Reprinted from JOURNAL OF FUNCTIONAL ANALYSIS
All Rights Reserved by Academic Press, New York and London

Vol. 102, No. 2, December 1991
Printed in Belgium

418

Interpolation between Vector-Valued Hardy Spaces

OSCAR BLASCO*

Departamento de Matemáticas, Universidad de Zaragoza, Zaragoza-50009, Spain

AND

QUANHUA XU

Université de Wuhan, Université Lille I, and Université Paris VI, Équipe d'Analyse, Université Paris VI 75230-Paris Cedex 05, France

Communicated by the Editors

Received June 1, 1989; revised November 23, 1990

Let $0 < p \leqslant \infty$ and let $\tilde{H}_p^h(X)$ denote the space of X-valued harmonic functions on the half-space with boundary values almost everywhere and Poisson maximal function in $L_p(\mathbb{R}^n)$, and $\tilde{H}_p(X)$ the closure of the X-valued analytic polynomials on the disc under the norm given by $\sup_{0<r<1} \|f_r\|_p$. It is shown that if $0 < p_0, p_1 \leqslant \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$, then $(\tilde{H}_{p_0}^h(X_0), \tilde{H}_{p_1}^h(X_1))_\theta = \tilde{H}_p^h(X_\theta)$. With the restriction $p > 1$ we prove $(\tilde{H}_{p_0}^h(X_0), \tilde{H}_{p_1}^h(X_1))_{\theta,p} = \tilde{H}_p^h(X_{\theta,p})$. A counterexample for the case $p = 1$ is given for the case of real interpolation. It is also proved that $(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_\theta$ is, in general, smaller than $\tilde{H}_p(X_\theta)$. Finally $BMO(X)$ is also considered as the end point for interpolation.

0. INTRODUCTION

The first results on interpolation of Hardy spaces of analytic functions on the disc go back to the 1950s. In [SZ] R. Salem and A. Zygmund showed, using interpolation methods, the boundedness of certain bilinear forms acting on Hardy spaces, and in [CZ] A. P. Calderón and A. Zygmund obtained some results on interpolation of $H^p(D)$, $0 < p < \infty$, based on result for L^p $(1 < p < \infty)$, using factorization and the boundedness of the Riesz projection for $1 < p < \infty$. Later real variable techniques were applied in the study of Hardy spaces (see [FS, CW]). Then thools like maximal functions and atomic decompositions turned out to be

* Partially supported by Grants C.A.I.C.Y.T. PB85-0338 and PS89-0106.

0022-1236/91 $3.00

decisive in solving questions on real and complex interpolation of Hardy spaces. The final solution was not attained until the 1980s. This was achieved combining the efforts of several people. Let us mention, among others, C. Fefferman, N. M. Riviere, and Y. Sagher for results concerning real interpolation (cf. [RS, FRS]) and also names such as C. Fefferman and E. Stein, A. P. Calderón and A. Torchinsky, S. Janson and P. Jones for most results on complex interpolation (cf. [FS, CT, J1, JJ]).

Let us denote by $H_p^h = H_p^h(\mathbb{R}^n)$, $0 < p < \infty$, the space of harmonic functions on $\mathbb{R}_+^{n+1}$ whose Poisson maximal functions belong to $L_p(\mathbb{R}^n)$. We shall use the notation $H_p = H_p(D)$ for the space of analytic functions f on the disc D such that

$$\|f\|_{H_p} = \sup_{0<r<1} \|f_r\|_{L_p(\mathbb{T})} < \infty,$$

where $\mathbb{T}$ is the circle and $f_r(t) = f(re^{it})$. We also write $BMO = BMO(\mathbb{R}^n)$ for the usual space of functions with bounded mean oscillation. If $(X_0, X_1)_\theta$ stands for the space obtained by the complex method of interpolation, we can summarize the results on complex interpolation as follows.

For $0 < p_0, p_1 \leqslant \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$

$$(H_{p_0}^h, H_{p_1}^h)_\theta = H_p^h \qquad \text{(see [CT, JJ])} \tag{0.1}$$

$$(H_{p_0}, H_{p_1})_\theta = H_p \qquad \text{(see [CZ, J1])}. \tag{0.2}$$

Also BMO was considered as an end point and the complete answer was given in [JJ] (see also [FS] for $1 < p_0 < \infty$)

$$(H_{p_0}^h, BMO)_\theta = H_p^h, \qquad \frac{1}{p} = \frac{1-\theta}{p_0}. \tag{0.3}$$

For real interpolation we refer the reader to [RS, FRS].

$$(H_{p_0}^h, H_{p_1}^h)_{\theta, p} = H_p^h. \tag{0.4}$$

The notation $(X_0, X_1)_{\theta,p}$ stands for the space obtained for the real method of interpolation and it should be mentioned that also results for $(H_{p_0}^h, H_{p_1}^h)_{\theta,q}$ with $q \neq p$ can be found in those previous papers.

It is well known that complex and real methods are very much connected. We refer the reader to [M, CMS] to get approaches to the complex interpolation results from the real interpolation ones, which allow us to get (0.1) from (0.4).

While interpolation theory for vector-valued L^p-spaces has been developed and lots of applications can be found in classical books of interpolation (for instance [BL, T]), not much is known for vector-valued H^p-spaces. The purpose of this paper is to deal with real and complex interpolation for vector-valued Hardy spaces. We shall try to extend (0.1) to (0.4) when functions are allowed to take values in two different Banach spaces.

Contrary to the classical case, different definitions of Hardy spaces lead to essentially different spaces when we allow our functions to take values in a general Banach space (see [B2]). Therefore we must set very precisely the Hardy space we shall be considering. We shall be working with $\mathbb{R}^{n+1}_+$ when dealing with harmonic functions and the unit disc D for the case of analytic ones. Given a complex Banach space X we denote by $H^h_p(X) = H^h_p(X, \mathbb{R}^n)$ and $H_p(X) = H_p(X, D)$ the natural extensions to the corresponding ones we have already mentioned in this introduction where we simply replace the absolute value by the norm in the space. It is known that in the vector-valued setting, functions in these spaces need not have boundary values almost everywhere. Hence we use the notation $\tilde{H}^h_p(X)$ and $\tilde{H}_p(X)$ for the closed subspaces of the previous ones where the functions have boundary limits (see Section 3 for concrete definitions of such spaces).

The paper is divided into six sections. The first two are devoted to recalling definitions and basic results on interpolation theory and vector-valued Hardy spaces, respectively. In Section 3 we deal with complex interpolation. The main result that we achieve there can be stated as follows: For $0 < p_0, p_1 \leqslant \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$,

$$(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_\theta = \tilde{H}^h_p(X_\theta),$$

where X_θ denotes $(X_0, X_1)_\theta$.

The proof uses atomic decompositions for functions in vector-valued H^p-spaces together with arguments in [JJ]. A similar approach was used in [B1] for the simpler case $1 \leqslant p_0, p_1 \leqslant \infty$.

The case $BMO(X)$ as end point is considered in Section 4, proving the following results: For $0 < p_0 < \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0$,

$$(L_{p_0}(X_0), BMO(X_1))_\theta = L_p(X_\theta)$$

$$(\tilde{H}^h_{p_0}(X_0), BMO(X_1))_\theta = \tilde{H}^h_p(X_\theta).$$

Section 5 is concerned with spaces of analytic functions on the disc. In this case the inclusion $(H_{p_0}(X_0), H_{p_1}(X_1))_\theta \subset H_p(X_\theta)$ remains valid in the vector-valued case but we show with an example due to G. Pisier that it is not equality in general. However, there are cases where the extension of (0.2) holds for functions with values in Banach spaces, namely when either both spaces have the UMD property or both spaces coincide. The last section is devoted to real interpolation. We prove that the analogous equation to (0.4) holds only when $p > 1$ and give a counterexample for $p = 1$. We also consider BMO as an end point and the case of analytic functions obtaining similar results to those for complex interpolation.

Let us finally remark that we do not consider Hardy spaces of vector-valued martingales in this paper, but similar results for these spaces can also be obtained. If $(\Omega, \mathscr{F}, P)$ is a probability space and $\mathscr{F}_n$ is a sequence of σ-fields with $\sigma(\bigcup \mathscr{F}_n)=\mathscr{F}$, we consider the space of X-valued martingales $f=(f_n)$ adecuated to $(\Omega, \mathscr{F}, \mathscr{F}_n, P)$ satisfying

$$\|f\|_{MH_p(X)}=\|\sup_n \|f_n(\omega)\|\,\|_{L_p(\Omega)}<\infty.$$

We denote by $MH_p(X)$ the completion of $L_2(X,\Omega)$ under this norm. $MBMO(X)$ will be the set of functions in $L_1(X,\Omega)$ such that

$$\|f\|_{MBMO(X)}=\sup_n \|\mathbb{E}(\|f-\mathbb{E}(f\mid\mathscr{F}_n)\|\mid\mathscr{F}_n)\|_{L_\infty(\Omega)}.$$

Similar ideas to the ones used in this paper for harmonic functions, but simpler, give the following results.

If $(\mathscr{F}_n)$ is "regular" in the sense of [G] and if $0<p_0<\infty$, $0<p_1\leqslant\infty$, $0<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$, and $1/q=(1-\theta)/p_0$ then

$$(MH_{p_0}(X_0), MH_{p_1}(X_1))_\theta=MH_p(X_\theta)$$

$$(MH_{p_0}(X_0), BMO(X_1))_\theta=MH_q(X_\theta).$$

Let us finish the introduction by mentioning some conventions that will be used throughout the paper. (X_0, X_1) denotes always an interpolation couple of complex or real Banach spaces and C stands for a positive constant which may depend on $n, p_0, p_1, \ldots$, but never on the functions considered and which may be different at each occurrence.

1. Basic Facts on Interpolation

Here we simply recall some basic facts on interpolation that will be used in the sequel. Some general references for interpolation theory are [C, LP, BL, T]. Since our main interest consists of interpolating $\tilde{H}^h_p(X)$ for $0<p\leqslant\infty$ by the complex method then we have to deal with some extension of Calderón's method to the context of quasi-Banach spaces. The difficulty arises from the failure of the maximum principle when functions are allowed to take values in a quasi-Banach space. We refer the reader to [CMS, JJ] for such extension to the quasi-Banach setting. Throughout this section (A_0, A_1) denotes an interpolation couple of complex or real quasi-Banach spaces.

Denote by $S=\{z\in\mathbb{C}: 0<\operatorname{Re} z<1\}$ and $A(S)$ the space of complex-valued functions which are analytic on S and bounded and continuous on

$\bar{S}$. Given an interpolation couple of complex quasi-Banach spaces (A_0, A_1) we define

$$\mathscr{F}(A_0, A_1) = \left\{ \sum_{k=1}^{m} a_k f_k \mid a_k \in A_0 \cap A_1, f_k \in A(S), m \in \mathbb{N} \right\}.$$

We set on this space the norm

$$\|f\|_{\mathscr{F}} = \max\{\sup_{\sigma \in \mathbb{R}} \|f(i\sigma)\|_{A_0}, \sup_{\sigma \in \mathbb{R}} \|f(1+i\sigma)\|_{A_1}\}$$

and for each $a \in A_0 \cap A_1$ we define for $0 < \theta < 1$

$$\|a\|_\theta = \inf\{\|f\|_{\mathscr{F}} : f \in \mathscr{F}(A_0, A_1) \text{ and } f(\theta) = a\}.$$

We denote by $(A_0, A_1)_\theta = A_\theta$ the completion of $A_0 \cap A_1$ with respect to this quasi-norm.

It is well known that this space coincides with the one defined by Calderón [C] when A_0 and A_1 are Banach spaces. A very useful fact shown by Calderón and which remains true for quasi-Banach spaces (cf. [CMS]) is the following: For f in $\mathscr{F}(A_0, A_1)$ and $0 < \theta < 1$

$$\operatorname{Log} \|f(\theta)\|_\theta \leqslant \sum_{j=0}^{1} \int_{-\infty}^{\infty} \operatorname{Log} \|f(j+i\sigma)\|_{A_j} P_j(\theta, \sigma)\, d\sigma, \tag{1.1}$$

where $P_0(\theta, \sigma)$ and $P_1(\theta, \sigma)$ stand for the Poisson kernel associated to S.

Let us now recall the K-method for real interpolation. For each $t > 0$ and x in $A_0 + A_1$ we define the functional

$$\begin{aligned} K(t, x) &= K(t, x, A_0, A_1) \\ &= \inf\{\|x_0\|_{A_0} + t\,\|x_1\|_{A_1} : x = x_0 + x_1,\, x_j \in A_j,\, j = 0, 1\}. \end{aligned}$$

For each $0 < q < \infty$, $0 < \theta < 1$ we consider

$$(A_0, A_1)_{\theta,q} = A_{\theta,q} = \left\{ x \in A_0 + A_1 : \|x\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(t, x))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

$$(A_0, A_1)_{\theta,\infty} = A_{\theta,\infty} = \{x \in A_0 + A_1 : \|x\|_{\theta,\infty} = \sup_{t>0} t^{-\theta} K(t, x) < \infty\}.$$

We refer the reader to [LP, S] for the following equivalent formulation. If $0 < p_0, p_1 < \infty$ and $1/q = (1-\theta)/p_0 + \theta/p_1$

$$\|x\|_{\theta,q} \approx \inf\left\{ \left(\sum_{-\infty}^{\infty} \|e^{-\theta n} x_{0n}\|_{A_0}^{p_0} \right)^{(1-\theta)/p_0} \times \left(\sum_{-\infty}^{\infty} \|e^{(1-\theta)n} x_{1n}\|_{A_1}^{p_1} \right)^{\theta/p_1} \right\}, \tag{1.2}$$

where the infimum is taken over all decompositions $x = x_{0n} + x_{1n}$.

The next theorem summarizes the basic results on interpolation of vector-valued L^p-spaces. Let (Ω, μ) be a measure space and for $0 < p \leqslant \infty$ we write $L_p(X) = L_p(X, \Omega)$ for the space of measurable functions with values in a real or complex quasi-Banach space X such that $\|f(\omega)\|$ belongs to $L_p(\Omega)$.

THEOREM A. *Let* $0 < p_0 < \infty$, $0 < p_1 \leqslant \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$,

$$(L_{p_0}(A_0), L_{p_1}(A_1))_\theta = L_p(A_\theta) \tag{1.3}$$

$$(L_{p_0}(A_0), L_{p_1}(A_1))_{\theta,q} = L_p(A_{\theta,q}). \tag{1.4}$$

The reader is referred to [BL] or [T] for a proof in the case of Banach spaces and $1 \leqslant p_0, p_1 \leqslant \infty$ and to [X1, S] for proofs of (1.3) and (1.4) in the quasi-Banach context, respectively. Let us finally mention that no reasonable generalization of (1.4) for different values of p can be expected (see [Cw]).

2. Basic Facts on Vector-Valued Hardy Spaces

Let X be a real or complex Banach space and denote by $\mathbb{R}^{n+1}_+$ the set $\{(x,t) : x \in \mathbb{R}^n, t > 0\}$. To each measurable function $f\colon \mathbb{R}^{n+1}_+ \to X$ we can associate the maximal function

$$f^*(x) = \sup_{t>0} \|f(x,t)\| \qquad (x \in \mathbb{R}^n).$$

For $0 < p \leqslant \infty$ we denote by $H^h_p(X) = H^h_p(X, \mathbb{R}^n)$ the space of harmonic functions from $\mathbb{R}^{n+1}_+$ into X with f^* belonging to $L_p(\mathbb{R}^n)$. The norm in this space is given by

$$\|f\|_{H^h_p(X)} = \|f^*\|_{L_p(\mathbb{R}^n)}.$$

For complex Banach X spaces and $0 < p \leqslant \infty$ we define $H_p(X) = H_p(X, D)$ the space of analytic functions f from the disc C into X such that

$$\|f\|_{H_p(X)} = \sup_{0<r<1} \left(\int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

(with the obvious modification for $p = \infty$).

These spaces become Banach spaces for $1 \leqslant p \leqslant \infty$ and quasi-Banach for $0 < p < 1$.

One of the main differences when working with vector-valued Hardy spaces comes from the fact that functions in $H_p^h(X)$ and $H_p(X)$ need not have boundary values even when $1 < p \leqslant \infty$. On the other hand we can still represent each function in those spaces in terms of a vector-valued distribution. Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ be the Schwartz class and denote by $\mathscr{S}'(X) = \mathscr{S}'(X, \mathbb{R}^n)$ the class of linear continuous maps from $\mathscr{S}$ into X. The elements in $\mathscr{S}'(X)$ are called X-valued tempered distributions. The following known result for the classical case (cf. [FS, GR]) goes over the vector-valued case.

PROPOSITION 2.1. *Let $0 < p \leqslant \infty$ and $f \in H_p^h(X)$. There exists the limit $\lim_{t \to 0} f(\cdot, t)$ in the sense of distributions in $\mathscr{S}'(X)$.*

Sketch of Proof. The case $0 < p \leqslant 1$ can be done similarly to the scalar-valued case (see [FS, p. 174]). For $p > 1$ we look at

$$T_t(\phi) = \int_{\mathbb{R}^n} f(x, t)\, \phi(x)\, dx \qquad (t > 0)$$

as a uniformly bounded family of operators from $L_{p'}(\mathbb{R}^n)$ into X (where $1/p + 1/p' = 1$). Since $\mathscr{L}(L_{p'}(\mathbb{R}^n), X^{**})$ is a dual space (namely it is identified to the dual of $L_{p'}(\mathbb{R}^n) \hat{\otimes} X^*$) we can find a subnet converging to 0 and an operator T in $\mathscr{L}(L_{p'}(\mathbb{R}^n), X^{**})$ such that $\langle T_t(\phi), \xi \rangle$ converges to $\langle T(\phi), \xi \rangle$ for all ϕ in $L_{p'}(\mathbb{R}^n)$ and ξ in X^*. Composing with functionals and using the scalar-valued result, it is easy to realize that $T_t(\phi) = T(\phi * P_t)$, where P_t stands for the Poisson Kernel in $\mathbb{R}_+^{n+1}$, which shows that in fact the range of T is in X, and then T_t converges to T, as $t \to 0$, in the strong topology of $\mathscr{L}(L_{p'}(\mathbb{R}^n), X)$ and therefore in $\mathscr{S}'(X)$. ∎

Remark 2.1. A similar result can be obtained for the unit disc by means of the conformal transformation of $\mathbb{R}_+^2$ into D.

Some distributions in Proposition 2.1 can came from functions. For instance, assume $1 \leqslant p \leqslant \infty$ and take f in $L_p(X, \mathbb{R}^n)$, if we consider the Poisson integral of f, that is, $Pf(x, t) = P_t * f(x)$ where P_t is the Poisson kernel in $\mathbb{R}_+^{n+1}$, then we get a harmonic function in $H_p^h(X)$ whose associated distribution is represented by the function f.

It is well known that for harmonic functions g and for $1 < p \leqslant \infty$ we have

$$\|g^*\|_{L_p} \leqslant C_p \sup_{t > 0} \|g(\cdot, t)\|_{L_p(X)}. \tag{2.1}$$

Therefore (2.1) allows us to identify $L_p(X)$, $1 < p \leqslant \infty$, with a closed subspace of $H_p^h(X)$ given by Poisson integrals of functions in $L_p(X)$. Let us denote by $\tilde{H}_p^h(X)$ such a subspace. Identifying the functions f with its Poisson integral Pf we can write

$$\|f\|_{L^p(X)} \leqslant \|f\|_{H_p^h(X)} \leqslant C_p \|f\|_{L^p(X)}. \tag{2.2}$$

For $0<p\leqslant 1$ define $\tilde{H}_p^h(X)$ as the closed subspace generated by $H_p^h(X)\cap L_1(X)$ (where we are identifying the function in $L_1(X)$ with the distribution in $H_p^h(X)$ which defines).

The situation for the disc is very similar. We assume now that X is a complex Banach space. Let $1\leqslant p\leqslant\infty$ and let f be a function in $L_p(X,\mathbb{T})$ with $\hat{f}(n)=0$ for $n<0$, we define the analytic function

$$Pf(re^{i\theta}) = P_r * f(\theta),$$

where P_r denotes the Poisson Kernel on the disc.

Denoting by

$$\tilde{H}_p(X) = \{Pf : f\in L_p(X,\mathbb{T}), \hat{f}(n)=0 \text{ for } n<0\}$$

we now have that for $1\leqslant p\leqslant\infty$

$$\|f\|_{L_p(X,\mathbb{T})} = \|Pf\|_{H_p(X)}.$$

For $0<p<1$ we use the notation $\tilde{H}_p(X)$ for the closure of the polynomials in $H_p(X)$.

The coincidence of $\tilde{H}_p^h(X)$ with $H_p^h(X)$ and $\tilde{H}_p(X)$ with $H_p(X)$ depends on some geometric properties of the Banach space X. It is known that $\tilde{H}_p^h(X)=H_p^h(X)$ for some $0<p\leqslant\infty$ (and equivalently for all $0<p\leqslant\infty$) if and only if X has the Radon–Nikodym property, and that the coincidence $\tilde{H}_p(X)=H_p(X)$ is equivalent to the analytic Radon–Nikodym property on the space X.

Let us now characterize $\tilde{H}_p(X)$ in a very useful way in terms of special "building blocks" called atoms. The reader is referred to [CW] or [GR] for general theory on atomic decompositions in the scalar-valued case. We say that a function $a\colon \mathbb{R}^n\to X$ is a p-atom $(0<p\leqslant 1)$ if

(i) $\operatorname{supp} a\subset Q$, Q being a cube in $\mathbb{R}^n$,

(ii) $\|a\|_{L_\infty(X)}\leqslant |Q|^{-1/p}$ (where $|\cdot|$ stands for the Lebesgue measure)

(iii) $\int_{\mathbb{R}^n} x^\alpha a(x)\,dx=0$ for $\alpha\in N^n$, $|\alpha|=\sum_{k=1}^n \alpha_k\leqslant[n(1/p-1)]$, and $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$.

Let us define

$$H_p^{at}(X)=\left\{\sum_{k\geqslant 0}\lambda_k a_k : \lambda=(\lambda_k)_{k\geqslant 0}\in l^p,\ a_k \text{ are } X\text{-valued } p\text{-atoms},\ k\geqslant 0\right\}$$

(where the series converges in $\mathscr{S}'(X)$).

For each f in $H_p^{at}(X)$ we define

$$\|f\|_{H_p^{at}(X)} = \inf\left\{\left(\sum_{k\geq 0} |\lambda_k|^p\right)^{1/p} : f = \sum_{k\geq 0} \lambda_k a_k\right\},$$

where the infimum is taken over all possible representations of f in terms of p-atoms. $(H_p^{at}(X), \|\ \|_{H_p^{at}(X)})$ is a Banach space for $p=1$ and a quasi-Banach for $0<p<1$.

It is very easy to show that all X-valued p-atoms belong to a fixed closed ball in $\tilde{H}_p^h(X)$, which allows us to write that $H_p^{at}(X) \subset \tilde{H}_p^h(X)$ (with continuity). The reverse inclusion is also true and it follows with the obvious modifications to the scalar-valued case (see [Co, L, LU] for a proof). Hence we can state the following theorem which implies that $\tilde{H}_p^h(X) = H_p^{at}(X)$ with equivalent norms.

THEOREM B. *Let* $0<p\leq 1$ *and* $f \in \tilde{H}_p^h(X)$. *There exists a sequence of* X*-valued* p*-atoms* $(a_k)_{k\geq 0}$ *and a sequence* $\lambda = (\lambda_k)_{k\geq 0} \in l^p$ *such that* $f = \sum_{k\geq 0} \lambda_k a_k$. *Moreover* $(\sum_{k\geq 0} |\lambda_k|^p)^{1/p} \leq C_p \|f\|_{H_p^h(X)}$.

3. COMPLEX INTERPOLATION BETWEEN $\tilde{H}_{p_0}^h(X_0)$ AND $\tilde{H}_{p_1}^h(X_1)$

Since we shall be dealing with complex interpolation in this and the next two sections, we consider complex Banach spaces X_0 and X_1, (X_0, X_1) stands for an interpolation couple, $X_\theta = (X_0, X_1)_\theta$, and $\|\ \|_0$, $\|\ \|_1$, and $\|\ \|_\theta$ denote the norms in X_0, X_1, and X_θ, respectively. The main result of this section is the following

THEOREM 3.1. *Let* $0<p_0, p_1<\infty$, $0<\theta<1$, *and* $1/p = (1-\theta)/p_0 + \theta/p_1$, *then*

$$(\tilde{H}_{p_0}^h(X_0), \tilde{H}_{p_1}^h(X_1))_\theta = \tilde{H}_p^h(X_\theta). \tag{3.1}$$

Remark 3.1. This is clearly true for $1<p_0, p_1\leq\infty$ because of (2.2) and (1.3). The case $p_0=1$ and $1<p\leq\infty$ was proved in [B1]. The proof will be based on the atomic decomposition provided by Theorem B and some ideas in [JJ].

Proof of Theorem 3.1. Let us show first the easy part

$$(\tilde{H}_{p_0}^h(X_0), \tilde{H}_{p_1}^h(X_1))_\theta \subset \tilde{H}_p^h(X_\theta).$$

From density arguments it suffices to see that if f belongs to $\tilde{H}_{p_0}^h(X_0)$ and $\tilde{H}_{p_1}^h(X_1)$ with norm in $(\tilde{H}_{p_0}^h(X_0), \tilde{H}_{p_1}^h(X_1))_\theta$, $\|f\|_\theta \leq 1$, then f belongs to $\tilde{H}_p^h(X_\theta)$ and $\|f\|_{H_p^h(X_\theta)} \leq 1$.

Given f under the above assumptions and $\varepsilon > 0$ we can find F in $\mathscr{F}(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))$ such that $F(\theta) = f$ and $\|F\|_{\mathscr{F}} \leqslant 1 + \varepsilon$. Now for each $(x, t) \in \mathbb{R}^{n+1}_+$ we have $F((x, t), \theta) = f(x, t)$ and $F((x, t), \cdot) \in \mathscr{F}(X_0, X_1)$, and therefore from (1.1) we get

$$\operatorname{Log} \sup_{t>0} \|f(x, t)\|_\theta^p \leqslant \sum_{j=0}^{1} \int_{-\infty}^{\infty} \operatorname{Log} \sup_{t>0} \|F((x, t), j + i\sigma)\|_j^p \, P_j(\theta, \sigma) \, d\sigma.$$

Using Hölder and Jensen's inequalities and denoting by $\theta_0 = 1 - \theta$ and $\theta_1 = \theta$ we can write

$$\begin{aligned}
\|f\|_{H^h_p(X)} &\leqslant \left(\int_{\mathbb{R}^n} \left\{ \prod_{j=0}^{1} \exp \int_{-\infty}^{\infty} \operatorname{Log} \sup_{t>0} \|F((x, t), j + i\sigma)\|_j^p \right.\right. \\
&\qquad \left.\left. \times P_j(\theta, \sigma) \, d\sigma \right\} dx \right)^{1/p} \\
&\leqslant \prod_{j=0}^{1} \left\{ \left(\int_{\mathbb{R}^n} \left[\exp \int_{-\infty}^{\infty} \operatorname{Log} \sup_{t>0} \|F((x, t), j + i\sigma)\|_j^{p_j} \right.\right.\right. \\
&\qquad \left.\left.\left. \times \theta_j^{-1} P_j(\theta, \sigma) \, d\sigma \right] dx \right) \right\}^{\theta_j/p_j} \\
&\leqslant \prod_{j=0}^{1} \left\{ \int_{-\infty}^{\infty} \left[\int_{\mathbb{R}^n} \sup_{t>0} \|F((x, t), j + i\sigma)\|_j^{p_j} \, dx \right] \theta_j^{-1} P_j(\theta, \sigma) \, d\sigma \right\}^{\theta_j/p_j} \\
&\leqslant \left(\sup_{\sigma \in \mathbb{R}} \|F(i\sigma)\|_{H^h_{p_0}(X_0)}\right)^{1-\theta} \left(\sup_{\sigma \in \mathbb{R}} \|F(1 + i\sigma)\|_{H^h_{p_1}(X_1)}\right)^{\theta} \\
&\leqslant \|F\|_{\mathscr{F}} \leqslant 1 + \varepsilon.
\end{aligned}$$

The second inclusion is much more delicate. From Remark 3.1 we may assume that $0 < p \leqslant 1$. We shall see that there is a constant such that for any f in $\tilde{H}^h_p(X_\theta)$ and $\|f\|_{H^h_p(X_\theta)} \leqslant 1$ there is a function F in $\mathscr{F}(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))$ with $\|F\|_{\mathscr{F}} \leqslant C$ and $\|f - F(\theta)\|_{H^h_p(X_\theta)} \leqslant 1/2$. Then a usual reiteration argument finishes the proof by showing that f belongs to $(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_\theta$ and its norm $\|f\|_\theta \leqslant C$.

Since $X_0 \cap X_1$ is dense in X_θ and simple functions with values in X_θ are dense in $\tilde{H}^h_p(X_\theta)$, we can assume that f is a simple function with values in $X_0 \cap X_1$. From Theorem B we would have an atomic decomposition for f, but we are going to be more explicit and to recall a procedure to get one. Denoting by $G(f)$ the Fefferman–Stein's grand maximal function, and considering $R_k = \{G(f) > 2^k\}$ $(k \in \mathbb{Z})$, we find a Whitney decomposition of R_k, say $\{Q_j^k\}$, such that Q_j^k are cubes verifying $R_k = \bigcup \{Q_j^k : j \geq 1\}$ and $\bar{Q}_j^k = 9/8 Q_j^k \subset R_k$. Moreover $\|\sum_{j \geq 1} \chi_{\bar{Q}_j^k}\|_{L_\infty(\mathbb{R}^n)} \leq C$.

Adecuated to this decomposition into cubes we can find a family of X_θ-valued functions $\{a_j^k\}$ having the properties

$$f = \sum_{k \in \mathbb{Z}} \sum_{j \geq 1} a_j^k \tag{3.2}$$

$$\operatorname{supp} a_j^k \subset \bar{Q}_j^k, \qquad \bar{Q}_j^k \text{ being a cube in } \mathbb{R}^n \tag{3.3}$$

$$\sum_{j \geq 0} \|a_j^k(x)\|_\theta \leq C 2^k \chi_{R_k}(x), \qquad x \in \mathbb{R}^n \leftarrow \tag{3.4}$$

$$\int_{\mathbb{R}^n} x^\alpha a_j^k(x)\, dx = 0 \qquad \text{for} \quad \alpha \in N^n, |\alpha| = \sum_{k=1}^n \alpha_k \leq N = N(p_0, p_1), \tag{3.5}$$

where $N = \max([n(1/p_0 - 1)], [n(1/p_1 - 1)])$.

In fact a_j^k are obtained from the function f as

$$a_j^k = (f - P_j^k)\, \phi_j^k - \sum_{i \geq 1} [(f - P_i^{k+1})\, \phi_i^{k+1} - P_{ji}^{k+1}]\, P_i^{k+1}, \tag{3.6}$$

where P_{ji}^k and P_i^k are polynomials with coefficients in X_θ and ϕ_j^k are C^∞ functions with values in $\mathbb{R}$ and supported in $\bar{Q}_j^k$. (We refer the reader to [L] or [LU] for a proof in the scalar-valued case that goes over the vector-valued setting with the obvious modifications.) By construction, since f is $X_0 \cap X_1$-valued, so P_{ji}^k and P_i^k are. According to (3.2) and (3.6) there exist k_0, k_1 in $\mathbb{Z}$ with $k_0 < k_1$ and for each $k_0 \leq k \leq k_1$ there exists J_k in $\mathbb{N}$ such that for each $k_0 \leq k \leq k_1$ and $1 \leq j \leq J_k$ we can find $N_j^k \in \mathbb{N}$ verifying that if we take

$$b_j^k = (f - P_j^k)\, \phi_j^k - \sum_{i=1}^{N_j^k} [(f - P_i^{k+1})\, \phi_i^{k+1} - P_{ji}^{k+1}]\, P_i^{k+1} \tag{3.7}$$

$$f_1 = \sum_{k=k_0}^{k_1} \sum_{j=1}^{J_k} b_j^k \tag{3.8}$$

then

$$\|f - f_1\|_{H_p^k(X)} \leq 1/4.$$

Note that b_j^k still have properties (3.3) to (3.5) for $k_0 \leq k \leq k_1$ and $1 \leq j \leq J_k$ and they are defined by finite sums. Observing that f is a $X_0 \cap X_1$-valued simple function, P_{ji}^k, P_i^k are polynomials and ϕ_{ji}^k, ϕ_i^k are compactly supported C^∞ functions, one can easily find simple functions c_j^k with values in $X_0 \cap X_1$ and supported in $\bar{Q}_j^k$ such that

$$\|b_j^k - c_j^k\|_{L_\infty(X_0 \cap X_1, \mathbb{R}^n)} \leq 1/2.$$

For such functions we can get $H_j^k \in \mathscr{F}(L_\infty(X_0, \bar{Q}_j^k), L_\infty(X_1, \bar{Q}_j^k))$ verifying that for all $x \in \bar{Q}_j^k$, $H_j^k(x, \theta) = c_j^k$ and $\|H_j^k(x, \cdot)\|_{\mathscr{F}(X_0, X_1)} \leqslant \|c_j^k(x)\|_\theta + 1/2$.

Consider now $G_j^k = H_j^k + (b_j^k - c_j^k)$.

Obviously $G_j^k \in \mathscr{F}(L_\infty(X_0, \bar{Q}_j^k), L_\infty(X_1, \bar{Q}_j^k))$ and for all $x \in \bar{Q}_j^k$,

$$G_j^k(x, \theta) = b_j^k \text{ and } \|G_j^k(x, \cdot)\|_{\mathscr{F}(X_0, X_1)} \leqslant \|b_j^k(x)\|_\theta + 3/2.$$

According to (3.4) one has

$$\|G_j^k(x, \cdot)\|_{\mathscr{F}(X_0, X_1)} \leqslant C 2^k \chi_{R_k}(x), \qquad x \in \mathbb{R}^n. \tag{3.9}$$

The next step consists of modifying a bit G_j^k to get a right order of vanishing moments. To do so we need the following procedure. Given a Banach space X, an integer $m \in \mathbb{N}$, a bounded measurable set E in $\mathbb{R}^n$ with positive measure, and a measurable function g defined on E with values in X, we shall denote $Q(g) = Q(g, E, m)$ the unique polynomial with values in X satisfying

$$\int_E x^\alpha Q(g)(x)\, dx = \int_E x^\alpha g(x)\, dx \qquad \text{for} \quad \alpha \in N^n, |\alpha| \leqslant m.$$

Let us state the following lemma whose proof depends only on an argument on finite dimensional Hilbert spaces and can be adapted to the vector-valued setting (cf. [FRS], L]).

LEMMA 3.1. *Let* $x_0 \in I$, I *being the unit cube in* $\mathbb{R}^n$. *If* $\|\int_I (x - x_0)^\alpha g(x)\, dx\| \leqslant 1$ *for* $|\alpha| \leqslant m$, *then* $\|Q(g)\|_{L_\infty(X, I)} \leqslant C$.

For each $z \in \bar{S}$ we consider $P_j^k(\cdot, z) = Q(G_j^k(\cdot, z), \bar{Q}_j^k, N)$. Now Lemma 3.1, together with elementary arguments of dilations and translations and (3.9), gives us the estimates

$$\|P_j^k(\cdot, i\sigma)\|_{L_\infty(X_0, \bar{Q}_j^k)} \leqslant C 2^k \qquad (\sigma \in \mathbb{R}) \tag{3.10}$$

$$\|P_j^k(\cdot, 1 + i\sigma)\|_{L_\infty(X_1, \bar{Q}_j^k)} \leqslant C 2^k \qquad (\sigma \in \mathbb{R}). \tag{3.11}$$

Define now the functions

$$F_j^k(z) = G_j^k(z) - P_j^k(\cdot, z)\, \chi_{\bar{Q}_j^k}.$$

Note that we still have $F_j^k \in \mathscr{F}(L_\infty(X_0, \bar{Q}_j^k), L_\infty(X_1, \bar{Q}_j^k))$ and from (3.10), (3.11), and (3.9) we have

$$\|F_j^k(x, \cdot)\|_{\mathscr{F}(X_0, X_1)} \leqslant C 2^k \chi_{R_k}(x), \qquad x \in \mathbb{R}^n. \tag{3.12}$$

Finally observe that $P_j^k(\cdot, \theta) = 0$ and then from the definition of $P_j^k(\cdot, z)$ we can also write

$$F_j^k(\theta) = b_j^k \tag{3.13}$$

$$\int_{\mathbb{R}^n} x^\alpha F_j^k(x, z)\, dx = 0 \qquad \text{for} \quad \alpha \in N^n,\ |\alpha| \leqslant N,\ z \in \bar{S}. \tag{3.14}$$

Denoting by $\alpha(z) = (1 - p/p_0)(z/\theta - 1)$, we build up the function

$$F(z) = \sum_{k_0}^{k_1} \sum_{j=1}^{J_k} 2^{k\alpha(z)} F_j^k(z).$$

Note that $F \in \mathscr{F}(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))$ and $F(\theta) = f_1$. Let us finally estimate the norm $\|F\|_{\mathscr{F}}$. Consider

$$f_j^k(x) = C^{-1} 2^{-k} |\bar{Q}_j^k|^{-1/p_0} F_j^k(z)$$

then (3.12) and (3.14) show that $f_j^k(i\sigma)$ are p_0-atoms with values in X_0, and therefore Theorem B gives that

$$\begin{aligned}\|F(i\sigma)\|_{H^h_{p_0}(X_0)} &\leqslant C \left(\sum_{k_0}^{k_1} \sum_{j=1}^{J_k} 2^{kp_0(1-1+p/p_0)} |\bar{Q}_j^k| \right)^{1/p_0} \\ &\leqslant C \left(\sum_{k_0}^{k_1} 2^{kp} |R_k| \right)^{1/p_0} \leqslant C(\|G(f)\|_{L_p(X_0)})^{p/p_0} \\ &\leqslant C(\|f\|_{H^h_p(X_0)})^{p/p_0}.\end{aligned}$$

(The reader is referred to [GR] to see how the space $\tilde{H}^h_p$ can also be defined in terms of $G(f)$ instead of f^*.)

Therefore we have shown that

$$\sup_{\sigma \in \mathbb{R}} \|F(i\sigma)\|_{H^h_{p_0}(X_0)} \leqslant C.$$

If $0 < p_1 \leqslant 1$, a similar argument would give

$$\sup_{\sigma \in \mathbb{R}} \|F(1 + i\sigma)\|_{H^h_{p_1}(X_1)} \leqslant C.$$

Assume now that $p_1 > 1$. According to (3.12) we have

$$\|F(x, 1 + i\sigma)\|_1 \leqslant C \sum_{k_0}^{k_1} \sum_{j=1}^{J_k} 2^{kp/p_1} \chi_{\bar{Q}_j^k}(x) \leqslant C \sum_{k \in \mathbb{Z}} 2^{kp/p_1} \chi_{R_k}(x).$$

For $p_1 < \infty$ we can now compute the norm in $L_{p_1}(X_1)$ and get

$$\|F(1+i\sigma)\|_{L_p(X_1)}^{p_1} \leqslant C \sum_{k \in \mathbb{Z}} 2^{kp} |R_k| \leqslant C \|f\|_{H_p^h(X_0)}^p$$

which implies $\sup_{\sigma \in \mathbb{R}} \|F(1+i\sigma)\|_{H_{p_1}^h(X_1)} \leqslant C$.

Hence the proof is completed since $\|F\|_{\mathscr{F}} \leqslant C$ and $\|F(\theta) - f\|_{H_p^h(X_\theta)} = \|f - f_1\|_{H_p^h(X_0)} \leqslant 1/4$. ∎

The case $p_1 = \infty$ was obtained in [B1] using Wolff's reiteration theorem [W]. Now in the quasi-Banach case we do not have it at our disposal, and we shall use the argument in the previous and ideas in [JJ] to get this extreme case.

THEOREM 3.2. *Let* $0 < p_0 < \infty$, $0 < \theta < 1$, *and* $1/p = (1-\theta)/p_0$. *Then*

$$(\tilde{H}_{p_0}^h(X_0), \tilde{H}_\infty^h(X_1))_\theta = \tilde{H}_p^h(X_\theta). \tag{3.15}$$

Sketch of Proof. The easy part is very similar and we omit it. To do the harder inclusion, we follow the same steps as before and get $f_1 = \sum_{k_0}^{k_1} \sum_{j=1}^{J_k} b_j^k$ where b_j^k are defined by (3.7). Now we use Lemma 5.1 in [JJ] to select a subfamily $\{Q_j\}_{j \in J}$ of $\{Q_j^k : 1 \leqslant j \leqslant J_k\, k_0 \leqslant k \leqslant k_1\}$ and certain functions b_j obtained as sums of functions in $\{b_j^k : 1 \leqslant j \leqslant J_k, k_0 \leqslant k \leqslant k_1\}$ which take values in $X_0 \cap X_1$ and are supported in Q_j and satisfy

$$f_1 = \sum_{j \in J} b_j. \tag{3.16}$$

For each $j \in J$ there is $m(j) \in \mathbb{N}$ such that

$$\textstyle\sum_{j \in J} 2^{pm(j)} |Q_j| \leqslant C_p. \tag{3.17}$$

If $Q_k \subset Q_j$ and $k \neq j$ then $m(k) > m(j)$, $Q_k \neq Q_j$, and

$$\sum_{Q_k \subset Q_j} |Q_k| \leqslant 2\, |Q_j|. \tag{3.18}$$

$$\|b_j\|_{L_\infty(X_\theta)} \leqslant C 2^{m(j)}. \tag{3.19}$$

$$\int_{\mathbb{R}^n} x^\alpha b_j(x)\, dx = 0 \qquad \text{for} \quad \alpha \in N^n,\ |\alpha| \leqslant N_0 = [n(1/p_0 - 1)]. \tag{3.20}$$

If $A \subset J$ then for every x there exists $j(x) \in A$ such that

$$\sum_{j \in A} \|b_j(x)\|_\theta \leqslant C_{p_0} 2^{m(j(x))} \chi_{\tilde{Q}_{j(x)}}(x). \tag{3.21}$$

As in [JJ] we consider

$$J(Q_j)=\{Q_k\subset 5Q_j,\ |Q_k|\leqslant |Q_j|\}\quad \text{and}\quad E_j=\Big\{x\in \bar{Q}_j\colon \sum_{Q_k\in J(Q_j)} \chi_{\bar{Q}_j}(x)\geqslant \lambda\Big\},$$

where λ will be chosen later. It is not hard to see that $|E_j|\leqslant (C/\lambda)\,|\bar{Q}_j|$. Hence $|E_j|\leqslant (1/2)\,|\bar{Q}_j|$ for λ large enough.

Now consider P_j being the $X_0\cap X_1$-valued polynomial satisfying

$$\int_{\bar{Q}_j-E_j} x^\alpha P_j(x)\,dx=\int_{\bar{Q}_j-E_j} x^\alpha b_j(x)\,dx \qquad \text{for}\quad \alpha\in N^n,\ |\alpha|\leqslant N_0.$$

It can be shown that

$$\|P_j\|_{L_\infty(X_0\cap X_1,\,\bar{Q}_j-E_j)}\leqslant C\,\frac{2^{m(j)}}{\lambda}. \tag{3.22}$$

Write now $\bar{b}_j=(b_j-P_j)\,\chi_{\bar{Q}_j-E_j}$ $(j\in J)$ and $f_2=\sum_{j\in J}\bar{b}_j$.

Taking λ large enough one gets $\|f_1-f_2\|_{H^h_p(X_\theta)}\leqslant 1/4$ (cf. [JJ]). Now we can repeat the arguments used in Theorem 3.1 by replacing b_j^k by $\bar{b}_j$ and f_1 by f_2. Finally properties (3.16) to (3.21) allow us to find F in $\mathscr{F}(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_\infty(X_1))$ and $F(\theta)=f_2$ and $\|F\|_{\mathscr{F}}\leqslant C$.

Therefore the proof is completed since

$$\|f-F(\theta)\|_{H^h_p(X_\theta)}\leqslant \|f-f_1\|_{H^h_p(X_\theta)}+\|f_1-f_2\|_{H^h_p(X_\theta)}\leqslant 1/2.$$

and the iteration argument can still be applied. ∎

4. Vector-Valued BMO as an End Point for Interpolation

We denote by $BMO(X)=BMO(X,\mathbb{R}^n)$ the set of locally integrable X-valued functions satisfying

$$\|f\|_{BMO(X)}=\sup_Q \frac{1}{|Q|}\int_Q \|f(x)-f_Q\|\,dx<\infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ and f_Q stands for the average $\int_Q f(x)\,dx/|Q|$.

$BMO(X)$ (modulo constant functions) is a Banach space and it has an equivalent norm given as

$$\|f\|'_{BMO(X)}=\sup_Q \inf_{a\in X} \frac{1}{|Q|}\int_Q \|f(x)-a\|\,dx.$$

It is quite easy to notice that

$$1/2\, \|f\|_{BMO(X)} \leqslant \|f\|'_{BMO(X)} \leqslant \|f\|_{BMO(X)}. \tag{4.1}$$

Following Fefferman and Stein [FS] we consider the "sharp function" $f^{\#}$ defined by

$$f^{\#}(x) = \sup_{x \in Q} \inf_{a \in X} \frac{1}{|Q|} \int_Q \|f(y) - a\|\, dy \qquad (x \in \mathbb{R}^n).$$

Hence (4.1) says that $f \in BMO(X)$ if and only if $f^{\#} \in L_\infty(\mathbb{R}^n)$. We shall need in the sequel the following generalizations of the "sharp function" and the Hardy–Littlewood maximal function Mf.

For $0 < r < \infty$ and $f \in L^r_{\text{loc}}(X, \mathbb{R}^n)$ let us define

$$f_r^{\#}(x) = \sup_{x \in Q} \inf_{a \in X} \left(\frac{1}{|Q|} \int_Q \|f(y) - a\|^r\, dy \right)^{1/r} \qquad (x \in \mathbb{R}^n).$$

$$M_r f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q \|f(y)\|^r\, dy \right)^{1/r} \qquad (x \in \mathbb{R}^n).$$

It is elementary to show that for $0 < r < \infty$ one has

$$C^{-1} f_r^{\#}(x) \leqslant \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q \|f(y) - f_Q\|^r\, dy \right)^{1/r} \leqslant C f_r^{\#}(x). \tag{4.2}$$

The following easy observations allow us to get results for values of $0 < r < \infty$ from the case $r = 1$. Given f and writing $g(x)$ for $\|f(x)\|^r$ then

$$Mg(x) = (M_r f(x))^r \qquad \text{and} \qquad g^{\#}(x) \leqslant (f_r^{\#}(x))^r. \tag{4.3}$$

The next result is an extension of the Hardy–Littlewood maximal theorem and Theorem 5 in [FS] to any value $0 < r < \infty$.

Lemma 4.1. *Let* $0 < r < \infty$.

$$\text{If } r < p \leqslant \infty \text{ then } \|M_r f\|_{L_p} \leqslant C\, \|f\|_{L_p(X)}. \tag{4.4}$$

$$\text{If } r < p < \infty \text{ and } r \leqslant p_0 \leqslant p \text{ then } \|M_r f\|_{L_p} \leqslant C\, \|f_r^{\#}\|_{L_p} \text{ for } f \in L_{p_0}(X). \tag{4.5}$$

Proof. Condition (4.4) is obvious from (4.3) and the classical result. Condition (4.5) was proved in [H] for $X = \mathbb{C}$ by modifying the original proof in [FS], but we shall show here that it certainly follows from the case $r = 1$. Note that if $r > 1$ then using (4.4) and the case $r = 1$ we get

$$\|M_r f\|_{L_p} \leqslant C\, \|f\|_{L_p(X)} \leqslant C\, \|Mf\|_{L_p} \leqslant C\, \|f^{\#}\|_{L_p} \leqslant C\, \|f_r^{\#}\|_{L_p}.$$

For $r<1$ put $g(x)=\|f(x)\|^r$ and use (4.3) together with the case $r=1$ to get

$$\|M_r f\|_{L_p} \leqslant \|Mg\|_{L_{p/r}} \leqslant C\,\|g^{\#}\|_{L_{p/r}} \leqslant C\,\|f_r^{\#}\|_{L_p}. \quad \blacksquare$$

Now we are ready to state the analogue to (0.2) in the vector-valued case.

THEOREM 4.1. *Let* $0<\theta<1$, $0<p_0<\infty$, *and* $1/p=(1-\theta)/p_0$. *Then*

$$(L_{p_0}(X_0),\, BMO(X_1))_\theta = L_p(X_\theta) \tag{4.6}$$

$$(\tilde{H}^h_{p_0}(X_0),\, BMO(X_1))_\theta = \tilde{H}^h_p(X_\theta). \tag{4.7}$$

Remark 4.1. This was shown in [B1] for $1 \leqslant p_0 < \infty$ under some additional assumptions coming from the use of duality in the proof. Here we present a different approach and extend the result to all values $0<p_0<\infty$.

Proof of Theorem 4.1. Using (1.3) and the fact $L_\infty(X_1)\subset BMO(X_1)$ we have $L_p(X_\theta)\subset(L_{p_0}(X_0), BMO(X_1))_\theta$.

For the other inclusion take r_0 and r_1 verifying $r_0<p_0$ and $0<r_0<1<r_1<\infty$. Write $1/r=(1-\theta)/r_0+\theta/r_1$, which obviously gives $r<p$. Let us take $f\in L_{p_0}(X_0)\cap BMO(X_1)$ with norm in $(L_{p_0}(X_0), BMO(X_1))_\theta$, $\|f\|_\theta \leqslant 1$, and choose F in $\mathscr{F}(L_{p_0}(X_0), BMO(X_1))$ with $F(\theta)=f$ and $\|F\|_{\mathscr{F}}\leqslant 2$. For each cube Q and z in $\bar{S}$ we define

$$G_Q(z)=F(z)-\frac{1}{|Q|}\int_Q F(x,z)\,dx.$$

Since $F(x,\cdot)\in\mathscr{F}(X_0,X_1)$ and $F(x,\theta)=f(x)$, then $G_Q(x,\cdot)\in\mathscr{F}(X_0,X_1)$ and $G_Q(x,\theta)=f(x)-f_Q$. To simplify the notation we write $f^{\#}_{r,k}$ to denote the function $f^{\#}_r$ when taking values in X_k for $k=0, 1, \theta$.

From (1.1) we can write

$$\operatorname{Log}\|f(x)-f_Q\|_\theta^r \leqslant \sum_{j=0}^{1}\int_{-\infty}^{\infty}\operatorname{Log}\|G_Q(x,j+i\sigma)\|_j^r\,P_j(\theta,\sigma)\,d\sigma.$$

The analogous argument to the one used in Theorem 3.1 in its easy part shows that if we denote $\theta_0=1-\theta$ and $\theta_1=\theta$

$$\left(\frac{1}{|Q|}\int_Q\|f(y)-f_Q\|^r\,dy\right)^{1/r} \leqslant \prod_{j=0}^{1}\left\{\int_Q\int_{-\infty}^{\infty}\|G_Q(x,j+i\sigma)\|_j^{r_j}\,\theta_j^{-1}P_j(\theta,\sigma)\,d\sigma\,\frac{dx}{|Q|}\right\}^{\theta_j/r_j}.$$

From (4.3) and using $r_0 < p_0$ we have

$$f^{\#}_{r,\theta}(x) \leqslant C \prod_{j=0}^{1} \left\{ \int_{-\infty}^{\infty} (F(j+i\sigma)^{\#}_{r_j,j}(x))^{r_j} \theta_j^{-1} P_j(\theta,\sigma)\, d\sigma \right\}^{\theta_j/r_j}$$
$$\leqslant C \sup_{\sigma \in \mathbb{R}} (F(1+i\sigma)^{\#}_{r_1,1}(x))^{\theta}$$
$$\times \left(\int_{-\infty}^{\infty} (F(i\sigma)^{\#}_{r_0,0}(x))^{p_0} (1-\theta)^{-1} P_0(\theta,\sigma)\, d\sigma \right)^{(1-\theta)/p_0}.$$

From this inequality we get

$$C^{-p} \|f^{\#}_{r,\theta}\|_p^p \leqslant (\sup_x \sup_\sigma (F(1+i\sigma)^{\#}_{r_1,1}(x)))^{p_\theta}$$
$$\times \left(\int_{\mathbb{R}^n} \int_{-\infty}^{\infty} (F(i\sigma)^{\#}_{r_0\,0}(x))^{p_0} (1-\theta)^{-1} P_0(\theta,\sigma)\, d\sigma\, dx \right)$$
$$\leqslant \sup_{\sigma \in \mathbb{R}} \|F(i\sigma)^{\#}_{r_0,0}\|^{p_0}_{L_{p_0}(\mathbb{R}^n)} \sup_{\sigma \in \mathbb{R}} \|F(1+i\sigma)^{\#}_{r_1,1}\|^{p_\theta}_{L_\infty(\mathbb{R}^n)}.$$

On the other hand it is well known that for $r_1 > 1$

$$\|F(1+i\sigma)^{\#}_{r_1,1}\|_{L_\infty(\mathbb{R}^n)} \leqslant C \|F(1+i\sigma)\|_{BMO(X_1)}. \tag{4.8}$$

Using now that $g^{\#}_r(x) \leqslant M_r g(x)$ and (4.4) then

$$\|F(i\sigma)^{\#}_{r_0,0}\|_{L_{p_0}(\mathbb{R}^n)} \leqslant C \|M_{r_0} F(i\sigma)\|_{L_{p_0}(\mathbb{R}^n)} \leqslant C \|F(i\sigma)\|_{L_{p_0}(X_0)}. \tag{4.9}$$

Therefore using (4.8), (4.9), and (4.5) in Lemma 4.1, we get

$$\|f\|_{L_p(X_\theta)} \leqslant C \sup_{\sigma \in \mathbb{R}} \|F(i\sigma)\|_{L_{p_0}(X_0)} \sup_{\sigma \in \mathbb{R}} \|F(1+i\sigma)\|_{BMO(X_0)}.$$

Consequently $\|f\|_{L_p(X_\theta)} \leqslant C \|F\|_{\mathscr{F}} \leqslant 2C$.
Now a density argument finishes the proof of (4.6).
Let us now prove (4.7). From Theorem 3.1,

$$\tilde{H}^h_p(X_\theta) = (\tilde{H}^h_{p_0}(X_0), L_\infty(X_1))_\theta \subset (\tilde{H}^h_{p_0}(X_0), BMO(X_1))_\theta.$$

For the reverse inclusion let us take $t > 0$ and consider $T_t : f \to f * P_t$. This is a bounded operator with norm $\leqslant 1$ from $\tilde{H}^h_{p_0}(X_0)$ into $L_{p_0}(X_0)$ and from $BMO(X_1)$ into itself. Hence (4.6) and Theorem 2 in [CMS] give

$$\|f * P_t\|_{L_p(X_\theta)} \leqslant C \|f\|_{(\tilde{H}^h_{p_0}(X_0), BMO(X_1))_\theta}.$$

Hence (2.1) implies for $1 < p$

$$\|f\|_{\tilde{H}^h_p(X_\theta)} \leqslant C \|f\|_{(\tilde{H}^h_{p_0}(X_0), BMO(X_1))_\theta}.$$

To prove the case $0 < p \leqslant 1$ we may use the reiteration theorem for complex interpolation (cf. [BL, Theorem 4.6.1]) (in fact the inclusion we need is the easy one and it is true also for quasi-Banach spaces). Choose $0 < \theta_1 < 1$ with $q = p_0(1-\theta_1)^{-1} > 1$, and write $\beta = \theta\theta_1^{-1}$.

$$(\tilde{H}^h_{p_0}(X_0), BMO(X_1))_\theta \subset (\tilde{H}^h_{p_0}(X_0), (\tilde{H}^h_{p_0}(X_0), BMO(X_1))_{\theta_1})_\beta. \tag{4.10}$$

Combining this with (4.6) for $q > 1$, (3.1), and reiteration again we get

$$\begin{aligned}(\tilde{H}^h_{p_0}(X_0), BMO(X_1))_\theta &\subset (\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_q(X_{\theta_1}))_\beta \\ &= \tilde{H}^h_p((X_0, X_{\theta_1})_\beta) = \tilde{H}^h_p(X_\theta). \quad \blacksquare\end{aligned}$$

5. Complex Interpolation for Vector-Valued Analytic H^p-Spaces

In this section we shall work with the unit disc D instead of $\mathbb{R}^{n+1}_+$ and we shall study the analogous results to Theorems 3.1, 3.2, and 4.1 for $\tilde{H}_p(X)$. We shall denote by $BMO_a(X) = BMO_a(X, \mathbb{T})$ the space of functions in $L_1(X, \mathbb{T})$ with $\hat{f}(n) = 0$ for $n < 0$ and

$$\|f\|_{BMO(X)} = \sup_J \frac{1}{|J|} \int_J \|f(x) - f_J\|\, dx < \infty,$$

where J stands for intervals in $\mathbb{T}$ and $|\cdot|$ will be the normalized Lebesgue measure on $\mathbb{T}$.

We refer the reader to [J1, J2] for interpolation results for the complex method in the scalar-valued case. We shall see that the situation differs very much from the setting of harmonic functions when the functions are allowed to take values in Banach spaces. First of all let us state the inclusions which always remain valid.

Proposition 5.1. *Let* $0 < \theta < 1$, $0 < p_0 < \infty$, $0 < p_1 \leqslant \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$, *and* $1/q = (1-\theta)/p_0$.

$$(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_\theta \subset \tilde{H}_p(X_\theta). \tag{5.1}$$

$$(\tilde{H}_{p_0}(X_0), BMO(X_1))_\theta \subset \tilde{H}_q(X_\theta). \tag{5.2}$$

Proof. For $f \in \tilde{H}_{p_0}(X_0) \cap \tilde{H}_{p_1}(X_1)$, consider $f_r(t) = f(re^{it})$ for $0 < r < 1$ and $t \in [0, 2\pi)$. Since $f_r \in L_{p_0}(X_0) \cap L_{p_1}(X_1)$, then from (1.3) we have

$$\|f_r\|_{L_p(X_\theta)} \leqslant \|f_r\|_{(L_{p_0}(X_0), L_{p_1}(X_1))_\theta} \leqslant \|f\|_{(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_\theta}.$$

This gives that $f \in \tilde{H}_p(X_\theta)$ and $\|f\|_{\tilde{H}_p(X_\theta)} \leqslant \|f\|_{(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_\theta}$.

We leave the proof of (5.2) to the reader. It uses similar arguments to those in Theorem 4.1 where we replace the use of (2.1) for the following maximal inequality

$$\int_0^{2\pi} \sup_{0<r<1} \|f_r(t)\|^p \, dt \leqslant C_p \, \|f\|_{H_p(X)} \qquad (0<p\leqslant\infty)$$

which follows easily from the subharmonicity of $g(z)=\|f(z)\|^p$ when f is analytic. ∎

The following result exhibits the difference with the case of harmonic functions.

PROPOSITION 5.2. *There exists an interpolation couple* (X_0, X_1) *of complex Banach spaces verifying that for any values* $0<\theta<1$, $0<p_0<\infty$, $0<p_1\leqslant\infty$, *writing* $1/p=(1-\theta)/p_0+\theta/p_1$ *and* $1/q=(1-\theta)/p_0$ *we have*

$$(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_\theta \neq \tilde{H}_p(X_\theta) \tag{5.3}$$

$$(\tilde{H}_{p_0}(X_0), BMO_a(X_1))_\theta \neq \tilde{H}_q(X_\theta). \tag{5.4}$$

Proof. We shall use the following example constructed by G. Pisier to show that the finite cotype does not pass to interpolation spaces by the complex method. Let $X_0=L_1(\mathbb{T})$ and $X_1=c_0(\mathbb{Z})$. (The injection given by the sequence of Fourier coefficients makes them an interpolation couple.)

LEMMA 5.1. (G. Pisier). *For* $0<\theta<1$, X_θ *contains* c_0.

In fact $f_k(t)=\cos 3^k t$, for $k\in\mathbb{N}$ and $t\in[0, 2\pi)$, defines a sequence in X_θ equivalent to the canonical basis of c_0 (see [D] for a proof).

We use this example to verify (5.3). Fix $0<\theta<1$, $0<p_0<\infty$, and $0<p_1\leqslant\infty$. Due to a certain factorization property that L_1 enjoys we have the following fact whose proof can be found in [HP]:

(*) There exists $C=C_{p_0}>0$ such that for any sequence $\{r_k\}_{k\geqslant 0}$ with $0\leqslant r_0<r_1<\cdots<1$ and any function $f\in\tilde{H}_{p_0}(X_0)$

$$\left(\sum_{k\geqslant 0} \|f_{r_k}-f_{r_{k-1}}\|^s_{L_{p_0}(X_0)}\right)^{1/s} \leqslant C\,\|f\|_{\tilde{H}_{p_0}(X_0)}, \tag{5.5}$$

where $s=\max(2, p_0)$ and $f_{r_{-1}}=0$.

This means that for any sequence r_n increasing to 1 the operator $T: f\to(f_{r_k}-f_{r_{k-1}})_{k\geqslant 0}$ is bounded from $\tilde{H}_{p_0}(X_0)$ into $l_s(L_{p_0}(X_0))$. On the other hand T is obviously bounded from $\tilde{H}_{p_1}(X_1)$ into $l_\infty(L_{p_1}(X_1))$. Assume for a moment that (5.3) is an equality, then an interpolation argument would give

$$\left(\sum_{k\geqslant 0} \|f_{r_k}-f_{r_{k-1}}\|^{s_\theta}_{L_p(X_\theta)}\right)^{1/s_\theta} \leqslant C_\theta \|f\|_{\tilde{H}_p(X_\theta)}, \tag{5.6}$$

where $1/s_\theta=(1-\theta)/s$ and C_θ is a constant independent of (r_n) and f. But note that (5.6) implies that X_θ has cotype s_θ with $2\leqslant s_\theta<\infty$ which would contradict Lemma 5.1.

Very little modification is needed to show (5.4). Simply mention that John–Nirenberg's Lemma assures that T is also bounded from $BMO_a(X_1)$ into $l_\infty(L_{p_1}(X_1))$ if $p_1\geqslant 1$, and therefore equality in (5.4) would give

$$\left(\sum_{k\geqslant 0} \|f_{r_k}-f_{r_{k-1}}\|^{s_\theta}_{L_p(X_\theta)}\right)^{1/s_\theta} \leqslant C_\theta \|f\|_{\tilde{H}_q(X_\theta)}, \tag{5.7}$$

where $1/s_\theta=(1-\theta)/s$, $1/p=(1-\theta)/p_0+\theta/p_1$, and $1/q=(1-\theta)/p_0$, which is still enough to imply that X_θ would have cotype $<\infty$.

Clearly Proposition 5.2 depends very heavily on the special properties of the interpolation couple (L_1, c_0), and one should expect that there are cases where the natural extensions remain valid. In fact Jones' proof [J1] can be adapted to the case where $X_0=X_1=X$ to get a vector-valued version of (0.2). Here we present a different approach which shows that the vector-valued case follows from the scalar one. We would like to thank Gilles Pisier who kindly communicated this argument to us.

PROPOSITION 5.3. *Let* $0<\theta<1$, $0<p_0<\infty$, $0<p_1\leqslant\infty$, $1/p=(1-\theta)/p_0+\theta/p_1$, *and* $1/q=(1-\theta)/p_0$ *and* X *a complex Banach space.*

$$(\tilde{H}_{p_0}(X), \tilde{H}_{p_1}(X))_\theta=\tilde{H}_p(X). \tag{5.8}$$

$$(\tilde{H}_{p_0}(X), BMO_a(X))_\theta=\tilde{H}_q(X). \tag{5.9}$$

Proof. Let us first notice that (5.9) follows from (5.2) and the case $p_1=\infty$ in (5.8). Therefore according to (5.1) it suffices to show

$$\tilde{H}_p(X)\subset(\tilde{H}_{p_0}(X), \tilde{H}_{p_1}(X))_\theta.$$

Take an X-valued polynomial f with $\|f\|_{H_p(X)}\leqslant 1$, and consider the outer function ϕ satisfying $|\phi(e^{it})|=\|f(e^{it})\|+1$ for all $t\in[0, 2\pi)$. Denoting by $g=\phi^{-1}f$, we have $f=\phi g$ with $g\in\tilde{H}_\infty(X)$ and $\phi\in H_{p_0}\cap H_{p_1}$ and verifying $\|g\|_{H_\infty(X)}\leqslant 1$ and $\|\phi\|_{H_p(X)}\leqslant C$. From the scalar-valued case there exists $\Phi\in\mathscr{F}(H_{p_0}, H_{p_1})$ with $\Phi(\theta)=\phi$ and $\|\Phi\|_{\mathscr{F}}\leqslant 2C$. Define now $F=g\Phi$ and observe that $F\in\mathscr{F}(\tilde{H}_{p_0}(X), \tilde{H}_{p_1}(X))$, $F(\theta)=f$, and $\|F\|_{\mathscr{F}}\leqslant 2C$. A density argument completes the proof. ∎

There are also conditions on X_0 and X_1 which allow us to have the vector-valued extension for couples of different Banach spaces, namely the UMD property.

Let us recall that one of the most useful characterizations of the UMD property refers to the boundedness of the vector-valued Hilbert transform (cf. [Bo, Bu]), namely

(**) X is a UMD space if and only if the Riesz projection is bounded on $L_p(X, \mathbb{T})$ for some (and equivalently for all) values $1 < p < \infty$.

In fact it can be also formulated in terms of the boundedness of the Riesz projection from $\tilde{H}_p^h(X)$ into $\tilde{H}_p(X)$ for all values of p, $0 < p < \infty$ (cf. [Bu2, B2]). Using these observations together with the analogous results for the disc in Theorem 3.1 one can get a proof of the next result, but we present here an elementary argument which does not use the very technical and delicate result in (3.1).

PROPOSITION 5.4. *Let X_0 and X_1 be UMD complex Banach spaces and let $0 < \theta < 1$, $0 < p_0, p_1 < \infty$, and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then*

$$(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_\theta = \tilde{H}_p(X_\theta). \tag{5.10}$$

Proof. We simply have to show $\tilde{H}_p(X_\theta) \subset (\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_\theta$.

From UMD assumption and L_p-results, this is clear for $1 < p_0, p_1 < \infty$. We shall use factorization to decrease the index 1. Let us take an integer k such that $kp_0, kp_1 > 1$, and assume that f is a polynomial with values in X_θ and $\|f\|_{H_p(X)} \leqslant 1$. Given $\varepsilon > 0$, let us consider now the outer function ϕ satisfying $|\phi(e^{it})| = \|f(e^{it})\|_\theta + \varepsilon$ for all $t \in [0, 2\pi)$. As above we consider $g = \phi^{-1} f$ which implies $g \in \tilde{H}_\infty(X)$ and $\|g\|_{H_\infty(X)} \leqslant 1$. We write then $f = f_1 f_2 \cdots f_k$ where $f_1 = g\phi^{1/k}$ and $f_2 = \cdots = f_k = \phi^{1/k}$. Obviously we get $f_1 \in \tilde{H}_{kp}(X_\theta)$ and $f_j \in H_{kp}$ for $2 \leqslant j \leqslant k$. Using that $kp_0, kp_1 > 1$ we can find $F_1 \in \mathscr{F}(\tilde{H}_{kp_0}(X_0), \tilde{H}_{kp_1}(X_1))$ and $F_j \in \mathscr{F}(H_{kp_0}, H_{kp_1})$ for $2 \leqslant j \leqslant k$, verifying

$$F_j(\theta) = f_j \qquad \text{and} \qquad \|F_j\|_{\mathscr{F}} \leqslant C(\|f_j\|_{kp} + \varepsilon), \quad 1 \leqslant j \leqslant k.$$

Finally take $F = F_1 F_2 \cdots F_k$. This gives a function $F \in \mathscr{F}(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))$ with $F(\theta) = f$ and $\|F\|_{\mathscr{F}} \leqslant \prod_{j=1}^k \|F_j\|_{\mathscr{F}} \leqslant C^k \prod_{j=1}^k (\|f_j\|_{kp} + \varepsilon)$.

We finish the proof by letting ε go to zero and using $\prod_{j=1}^k \|f_j\|_{kp} = \|f\|_{H_p(X_\theta)}$. ∎

Remark 5.1. Standard techniques involving singular integrals show that the UMD property for X also implies the boundedness of the Riesz projection from $BMO(X, \mathbb{T})$ into $BMO_a(X, \mathbb{T})$. Unfortunately the previous argument does not work for BMO since the product of BMO functions need not be a BMO-function, but using (4.7) in Theorem 4.1 (its version for the disc) we can also establish that if X_0 and X_1 are UMD complex Banach spaces, $0<\theta<1$, $0<p_0<\infty$, and $1/p=(1-\theta)/p_0$ then

$$(\tilde{H}_{p_0}(X_0), BMO_a(X_1))_\theta = \tilde{H}_p(X_\theta). \tag{5.11}$$

Remark 5.2. In the particular case of L_p-spaces one can also extend (0.2) quite easily. Assume $X_0=L_{q_0}(\Omega, \mu)$ and $X_1=L_{q_1}(\Omega, \mu)$ for some measure space (Ω, μ).

For $1\leqslant q_0, q_1<\infty$ $0<\theta<1$, $0<p_0, p_1<\infty$, $1/p=(1-\theta)/p_0+\theta/p_1$, and $1/q=(1-\theta)/q_0+\theta/q_1$ we have

$$(H_{p_0}(L_{q_0}), H_{p_1}(L_{q_1}))_\theta = H_p(L_q). \tag{5.12}$$

First of all let us mention that $H_p(L_q)=\tilde{H}_p(L_q)$ for $0<p, q<\infty$, which allows us to write (5.12) without "~" (this is a well known result and says that L_q has the analytic Radon–Nikodym property). Secondly let us point out that (5.12) is not a simple consequence of Proposition 5.4 since we are including the case $q_0=1$, and $L_1(\Omega, \mu)$ fails UMD property. The proof of (5.12) is done in two steps. First one shows

$$(H_{q_0}(L_{q_0}), H_{q_1}(L_{q_1}))_\theta = H_q(L_q) \tag{5.13}$$

which follows from Fubini's theorem, (0.3), and (1.3), and then one uses a factorization argument like that in Proposition 5.4 to extend to the other values of p_0 and p_1.

Remark 5.3. In fact (5.12) can be extended to all values $0<q_0, q_1<\infty$ and even for the non-commutative L_p-spaces. The general result can be stated as follows (we refer to [X1] for a proof).

If $L_q(M, \tau)$ denotes a non-commutative L_q-space associated to a semi-finite von Neumann algebra (M, τ) and we take $0<\theta<1$, $0<p_0, p_1, q_0, q_1<\infty$, $1/p=(1-\theta)/p_0+\theta/p_1$, and $1/q=(1-\theta)/q_0+\theta/q_1$, then

$$(H_{p_0}(L_{q_0}(M, \tau)), H_{p_1}(L_{q_1}(M, \tau)))_\theta = H_p(L_q(M, \tau)). \tag{5.14}$$

6. Real Interpolation of Vector-Valued H^p-Spaces

Throughout this section (X_0, X_1) will stand for an interpolation couple of real or complex Banach spaces and we shall write $X_{\theta, p}$ for $(X_0, X_1)_{\theta, p}$ when $0<\theta<1$, and $0<p\leqslant\infty$. Since $X_{\theta, p}$ is not a Banach space for $0<p<1$, and we shall deal with $X_{\theta, p}$ valued harmonic functions then it is natural to work only with $1\leqslant p\leqslant\infty$.

We shall show that surprisingly we can not even extend (0.4) for vector-valued harmonic functions unless we restrict ourselves to values $p>1$. Let us begin with the following elementary fact.

PROPOSITION 6.1. *Let $0<\theta<1$, $0<p_0, p_1\leqslant\infty$, and $1/p=(1-\theta)/p_0+\theta/p_1$ such that $p\geqslant 1$. Then*

$$(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,p}\subset \tilde{H}^h_p(X_{\theta,p}). \tag{6.1}$$

Proof. Given f in $(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,p}$ with norm $\leqslant 1$, and using (1.2) we find $g_n\in \tilde{H}^h_{p_0}(X_0)$ and $h_n\in\tilde{H}^h_{p_1}(X_1)$ such that $f=g_n+h_n$ $(n\in\mathbb{Z})$ and

$$\left(\sum_{-\infty}^{\infty}\|e^{-n}g_n\|^{p_0}_{H^h_{p_0}(X_0)}\right)^{(1-\theta)/p_0}\left(\sum_{-\infty}^{\infty}\|e^{(1-\theta)n}h_n\|^{p_0}_{H^h_{p_0}(X_1)}\right)^{\theta/p_1}\leqslant C. \tag{6.2}$$

Since $f(x,t)=g_n(x,t)+h_n(x,t)\in X_0+X_1$ for $(x,t)\in\mathbb{R}^{n+1}_+$ and $n\in\mathbb{Z}$, then

$$\|f(x,t)\|\leqslant C\left(\sum_{-\infty}^{\infty}\|e^{-\theta n}g_n(x,t)\|_0^{p_0}\right)^{(1-\theta)/p_0}$$
$$\times\left(\sum_{-\infty}^{\infty}\|e^{(1-\theta)n}h_n(x,t)\|_1^{p_1}\right)^{\theta/p_1}.$$

Using Hölder's inequality and (6.2), we get $\|f\|_{\tilde{H}^h_p(X)}\leqslant C$, which proves (6.1). ∎

Obviously (6.1) becomes equality when $1<p_0, p_1\leqslant\infty$. It was also shown in [B1] that this also happens for $p_0=1$ and $1<p_1\leqslant\infty$. We shall prove that the result can be extended to other values of p_0, p_1 but only if the value p given by $1/p=(1-\theta)/p_0+\theta/p_1$ happens to be bigger than 1.

PROPOSITION 6.2. *Let $0<\theta<1$, $0<p_0, p_1\leqslant\infty$, and $1/p=(1-\theta)/p_0+\theta/p_1$ such that $p>1$. Then*

$$(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,p}=\tilde{H}^h_p(X_{\theta,p}). \tag{6.3}$$

Proof. Let us first state the following lemma which, in principle, is weaker than (6.3) but that will imply it.

LEMMA 6.1. *Under the conditions of Proposition 6.2, we have*

$$\tilde{H}^h_p(X_{\theta,p})\subset(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,\infty}. \tag{6.4}$$

Before proving the lemma, let us use it to finish the proof of Proposition 6.2. Since $p>1$, we can find $0<\theta_0<\theta<\theta_1$ such that $1/q_j=$

$(1-\theta_j)/p_0+\theta_j/p_1$ with $q_j>1$ $(j=0, 1)$. Take β such that $\theta=(1-\beta)\,\theta_0+\beta\theta_1$. Using the reiteration theorem (see [BL, p. 67]) and (6.4), we get

$$\begin{aligned}(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,p}\\ &=((\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta_0,\infty}, (\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta_1,\infty})_{\beta,p}\\ &\supset (\tilde{H}^h_{p_0}(X_{\theta_0,q_0}), \tilde{H}^h_{p_1}(X_{\theta_1,q_1}))_{\beta,p}=\tilde{H}^h_p((X_{\theta_0,q_0}, X_{\theta_1,q_1})_{\beta,p})=\tilde{H}^h_p(X_{\theta,p})\end{aligned}$$

which together with (6.1) shows (6.3). ▮

Proof of Lemma 6.1. We need only to consider the case $0<p_0\leqslant 1$. In this case we would have $p_1>1$. Let f belong to $\tilde{H}^h_p(X_{\theta,p})$ with $\|f\|_{\tilde{H}^h_p(X_{\theta,p})}\leqslant 1$. Without lost of generality we assume f is a simple function with values in $X_0\cap X_1$. We shall use an argument like that one used in Theorem 3.1 to show that there exists a function g in $(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,\infty}$ such that

$$\|g\|_{(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,\infty}}\leqslant C \qquad \text{and} \qquad \|f-g\|_{\tilde{H}^h_p(X_{\theta,p})}\leqslant 1/2.$$

for some absolute constant C. An iteration argument will then complete the proof. We shall do only the case $p_1<\infty$. The proof of the case $p_1=\infty$ needs simply some modifications.

Keeping all the notations in the proof of Theorem 3.1, we consider the function f_1 associated to f defined by (3.8). Let $t>0$ and $\alpha=(1/\theta)(p/p_0-1)$. By elementary properties of $X_{\theta,p}$ and the special form of the functions b^k_j defined by (3.7), we find easily functions c^k_j, d^k_j with values in X_0 and X_1, respectively, supported in $\bar{Q}^k_j$, such that for all $x\in\bar{Q}^k_j$

$$b^k_j(x)=c^k_j(x)+d^k_j(x),$$
$$\|c^k_j(x)\|_0\leqslant C(t2^{\alpha k})^{\theta}\,\|b^k_j(x)\|_{\theta,p},$$

and

$$\|d^k_j(x)\|_1\leqslant C(t2^{\alpha k})^{-(1-\theta)}\,\|b^k_j(x)\|_{\theta,p}.$$

Hence

$$\|c^k_j\|_{L_\infty(X_0,\bar{Q}^k_j)}\leqslant C(t2^{\alpha k})^{\theta}\,2^k=Ct^{\theta}2^{kp/p_0} \tag{6.5}$$
$$\|d^k_j\|_{L_\infty(X_1,\bar{Q}^k_j)}\leqslant C(t2^{\alpha k})^{-(1-\theta)}\,2^k=Ct^{-(1-\theta)}2^{kp/p_1}. \tag{6.6}$$

Let $Q(c^k_j)=Q(c^k_j, \bar{Q}^k_j, N)$ and $Q(d^k_j)=Q(d^k_j, \bar{Q}^k_j, N)$ be the polynomials associated with c^k_j and d^k_j, respectively, by Lemma 3.1. Then we have

$$b^k_j=c^k_j-Q(c^k_j)+d^k_j-Q(d^k_j)$$

and by (6.5), (6.6), and Lemma 3.1,

$$\|Q(c_j^k)\|_{L_\infty(X_0,\, \bar{Q}_j^k)} \leqslant C t^{\theta} 2^{kp/p_0} \tag{6.7}$$

$$\|Q(d_j^k)\|_{L_\infty(X_1,\, \bar{Q}_j^k)} \leqslant = C t^{-(1-\theta)} 2^{kp/p_1}. \tag{6.8}$$

Writing now

$$g_1 = \sum_{k=k_0}^{k_1} \sum_{j=1}^{J_k} c_j^k - Q(c_j^k) \qquad \text{and} \qquad h_1 = \sum_{k=k_0}^{k_1} \sum_{j=1}^{J_k} d_j^k - Q(d_j^k)$$

then $g_1 \in \tilde{H}^h_{p_0}(X_0)$, $\tilde{H}^h_{p_1}(X_1)$ and $f_1 = g_1 + h_1$.

Using (6.5)–(6.8) and some estimates in the proof of Theorem 3.1, we easily get

$$\|g_1\|_{\tilde{H}^h_{p_0}(X_0)} \leqslant C t^{\theta} \qquad \text{and} \qquad \|h_1\|_{\tilde{H}^h_{p_1}(X_1)} \leqslant C t^{-(1-\theta)}$$

which proves that $K(t, f_1, \tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1)) \leqslant C t^{\theta}$, or in other words that $f_1 \in (\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,\infty}$ and $\|f\|_{(\tilde{H}^h_{p_0}(X_0), (\tilde{H}^h_{p_1}(X_1))_{\theta,\infty}} \leqslant C$. ∎

The restriction $p > 1$ might look due to the method of proof but the next remark shows that this is not the case.

Remark 6.1. Let $0 < \theta < 1$, $0 < p_0, p_1 \leqslant \infty$, and $(1-\theta)/p_0 + \theta/p_1 = 1$. Let $X_0 = L_2(0, 1)$, $X_1 = L_1(0, 1)$. Then

$$(\tilde{H}^h_{p_0}(X_0), \tilde{H}^h_{p_1}(X_1))_{\theta,1} \neq \tilde{H}^h_p(X_{\theta,1}). \tag{6.9}$$

The proof of (6.9) is based on the following characterization of superreflexivity.

THEOREM C (cf. [X2]). *Let X be a Banach space. X is superreflexive if and only if for some $0 < p < \infty$ (and equivalently for all $0 < p < \infty$) there exist two constants $0 < C, q < \infty$ such that for every function $f \in \tilde{H}^h_p(X, \mathbb{R}^{n+1}_+)$ and every sequence $t_0 > t_1 > \cdots > 0$, $t_n \to 0$, we have*

$$\left(\sum_{k \geqslant 0} \|f_{t_k} - f_{t_{k-1}}\|^q_{L_p(X, \mathbb{R}^n)} \right)^{1/q} \leqslant C \|f\|_{\tilde{H}^h_p(X)},$$

where $f_t(x) = f(x, t)$, $x \in \mathbb{R}^n$ and $f_{-1} = 0$.

Theorem C was proved in [X2] only for the case of the disc, but with some modifications the argument is also valid for the above situation.

Let us now fix a decreasing sequence (t_n) converging to zero. It follows from Theorem C that

$$\left(\sum_{k \geqslant 0} \|f_{t_k} - f_{t_{k-1}}\|^q_{L_{p_0}(X_0, \mathbb{R}^n)} \right)^{1/q} \leqslant C \|f\|_{\tilde{H}^h_{p_0}(X_0)} \tag{6.10}$$

for some $0 < C, q < \infty$ (in fact q can be shown to be $\max(p_0, 2)$).

Clearly, we also have

$$\sup_{k \geq 0} \{ \|f_{t_k} - f_{t_{k-1}}\|_{L_{p_1}(X_1)} \} \leq C \|f\|_{\tilde{H}^h_{p_1}(X_1)}. \tag{6.11}$$

If (6.9) were equality, an interpolation argument and Theorem C would imply that $X_{\theta,1}$ would be superreflexive, but $X_{\theta,1} = (L_2, L_1)_{\theta,1} = L_{r,1}$ for $1/r = (1-\theta)/2 + \theta/1$, which is not even reflexive. ∎

Remark 6.2. Propositions 6.1 and 6.2 have analogous formulations when taking $BMO(X_1)$ as an end point.

For $0 < \theta < 1$, $0 < p_0 < \infty$, and $1/p = (1-\theta)/p_0$ then

$$(\tilde{H}^h_{p_0}(X_0), BMO(X_1))_{\theta,p} \subset \tilde{H}^h_p(X_{\theta,p}). \tag{6.12}$$

with equality for $p > 1$ and, in general, strict inclusion for $p = 1$.

However, when we replace $\tilde{H}^h_{p_0}(X_0)$ by $L_{p_0}(X_0)$ in (6.12) we get equality for all values of p.

PROPOSITION 6.3. *Let* $0 < \theta < 1$, $0 < p_0 < \infty$, *and* $1/p = (1-\theta)/p_0$. *Then*

$$(L_{p_0}(X_0), BMO(X_1))_{\theta,p} = L_p(X_{\theta,p}). \tag{6.13}$$

Proof. The inclusion $L_p(X_{\theta,p}) \subset (L_{p_0}(X_0), BMO(X_1))_{\theta,p}$ is trivial. The other one can be done using Lemma 4.1 and some arguments in the proofs of Theorem 4.1 and Proposition 6.1, and the details are left to the reader. ∎

Remark 6.3. For the case of analytic functions (the Banach spaces considered being now assumed over the complex field), we only have the analogue to Proposition 6.1, that is, if $0 < p_0, p_1 \leq \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$, then

$$(\tilde{H}_{p_0}(X_0), \tilde{H}_{p_1}(X_1))_{\theta,p} \subset \tilde{H}_p(X_{\theta,p}). \tag{6.14}$$

The example presented in Proposition 5.2 works also in this real interpolation case since we also have that for any value of θ, $0 < \theta < 1$, and any value of p, $0 < p < \infty$, C_0 is contained in $X_{\theta,p}$ when $X_0 = L_1(\mathbb{T})$ and $c_0(\mathbb{Z})$. (This fact follows from the relation between the real and complex methods of interpolation and Lemma 5.1.) Therefore we can state that the equality in (6.14) does not hold for general spaces and for any value $0 < p < \infty$.

A Final Remark: The Case $(H^h_{p_0}(X_0), H^h_{p_1}(X_1))_\theta$

In a recent paper by D. J. H. Garling and S. J. Montgomery-Smith (see [GM]) the following result is obtained.

THEOREM D. *Let $0<p<\infty$ and $0<\theta<1$. There exists a couple of Banach spaces (X_0, X_1) such that X_0 and X_1 are isometric to l_1 and $(X_0, X_1)_\theta$ and $(X_0, X_1)_{\theta,p}$ contains a complemented subspace isomorphic to c_0.*

As a consequence the RNP does not pass to intermediate spaces by the complex or the real method of interpolation.

Using the known fact that X having RNP is equivalent to $H_p^h(X)=\tilde{H}_p^h(X)$ for any value $0<p\leqslant\infty$, together with the results obtained for spaces $\tilde{H}_p^h(X)$, one can easily conclude the following: For $0<p_0, p_1\leqslant\infty$, $0<\theta<1$, and $1/p=(1-\theta)/p_0+\theta/p_1$, there exists a couple of Banach spaces (X_0, X_1) such that

$$(H_{p_0}^h(X_0), H_{p_1}^h(X_1))_\theta \neq H_p^h(X_\theta)$$

and

$$(H_{p_0}^h(X_0), H_{p_1}^h(X_1))_{\theta,p} \neq H_p^h(X_{\theta,p}).$$

REFERENCES

[BL] J. BERGH AND J. LOFSTROM, "Interpolation Spaces, an Introduction," Springer-Verlag, Berlin, 1976.

[B1] O. BLASCO, Interpolation between $H_{B_0}^1$ and $L_{B_1}^p$, *Studia Math.* **92**, No. 3 (1989), 295–210.

[B2] O. BLASCO, Boundary values of functions in vector-valued Hardy spaces and geometry of Banach spaces, *J. Funct. Anal.* **78** (1988), 346–364.

[Bo] J. BOURGAIN, Some remarks on Banach spaces in which martingale differences are unconditional, *Ark. Mat.* **21** (1983), 163–168.

[Bu1] D. L. BURKHOLDER, A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions, *in* "Conf. Harmonic Analysis in Honor of A. Zygmund," pp. 270–286, Wadsworth, Belmont, CA, 1982.

[Bu2] D. L. BURKHOLDER, Martingales and Fourier analysis in Banach spaces, *in* "C.I.M.E. Lectures, Varenna, Italy, 1985," Lecture Notes in Math., Vol. 1206, Springer-Verlag, Berlin, 1986.

[C] A. P. CALDERON, Intermediate spaces and interpolation: The complex method, *Studia Math.* **24** (1964), 113–190.

[CT] A. P. CALDERON AND A. TORCHINSKY, Parabolic maximal functions associated with a distribution, II, *Adv. in Math.* **24** (1977), 101–171.

[CZ] A. P. CALDERON AND A. ZYGMUND, On the theorem of Hausdorff–Young and its extension, *in* "Ann. of Math. Stud.," Vol. 25, pp. 166–188, Princeton Univ. Press, Princeton, NJ, 1950.

[Co] R. R. COIFMAN, A real characterization of H^p, *Studia Math.* **51** (1974), 269–274.

[CW] R. R. COIFMAN AND G. WEISS, Extension of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[Cw] M. CWIKEL, On $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}$, *Proc. Amer. Math. Soc.* **44** (1974), 286–292.

[CMS] M. CWIKEL, M. MILMAN, AND Y. SAGHER, Complex interpolation of some quasi-Banach spaces, *J. Funct. Anal.* **65** (1986), 339–347.

[D] S. Dilworth, Complex convexity and geometry of Banach spaces, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 495–506.

[FS] C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193.

[FRS] C. Fefferman, N. M. Riviere, and Y. Sagher, Interpolation between H^p spaces: The real method, *Trans. Amer. Math. Soc.* **191** (1974), 75–81.

[GR] J. Garcia-Cuerva and J. L. Rubio de Francia, "Weighted Norm Inequalities and Related Topics," North-Holland, Amsterdam, 1985.

[GM] D. J. H. Garling and S. J. Montgomery-Smith, Complemented subspaces of spaces obtained by interpolation, preprint.

[G] A. M. Garsia, "Martingale Inequalities," Benjamin, New York, 1973.

[HP] U. Haagerup and G. Pisier, Factorization of analytic functions with values in non-conmutative L_1-spaces, *Canad. J. Math.* **41** (1989), 882–906.

[H] R. Hanks, Interpolation by real method between *BMO*, $L^\alpha(0<\alpha<\infty)$ and $H^\alpha(0<\alpha<\infty)$, *Indiana Univ. Math. J.* **26** (1977), 679–690.

[JJ] S. Janson and P. W. Jones, Interpolation between H^p-spaces: The complex method, *J. Funct. Anal.* **48** (1982), 58–80.

[J1] P. W. Jones, L^∞ estimates for the $\bar{\partial}$ problem in a half-plane, *Acta Math.* **150** (1983), 137–152.

[J2] P. W. Jones, Interpolation between Hardy spaces, *in* "Proceedings, Conference on Harmonic Analysis in Honor of Antoni Zygmund, 1981," Wadsworth, Belmont, CA, 1983.

[L] R. H. Latter, A characterization of $H^p(R^n)$ in terms of atoms, *Studia Math.* **62** (1978), 93–101.

[LU] R. H. Latter and A. Uchiyama, The atomic decomposition for parabolic H^p-spaces, *Trans. Amer. Math. Soc.* **253** (1979), 391–398.

[LP] J. L. Lions and J. Peetre, Sur une classe d'espaces d'interpolation, *Inst. Hautes Études Sci. Publ. Math.* **19** (1964), 5–68.

[M] M. Milman, Fourier type and complex interpolation, *Proc. Amer. Math. Soc.* **89** (1983), 246–248.

[RS] N. M. Riviere and Y. Sagher, Interpolation between L^∞ and H^1, the real method, *J. Funct. Anal.* **14** (1973), 401–409.

[S] Y. Sagher, Interpolation of r-Banach spaces, *Studia Math.* **41** (1972), 45–70.

[SZ] R. Salem and A. Zygmund, A convexity theorem, *Proc. Nat. Acad. Sci. U.S.A.* **34** (1948), 443–447.

[T] H. Triebel, "Interpolation Theory, Function Spaces, Different Operators," North-Holland, Amsterdam, 1978.

[W] T. H. Wolff, A note on interpolation spaces, *in* "Lecture Notes in Math.," Vol. 908, pp. 199–204, Springer-Verlag, New York/Berlin, 1982.

[X1] Q. Xu, Applications du théorème de factorization pour des fonctions à valeurs opérateurs, *Studia Math.* **95**, No. 3 (1990), 273–292.

[X2] Q. Xu, Convexités uniformes et inégalités de martingales, *Math. Ann.* **287** (1990), 193–211.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium