s,ℓ_p)-summing operators. Next result is the extension of Theorem 1.2.3 in [6] to our setting.

Theorem 4.3 *Let X, Y, Z and W be Banach spaces and $2 < p < \infty$. Assume that Y has type 2 and Z has cotype 2. Then*

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z) = \Pi_{(\ell_2^s,\ell_2)}(\mathcal{L}(X,Y),Z).$$

PROOF. Let $\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$ and $T_1,\dots,T_n$ be a finite sequence of operators in $\mathcal{L}(X,Y)$. Using that Z has cotype 2 one has

$$\begin{aligned}\Big(\sum_{i=1}^n \|\Phi(T_i)\|_Z^2\Big)^{\frac{1}{2}} &\leq C_2(Z)\int_0^1 \Big\|\sum_{i=1}^n r_i(t)\Phi(T_i)\Big\|_Z dt\\ &\leq C_2(Z)\Big(\int_0^1 \Big\|\Phi\Big(\sum_{i=1}^n r_i(t)T_i\Big)\Big\|_Z^p dt\Big)^{\frac{1}{p}}.\end{aligned}$$

Observe that $\sum_{i=1}^n r_i(t)T_i$ is a simple function $\sum_{j=1}^{2^n} S_j\chi_{[\frac{j-1}{2^n},\frac{j}{2^n})}$ for some operators S_j and then

$$\begin{aligned}\int_0^1 \Big\|\Phi\Big(\sum_{i=1}^n r_i(t)T_i\Big)\Big\|_Z^p dt &= 2^{-n}\sum_{j=1}^{2^n}\|\Phi(S_j)\|_Z^p\\ &\leq (\pi_{(\ell_p^s,\ell_p)}(\Phi))^p 2^{-n}\sup_{x\in B_X}\sum_{j=1}^{2^n}\|S_j(x)\|_Y^p\\ &\leq (\pi_{(\ell_p^s,\ell_p)}(\Phi))^p \sup_{x\in B_X}\int_0^1\Big\|\sum_{i=1}^n r_i(t)T_i(x)\Big\|_Y^p dt.\end{aligned}$$

Hence, using Kahane's inequality and the type 2 condition on Y, it yields

$$\begin{aligned}\Big(\sum_{i=1}^n \|\Phi(T_i)\|_Z^2\Big)^{\frac{1}{2}} &\leq \pi_{(\ell_p^s,\ell_p)}(\Phi)C_2(Z)\sup_{x\in B_X}\Big(\int_0^1\Big\|\sum_{i=1}^n r_i(t)T_i(x)\Big\|_Y^p dt\Big)^{\frac{1}{p}}\\ &\leq \pi_{(\ell_p^s,\ell_p)}(\Phi)C_2(Z)K_p\sup_{x\in B_X}\int_0^1\Big\|\sum_{i=1}^n r_i(t)T_i(x)\Big\|_Y dt\\ &\leq \pi_{(\ell_p^s,\ell_p)}(\Phi)C_2(Z)T_2(Y)K_p\sup_{x\in B_X}\Big(\sum_{i=1}^n\|T_i(x)\|_Y^2\Big)^{\frac{1}{2}}\end{aligned}$$

and the proof is finished. $\square$

Corollary 4.4 *Let $2<p<\infty$, X, Y, Z and W be Banach spaces. If Y has type 2 and W has cotype 2, then*

$$\Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,W)) = \Pi_{(\ell_2^s,\ell_2^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,W)).$$

In particular we get the following applications to operators acting on $\ell_r^w(X) = \mathcal{L}(\ell_{r'}, X)$.

Corollary 4.5 *Let X and Y be Banach spaces of type 2 and cotype 2 respectively. If $2 < p < \infty$ and $1 \leq r, q < \infty$, then*
(i) $\Pi_{(\ell_p^s, \ell_p)}(\ell_r^w(X), Y) = \Pi_{(\ell_2^s, \ell_2)}(\ell_r^w(X), Y)$.
(ii) $\Pi_{(\ell_p^s, \ell_p^s)}(\ell_r^w(X), \ell_q^w(Y)) = \Pi_{(\ell_2^s, \ell_2^s)}(\ell_r^w(X), \ell_q^w(Y))$.

5 Open problems

Recall the domination theorem in the setting of (p, Y)-summing operators proved by S. Kisliakov.

Theorem 5.1 *(see [6]) Let $1 \leq p < \infty$ and X, Y and Z be Banach spaces. An operator $T : X \check{\otimes} Y \to Z$ is (p, Y)-summing if and only if there are a probability measure μ on $(B_{X^*}, \sigma(X^*, X))$ and a constant $C > 0$ such that for all $u \in X \otimes Y$ one has*

$$\|T(u)\|_Z^p \leq C^p \int_{B_{X^*}} \|u(x^*)\|_Y^p d\mu(x^*).$$

Moreover $\pi_p^Y(T)$ is the least of the constants verifying the previous estimate.

Question 1. Let $1 \leq p < \infty$ and X, Y and Z be Banach spaces. Assume $T : \mathcal{L}(X^*, Y) \to Z$ is (ℓ_p^s, ℓ_p)-summing operator.

Does there exist a probability measure μ on $(B_{X^*}, \sigma(X^*, X))$ and a constant $C > 0$ such that

$$\|T(u)\|_Z^p \leq C^p \int_{B_{X^*}} \|u(x^*)\|_Y^p d\mu(x^*)$$

for all $u \in \mathcal{L}(X^*, Y)$?

The reader should be aware that the classical proofs can be repeated, under certain assumptions of reflexivity and separability on the spaces X and Y. The difficulty appears when dealing with general Banach spaces.

Now let us point out that the authors, relying upon Theorem 5.1, have been able to show the following result about composition operators for (p, Y)-summing operators.

Theorem 5.2 *(see [1]) Let X, Y, Z and W be Banach spaces and $1 \leq p, q, s \leq \infty$ where $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} \leq 1$. If $T \in \Pi_p^Y(X\check{\otimes}Y, Z)$ and $S \in \Pi_s(Z, W)$ then the operator $ST \in \Pi_q^Y(X\check{\otimes}Y, W)$ and $\pi_q^Y(ST) \leq \pi_s(S) \cdot \pi_p^Y(T)$.*

Since we do not have at our disposal such a domination theorem in general we do not know the answer to the following question.

Question 2. Let X, Y, Z and W be Banach spaces and $1 \leq p, q, r \leq \infty$ where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} \leq 1$. Let $T \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), Z)$ and $S \in \Pi_r(Z, W)$.

Does ST belong to $\Pi_{(\ell_q^s, \ell_q)}(\mathcal{L}(X, Y), W)$?

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References

[1] O. Blasco and T. Signes. *Remarks on (p, q, Y)-summing operators.* Submitted.

[2] A. Defant and K. Floret. *"Tensor norms and operator ideals"*. Noth-Holland, Amsterdam, 1993.

[3] J. Diestel, H. Jarchow y A. Tonge, *"Absolutely Summing Operators"*. Cambridge University Press, 1995.

[4] N. Dinculeanu, *"Vector Measures"*. Pergamon Press, New York, 1967.

[5] H. Jarchow, *"Locally convex spaces"*. Teuber, Stuttgart, 1981.

[6] S. V. Kislyakov, *Absolutely summing operators on the disk algebra*, St. Petersburg Math. J. **3** (4) (1992), 705-774.

[7] J. Lindenstrauss and A. Pelczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. **29** (1968), 275-326.

[8] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces I*, Lecture Notes in Mathematics, vol. 338, Springer-Verlag (1973)

[9] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, New York (1979).

[10] B. Maurey, *"Théorèmes de factorisation pour les opérateurs à valeurs dans les espaces L^p"*. Soc. Math. France, Astérisque 11, Paris 1974.

[11] A. Pietsch, *"Operator Ideals"*. Noth-Holland, Amsterdam, 1980.

[12] G. Pisier, *"Factorization of linear operators and geometry of Banach spaces"*. Amer. Math. Soc. Vol 60 , Providence,R.I. 1986.

[13] C. Swartz, *Absolutely summing and dominated operators on spaces of vector-valued continuos functions*, Trans. Amer. Math. Soc. **179** (1973), 123-131.

[14] P. Wojtaszczyk, *"Banach Spaces for Analysts"*. Cambridge University Press, Cambridge, 1991.

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