K-CONVEXITY AND DUALITY FOR ALMOST SUMMING OPERATORS

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Abstract

For a fixed sequence $f_{\cdot} = (f_n)$ of independent identically distributed symmetric random variables with $\mathbf{E}f_1^2 = 1$, we introduce the notion of $K^{f_{\cdot}}$-convex Banach space and the notions (f_n)-bounding and (f_n)-converging operators acting between Banach spaces. It is shown that the dual of the space of (f_n)-converging operators between a Hilbert space and a $K^{f_{\cdot}}$-convex Banach space admits a precise description in terms of trace duality. The obtained results recover similar formulations for almost summing and γ-Radonifying operators.

1 Introduction

Given two Banach spaces X and Y we shall be dealing with the class of operators $u : X \to Y$ which map sequences (x_n) in $l_2^{weak}(X)$ into series $\sum u(x_n)f_n$ which converge in $L_2(\Omega; Y)$,where (f_n) is a sequence of independent identically distributed symmetric random variables with $\mathbf{E}f_1^2 = 1$. We shall call these operators (f_n)-converging operators and the class of all (f_n)-converging operators will be denoted by $\mathfrak{R}_{(f_n)}(X, Y)$. In the case (f_n) being the Rademacher sequence (r_n) they are called almost summing operators and denoted by $\Pi_{as}(X, Y)$ (see [**?**]) and for (f_n) being the standard gaussian sequences (γ_n) they are called γ-Radonifying operators and denoted by $\mathfrak{R}_\gamma(X, Y)$.

Our aim is to describe the dual of the $\mathfrak{R}_{(f_n)}(H, X)$ for an infinite dimensional separable Hilbert space H. Previous results for finite dimensional Hilbert space and (γ_n) were achieved in [**?**]. Motivated from her results we

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are introducing the space $\mathfrak{R}^{dual}_{(f_n)}(X, H)$ given by those continuous linear operators $v : X \longrightarrow H$ whose adjoint $v^* \in \mathfrak{R}_{(f_n)}(H, X^*)$. We are showing that for an arbitrary $u \in \mathfrak{R}_{(f_n)}(H, X)$ and $v \in \mathfrak{R}^{dual}_{(f_n)}(X, H)$, the operator $vu : H \longrightarrow H$ is nuclear and the linear functional $u \longrightarrow tr(vu)$ is continuous on $\mathfrak{R}_{(f_n)}(H, X)$. In this way $\mathfrak{R}^{dual}_{(f_n)}(X, H)$ can be identified with a subspace of $(\mathfrak{R}_{(f_n)}(H, X))^*$.

We shall be able to give a complete characterisation of the dual in terms of the trace duality only for $K^{f.}$-convex spaces X (see definition below). Namely, we are showing that the equality $(\mathfrak{R}_{(f_n)}(H, X))^* = \mathfrak{R}^{dual}_{(f_n)}(X, H)$ holds isomorphically in the sense of trace duality if and only if X is $K^{f.}$-convex.

As a corollary of our results we get that it is equivalent that X is a K-convex Banach space to the fact $(\Pi_{as}(H, X))^* = \Pi^{dual}_{as}(X, H)$ (cf. [?], p. 280) or to the fact $(\mathfrak{R}_\gamma(H, X))^* = \mathfrak{R}^{dual}_\gamma(X, H)$.

We are not using the general theory of conjugate operators ideals. Our arguments are based upon the study of the dual of the space $s_2[(f_n), X]$, which is given by sequences (x_n) such that $\sum x_n f_n$ is convergent in $L^2(\Omega, X)$. This dual space can be represented as $b_2[(f_n), X^*]$, the space of sequences (x_n^*) in the dual X^* such that the series $\sum x_n^* f_n$ has bounded partial sums in $L^2(\Omega, X^*)$, only in the case that X is $K^{f.}$-convex.

In particular we shall show that for any $(x_n) \in Rad(X)$ and $(x_n^*) \in Rad(X^*)$ we have $\sum |x_n^*(x_n)| < \infty$ and the linear functional $(x_n) \longrightarrow \sum x_n^*(x_n)$ is continuous on $Rad(X)$. Again the equality $(Rad(X))^* = Rad(X^*)$ holds if and only if X is K-convex. The equality $(Rad(X))^* = Rad(X^*)$ for the separable Banach spaces having type 2 was obtained in [?] and it was already pointed out in [?] for the general case.

The paper is divided into three sections. In the first one we are recalling the facts that will be used in the sequel and introduce a new notion of (f_n)-contractive Banach space (see definition below), that plays a particular role because it allows to connect the vector-valued sequence spaces with the spaces of operators which are (f_n)-bounding. The second section is devoted to analyse the duality for sequence spaces and in the last section we prove duality results for the spaces of (f_n)-converging operators between Hilbert and Banach spaces.

2 Notation and Auxiliary Results

2.1 Random vector series and (f_n)-contractivity.

For a given Banach space X the notations $l_p^{strong}(X)$ and $l_p^{weak}(X)$, $0 < p < \infty$ have the same meaning as in [?]. If $(\Omega, \mathfrak{A}, P)$ is a probability space and X is a Banach space then $L_p(\Omega, \mathfrak{A}, P; X)$ or shortly $L_p(\Omega; X)$ denote the ordinary space of X-valued strongly measurable functions $\xi : \Omega \longrightarrow X$, such that $\|\xi\|_p := \left(\int_\Omega \|\xi(\omega)\|^p dP(\omega)\right)^{1/p} < \infty$.

For a scalar or a vector-valued integrable function f, $\mathbf{E}f$ will denote the integral $\int_\Omega f dP$.

Throughout the paper (f_n) will stand for a sequence of independent identically distributed symmetric random variables on $(\Omega, \mathfrak{A}, P)$ such that $\mathbf{E}f_1^2 = 1$.

Let us recall the following notations from [?] (p. 316)

$$b_2[(f_n); X] = \Big\{(x_n) \in X^{\mathbb{N}} : \sup_n \Big\| \sum_{k=1}^n f_k x_k \Big\|_2 < \infty \Big\},$$

$$s_2[(f_n); X] = \Big\{(x_n) \in X^{\mathbb{N}} : \sum x_n f_n \text{ is convergent in } L_2(\Omega; X)\Big\},$$

$$S_2[(f_n); X] = \Big\{\sum x_n f_n : (x_n) \in s_2[(f_n); X]\Big\}.$$

Notice that $b_2[(f_n); X]$ is a Banach space with respect to the norm

$$|||(x_n)|||_{(f_n)} = \sup_n \Big\| \sum_{k=1}^n f_k x_k \Big\|_2 = \lim_n \Big\| \sum_{k=1}^n f_k x_k \Big\|_2.$$

The space $s_2[(f_n); X]$ is a closed subspace of $b_2[(f_n); X]$ and

$$|||(x_n)|||_{(f_n)} = \Big\| \sum_n x_n f_n \Big\|_2, \qquad (x_n) \in s_2[(f_n); X],$$

and also $S_2[(f_n); X]$ is a closed subspace of $L_2(\Omega; X)$. Evidently $s_2[(f_n); X]$ and $S_2[(f_n); X]$ are isometric.

The cases that have been very deeply studied correspond to (f_n) being either the sequence (r_n) of Rademacher functions on $[0, 1]$ with the Lebesgue measure or the sequence (γ_n) of independent standard Gaussian random variables on a probability space $(\Omega, \mathfrak{A}, P)$. We shall denote the space $s_2[(r_n); X]$ by $Rad(X)$, although in the literature the notation $Rad(X)$ is sometimes used for the space $S_2[(r_n); X]$.

Remark 2.1. In general the sets $b_2[(f_n); X]$ and $s_2[(f_n); X]$ are different; Actually (see [?], p. 347-348) $b_2[(f_n); X] = s_2[(f_n); X]$ if and only if X does not contain a subspace isomorphic to c_0.

Fix two numbers p, q, $1 < p \leq 2 \leq q < \infty$. We recall that a Banach space X is said to have (f_n)-type p, resp. (f_n)-cotype q if

$$l_p^{strong}(X) \subset s_2[(f_n); X],$$

resp. if

$$s_2[(f_n); X] \subset l_q^{strong}(X).$$

If X has (f_n)-type p (resp. (f_n)-cotype q), then the norm of inclusion operator $l_p^{strong}(X) \subset s_2[(f_n); X]$ (resp. $s_2[(f_n); X] \subset l_q^{strong}(X)$) is denoted by $t_p(f_., X)$, (resp. $c_q(f_., X)$) and is called the type p constant, (resp. cotype q constant of X).

The spaces of (r_n)-type p, (resp. (r_n)-cotype q) are named simply type p (resp. cotype q).

Remark 2.2. (a) It is known that

$$||\sum_{k=1}^{n} r_k x_k||_2 \leq c_1 ||\sum_{k=1}^{n} f_k x_k||_2 \tag{2.1}$$

for all $x_1, x_2, ..., x_n \in X$ and $n \in \mathbb{N}$, where $c_1 = (\mathbf{E}|f_1|)^{-1}$ (see [**?**], pp. 323-324). Therefore

$$b_2[(f_n); X] \subset b_2[(r_n); X].$$

and

$$s_2[(f_n); X] \subset Rad(X).$$

(b) It follows from (a) that if X is of (f_n)-type p, then X is type p and $t_p(r_., X) \leq c_1 t_p(f_., X)$. Conversely, if X is type p, then X is also (f_n)-type p and $t_p(f_., X) \leq t_p(r_., X)$. (In fact, fix $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $t \in [0, 1]$. Since (f_n) is i.i.d. symmetric sequence, we can write

$$||\sum_{k=1}^{n} x_k f_k||_2^2 = \mathbf{E}||\sum_{k=1}^{n} x_k f_k||^2 = \mathbf{E}||\sum_{k=1}^{n} x_k f_k r_k(t)||^2, \ \forall t \in [0, 1]$$

Integrating with respect to t, using Fubini's theorem and Minkowski's inequality, we obtain

$$||\sum_{k=1}^{n} x_k f_k||_2^2 = \mathbf{E} \int_0^1 ||\sum_{k=1}^{n} x_k f_k r_k(t)||^2 dt \leq$$

$$\leq t_p^2(r_n, X)\mathbf{E}(\sum_{k=1}^{n} ||x_k||^p |f_k|^p)^{2/p} \leq t_p^2(r_n, X)(\sum_{k=1}^{n} ||x_k||^p)^{2/p}.)$$

(c) Again using (a) we have that if X has cotype q, then$X$ has(f_n)-cotype q and $c_q(f_\cdot, X) \leq c_q(r_\cdot, X)$. The question of validity converse statement is more delicate (see (e) below).

Suppose X has (f_n)-cotype q. Suppose additionally that $\mathbf{E}|f_1|^r < \infty$, $\forall r > 0$. Observe that then X does not contain l^n_∞ uniformly (this is not difficult to check). Using this and (d) (see below) we can conclude that then X has cotype q and $c_q(r_\cdot, X) \leq C_{f_1}(X) c_q(f_\cdot, X)$.

(d) When a Banach space X does not contain l^n_∞ uniformly and $\mathbf{E}|f_1|^r < \infty$ for all r, $0 < r < \infty$, then there is a constant $C_{f_1}(X)$ such that

$$||\sum_{k=1}^{n} f_k x_k||_2 \leq C_{f_1}(X)||\sum_{k=1}^{n} r_k x_k||_2$$

for all $x_1, x_2, ..., x_n \in X$ and $n \in \mathbb{N}$. This is an important result of [**?**] (see Cor. 1.3 and Remark 1.5 (d) of that paper).

Remark 2.3. Let us recall that from the Contraction Principle (see [**?**], page 301) we have that

$$||\sum_{k=1}^{n} \alpha_k x_k f_k||_2 \leq max_{1 \leq k \leq n}|\alpha_k|||\sum_{k=1}^{n} f_k x_k||_2 \tag{2.2}$$

for all $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$, $x_1, x_2, ..., x_n \in X$ and $n \in \mathbb{N}$.

Therefore it follows that if $(x_n) \in b_2[(f_n); X]$ (respect. $(x_n) \in s_2[(f_n); X]$) and $(\alpha_n) \in l^\infty$ then $(\alpha_n x_n) \in b_2[(f_n); X]$ (respect. $(\alpha_n x_n) \in s_2[(f_n); X]$).

We shall need the following stronger contractivity property.

Definition 2.1 *A Banach space X is (f_n)-contractive if there exists a constant $c > 0$ such that for any $(x_n) \in b_2[(f_n); X]$ and any sequence (y_n) in X verifying*

$$\sum |\langle y_n, x^* \rangle|^2 \leq \sum |\langle x_n, x^* \rangle|^2, \ \forall x^* \in X^*,$$

we have that $(y_n) \in b_2[(f_n); X]$ and

$$|||(y_n)|||_{(f_n)} \leq c|||(x_n)|||_{(f_n)}.$$

The infimum of all constants c for which the last inequality holds is called the (f_n)-contractivity constant of X and it will be denoted $b_{f_\cdot}(X)$.

This new notion will be very relevant for our purposes. It is justified by the following two assertions, first of which is well known (see, e.g., [**?**], Th. 8)

Proposition 2.1 *Any Banach space is (γ_n)-contractive and $b_{\gamma_\cdot}(X) = 1$.*

Proposition 2.2 *(see [?]) Let X be a Banach space. The following are equivalent:*

(i) X does not contain l_∞^n *uniformly.*

(ii) X is (r_n)*-contractive.*

(iii) X is (f_n)*-contractive for any* (f_n) *such that* $\mathbf{E}|f_1|^p < \infty$ *for all* p, $0 < p < \infty$.

Moreover, (i) implies that $b_{r.}(X) \le C_{\gamma_1}(X)$*, where* $C_{\gamma_1}(X)$ *is the constant from Remark 2.2 (b).*

2.2 Converging and bounding operators.

Let us now recall some definitions and notation on operators to be used later on. Let X, Y be Banach spaces. Let us say that a continuous linear operator $u : X \longrightarrow Y$ is (f_n)-*bounding* (respectively (f_n)-*converging*) if for any $(x_n) \in l_2^{weak}(X)$ we have $(ux_n) \in b_2[(f_n);Y]$ (respectively $(ux_n) \in s_2[(f_n);Y)]$). Denote by $\Pi_{(f_n)}(X,Y)$ (respectively by $\mathfrak{R}_{(f_n)}(X,Y)$) the set of all (f_n)-bounding (respectively of all (f_n)-converging) operators $u : X \longrightarrow Y$.

In the standard way it can be shown, that a linear operator $u : X \longrightarrow Y$ is (f_n)-bounding if and only if it is (f_n)-summing, i.e., there is a constant $c > 0$ such that the inequality

$$\Big\| \sum_{k=1}^{n} ux_k f_k \Big\|_2 \le c \sup_{\|x^*\|\le 1} \left(\sum_{k=1}^{n} |x^*(x_k)|^2 \right)^{1/2}$$

holds for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$. If $u \in \Pi_{(f_n)}(X,Y)$ then the infimum of the constants c for which the above inequality holds shall be denoted by $\|u\|_{(f_n)}$ and called the (f_n)-bounding norm of u.

It can be shown that $(\Pi_{(f_n)}(X,Y),\ \| \ \|_{(f_n)})$ is a Banach space and $\mathfrak{R}_{(f_n)}(X,Y)$ is a closed subspace of it.

In [?] (r_n)-converging operators are called *almost summing* and the corresponding space is denoted by $\Pi_{as}(X,Y)$; the norm $\|u\|_{(r_n)}$ is denoted there as $\pi_{as}(u)$. Therefore in our notations $\Pi_{as}(X,Y)$ is $\mathfrak{R}_{(r_n)}(X,Y)$.

The (γ_n)-bounding operators with the name of γ-*summing* operators were introduced in [?]; the notion was discovered independently in [?]. The (γ_n)-converging operators sometimes are called γ-Radonifying operators. Already in [?] is remarked that $\Pi_{(\gamma_n)}(H,c_0) \ne \mathfrak{R}_{(\gamma_n)}(H,c_0)$.

Remark 2.4. Notice, that in [?] it is stated incorrectly, that $\Pi_{as}(X,Y) = \Pi_{(r_n)}(X,Y)$; in fact it can be shown that if H is an infinite dimensional Hilbert space, then for a Banach space Y the equality $\Pi_{as}(H,Y) = \Pi_{(r_n)}(H,Y)$ holds if and only if Y does not contain a subspace isomorphic to c_0 (see [?]).

Notice also that the notion of (f_n)-summing operators, where (f_n) is an arbitrary orthonormal sequence, is introduced and studied in [?] and [?].

It follows from Remark 2.2 that, in general,

$$\Pi_{(f_n)}(X,Y) \subset \Pi_{(r_n)}(X,Y). \tag{2.3}$$

The following result, obtained in [?], will be important in further considerations. We formulate it in our notations.

Theorem 2.1 *(See [?], p. 240, theorem 12.12). Let X, Y be Banach spaces. Then $\Pi_{(r_n)}(X,Y) = \Pi_{(\gamma_n)}(X,Y)$ and*

$$\sqrt{2/\pi}\pi_{as}(u) \leq \|u\|_{(\gamma_n)} \leq \pi_{as}(u)$$

for any $u \in \Pi_{(r_n)}(X,Y)$.

Let us collect some easy relationships between (f_n)-summing operators and other well-known classes of operators.

Remark 2.5. Denoting $\Pi_p(X,Y)$ and $\Pi_{p,q}(X,Y)$ the space of p-summing operators and (p,q)-summing operators. An application of Pietch's domination theorem allows us to get the following observations:

(a) $\Pi_2(X,Y) \subset \Pi_{(f_n)}(X,Y)$.
Moreover $\|u\|_{(f_n)} \leq \pi_2(u)$ for all $u \in \Pi_2(X,Y)$.

(b) If Y is of cotype $q \geq 2$ then $\Pi_{(f_n)}(X,Y) \subset \Pi_{q,2}(X,Y)$.
Moreover $\pi_q(u) \leq (\mathbf{E}|f_1|)^{-1}\|u\|_{(f_n)}c_q(r_., Y)$ for all $u \in \Pi_{(f_n)}(X,Y)$ where $c_q(r_., Y)$ is the cotype q-constant of Y.

(c) If H is an infinite dimensional separable Hilbert space and
$\Pi_{(f_n)}(X,Y) \subset \Pi_2(X,Y)$, then Y is of cotype 2. (In fact, by remark 2.2 (a) we have also $\Pi_{(r_n)}(X,Y) \subset \Pi_2(X,Y)$, this and Theorem 2.1 imply $\Pi_{(\gamma_n)}(X,Y) \subset \Pi_2(X,Y)$. The last inclusion implies that Y is of cotype 2 (see [?])).

(d) If we assume (f_n) is such that $\mathbf{E}|f_1|^p < \infty$ for some $p > 2$ then $\Pi_p(X,Y) \subset \Pi_{(f_n)}(X,Y)$ for all $u \in \Pi_p(X,Y)$.
Moreover $\|u\|_{(f_n)} \leq \pi_p(u)\|f_1\|_p B_p$ where B_p is the constant appearing in Kintchine's inequality.

When dealing with the particular case X being a separable Hilbert space much easier descriptions of $\Pi_{(f_n)}(H,Y)$ can be obtained, at least for some spaces Y. To formulate the corresponding result we need some more notations.

Let X be a Banach space and (e_n) be an orthonormal bases in H. Denote by $\Pi^{(e_n)}_{(f_n)}(H,X)$, (respectively $\mathfrak{R}^{(e_n)}_{(f_n)}(H,X)$),the set of continuous linear operators $u : H \to X$ such that $(ue_n) \in b_2[(f_n),X]$, (respectively $(ue_n) \in s_2[(f_n),X]$). Evidently these set are vector subspaces of $L(H,X)$. The functional $|| \, ||^{(e_n)}_{(f_n)}$ defined by the equality

$$||u||^{(e_n)}_{(f_n)} = |||(ue_n)|||_{(f_n)}$$

is a norm on $\Pi^{(e_n)}_{(f_n)}(H,X)$,and$\Pi^{(e_n)}_{(f_n)}(H,X)$ is a Banach space with this norm and also $\mathfrak{R}^{(e_n)}_{(f_n)}(H,X)$ a closed subspace of it.

Notice that $\mathfrak{R}_{(f_n)}(H,X) \subset \mathfrak{R}^{(e_n)}_{(f_n)}(H,X)$, $\Pi_{(f_n)}(H,X) \subset \Pi^{(e_n)}_{(f_n)}(H,X)$ and corresponding inclusion maps have norms one.

We have the following well-known characterisation of (γ_n)-bounding and (γ_n)-converging operators.

Proposition 2.3 *Let H be a separable Hilbert space, X be a Banach space and (e_n) be an orthonormal bases of H. Then following assertions are valid.*

(a) $\Pi_{(\gamma_n)}(H,X) = \Pi^{(e_n)}_{(\gamma_n)}(H,X)$ and the equality

$$\|u\|_{(\gamma_n)} = \sup_n \Big\| \sum_{k=1}^{n} \gamma_k u e_k \Big\|_2 = \lim_n \Big\| \sum_{k=1}^{n} \gamma_k u e_k \Big\|_2$$

holds for any $u \in \Pi_{(\gamma_n)}(H,X)$.

(b)$\mathfrak{R}_{(\gamma_n)}(H,X) = \mathfrak{R}^{(e_n)}_{(\gamma_n)}(H,X)$ and the equality

$$\|u\|_{(\gamma_n)} = \Big\| \sum \gamma_k u e_k \Big\|_2 \tag{2.4}$$

holds for any $u \in \mathfrak{R}_{(\gamma_n)}(H,X)$.

Remark 2.6. In page 82 of [?] the norm $\|u\|_{(\gamma_n)}$ denoted as $l(u)$ and it is stated incorrectly that the equality (2.4) holds for all $u \in L_\gamma(H,X) := \Pi_{(\gamma_n)}(H,X)$.

Remark 2.7. It is interesting to note that if we replace in Proposition 2.3 (γ_n) by (r_n), then the corresponding conclusions (without the equalities for norms) remain valid if and only if X is of finite (Rademacher) cotype (see [?] theorem 1.7).

Remark 2.8. If X has cotype 2 then X is (f_n)-contractive for any sequence (f_n).

Indeed, let us take $(x_n) \in b_2[(f_n); X]$ and $(y_n) \in l_2^{weak}(X)$ such that

$$\sum |\langle y_n, x^* \rangle|^2 \leq \sum |\langle x_n, x^* \rangle|^2, \ \forall x^* \in X^*.$$

We need to show that $(y_n) \in b_2[(f_n); X]$ and

$$|||(y_n)|||_{(f_n)} \leq C|||(x_n)|||_{(f_n)}$$

for certain constant $C > 0$.

Since $(x_n) \in b_2[(f_n); X]$, then Remark 2.2 (a) shows that $(x_n) \in b_2[(r_n); X]$.

Let us fix a Hilbert space H and an orthonormal bases (e_n) and consider the operator $u : H \to X$ given by $u(e_n) = x_n$. Since Remark 2.7 gives that $\Pi_{(r_n)}(H, X) = \Pi_{(r_n)}^{(e_n)}(H, X)$ we have that $u \in \Pi_{(r_n)}(H, X)$.

In particular (see Remark 2.5 (c)) $u \in \Pi_2(H, X)$. On the other hand if $v : H \to X$ is given by $v(e_n) = y_n$, then there exists $w : H \to H$ such that $v = uw$. Hence we get that $v \in \Pi_2(H, X)$ and therefore (see Remark 2.5 (a)) we have that $v \in \Pi_{(f_n)}(H, X)$ what, in particular, shows that $(y_n) \in b_2[(f_n); X]$. $\blacksquare$.

In general the following assertion is true:

Theorem 2.2 *Let X be a Banach space, H be a separable Hilbert space and (e_n) a fixed orthonormal basis of H. The following are equivalent:*

(i) X is (f_n)-contractive.

(ii) $\Pi_{(f_n)}(H, X) = \Pi_{(f_n)}^{(e_n)}(H, X)$ and there is a constant c_2 such that

$$||u||_{(f_n)} \leq c_2 \sup_n \Big\| \sum_{k=1}^{n} f_k u e_k \Big\|_2 = c_2 \lim_n \Big\| \sum_{k=1}^{n} f_k u e_k \Big\|_2, \ \forall u \in \Pi_{(f_n)}(H, X).$$

Moreover, (i) implies that in (ii) we can put $c_2 = b_{f.}(X)$ and (ii) implies that $b_{f.}(X) \leq c_2$.

Proof. (i)$\Rightarrow$(ii). It is enough to prove that

$$\Pi_{(f_n)}^{(e_n)}(H, X) \subset \Pi_{(f_n)}(H, X).$$

Take arbitrarily $u \in \Pi_{(f_n)}^{(e_n)}(H, X)$. Let us show that $u \in \Pi_{(f_n)}(H, X)$. Take $(h_n) \in l_2^{weak}(H)$ such that $||(h_n)||_2^w = 1$ and denote by $B : H \to H$ the norm one operator for which $B^* e_n = h_n$ for all $n \in \mathbb{N}$, that is

$$Bh = \sum (h|h_n) e_n.$$

Observe now that if $y_n = uh_n = uB^*e_n$ then we have

$$\sum |\langle y_n, x^*\rangle|^2 = ||Bu^*x^*||^2 \leq ||u^*x^*||^2 = \sum |\langle ue_n, x^*\rangle|^2$$

for all $x^* \in X^*$. Therefore, since $(ue_n) \in b_2[(f_n); X]$ and X is (f_n)-contractive, it follows that $(y_n) \in b_2[(f_n); X]$ and

$$|||(uh_n)|||_{(f_n)} \leq b_{f.}(X)|||(ue_n)|||_{(f_n)}.$$

Consequently

$$||u||_{(f_n)} \leq b_{f.}(X)||u||_{(f_n)}^{(e_n)}.$$

$(ii) \Rightarrow (i)$. Take $(x_n) \in b_2[(f_n); X]$ and $(y_n) \in l_2^{weak}(X)$ such that

$$\sum |\langle y_n, x^*\rangle|^2 \leq \sum |\langle x_n, x^*\rangle|^2, \ \forall x^* \in X^*. \tag{2.5}$$

We need to show that $(y_n) \in b_2[(f_n); X]$ and

$$|||(y_n)|||_{(f_n)} \leq c_2|||(x_n)|||_{(f_n)}.$$

Since $(x_n) \in b_2[(f_n); X]$ there is $u \in \Pi_{(f_n)}^{(e_n)}(H, X)$ such that $ue_n = x_n, \ \forall n \in \mathbb{N}$.

According to (ii) we have $u \in \Pi_{(f_n)}(H, X)$. Therefore it is sufficient to find $(h_n) \in l_2^{weak}(H)$ such that $uh_n = y_n, \ \forall n \in \mathbb{N}$. For this we shall use (2.5). There is a continuous linear operator $v : H \to X$ such that $ve_n = y_n, \ \forall n \in \mathbb{N}$. So we have

$$v^*x^* = \sum_n \langle y_n, x^*\rangle e_n, \ \forall x^* \in X^*.$$

Observe that

$$||v^*x^*||^2 = \sum_n |\langle y_n, x^*\rangle|^2 \leq \sum_n |\langle x_n, x^*\rangle|^2 = ||u^*x^*||^2, \ \forall x^* \in X^*$$

The last inequality implies there exists a continuous linear operator $B : H \to H$ such that $||B|| \leq 1$ and $Bu^* = v^*$. This implies that $v = uB^*$. Denote $h_n = B^*e_n, \ \forall n \in \mathbb{N}$. Then evidently $(h_n) \in l_2^{weak}(H), \ ||(h_n)||_2^w \leq 1$ and

$$uh_n = uB^*e_n = ve_n = y_n, \ \forall n \in \mathbb{N}$$

Therefore $(y_n) \in b_2[(f_n); X]$. Also we can write

$$|||(y_n)|||_{(f_n)} = |||(uh_n)|||_{(f_n)} \leq ||u||_{(f_n)} \leq c_2||u||_{(f_n)}^{(e_n)} = c_2|||(ue_n)|||_{(f_n)} = c_2|||(x_n)|||_{(f_n)}.$$

So we obtain that

$$|||(y_n)|||_{(f_n)} \leq c_2|||(x_n)|||_{(f_n)}.$$

Consequently (f_n)-contractivity constant of X is less or equal than c_2. ■

Corollary 2.1 *Suppose X is (f_n)-contractive Banach space, H be a separable Hilbert space and (e_n) a fixed orthonormal basis of H. Then*

(a) For any $u \in \mathfrak{R}_{(f_n)}(H;X)$ we have $||u||_{(f_n)} \leq b_{f.}(X)||\sum_k ue_k f_k||_2$.

(b) If X does not contains a subspace isomorphic to c_0, then $\mathfrak{R}_{(f_n)}(H;X) = \mathfrak{R}^{(e_n)}_{(f_n)}(H;X)$.

Proof. (a) Since $u \in \mathfrak{R}_{(f_n)}(H;X)$ the series $\sum_k ue_k f_k$ is convergent, so we can apply the inequality from Theorem 2.2.

(b) By Theorem 2.2 we have $\Pi_{(f_n)}(H;X) = \Pi^{(e_n)}_{(f_n)}(H;X)$.

Since by our assuption X does not contains a subspace isomorphic to c_0, we have also $\mathfrak{R}_{(f_n)}(H;X) = \Pi_{(f_n)}(H;X)$ and $\mathfrak{R}^{(e_n)}_{(f_n)}(H;X) = \Pi^{(e_n)}_{(f_n)}(H;X)$. So this implies the assertion.■

2.3 $K^{f.}$-convex spaces.

Let us introduce now the notion of $K^{f.}$-convexity of a Banach space X. We use the method of [**?**]. Fix a natural number n and consider the operator

$$R^{f.}_n : L_2(\Omega;X) \longrightarrow S_2[(f_n);X]$$

defined by the equality

$$R^{f.}_n \xi = \sum_{k=1}^{n} (\mathbf{E}\xi f_k) f_k; \quad \xi \in L_2(\Omega;X).$$

Set, $K^{f.}_n(X) = \|R^{f.}_n\|$ and define the $K^{f.}$-convexity constant, $K^{f.}(X)$ by

$$K^{f.}(X) = \sup_n K^{f.}_n(X).$$

Definition 2.2 *A Banach space X is called $K^{f.}$-convex, if $K^{f.}(X) < \infty$.*

Recall that a Banach space is called K-convex if it is $K^{r.}$-convex.

Let us formulate different characterisations of $K^{f.}$-convexity, whose elementary proofs are left to the reader.

Proposition 2.4 *Let X be a Banach space. The following assertions are equivalent:*

(i) X is $K^{f.}$-convex.

(ii) For any $\xi \in L_2(\Omega;X)$ we have

$$(\mathbf{E}\xi f_n) \in b_2[(f_n);X].$$

(iii) For any $\xi \in L_2(\Omega; X)$ we have

$$(\mathbf{E}\xi f_n) \in s_2[(f_n); X].$$

(iv) For any $\xi \in L_2(\Omega; X)$ we have $(\mathbf{E}\xi f_n) \in s_2[(f_n); X]$ and the operator $\xi \longrightarrow \sum_n (\mathbf{E}\xi f_n) f_n$ is a continuous linear projection of $L_2(\Omega; X)$ onto $S_2[(f_n); X]$ (i.e. $S_2[(f_n); X]$ is complemented in $L_2(\Omega; X)$).

Proposition 2.5 *Let X be a Banach space. Then:*

(a) For any fixed natural number n we have $K_n^{f.}(X) = K_n^{f.}(X^*)$.

(b) X is $K^{f.}$-convex if and only if X^* is $K^{f.}$-convex and $K^{f.}(X) = K^{f.}(X^*)$.

Proof.

(a) Let $\xi^* \in L^2(\Omega, X^*)$, $\phi \in L^2(\Omega, X)$ and $n \in \mathbb{N}$. Note that

$$\int_\Omega \langle \sum_{k=1}^n \mathbf{E}(\xi^* f_k) f_k(w), \phi(w) \rangle dP(w) = \int_\Omega \langle \xi^*(w), \sum_{k=1}^n \mathbf{E}(\phi f_k) f_k(w) \rangle dP(w).$$

This clearly gives that

$$\left| \int_\Omega \langle \sum_{k=1}^n \mathbf{E}(\xi^* f_k) f_k(w), \phi(w) \rangle dP(w) \right| \le ||\xi^*||_2 K_n^{f.}(X) ||\phi||_2.$$

Hence $\| \sum_{k=1}^n \mathbf{E}(\xi^* f_k) f_k \|_2 \le ||\xi^*||_2 K_n^{f.}(X)$.

Therefore

$$K_n^{f.}(X^*) \le K_n^{f.}(X). \tag{2.6}$$

The converse follows from (2.6) and the embedding $X \subset X^{**}$.

(b) Follows from (a).■

Proposition 2.6 *Let X be a Banach space and $(g_n)_{n\in\mathbb{N}}$ be an another sequence of independent identically distributed symmetric random variables such that $\mathbf{E}g_1^2 = 1$. Supose further that there are constants C_1 and C_2 such that for any $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n \in X$ and $x_1^*, x_2^*, \dots, x_n^* \in X^*$*

$$\| \sum_{k=1}^n x_k g_k \|_2 \le C_1 \| \sum_{k=1}^n x_k f_k, \text{ and } \| \sum_{k=1}^n x_k^* g_k \|_2 \le C_2 \| \sum_{k=1}^n x_k^* f_k \|_2.$$

Then the following statement are valid:

(a) For any natural number n we have $K_n^{g.}(X) \le C_1 C_2 K_n^{f.}(X)$.

(b) If X is $K^{f.}$-convex then X is $K^{g.}$-convex, and $K^{g.}(X) \le C_1 C_2 K^{f.}(X)$.

(c) For any natural number n we have $K_n^{r.}(X) \le c_1^2 K_n^{f.}(X)$.

(d) If X is $K^{f.}$-convex then X is K-convex, and $K^{r.}(X) \le c_1^2 K^{f.}(X)$, where $c_1 = (\mathbf{E}(|f_1|))^{-1}$.

Proof. (a) This is, up to notations, a particular case of Lemma 12.6 in [**?**].

(b) Follows from (a).

(c) Follows from (a) and Remark 2.2 (a).

(d) Follows from (c).■

Remark 2.9 Consider the following assertions concerning a Banach space X:

(i) X is K-convex.

(ii) X does not contain l_1^n uniformly.

(iii) X does not contain l_∞^n uniformly.

It is not difficult to see that $(i) \Rightarrow (ii) \Rightarrow (iii)$ (see [**?**], p. 260). This already implies that the spaces $c_0,\ l_1,\ L_1[0,1]$ are not K-convex. From this also follows that if X is K-convex Banach space then X does not contain a subspace isomorphic to c_0. An important result of G. Pisier asserts that the implication $(ii) \Rightarrow (i)$ also is valid (see [**?**], p. 260). We shall not make use of this implication in the sequel. It is not difficult to show that any type 2 Banach space is K-convex (see Proposition 2.8 (c)).

Corollary 2.2 *(a) If X is a K-convex Banach space then X is $K^{g.}$-convex for any sequence (g_n) be such that $\mathbf{E}|g_1|^p < \infty$ for any p, $0 < p < \infty$.*

Moreover $K^{g.}(X) \leq C_{g_1}(X)C_{g_1}(X^*)K^{r.}(X)$, where $C_{g_1}(X)$ and $C_{g_1}(X^*)$ are constants from Remark 2.2 (b).

(b) In particular X is $K^{\gamma.}$-convex if and only if X is K-convex.

Proof. (a) By Proposition 2.5 X and X^* are both K-convex. Hence Remark 2.9 gives that X and X^* do not contain l_∞^n uniformly. Now Remark 2.2 (b) allows to have the assumptions of Propositions 2.6 satisfied for $(f_n) = (r_n)$ and (g_n) and then (a) follows from Proposition 2.6 (b) .

(b) Use part (a) and Proposition 2.6 (d).■

Notice that Corollary 2.2 is known for $(g_n) = (\gamma_n)$, see [**?**], p. 88, where a better estimate $K^{\gamma.}(X) \leq K^{r.}(X)$ is obtained.

Proposition 2.7 *Let X be a Banach space. Then*

(a) For any $n \in \mathbb{N}$ and $y_1, \dots, y_n \in X$

$$\|\sum_{k=1}^n y_k f_k\|_2 \leq (\mathbf{E}|f_1|)^{-1} \cdot K_n^{f.}(X) \|\sum_{k=1}^n y_k r_k\|_2.$$

(b) If X is $K^{f.}$-convex then $Rad(X) = s_2[(f_n), X] = b_2[(f_n), X]$.

(c) If X is $K^{f.}$-convex, then X is (f_n)-contractive.

Moreover $b_{f.}(X) \leq (\mathbf{E}|f_1|)^{-2} \cdot K^{f.}(X) \cdot C_{\gamma_1}(X)$ where $C_{\gamma_1}(X)$ is the constant in Remark 2.2 (b).

Proof. (a) Given $\xi^* \in L^2(\Omega, X^*)$ and $n \in \mathbb{N}$ we can write

$$|\int_\Omega \langle \xi^*(w), \sum_{k=1}^n y_k f_k(w)\rangle dP(w)| = |\sum_{k=1}^n \langle \mathbf{E}(\xi^* f_k), y_k\rangle|$$

$$= |\int_0^1 < \sum_{k=1}^n \mathbf{E}(\xi^* f_k) r_k(t), \sum_{k=1}^n y_k r_k(t) > dt|$$

$$\leq \|\sum_{k=1}^n \mathbf{E}(\xi^* f_k) r_k\|_2 \|\sum_{k=1}^n y_k r_k\|_2.$$

From this and Remark 2.2 (a) we obtain

$$|\int_\Omega \langle \xi^*(w), \sum_{k=1}^n y_k f_k(w)\rangle dP(w)| \leq (\mathbf{E}|f_1|)^{-1} \cdot K^{f.}(X) \|\sum_{k=1}^n y_k r_k\|_2.$$

As ξ^* was arbitrary, the last inequality implies (a).

(b) It follows from (a) and Remark 2.2 (a) that $Rad(X) = s_2[(f_n), X]$. Now use Proposition 2.6 (b) together with Remarks 2.7 and 2.1 to get $s_2[(f_n), X] = b_2[(f_n), X]$.

(c) Let us take $(x_n) \in b_2[(f_n); X]$ and a sequence (y_n) in X satisfying

$$\sum |\langle y_n, x^*\rangle|^2 \leq \sum |\langle x_n, x^*\rangle|^2$$

for all $x^* \in X^*$.

Note first that using (a) and Remark 2.2 (a) we can write

$$\sup_n \|\sum_{k=1}^n y_k f_k\|_2 \leq (\mathbf{E}|f_1|)^{-1} \cdot K^{f.}(X) \sup_n \|\sum_{k=1}^n y_k r_k\|_2.$$

Since X is $K^{f.}$-convex, according to Remark 2.9 it does not contain l_∞^n uniformly, so by Proposition 2.2, X is (r_n)-contractive with constant $C_{\gamma_1}(X)$. This implies

$$\sup_n \|\sum_{k=1}^n y_k r_k\|_2 \leq C_{\gamma_1}(X) \sup_n \|\sum_{k=1}^n x_k r_k\|_2.$$

Now using Remark 2.2 (a) again we have

$$\sup_n \|\sum_{k=1}^n y_k r_k\|_2 \leq (\mathbf{E}|f_1|)^{-1} \cdot C_{\gamma_1}(X) \sup_n \|\sum_{k=1}^n x_k f_k\|_2.$$

Consequently,

$$\sup_n \|\sum_{k=1}^n y_k f_k\|_2 \leq (\mathbf{E}|f_1|)^{-2} \cdot K^{f.}(X) \cdot C_{\gamma_1}(X) \sup_n \|\sum_{k=1}^n x_k f_k\|_2$$

and (c) is proved.■

Regarding the converse of the implication of part (c) in Proposition 2.7, let us observe that c_o is (γ_n)-contractive while it is not $K^{\gamma_\cdot}$-convex or that $L^1(\mu)$ is (r_n)-contractive according to Proposition 2.2 but it is not K- convex.

Definition 2.3 *Let (f_n) be a sequence of independent identically distributed symmetric random variables such that $\mathbf{E}f_1^2 = 1$ and let $2 < r < \infty$. We shall say that (f_n) is r-regular if*

$$\liminf_{t \to \infty} t^r P\{\omega \in \Omega : |f_1(\omega)| > t\} > 0. \tag{2.8}$$

It is easy to construct r-regular sequences for any r, by using,for instance independent standard Cauchy random variables.

Remark 2.10 It is rather simple to see that the condition (2.8) implies

$$b_2[(f_n); X] \subset l_r^{strong}(X). \tag{2.9}$$

Let us now show that, in general, the notions of K-convex and $K^{f_\cdot}$-convex Banach spaces are different. Note that

Corollary 2.3 *Let $2 < r < \infty$ and (f_n) be r-regular sequence.*

If X is $K^{f_\cdot}$-convex, then X is of cotype r.

In particular l_p is not $K^{f_\cdot}$-convex for $r < p < \infty$, while it is of type 2 and hence is K-convex (see, e. g., Remark 2.11 below).

Proof. Observe that cotype r means $s_2[(r_n); X] \subset l_r^{strong}(X)$., then the result follows from (2.9) and by Proposition 2.7 (b). ■

Let us present some extra assumption to get $K^{f_\cdot}$-convexity out of (f_n)-contractivity.

Proposition 2.8 *Let X be a Banach space which is type 2 and (f_n)-contractive. Then the following assertion are valid:*

(a) For any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$

$$\| \sum_{k=1}^n x_k f_k \|_2 \leq b_{f_\cdot}(X) \cdot t_2(f_\cdot, X) \| \sum_{k=1}^n x_k r_k \|_2.$$

(b) $Rad(X) = s_2[(f_n), X]$.

(c) X is $K^{f_\cdot}$-convex.

Moreover $K^{f_\cdot}(X) \leq b_{f_\cdot}(X) \cdot t_2(f_\cdot, X)$.

Proof.

(a) Put

$$y_\theta = \frac{1}{2^{n/2}} \sum_{k=1}^{n} \theta_k x_k,\ \theta = (\theta_1, \dots, \theta_n) \in \{-1, 1\}^n.$$

Then we have

$$\| \sum_{k=1}^{n} x_k r_k \|_2 = \Big(\sum_\theta \|y_\theta\|^2 \Big)^{1/2}$$

and

$$\sum_{k=1}^{n} \langle x_k, x^* \rangle^2 = \sum_\theta \langle y_\theta, x^* \rangle^2,\ \forall x^* \in X^*.$$

We write $(y_\theta) = (y_1, \dots, y_{2^n})$. Since the above equality holds and X is (f_n)-contractive, we can write

$$\| \sum_{k=1}^{n} x_k f_k \|_2 \leq b_{f.}(X) \| \sum_{k=1}^{2^n} y_k f_k \|_2.$$

Since X is of type 2, it is also of (f_n)-type 2 (see Remark 2.2 (b)), we also have

$$\| \sum_{k=1}^{2^n} y_k f_k \|_2 \leq t_2(f., X) \Big(\sum_{k=1}^{2^n} \|y_k\|^2 \Big)^{1/2} = t_2(f., X) \| \sum_{k=1}^{n} x_k r_k \|_2.$$

These two inequalites imply (a).

(b) It follows from (a) and Remark 2.2 (a).

(c) It is sufficient to show that for any simple function $\xi \in L_2(\Omega, X)$ we have

$$\|R_n^{f.} \xi\| \leq b_{f.}(X) \cdot t_2((f_n), X) \|\xi\|_2. \tag{2.7}$$

Given a simple function ξ, we can find $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$ such that

$$\|\xi\|_2 = \Big(\sum_{k=1}^{m} \|x_k\|^2 \Big)^{1/2}$$

and

$$\mathbf{E} \langle \xi, x^* \rangle^2 = \sum_{k=1}^{m} \langle x_k, x^* \rangle^2,\ \forall x^* \in X^*.$$

Denote now $y_k = \mathbf{E} \xi f_k,\ k = 1, \dots, n.$ Observe that

$$\sum_{k=1}^{n} \langle y_k, x^* \rangle^2 \leq \mathbf{E} \langle \xi, x^* \rangle^2 = \sum_{k=1}^{m} \langle x_k, x^* \rangle^2,\ \forall x^* \in X^*$$

Using now (f_n)-contractivity of X, we have

$$\|R_n^{f.}\xi\| = \|\sum_{k=1}^{n} y_k f_k\|_2 \leq b_{f.}(X)\|\sum_{k=1}^{m} x_k f_k\|_2.$$

Since X is of (f_n)-type 2, we have also

$$\|\sum_{k=1}^{m} x_k f_k\|_2 \leq t_2((f_n), X)\Big(\sum_{k=1}^{m} \|x_k\|^2\Big)^{1/2} = t_2(f., X)\|\xi\|_2.$$

These two inequalities imply (2.7) and (c) is proved.■

Remark 2.11. It follows from Proposition 2.1 (respec.Proposition 2.2) and Proposition 2.8 (c) that if X is a Banach space of γ_n-type 2 (respect. r_n-type 2)then X is K-convex.

Moreover $K^{\gamma.}(X) \leq t_2(\gamma_n, X)$ (respec. $K(X) \leq t_2(r_n, X)$).

3 Duality Results for the Sequence Spaces

Let X be a Banach space and let E be a vector subspace of $X^{\mathbb{N}}$. Denote by $E^{\times}$ the Köthe's dual of E, i.e. $E^{\times}$ is the set of all sequences $(x_n^*) \in (X^*)^{\mathbb{N}}$ such that $\sum_n |x_n^*(x_n)| < \infty, \quad \forall (x_n) \in E$.

Let us assume that E is a vector space containing the set $X_0^{\mathbb{N}}$ of all sequences with finite support. Then for any fixed $(x_n^*) \in E^{\times}$ let us denote by $l_{(x_n^*)}$ the linear functional on E defined by the relation

$$(x_n) \longrightarrow l_{(x_n^*)}(x_n) = \sum_n x_n^*(x_n), \quad (x_n) \in E.$$

It is clear that if two sequences (x_n^*) and (y_n^*) in $E^{\times}$ verify $l_{(x_n^*)} = l_{(y_n^*)}$, then $x_n^* = y_n^*$ $\forall n \in \mathbb{N}$. Hence whenever $X_0^{\mathbb{N}} \subset E$ and $(x_n^*) \in E^{\times}$, we shall identify (x_n^*) with the linear functional $l_{(x_n^*)}$

Lemma 3.1 *Let X be a Banach space, $(x_n) \in b_2[(f_n); X]$ and $(x_n^*) \in b_2[(f_n); X^*]$. Then:*

$$\sum_{k=1}^{\infty} |x_k^*(x_k)| \leq \sup_n \|\sum_{k=1}^{n} x_k f_k\|_2 \sup_n \|\sum_{k=1}^{n} x_k^* f_k\|_2. \tag{3.1}$$

Proof. Denote $\alpha_k = sign(x_k^*(x_k))$ for any natural k. Then by (2.2) we have that $(\alpha_n x_n) \in b_2[(f_n); X]$ and

$$\sup_n \Big\|\sum_{k=1}^{n} \alpha_k x_k f_k\Big\|_2 \leq \sup_n \Big\|\sum_{k=1}^{n} x_k f_k\Big\|_2.$$

Fix a natural number n and put $\xi_n = \sum_{k=1}^n \alpha_k x_k f_k$ and $\eta_n = \sum_{k=1}^n x_k^* f_k$. Then

$$\sum_{k=1}^n |x_k^*(x_k)| = \sum_{k=1}^n \alpha_k x_k^*(x_k) = \mathbf{E}\langle \xi_n, \eta_n \rangle \le \|\xi_n\|_2 \|\eta_n\|_2 \le$$

$$\le \sup_n \|\sum_{k=1}^n x_k f_k\|_2 \sup_n \|\sum_{k=1}^n x_k^* f_k\|_2.$$

Since n was arbitrary this inequality implies the assertion.■

Proposition 3.1 *Let X be a Banach space. Then*
(a) $b_2[(f_n); X^*] \subset (b_2[(f_n); X])^\times \subset (b_2[(f_n); X])^*$.
Moreover, for any $(x_n^*) \in b_2[(f_n); X^*]$ we have

$$\|l_{x_\cdot^*}\| \le |||(x_n^*)|||_{(f_n)}. \tag{3.2}$$

(b) $(s_2[(f_n); X])^\times = (s_2[(f_n); X])^*$.

Proof. (a) The first inclusion and (3.2) follows from Lemma 3.1 and (3.1) respectively.

To see the second inclusion, let us fix $(x_n^*) \in (b_2[(f_n); X])^\times$. We need to show that the linear functional $l_{(x_n^*)}$ is continuous on $b_2[(f_n); X]$. Now for any natural number n the functional l_n on $b_2[(f_n); X]$ defined by

$$(x_k) \longrightarrow l_n(x_k) = \sum_{k=1}^n x_k^*(x_k)$$

is obviously continuous. Since the sequence (l_n) converges to $l_{(x_n^*)}$ at any point of $b_2[(f_n); X]$ then Banach-Steinhaus theorem gives that $l_{(x_n^*)}$ is continuous.

(b) The inclusion $(s_2[(f_n); X])^\times \subset (s_2[(f_n); X])^*$ can be shown as above.

Fix now a continuous linear map $l : s_2[(f_n); X] \longrightarrow \mathbb{R}$ and let us find $(x_n^*) \in (s_2[(f_n); X])^\times$ such that $l = l_{(x_n^*)}$. Take a natural number n and consider the mapping $j_n : X \longrightarrow s_2[(f_n); X]$ defined by the rule: $x \longrightarrow (0, ..., x, 0, ...)$, where x is on n-th place. Evidently j_n is an isometric linear operator. Therefore $x_n^* = l j_n \in X^*$.

Let us first show that $(x_n^*) \in (s_2[(f_n); X])^\times$. Take arbitrary $(x_n) \in s_2[(f_n); X]$ and fix $n \in \mathbb{N}$. Then if $\alpha_k = sign(x_k^*(x_k))$ and $y_k = \alpha_k x_k$ we have

$$\sum_{k=1}^n |x_k^*(x_k)| = \sum_{k=1}^n x_k^*(y_k) = l(\sum_{k=1}^n j_k(y_k)).$$

Hence, using (2.2),

$$\sum_{k=1}^n |x_k^*(x_k)| \le \|l\| \|\sum_{k=1}^n y_k f_k\|_2 \le \|l\| \|\sum_{k=1}^n x_k f_k\|_2 \le \|l\| |||(x_n)|||_{(f_n)}.$$

Consequently $\sum_k |x_k^*(x_k)| < \infty$ and so $(x_n^*) \in (s_2[(f_n); X])^\times$.

Finally, since the sequence $(x_1, ..., x_n, 0, ...)$, $n \in \mathbb{N}$ tends to (x_n) in the topology of $s_2[(f_n); X]$ and l is continuous, we obtain

$$l(x_n) = \lim_n l(x_1, ..., x_n, 0, ...) = \lim_n \sum_{k=1}^{n} x_k^*(x_k).$$

Therefore $l = l_{(x_n^*)}$.■

Lemma 3.2 *Let X be a $K^{f.}$-convex Banach space, $l \in (s_2[(f_n); X])^*$. Then there exists $(x_n^*) \in b_2[(f_n); X^*]$, such that $l = l_{(x_n^*)}$ and*

$$\sup_n \| \sum_{k=1}^{n} x_k^* f_k \|_2 \leq K^{f.}(X) \|l\|. \tag{3.3}$$

Proof. By Proposition 3.1 (b) there exists $(x_n^*) \in (s_2[(f_n); X])^\times$ such that $l = l_{(x_n^*)}$. The proof will be finished if we show that $(x_n^*) \in b_2[(f_n); X^*]$ and (3.3) holds.

Step 1. Fix $n \in \mathbb{N}$ and consider $l_n : s_2[(f_n); X] \longrightarrow \mathbb{R}$ defined by the sequence $(x_1^*, ..., x_n^*, 0, ...)$. Then using the contraction principle, it is easy to show that

$$\|l_n\| \leq \|l\| \qquad \forall n \in \mathbb{N}. \tag{3.4}$$

Step 2. Put $\eta_n = \sum_{k=1}^{n} x_k^* f_k$ and take r, $0 < r < 1$. Then, since $\eta_n \in L_2(\Omega; X^*) \subset (L_2(\Omega; X))^*$, we can find $\xi_n \in L_2(\Omega; X)$, $\|\xi_n\|_2 = 1$, such that

$$r\|\eta_n\|_2 < \mathbf{E}(\xi_n, \eta_n) = \sum_{k=1}^{n} \langle \mathbf{E}\xi_n f_k, x_k^* \rangle . \tag{3.5}$$

Now, since X is $K^{f.}$-convex, by Proposition 2.4

$$(\mathbf{E}\xi_n f_1, \mathbf{E}\xi_n f_2, ...) \in s_2[(f_n); X]$$

and

$$\Big\| \sum_k (\mathbf{E}\xi_n f_k) f_k \Big\|_2 \leq K^{f.}(X) \|\xi_n\|_2 = K^{f.}(X).$$

Using this and (3.5) we can write as follows

$$r\|\eta_n\|_2 \leq \sum_{k=1}^{n} \langle \mathbf{E}\xi_n f_k, x_k^* \rangle = l_n(\mathbf{E}\xi_n f_1, \mathbf{E}\xi_n f_2, ...).$$

Now we can use (3.4) and (3.5) and write

$$r\|\eta_n\|_2 \leq K^{f.}(X) \|l\|,$$

The last inequality, since n and r were arbitrary implies $(x_n^*) \in b_2[(f_n); X^*]$ and (3.3) hods.■

Our first duality result can be formulated as follows.

Theorem 3.1 *Let X be a Banach space. The following assertions are equivalent:*

(i) X is $K^{f.}$-convex.

(ii) $T : b_2[(f_n); X^*] \longrightarrow (s_2[(f_n); X])^*$ defined by the equality $T(x_n^*) = l_{(x_n^*)}$ is a Banach-space-isomorphism, with $\|T\| = 1$.

Moreover, (i) implies that $\|T^{-1}\| \le K^{f.}(X)$ and (ii) implies that $K^{f.}(X) \le \|T^{-1}\|$.

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.1 (a) we have that T is a continuous linear operator with $\|T\| \le 1$. Lemma 3.2 implies that T is onto and $\|T^{-1}\| \le K^{f.}(X)$.

(ii) $\Rightarrow$ (i). Fix arbitrarily $\xi \in L_2(\Omega; X)$ with $\|\xi\|_2 = 1$. and write $x_n = \mathbf{E}\xi f_n,\ n \in \mathbb{N}$. According to Proposition 2.4 it is sufficient to show that $(x_n) \in b_2[(f_n); X]$. Actually we shall show that

$$\sup_n \Big\| \sum_{k=1}^n x_k f_k \Big\|_2 \le \|T^{-1}\|. \tag{3.6}$$

Fix $n \in \mathbb{N}$. By the Hahn-Banach theorem there exists $l \in (s_2[(f_n); X])^*$ such that $\|l\| = 1$ and

$$\Big\| \sum_{k=1}^n x_k f_k \Big\|_2 = l(x_1, ..., x_n, 0, ...).$$

According to the assumption $l = T(x_n^*)$ for some $(x_n^*) \in b_2[(f_n); X^*]$, hence

$$\sup_n \Big\| \sum_{k=1}^n x_k^* f_k \Big\|_2 \le \|T^{-1}\|.$$

Note that

$$\Big\| \sum_{k=1}^n x_k f_k \Big\|_2 = \sum_{k=1}^n x_k^*(x_k) = \mathbf{E}\langle \xi, \sum_{k=1}^n x_k^* f_k \rangle \le \Big\| \sum_{k=1}^n x_k^* f_k \Big\|_2 \le \|T^{-1}\|.$$

This, since n was arbitrary, implies (3.6) and the theorem is proved.■

Corollary 3.1 *Let X be a Banach space. The following are equivalent:*

(i) X is $K^{f.}$-convex

(ii) $(s_2[(f_n); X])^* = s_2[(f_n); X^*]$

(iii) $(S_2[(f_n); X])^* = S_2[(f_n); X^*]$

Proof. (i)$\Rightarrow$ (ii). By Theorem 3.1 we can write $(s_2[(f_n);X])^* = b_2[(f_n);X^*]$. According to Propositions 2.5 (b) and 2.7 (b) we have $b_2[(f_n);X^*] = s_2[(f_n);X^*]$ and the implication is proved.

(ii) $\Rightarrow$ (i) By Proposition 3.1 we always have $b_2[(f_n);X^*] \subset (s_2[(f_n);X])^*$. This and (ii) imply that $s_2[(f_n);X^*] = b_2[(f_n);X^*]$ and by Theorem 3.1 X is $K^{f.}$-convex.

(ii) $\Leftrightarrow$ (iii) It is obvious.■

Remark 3.1. The implication (i)$\Rightarrow$(iii) of Corollary 3.1 for $(f_n) = (r_n)$ was pointed out in [**?**]. The same implication for $(f_n) = (\gamma_n)$ and for the separable Banach space having type 2 was obtained in [**?**].

4 Duality Results for Almost Summing Operators

In this section H will denote an infinite dimensional separable Hilbert space, X will be a Banach space. $L(Y_1, Y_2)$ denotes the space of all continuous linear operators between the Banach spaces Y_1 and Y_2. $\mathfrak{N}_p(Y_1, Y_2)$ is the space of all p-nuclear operators and ν_p denotes p-nuclear norm (see [**?**], p. 112). We put also $\mathfrak{N}(H) = \mathfrak{N}_1(H, H)$.

It is well-known that for any $w \in \mathfrak{N}(H)$ and any orthonormal bases (e_n) in H the series $\sum_n (we_n|e_n)$ is convergent, its sum does not depend on particular choice of (e_n) and it is denoted by trw. The number trw is called the trace of w and the inequality $|trw| \le \nu_1(w)$ holds.

Let us denote also

$$\Pi^{dual}_{(f_n)}(X, H) = \{v \in L(X, H) : v^* \in \Pi_{(f_n)}(H, X^*)\}$$

and

$$\mathfrak{R}^{dual}_{(f_n)}(X, H) = \{v \in L(X, H) : v^* \in \mathfrak{R}_{(f_n)}(H, X^*)\}.$$

We shall endow $\Pi^{dual}_{(f_n)}(X, H)$ and $\mathfrak{R}^{dual}_{(f_n)}(X, H)$ with the norm

$$\|v\|^{dual}_{(f_n)} = \|v^*\|_{(f_n)}, \quad v \in \Pi^{dual}_{(f_n)}(X, H).$$

Evidently $\mathfrak{R}^{dual}_{(f_n)}(X, H) \subset \Pi^{dual}_{(f_n)}(X, H)$, and if $c_0 \not\subset X^*$, then Remark 2.1 gives $\Pi^{dual}_{(f_n)}(X, H) = \mathfrak{R}^{dual}_{(f_n)}(X, H)$.

Lemma 4.1 *Let X be a Banach space, $u \in \Pi_{(f_n)}(H, X)$ and $v \in \Pi^{dual}_{(f_n)}(X, H)$. Then vu is nuclear and*

$$\nu_1(vu) \le \|u\|_{(f_n)} \|v^*\|_{(f_n)}. \tag{4.1}$$

Proof. It is needed to see that $vu \in \mathfrak{N}(H)$ and (4.1) holds . For this it is enough to show that for any two orthonormal basis $(e_n^{'})$ and $(e_n^{''})$ of H we have

$$\sum_n |(vue_n^{'}|e_n^{''})| \leq \|u\|_{(f_n)}\|v^*\|_{(f_n)}$$

(see [?], p. 118). Evidently we have $(ue_n^{'}) \in b_2[(f_n);X]$ and $(v^*e_n^{''}) \in b_2[(f_n);X^*]$. So, by Lemma 3.1 we have

$$\sum_n |(vue_n^{'}|e_n^{''})| = \sum_n |\langle ue_n^{'}, v^*e_n^{''}\rangle| \leq$$

$$\leq \sup_n \Big\| \sum_{k=1}^n ue_k^{'}f_k \Big\|_2 \sup_n \Big\| \sum_{k=1}^n v^*e_k^{''}f_k \Big\|_2 \leq \|u\|_{(f_n)}\|v^*\|_{(f_n)}.$$

From this (4.1) easily follows.■

Lemma 4.2 *Let X be a $K^{f.}$-convex Banach space and $F \in (\mathfrak{R}_{(f_n)}(H,X))^*$. Then there is $v \in \Pi_{(f_n)}^{dual}(X,H)$ such that $F(u) = tr(vu)$, $\forall u \in \mathfrak{R}_{(f_n)}(H,X)$.*

Moreover $\|v^*\|_{(f_n)} \leq b_{f.}(X) \cdot b_{f.}(X^*) \cdot K^{f.}(X) \cdot \|F\|$.

Proof. Fix an orthonormal basis (e_n) of H, consider the operator

$$A : \mathfrak{R}_{(f_n)}(H,X) \to s_2[(f_n),X]$$

defined by the relation $Au = (ue_1, ue_2, \ldots)$. Since X is $K^{f.}$-convex, then by Proposition 2.7, X is (f_n)-contractive. This implies, by Theorem 2.2, that A is an isomorphism between corresponding spaces such that $\|A\| \leq 1$ and $\|A^{-1}\| \leq b_{f.}(X)$.

Consider $l = F \circ A^{-1}$, then $l \in (s_2[(f_n),X])^*$. So, by Lemma 3.2, there exists $(x_n) \in b_2[(f_n),X^*]$ such that $l = l_{(x_n^*)}$ and

$$\sup_n \| \sum_{k=1}^n x_k^* f_k \|_2 \leq K^{f.}(X)\|l\| \leq K^{f.}(X)\|F\|\|A^{-1}\| \leq c_1 K^{f.}(X)\|F\|.$$

Define now another operator $v : X \to H$ by the equality

$$vx = \sum_k x_k^*(x)e_k.$$

Evidently $v^*e_k = x_k^*$ for any k. Since now by Proposition 2.5 (b) X^* is $K^{f.}$-convex, by Proposition 2.7 (c) it is also (f_n)-contractive, what allows us to use again Theorem 2.2 to conclude that $v^* \in \Pi_{(f_n)}(H,X^*)$ and

$$\|v^*\|_{(f_n)} \leq b_{f.}(X^*) \cdot \sup_n \| \sum_{k=1}^n x_k^* f_k \|_2.$$

Consequently $v \in \Pi^{dual}_{(f_n)}(X, H)$. Now is easy to see that $F(u) = tr(vu)$ for any u and

$$\|v^*\|_{(f_n)} \leq b_{f.}(X) b_{f.}(X^*) K^{f.}(X) \|F\|,$$

and the lemma is proved.■

Theorem 4.1 *Let X be a Banach space. The following are equivalent*

(i) X is $K^{f.}$-convex

(ii) $(\mathfrak{R}_{(f_n)}(H, X))^* = \Pi^{dual}_{(f_n)}(X, H)$ (with equivalent norms).

Proof. (i) ⇒ (ii). This follows at once from Lemma 4.1 and Lemma 4.2.

(ii)⇒(i). By Theorem 3.1 is enough to show that (ii) implies the equality

$$(s_2[(f_n); X])^* = b_2[(f_n); X^*]. \tag{4.2}$$

The inclusion $(s_2[(f_n); X])^* \supset b_2[(f_n); X^*]$ is valid for all Banach spaces according to Proposition 3.1 (a).

Take now $l \in (s_2[(f_n); X])^*$ and let us find $(x_n^*) \in b_2[(f_n); X^*]$ such that $l = l_{(x_n^*)}$. Fix any orthonormal basis (e_n) of H. Observe that the equality

$$F(u) = l(ue_1, ue_2, \ldots, ue_n, \ldots),\ u \in \mathfrak{R}_{(f_n)}(H, X)$$

defines an element in $(\mathfrak{R}_{(f_n)}(H, X))^*$. From the assumption there is $v \in \Pi^{dual}_{(f_n)}(H, X)$ such that

$$F(u) = tr(vu) = \sum_n \langle ue_n, v^* e_n \rangle,\ \forall u \in \mathfrak{R}_{(f_n)}(H, X).$$

Put $x_n^* = v^* e_n$, $\forall n \in \mathbb{N}$. Since $v^* \in \Pi_{(f_n)}(H; X^*)$ we have $(x_n^*) \in b_2[(f_n); X^*]$. Let us see that $l = l_{x^*}$.

Fix arbitrary $(x_n) \in s_2[(f_n); X]$ and let us show that $l(x_n) = l_{(x_n^*)}(x_n)$. According to the above equality we can write

$$l(ue_1, \ldots, ue_n, \ldots) = l_{x^*_.}(ue_1, \ldots, ue_n, \ldots),\ \forall u \in \mathfrak{R}_{(f_n)}(H, X). \tag{4.3}$$

Consider for a fixed $n \in \mathbb{N}$ a finite rank operator $u_n : H \to X$ defined as follows: $ue_k = x_k$, for $k \leq n$ and $ue_k = 0$ for $k > n$. We have that $u_n \in \mathfrak{R}_{(f_n)}(H, X)$. Consequently, the equality (4.8) holds for u_n. Using this, the fact that the sequence $(x_1, \ldots, x_n, 0, \ldots)$, $n = 1, \ldots$ converges to (x_n) in $s_2[(f_n), X]$ and the continuity on $s_2[(f_n), X]$ of the functionals l and $l_{x^*_.}$, we get $l(x_n) = l_{x^*_.}(x_n)$, and the proof is finished.■

Corollary 4.1 *Let X be a Banach space. The following are equivalent:*

(i) X is $K^{f.}$-convex.

(ii) $(\mathfrak{R}_{(f_n)}(H, X))^* = \mathfrak{R}^{dual}_{(f_n)}(X, H)$.

Proof. (i)⇒(ii) Follows from Theorem 4.1, Proposition 2.5 and Remark 2.1.

(ii)⇒(i) The condition (ii) together with Proposition 4.1 (a) implies that $(\mathfrak{R}_{(f_n)}(H,X))^* = \Pi^{dual}_{(f_n)}(X,H)$. So, by Theorem 4.1, X is $K^{f.}$-convex. ■

Corollary 4.2 *Let X be a Banach space. The following are equivalent:*

(i) X is K-convex.

(ii) $(\Pi_{as}(H,X))^* = \Pi^{dual}_{(r_n)}(X,H)$.

(iii) $(\mathfrak{R}_{(\gamma_n)}(H,X))^* = \Pi^{dual}_{(\gamma_n)}(X,H)$.

(iv) X is $K^{\gamma.}$-convex.

Proof. (i)⇔(ii) and (iii)⇔(iv) by Theorem 4.1 applied for (r_n) and (γ_n).

(ii)⇒(iii) We have that X and X^* are K-convex, therefore they do not contain c_0, this implies that $\Pi_{as}(H,X) = \Pi_{(r_n)}(H,X)$ which coincides with $\Pi_{(r_n)}(H,X) = \Pi_{(\gamma_n)}(H,X)$ (by Theorem 2.1).

On the other hand, we have $\Pi_{(\gamma_n)}(H,X) = \mathfrak{R}_{(\gamma_n)}(H,X)$. So $\Pi_{as}(H,X) = \mathfrak{R}_{(\gamma_n)}(H,X)$. And so $(\mathfrak{R}_{(\gamma_n)}(H,X))^* = \Pi^{dual}_{(r_n)}(X,H)$. Now let us show that $\Pi^{dual}_{(r_n)}(X,H) = \Pi^{dual}_{(\gamma_n)}(X,H)$.

The inclusion $\Pi^{dual}_{(r_n)}(X,H) \supset \Pi^{dual}_{(\gamma_n)}(X,H)$ is clear.

For the other inclusion let us take $v \in \Pi^{dual}_{(r_n)}(X,H)$, then $v^* \in \Pi_{(r_n)}(H,X^*)$ and $v^* \in \Pi_{(\gamma_n)}(H,X^*)$ by Theorem 2.1, consequently $v^* \in \Pi^{dual}_{(\gamma_n)}(X,H)$.

(iii)⇒(ii) Is true by similar reason, and the corollary is proved.■

Remark 4.1. It is known that if a Banach space X is a GL-space (see [**?**] for definition and properties) and X^* has a finite cotype, then $\Pi^{dual}_{as}(X,H) = \Pi_1(X,H)$. If X is K-convex, X^* is also K-convex, and so has a finite cotype. From this observations and from Corollary 4.1 it follows that if X is a K-convex GL-space, then $(\Pi_{as}(H,X))^* = \Pi_1(X,H)$.

Proposition 4.1 *Let X be a Banach space. The following assertions are equivalent.*

(i) X is of cotype 2.

(ii) $(\Pi_{as}(H,X))^* = \Pi_2(X,H)$.

Proof. (i)⇒ (ii) Since X is of cotype 2 we have

$$\Pi_{as}(H,X) = \Pi_2(H,X) \tag{4.4}$$

and for any $u \in \Pi_{as}(H,X)$

$$c(r_.,X)\pi_2(u) \le \pi_{as}(u) \le \pi_2(u)$$

where $c(r_.,X)$ is the cotype 2 constant of X (see Remark 2.5 (a), (b)).

On the other hand, for any Banach space X, we have

$$\Pi_2(H,X) = \mathfrak{N}_2(H,X) \tag{4.5}$$

and for all $u \in \Pi_2(H,X)$, $\pi_2(u) = \nu_2(u)$.

It is known that the equality

$$(\mathfrak{N}_2(H,X))^* = \Pi_2(X,H) \tag{4.6}$$

holds isometrically (see [?], p. 448). From (4.6), (4.5), and (4.4) follows the statement (ii).

(ii)⇒(i). Let us show that X has the Gaussian cotype 2. Take arbitrarily $(x_n) \in s_2[(\gamma_n);X]$. It is needed to show that $(x_n) \in l_2^{strong}(X)$. Fix an orthonormal basis (e_n) of H. Consider the operator $u \in L(H,X)$ such that $ue_n = x_n$ for all $n \in \mathbb{N}$. By Proposition 2.3 $u \in \mathfrak{R}_{(\gamma_n)}(H,X)$, hence $u \in \Pi_{as}(H,X)$.

Take now a sequence $(x_n^*) \in l_2^{strong}(X^*)$ and consider the operator $v : X \to H$ defined by the equality

$$vx = \sum_n x_n^*(x)e_n, \quad \forall x \in X.$$

It is easy to see that $v \in \Pi_2(X,H)$. This, according with (ii), implies that the operator vu is nuclear; hence,

$$\sum_n |(vue_n|e_n)| = \sum_n |\langle ue_n, v^*e_n\rangle| = \sum_n |\langle x_n, x_n^*\rangle| < \infty.$$

Since $(x_n^*) \in l_2^{strong}(X^*)$ was arbitrary, the last relation implies that $(x_n) \in l_2^{strong}(X)$. Therefore X is of Gaussian cotype 2 and then X is, at it is known, of Rademacher cotype 2.■

References

[1] Baur F. *Banach Operators Ideals Generated by Orthonormal Systems.* Inaugural-Dissertation Zurich 1997.

[2] Chobanyan S.A., Tarieladze V.I. *Gaussian Characterizations of Certain Banach Spaces.* J. Mult. Anal. 7(1977), 183-203.

[3] Diestel J., Jarchow H., Tonge A. *Absolutely Summing Operators.* Cambridge University Press (1995).

[4] Figiel T., Tomczak-Jaegermann N. *Projections onto Hilbertian Subspaces of Banach Spaces.* Israel J. Math. 33(1979), 155-171.

[5] Gohberg I.C., Krein M.G. *Introduction to the Theory of Linear Non selfadjoint Operators in Hilbert Space.* Translations A.M.S. 18, 1969.

[6] Jarchow H. *Locally Convex Spaces.* B.G. Teubner Stuttgart 1.994.

[7] Linde W., Pietsch A. *Mappings Gaussian Cylindrical Measures in Banach Spaces.* Theory Probability Appl. 19(1974), 445-460.

[8] Maurey B., Pisier G. *Serie de Variables Aléatoires Vectorielles Indepéndantes et Propietés Géometriques des Espaces de Banach.* Studia Math. T. LVIII. (1976) 45-90.

[9] Pietsch A., Wenzel J. *Orthonormal Systems and Banach Space Geometry.* Cambridge University Press (1998).

[10] Pisier G. *On the duality Between Type and Cotype.* Lecture Notes in Math. 939 (1981).

[11] Pisier G. *Semigroupes Holomorphes et K-convexite.* Seminaire D'Analyse Functiounelle, 1980-1981. Exposè No II. (28 November 1980).

[12] Nguyen Duy Tien, Tarieladze V.I., Vidal R. *On Summing and Related Operators Acting From a Hilbert Space.* Bull. Polish Acad. Sci. 1998, Vol.46, no. 4.

[13] Nguyen Duy Tien, Vidal R. *Almost Summing Operators in Banach Spaces.* Preprint, 1996.

[14] Nguyen Duy Tien, Vidal R. *Comparison Theorems for Random series in Banach spaces.* Preprint, 1996.

[15] Tomczak-Jaegermann N. *Banach-Mazur Distance and Finite Dimensional Operators Ideals.* Longman Scientific Technical 1989.

[16] Vakhania N.N., Tarieladze V.I., Chobanyan S.A. *Probability Distributions on Banach Spaces.* D. Reidel, 1987.