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Convolution of Operators and Applications

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1. Convolution of Functions and Operators

The objective of this paper is to apply the notion of convolution between functions and a special kind of operators, the cone absolutely summing operators, to characterize the Radon-Nikodym property and the analytic Radon-Nikodym property.

Throughout this paper L^p will denote $L^p(\mathscr{T})$ where $\mathscr{T}$ is the circle $\{z: |z|=1\}$ and m denotes normalized Lebesgue measure, B will be a Banach space and $\mathscr{L}(L^p, B)$ will stand for the bounded linear operators from L^p into B.

Definition 1.1. Let $1<p<\infty$. Given an operator T in $\mathscr{L}(L^p, B)$ and a function g in L^1 we shall define the operator $g * T$ by

$$g * T(\psi) = T(g * \psi) \qquad \text{for all } \psi \text{ in } L^p. \tag{1.1}$$

Obviously the classical result about convolutions implies that $\|g * T\| \leqq \|g\|_1 \|T\|$.

We shall deal with special classes of operators which will be invariant under the action of convolution with functions:

a) Representable Operators. An operator T in $\mathscr{L}(L^p, B)$ is called representable if there exists a function f in $L^{p'}(B)$, $1/p+1/p'=1$, such that

$$T(\varphi) = \int f(t)\, \varphi(t)\, dt. \tag{1.2}$$

In this case we shall write $T=T_f$ and clearly from Holder's inequality we have $\|T_f\| < \|f\|_p$.

b) r-absolutely Summing Operators $\Pi_r(L^p, B)$. Let $1 \leqq r < \infty$. An operator T in $\mathscr{L}(L^p, B)$ is called r-absolutely summing operator if there is a constant C such that for every finite family $\psi_1, \psi_2, \ldots, \psi_n$ of functions in L^p it verifies

$$\left(\sum \|T(\psi_i)\|^r\right)^{1/r} \leqq C \sup\left\{\left(\sum \left|\int \psi_i(t)\, \phi(t)\, dt\right|^r\right)^{1/r} : \|\phi\|_{p'} = 1\right\}. \tag{1.3}$$

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c) Positive r-Summing Operators $\Lambda_r(L^p, B)$. Let $1 \leqq r < \infty$. An operator T in $\mathscr{L}(L^p, B)$ is called positive r-summing operator if there is a constant C such that for every finite family $\psi_1, \psi_2, \ldots, \psi_n$ of *positive* functions in L^p it verifies

$$\left(\sum \|T(\psi_i)\|^r\right)^{1/r} \leqq C \sup\left\{\left(\sum \left|\int \psi_i(t)\, \phi(t)\, dt\right|^r\right)^{1/r} : \|\phi\|_{p'} = 1\right\}. \tag{1.4}$$

The norm in both last spaces is given by the infimum of the constants verifying (1.3) and (1.4) respectively. The reader is referred to [6] and [1, 2] to see some properties of these classes of operators respectively. A remarkable fact is the coincidence of the spaces $\Lambda_r(L^p, B)$ and $\Lambda_1(L^p, B)$ when $1 \leqq r \leqq p'$. This last space $\Lambda_1(L^p, B)$ was considered by Schaefer [9] who denoted it by space of cone absolutely summing operators. Let us denote by

$$|||T|||_{p,r} = \inf\{C: \text{verifying (1.3)}\}$$

and

$$\|T\|_{p,r} = \inf\{C: \text{verifying (1.4)}\}.$$

Let us recall the following fact (see [9], page 275):

$$\text{If } f \text{ belongs to } L^p(B) \text{ then } \|T_f\|_{p,1} = \|f\|_p. \tag{1.5}$$

A very easy computation shows that $g * T_f = T_{g*f}$, where $g*f$ stands for $g*f(t) = \int f(s)\, g(t-s)\, ds$, and therefore, using (1.5), we can rewrite the result $\|g*f\|_{p'} \leqq \|g\|_1 \|f\|_{p'}$ as follows: $\|g * T_f\|_{p,1} \leqq \|g\|_1 \|T_f\|_{p,1}$. This inequality also holds for general operators in the following sense.

Theorem 1.1. *Let* $1 \leqq r < \infty$, $1 \leqq p < \infty$, *and* $g \geqq 0$ *in* L^1.

a) *If* T *belongs to* $\Pi_r(L^p, B)$ *then* $g * T$ *belongs to* $\Pi_r(L^p, B)$. *Moreover* $|||g * T|||_{p,r} \leqq \|g\|_1 |||T|||_{p,r}$.

b) *If* T *belongs to* $\Lambda_r(L^p, B)$, *and* g *is positive then* $g * T$ *belongs to* $\Lambda_r(L^p, B)$. *Moreover* $\|g * T\|_{p,r} \leqq \|g\|_1 \|T\|_{p,r}$.

c) *If* T *belongs to* $\mathscr{L}(L^p, B)$ *and* g *belongs to* L^p *then there is a continuous function* h *such that* $g * T = T_h$. *If, in addition,* $T \in \Lambda_1(L^p, B)$ *then* $\|h\|_{p'} \leqq \|g\|_1 \|T\|_{p,1}$.

Proof. Since $g * T$ is nothing but the composition of two operators T and Δ, being $\Delta(\phi) = g * \phi$, then a) and b) follow from general properties of r-absolutely summing operators (see [6]) and positive r-summing ones (see [1]). To show c) let us take $h(t) = T(g_t)$ where $g_t(s) = g(t-s)$. Notice that h is well defined since g_t belongs to L^p for all values of t in $\mathscr{T}$. In addition we have

$$\|h(t) - h(s)\| \leqq \|T\|\, \|g_t - g_s\|_p$$

what clearly implies the continuity of h. Moreover we can write

$$\|h\|_\infty = \sup \|T(g_t)\| \leqq \|T\| \sup \|g_t\|_p = \|T\|\, \|g\|_p.$$

Let us assume that T belongs to $\Lambda_1(L^p, B)$ and look at h as element in $L^{p'}(B)$, then from (1.5) it suffices to show that $g * T = T_h$.

Now to see this we just have to apply Hille's theorem (see [4], page 47)

$$g * T(\phi) = T(g * \phi) = T(\int g_t \cdot \phi(t)\, dt) = \int T(g_t)\, \phi(t)\, dt = \int h(t)\, \phi(t)\, dt = T_h(\phi). \quad \square$$

Let us now use the convolution to give some approximation results. To do that let us recall the concept of approximate identity.

Definition 1.2. A sequence of integrable functions g_n in L^1 is called an approximate identity (a.i.) if it verifies

a) $\int g_n(t)\, dt = 1$ for all n

b) $\int |g_n(t)|\, dt \leq C$ for all n

c) For each $\delta > 0$ $\int_{|t|>\delta} g_n(t)\, dt$ converges to zero as n goes to ∞.

The next objective is to extend to operators the following well known result: If g_n is a.i. and f belongs to $L^p(B)$ for some $1 \leq p < \infty$, then

$$g_n * f \text{ converges to } f \text{ in } L^p(B). \tag{1.6}$$

Theorem 1.2. *Let $1 < p < \infty$, T belong to $\mathscr{L}(L^p, B)$ and g_n be an a.i. in L^p.*

a) *$g_n * T$ converges to T in the strong topology.*

b) *$g_n * T$ converges to T in norm if and only if T is compact.*

Proof. a) easily follows from (1.6).

To prove b) let us observe that from Theorem 1.1.c. we have that $g_n * T$ is represented by a function in $L^{p'}(B)$ which implies that it is compact operator. Therefore if we have norm convergence then T must be compact. On the other hand since L^p does have the approximation property (see [4], page 242) we can approach any compact operator with finite rank operators which are represented by simple functions. Hence given $\varepsilon > 0$ let us take T_ε the operator of finite rank represented by s_ε such that $\|T - T_\varepsilon\| < \varepsilon$. Therefore

$$\begin{aligned}\|g_n * T - T\| &\leq \|g_n * T - g_n * T_\varepsilon\| + \|g_n * T_\varepsilon - T_\varepsilon\| + \|T_\varepsilon - T\| \\ &\leq (\sup \|g_n\|_1 + 1)\, \|T - T_\varepsilon\| + \|g_n * s_\varepsilon - s_\varepsilon\|_{p'}.\end{aligned}$$

And now the result follows from (1.6). $\square$

2. Application to Geometry of Banach Spaces

We shall apply the convolution to deal with the Radon-Nikodym and the analytic Radon-Nikodym properties. The reader is referred to [4] for the first one, the formulation we are going to use being as follows:

(*) B has the RNP if any operator T in $\mathscr{L}(L^1, B)$ is representable by a function f in $L^1(B)$.

The other property we are concerned with was introduced in [3] by Bukhvalov and Danilevich,

(**) B has analytic Radon-Nikodym property (ARNP) if every B-valued bounded holomorphic function on the disc has limits at the boundary almost everywhere.

An equivalent formulation of (**) is the following (see [3])

(**') Let $1 \leqq p < \infty$, B has ARNP if for every F in $H^p_B(D)$ it verifies that $F_r(t) = F(r e^{it})$ converges in $L^p(B)$ as $r \uparrow 1$.

($H^p_B(D)$ stands for the classical Hardy space but for B-valued functions and B will be a complex Banach space in this case).

Theorem 2.1. *Let $1 < p < \infty$, and let g_n be an a.i. of positive functions in L^p. The following statements are equivalent:*

a) *B has the Radon-Nikodym property*

b) *For every operator T in $A_1(L^p, B)$ the convolution $g_n * T$ converges to T in $A_1(L^p, B)$.*

Proof. Let us assume that B has RNP and take T in $A_1(L^p, B)$. Define now the following vector valued measure

$$G(E) = T(\chi_E) \quad \text{for all measurable set } E. \tag{2.1}$$

We shall prove that G is absolutely continuous with respect to the Lebesgue measure and that it has bounded variation. Let us take a measurable set A and a partition $E_1, E_2, \ldots, E_n$ of A with $m(E_i) > 0$ (m stands for the normalized Lebesgue measure).

$$\begin{aligned}\sum \|G(E_i)\| &\leqq \sum \|T(\chi_{E_i})\| \\ &\leqq \|T\|_{p,1} \sup\{\sum |\int_{E_i} \phi(t)\,dt| : \|\phi\|_{p'} = 1\} \\ &\leqq \|T\|_{p,1} \sup\{\int_A |\phi(t)|\,dt : \|\phi\|_{p'} = 1\} \\ &\leqq \|T\|_{p,1}\, m(A)^{1/p}.\end{aligned}$$

From here the RNP implies the existence of a function f in $L^1(B)$ such that $G(E) = \int_E f(t)\,dt$. A standard argument shows now that f belongs to $L^{p'}(B)$ and that T is represented by f. Therefore $g_n * T$ is represented by $g_n * f$ and this sequence converges to f in $L^{p'}(B)$ which means that $g_n * T$ converges to T in $A_1(L^p, B)$.

Conversely, let us take an operator T in $\mathscr{L}(L^1, B)$. First we shall show that T belongs to $A_1(L^p, B)$. Consider $\psi_1, \psi_2, \ldots, \psi_n$ positive functions in L^p and observe the following

$$\begin{aligned}\sum \|T(\psi_i)\| &\leqq \|T\| \sum \|\psi_i\|_1 = \|T\|\, \|\sum \psi_i\|_1 \leqq \|T\|\, \|\sum \psi_i\|_p \\ &= \|T\| \sup\{|\int \sum \psi_i(t)\, \phi(t)\,dt| : \|\phi\|_{p'} = 1\} \\ &\leqq \|T\| \sup\{\sum |\int \phi(t)\, \psi_i(t)\,dt| : \|\phi\|_{p'} = 1\}.\end{aligned}$$

Now according to Theorem 1.1.c. and the assumption we have that $g_n * T$ are represented by functions f_n in $L^{p'}(B)$ and they form a Cauchy sequence in $L^{p'}(B)$. The proof is finished by showing that the limit function f represents the operator T, which is simply a computation. □

Let $r(n)$ be a sequence in (0, 1) converging to 1. Let us write P_n for $P_{r(n)}$, where P_r stands for the Poisson Kernel on the disc

$$P_r(t) = (1-r^2)/(1+r^2-2r\cos t).$$

P_n is then an approximate identity (a.i.) of positive and continuous functions.

Theorem 2.2. *Let $1<p<\infty$, $e_n(t)=e^{-int}$ for any integer n. The following statements are equivalent:*

a) *B has the ARNP*

b) *For every T in $A_1(L^p, B)$ with $T(e_n)=0$ for $n<0$, the convolution $P_n * T$ converges to T in $A_1(L^p, B)$.*

Proof. Suppose B has ARNP and take T in $A_1(L^p, B)$ with $T(e_n)=0$ for $n<0$.

Let us define

$$F(z) = T(P_z) \quad \text{where} \quad z = r\,e^{it} \quad \text{and} \quad P_z(s) = P_r(s-t). \tag{2.2}$$

It is easy to verify that F is a holomorphic function on the disc with values in B. Observe that $P_n * T$ is represented by the function $F_{r(n)}$. Therefore F is a function in $H_B^{p'}(D)$ since $\|F_{r(n)}\|_{p'} \leq \|P_n\|_1 \, \|T\|_{p,1} = \|T\|_{p,1}$. Consequently we get the result from (**').

To prove the converse let us take a bounded holomorphic function on the disc, and write it as $F(z) = \sum_{k\geq 0} a_k z^k$. Then for any trigonometrical polynomial $q = \sum_{-M}^{N} \lambda_k e_k$ we can define

$$T(q) = \sum_{k=0}^{N} \lambda_k \cdot a_k. \tag{2.3}$$

Since $a_j = \lim \int F_r(t)\, e_j(t)\, dt$ then we can write $T(q) = \lim \int F_r(t)\, q(t)\, dt$. This clearly implies that

$$\|T(q)\| \leq \sup \|F(z)\| \, \|q\|_1 .$$

Hence, extending by density, we have got an operator in $\mathscr{L}(L^1, B)$ which obviously satisfies $T(e_n)=0$ for $n<0$. As in Theorem 2.1, we have that T, in fact, belongs to $A_1(L^p, B)$ and therefore $P_n * T$, which is represented by $F_{r(n)}$, converges to T in $A_1(L^p, B)$. From here we have that $F_{r(n)}$ is a convergent sequence in $L^{p'}(B)$ to some f in $L^{p'}(B)$ which is the end of the proof since then $F_{r(n)} = P_n * f$, and thus F has boundary limits almost everywhere. □

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