INTRODUCTION.

The duality between $H^1$ and $BMO$, the space of functions of bounded mean oscillation (see [JN]), was first proved by C. Fefferman (see [F], [FS]) and then other proofs of it were obtained. Using the atomic decomposition approach ([C], [L]) the author studied the problem of characterizing the dual space of $H^1$ of vector-valued functions. In [B2] the author showed, for the case $\Omega = \{|z| = 1\}$, that the expected duality result $H^1$-$BMO$ holds in the vector valued setting if and only if $X^*$ has the Radon-Nikodym property. If we want to get a duality result valid for all Banach spaces we may consider vector valued measures (see [BT], where the vector valued $L_p$ case is treated, for an explanation) and therefore to deal with the general case it was necessary to consider a new space of vector valued measures closely related to $BMO$ (see [B1]).

In this paper we shall study such space in little more detail and we shall consider the $H^1$-$BMO$ duality for vector-valued functions in the more general setting of spaces of homogeneous type (see [CW]).

Throughout the paper $X$ will stand for a Banach space, $\Omega$ will be a space of homogeneous type (see definition in the preliminary section) and we write $L_p(\Omega, X)$ for the space of measurable functions on $\Omega$ with values in $X$ such that $\|f(x)\|$ belongs to $L_p(\Omega)$. As usual C will denote a constant not necessarily the same at each occurrence.

PRELIMINARIES

A space of homogeneous type $\Omega$ is a topological space endowed with a Borel measure $m$ and a quasi-distance $d$, that is $d : X \times X \to \mathbb{R}^+$ with

\begin{align*}
a) & \quad d(x, y) = d(y, x), \\ b) & \quad d(x, y) = 0 \quad \text{if and only if} \quad x = y, \\ c) & \quad d(x, y) \leq K(d(x, z) + d(z, y)).
\end{align*}

and we assume that the balls $B_r(x) = \{y \in \Omega : d(x, y) < r\}$ form a basis of open neighborhoods of the point $x$ and there exists a constant $A$ satisfying

\begin{equation}
m(B_r(x)) \leq A m(B_{r/2}(x))
\end{equation}

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From (1.0) we can assume that $0 < m(B) < \infty$ for every ball $B$ (otherwise $m$ would be identically 0 or $\infty$) and therefore $m$ is a $\sigma$-finite measure on $\Omega$. Denote by $\Sigma_0$ the ring of bounded measurable sets. The $\sigma$-finiteness condition implies that the $\sigma$-algebra generated by $\Sigma_0$ coincides with the Borel $\sigma$-algebra that we shall denote by $\Sigma$.

Let us now recall the notion of atom with values is $X$. Given $1 < p \leq \infty$, a function $a$ in $L_p(\Omega, X)$ is called $(X,p)$-atom if

a) the support is contained in a ball $B = B_r(x_o)$

(b) $\left( \frac{1}{m(B)} \int_B \|a(x)\|^p \, dm(x) \right)^{1/p} \leq \frac{1}{m(B)} \quad (p < \infty)$

(b') $\|a(x)\| \leq \frac{1}{m(B)} \quad m - a.e. \quad (p = \infty)$

c) $\int_B a(x) \, dm(x) = 0$

In the case $m(\Omega) < \infty$ the constant function $\frac{1}{m(\Omega)} \, b$, where $b \in X$ with $\|b\| = 1$, is also considered as a $(X,p)$-atom.

Note that the atoms are in the unit ball of $L_1(\Omega, X)$.

Following [CW] we define $H_p^1(\Omega, X)$ as the space of functions $f$ in $L_1(\Omega, X)$ admitting an atomic decomposition

(1.1) $f = \sum_{j=0}^{\infty} \lambda_j a_j$

where the $a_j$’s are $(X,p)$-atoms and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$ .(The convergence of (1.1) is taken in $L_1(\Omega, X)$ ).

We get a Banach space if we consider the norm

$\|\cdot\|_{H_p^1} = \inf \sum_{j=0}^{\infty} |\lambda_j|$

where the infimum is taken over all representations $f = \sum_{j=0}^{\infty} \lambda_j a_j$ .

The same arguments as in [CW] show that, in fact, for $1 < p, r \leq \infty$

(1.2) $H_p^1(\Omega, X) = H_r^1(\Omega, X)$ (with equivalent norms) .
Let us also recall the definition of vector-valued BMO. Let \(1 \leq q < \infty\), a \(X\)-valued function which is locally in \(L^q(\Omega, X)\) is said to belong to \(BMO_q(\Omega, X)\) provided that

\[
\left( \sup_{\text{ball } B} \left( \frac{1}{m(B)} \int_B \|g(x) - g_B\|^q \, dm(x) \right) \right)^{1/q} \leq C
\]

where \(g_B = \frac{1}{m(B)} \int_B g(x) \, dm(x)\).

Let us denote by

\[
\|g\|_{*,q} = \sup\left\{ \left( \frac{1}{m(B)} \int_B \|g(x) - g_B\|^q \, dm(x) \right)^{1/q} : \text{B ball} \right\}
\]

When \(m(\Omega) = \infty\) then \(\|g\|_{BMO_q} = \|g\|_{*,q}\) gives a norm on the set of equivalence classes of functions which differ by a constant in \(X\).

For \(m(\Omega) < \infty\) we consider the norm \(\|g\|_{BMO_q} = \|g\|_{*,q} + \| \int_{\Omega} g(x) \, dm(x) \|\).

Let us recall now a few definitions about vector-valued measures we shall use later on. Let \((\Omega, \Sigma, m)\) be any \(\sigma\)-finite measure space, \(A\) a measurable set and \(1 < p < \infty\). Given a vector valued measure \(G\), we denote by \(|G|\) the variation of \(G\), that is

\[
|G|(A) = \sup\left\{ \sum_{i=1}^{n} \|G(E_i)\| : (E_i) \text{ partition of } A \right\}
\]

and by \(|G|_p(A)\) the \(p\)-variation on \(A\), that is

\[
|G|_p(A) = \sup\left\{ \left( \sum_{i=1}^{n} \frac{\|G(E_i)\|^p}{m(E_i)^{p-1}} \right)^{1/p} \right\}
\]

where the supremum is taken over all finite partitions \((E_i)\) of disjoint measurable sets contained in \(A\) with \(m(E_i) > 0\).

For the case \(p = \infty\) we shall denote by \(V^\infty(\Omega, X)\) the space of \(X\)-valued measures \(G\) satisfying

\[
\|G(E)\| \leq C \, m(E) \text{ for all measurable set } E
\]

Defining the norm by the infimum of the constants satisfying (1.6) we get a Banach space.

Remark 1.1. It is not hard to see that in fact \(\|G(E_i)\|\) can be replaced by \(|G|(E_i)\) in the definition of \(p\)-variation. (See Lemma 1 in [B3])

Remark 1.2. If \(G\) is a vector valued measure defined on \(\Sigma_0\) which is absolutely continuous with respect to \(m\), that is \(\lim_{m(E)\to 0} G(E) = 0\), then it can be extended to a measure on \(\Sigma\), being still absolutely continuous with respect to \(m\). (See [D],[DU])
We refer the reader to ([DU], [D]) and to ([J], [GC-RF]) for general theory and the properties we shall use about vector valued measures and Hardy spaces respectively.

**VECTOR VALUED MEASURES OF BOUNDED MEAN OSCILLATION.**

**Definition 2.1.** Let $1 \leq q < \infty$. Given a countably additive measure $G$ defined on $\Sigma$ and with values in $X$, it is said that $G$ belongs to $MBMO_q(\Omega, X)$ if

\[
|G|_{*,q} = \sup \left\{ \left( \sum_{i=1}^{n} \left| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right|^q \frac{m(E_i)}{m(B)} \right)^{1/q} \right\} < \infty
\]

where the supremum is taken over all balls $B$ and over all finite partitions of $B$ in pairwise disjoint measurable sets $E_i$ with $m(E_i) > 0$.

When $m(\Omega) = \infty$ then $\|G\|_{MBMO_q} = |G|_{*,q}$ gives a norm on the set of equivalence classes of measures: $G_1 \sim G_2$ if there is $b$ in $X$ such that $G_1(E) - G_2(E) = bm(E)$ for all measurable set $E$.

For $m(\Omega) < \infty$ we consider the norm $\|G\|_{MBMO_q} = |G|_{*,q} + \|G(\Omega)\|$.

It is obvious that if $1 < q_1 < q_2 < \infty$ then

\[
V^\infty(\Omega, X) \subset MBMO_{q_2}(\Omega, X) \subset MBMO_{q_1}(\Omega, X)
\]

**Remark 2.1.** Let us assume $G$ belong to $MBMO_q(\Omega, X)$. Given a ball $B$ and a measurable set $E \subset B$, it is quite immediate to find a constant $C_B$ depending on $B$ satisfying

\[
\|G(E)\| \leq C_B \max(m(E), m(E)^{1-1/q})
\]

Suposse we consider $B_n = \{y \in \Omega : d(x_0, y) < n\}$ and denote by $G_{B_n}$ the measure $G$ concentrated on $B_n$, that is $G_{B_n}(E) = G(E \cap B_n)$. A glance at (2.3) allows us to say that for any $1 < q < \infty$ if $G$ belongs to $MBMO_q(\Omega, X)$ then $G_{B_n}$ are necessarily absolutely continuous with respect to $m$ and this clearly implies that also $G$ is absolutely continuous with respect to $m$. (Recall that for vector-measures on $\sigma$-algebras it suffices to check that they vanish on $m$-null sets).

**Proposition 2.1.** Let $1 \leq q < \infty$, $g$ be locally in $L_q(\Omega, X)$ and $G$ be an $X$-valued measure such that $G(E) = \int_E g(x) \, dm(x)$ for all measurable bounded set $E$.

Then $g$ belongs to $BMO_q(\Omega, X)$ if and only if $G$ belongs to $MBMO_q(\Omega, X)$.

Moreover $\|G\|_{MBMO_q} = \|g\|_{BMO_q}$.

**Proof.-** Given any ball $B$, consider $G_B(E) = G(E \cap B) - \frac{G(B)}{m(B)} m(E \cap B)$. Observe that

\[
\sup \left\{ \left( \sum_{i=1}^{n} \left| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right|^q \frac{m(E_i)}{m(B)} \right)^{1/q} : (E_i) \text{ partition of } B \right\}
\]
coincides with the q-variation of $G_B$ on $\Omega$ divided by $m(B)^{1/q}$ and $G_B$ is a measure represented by the function $(g - g_B)\chi_B$, that is

$$G_B(E) = \int_E (g(x) - g_B)\chi_B \, dm(x).$$

Therefore the proposition follows from the equality between the q-variation and the norm in $L_q$ of the function which represents the measure (see [D]).

**Remark 2.2.** In general it is not true that any measure in $MBMO_q(\Omega, X)$ is representable by a function, this depends on the Radon-Nikodym property. We refer the reader to [B1] for the case $\Omega = \{|z| = 1\}$, but a similar result and proof can be established also in this general setting.

**Proposition 2.2.** Let $1 \leq q < \infty$. $G$ belongs to $MBMO_q(\Omega, X)$ if and only if there exists a family of vectors in $X$, say $\{a_B: B \text{ ball}\}$, such that

$$(2.4) \quad \sup \left\{ \left( \sum_{i=1}^n \| G(E_i) - a_B \|^{q \frac{m(E_i)}{m(B)}} \right)^{1/q} \right\} < \infty$$

where the supremum is taken over all balls $B$ and over all finite partitions of $B$ in pairwise disjoint measurable sets $E_i$ with $m(E_i) > 0$.

**Proof.**- The direct implication is obvious by taking $a_B = \frac{G(B)}{m(B)}$. To show the converse let us assume that we have $\{a_B: B \text{ sphere}\}$ with the above property, and notice that

$$\|a_B - \frac{G(B)}{m(B)}\| \leq C$$

for all $B$ (simply take the partition of $B$ given only by $B$).

Therefore for any $B$ and any partition

$$\left( \sum_{i=1}^n \| G(E_i) - a_B \|^{q \frac{m(E_i)}{m(B)}} \right)^{1/q} \leq \left( \sum_{i=1}^n \| G(E_i) - a_B \|^{q \frac{m(E_i)}{m(B)}} \right)^{1/q} + \left( \sum_{i=1}^n \| a_B - \frac{G(B)}{m(B)} \|^{q \frac{m(E_i)}{m(B)}} \right)^{1/q} \leq C \quad \diamondsuit$$

As in the case of functions we can define an equivalent norm in $MBMO_q(\Omega, X)$.

$$(2.5) \quad |G|_{*, q} = \sup_{\text{ball } B} \left\{ \inf_{a \in X} \frac{1}{m(B)^{1/q}} |G - a m|_q(B) \right\}.$$
Note that essentially the same argument as in Proposition 2.2. shows the following

\[(2.6) \quad |G|_{*q} \leq |G|_{*q} \leq C|G|_{*q}\]

**Proposition 2.3.** Let \(1 < q < \infty\). If \(G\) belongs to \(MBMO_q(\Omega, X)\) then there exists a non negative function \(\phi\) in \(BMO_q(\Omega)\) such that

\[|G|(E) = \int_E \phi(x) \, dm(x).\]

Moreover \(\|\phi\|_{BMO_q} \leq C\|G\|_{MBMO_q}\).

**Proof.** Since \(G\) is countably additive and \(m\)-continuous then the same is true for the variation of \(G\), \(|G|\). Therefore using the Radon-Nikodym theorem there exists a non negative measurable function \(\phi\) which represents the measure \(|G|\). To show that \(\phi\) belongs to \(BMO_q(\Omega)\), we shall use Propositions 2.2 and 2.1. We simply have to find a family of real numbers \(\{a_B\}\) such that

\[\sup\left\{ \left( \sum_{i=1}^{n} \frac{|G|(E_i) \cdot m(E_i)}{m(B)} - a_B \left| \frac{m(E_i)}{m(B)} \right|^{1/q} \right) \right\} < \infty\]

Take \(a_B = \frac{\|G(B)\|_{m(B)}}{m(B)}\), and observe that

\[\left| G(E) - \frac{\|G(B)\|_{m(B)}}{m(B)} m(E) \right| \leq |G - \frac{G(B)}{m(B)} m(E)|\]

Then

\[\sup\left\{ \left( \sum_{i=1}^{n} \frac{|G|(E_i) \cdot m(E_i)}{m(B)} - \left| \frac{G(B)}{m(B)} \right| \left| \frac{m(E_i)}{m(B)} \right|^{1/q} \right) \right\} \leq \sup\left\{ \frac{1}{m(B)^{1/q}} \left( \sum_{i=1}^{n} \left| G - \frac{G(B)}{m(B)} m(E_i) \right| q m(E_i)^{1-q} \right)^{1/q} \right\} \leq |G|_{*q}\]

The last inequality follows from Remark 1.1. ♦

**THEOREM AND ITS PROOF.**

In the sequel \(1 < p, q < \infty\), with \(\frac{1}{p} + \frac{1}{q} = 1\). In this section we shall achieve the duality result between \(H^1_p(\Omega, X)\) and \(MBMO_q(\Omega, X^*)\). We shall need several lemmas before we prove the result. The next result was done in [B1] for the circle and for \(q = 2\), and here we present a different approach which is valid for general spaces of homogeneous type. The author would like to point out that a similar and independent proof of the following lemma has been obtained by T. Wolniewicz (personal communication).
**Lemma 3.1.** Let $G$ be a measure in $MBMO_q(\Omega, X)$. Then for each integer $n \in \mathbb{N}$ we can find a measure $G_n$ in $V^\infty(\Omega, X)$ and a constant $C_n$ satisfying $|G_n|_{*,q} \leq C_n$ and such that

\begin{equation}
|G|_{*,q} \leq \lim_{n \to \infty} C_n \leq K|G|_{*,q}
\end{equation}

\begin{equation}
\lim_{n \to \infty} G_n(E) = G(E) \quad \text{for all measurable bounded set } E.
\end{equation}

**Proof.** Using Proposition 2.3 we first get a function $\phi$ in $BMO_q(\Omega)$. Denote by $\Omega_n = \{ x \in X : \phi(x) > n \}$ and $\phi_n(x) = \min(1, n/\phi(x))$. Let us define now

\begin{equation}
G_n(E) = \int_E \phi_n(x) \, dG(x) \quad (E \in \Sigma_0)
\end{equation}

Notice that

\[
\|G_n(E)\| \leq |G_n|(E) \leq \int_E \phi_n(x) \, d|G|(x) \leq \int_E \phi_n(x) \, \phi(x) \, dm(x) \leq n \, m(E)
\]

This, using Remark 1.2., allows to extend $G_n$ to $\Sigma$ and shows that $G_n$ belongs to $V^\infty(\Omega, X)$. On the other hand

\begin{equation}
G(E) - G_n(E) = \int_{E \cap \Omega_n} (1 - \phi_n(x)) \, dG(x)
\end{equation}

Therefore if $E$ is contained in some ball $B$

\[
\|G(E) - G_n(E)\| \leq 2 \int_{E \cap \Omega_n} \phi(x) \, dm(x)
\]

Since $\phi \chi_B$ is in $L_1(\Omega)$ then taking limit as $n \to \infty$ shows (3.2).

From (2.6) we have finally to estimate $m(B)^{-1/q} |G_n - a m_q|(B)$ for all balls $B$. Using (3.4) we have that for any $E \subset B$

\[
\|G(E) - G_n(E)\| \leq \int_{E \cap \Omega_n} (1 - n/\phi(x)) \, d|G|(x)
\]

If $\|a\| \leq n$ then

\[
\|G(E) - G_n(E)\| \leq \int_{E \cap \Omega_n} (\phi(x) - n) \, dm(x) \leq \int_{E \cap \Omega_n} (\phi(x) - \|a\|) \, dm(x)
\]

Therefore we have

\begin{equation}
|G_n - G|_q(B) \leq |G - a m_q|(B \cap \Omega_n)
\end{equation}
Though $|G|^q$ is not a measure for $q > 1$ the q-variation es subadditive and therefore we get that for all $\|a\| \leq n$

\[(3.6)\quad m(B)^{-1/q}|G_n - a m|_q(B) \leq 2 m(B)^{-1/q}|G - a m|_q(B)\]

Denoting now by

\[D_n = \sup_{ballB} \inf_{\|a\| \leq n} \{m(B)^{-1/q}|G - a m|_q(B)\}\]

we get (3.1) for $C_n = 2 C D_n$ where $C$ is the constant appearing in (2.6). ♦

Notice that $V^\infty(\Omega, X^*)$ can be obviously identified with the dual of $L_1(\Omega, X)$. Indeed any measure $G$ in $V^\infty(\Omega, X^*)$ defines a functional $T_G$ acting on $X$-valued simple functions (which are dense in $L_1(\Omega, X)$ ) by the formula

\[(3.7)\quad T_G(\sum_{i=1}^{n} a_i \chi_{E_i}) = \sum_{i=1}^{n} < G(E_i), a_i >\]

where $<,>$ means duality between $X$ and $X^*$.

**Lemma 3.2.** Let $1 < p, q < \infty$, \( \frac{1}{p} + \frac{1}{q} = 1 \) and $G$ belong to $V^\infty(\Omega, X^*)$. Then

\[(3.8)\quad |T_G(f)| \leq C \|G\|_{MBMO_q} \|f\|_{H^1_p} \text{ for all } f \text{ in } H^1_p(\Omega, X).\]

**Proof.** Let us first take a “simple atom” in $H^1_1(\Omega, X)$, that is

$s = \sum_{i=1}^{n} b_i \chi_{E_i}$, $E_i \subset B$ for some sphere $B$, $\sum_{i=1}^{n} b_i m(E_i) = 0$

and $\sum_{i=1}^{n} \|b_i\|_X m(E_i) \leq m(B)^{1-p}$.

For such an atom we can write

\[T_G(\sum_{i=1}^{n} b_i \chi_{E_i}) = \sum_{i=1}^{n} < G(E_i), b_i > = \sum_{i=1}^{n} < G(E_i) - \frac{G(B)}{m(B)} m(E_i), b_i >\]

Therefore

\[|T_G(s)| \leq \sum_{i=1}^{n} \|G(E_i) - \frac{G(B)}{m(B)} m(E_i)\|_X m(E_i)^{-1/p} m(E_i)^{1/p} \|b_i\|_X \leq\]

\[\leq \sum_{i=1}^{n} \left( \frac{G(E_i)}{m(E_i)} \right) - \left( \frac{G(B)}{m(B)} \right) \|X, m(E_i)\|^{1/q} \left( \sum_{i=1}^{n} \|b_i\|_X \right) \|m(E_i)\|^{1/p} \leq\]

\[\leq \left( \sum_{i=1}^{n} \left( \frac{G(E_i)}{m(E_i)} \right) \right) - \left( \frac{G(B)}{m(B)} \right) \|X, m(E_i)\|^{1/q} \leq |G|_{s,q}\]

For a general atom $a$ supported in $B$ in $H^1_1(\Omega, X)$ we can use approximation by simple functions in $L_p(\Omega, X)$, and find a sequence of simple functions $d_k$ supported in $B$
converging to \(a\) in \(L^p(\Omega, X)\), and take the sequence \(s_k = (d_k - \int_B d_k(x) dm(x)) \chi_B\) which clearly also converges to \(a\) in \(L^p(\Omega, X)\). Hence \(\|s_k\|_p \leq 2\|a\|_p\) for \(k\) large enough, and therefore \(s_k/2\) are “simple atoms”.

Using now that \(T_G\) is continuous as operator on \(L^1(\Omega, X)\), and that \(s_k\) converges to \(a\) in \(L^1(\Omega, X)\), then

\[
\lim_{k \to \infty} |T_G(s_k)| = \lim_{k \to \infty} |T(s_k/2)| \leq 2 |G|_{s,q}.
\]

For a general function \(f\), take any representation of \(f\) in \(H^1_p(\Omega, X)\), say \(f = \sum_{j=0}^{\infty} \lambda_j a_j\), where the \(a_j\) are \((X,p)\)-atom and \(\sum_{j=0}^{\infty} |\lambda_j| < \infty\) and notice that (3.8) follows from (3.9) and the fact that the series \(f = \sum_{j=0}^{\infty} \lambda_j a_j\) is absolutely convergent in \(L^1(\Omega, X)\) what implies that \(T_G(f) = \sum_{j=0}^{\infty} \lambda_j T_G(a_j)\).

**Theorem 3.1.** Let \(1 < p, q < \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Then

\[
(H^1_p(\Omega, X))^* = MBMO_q(\Omega, X^*)\ (\text{equivalent norms})
\]

**Proof.**- Let us take \(G\) in \(MBMO_q(\Omega, X^*)\), and define as above

\[
T_G(\sum_{i=1}^{n} b_i \chi_{E_i}) = \sum_{i=1}^{n} <G(E_i), b_i>
\]

From the definition of \(H^1_p(\Omega, X)\) we can easily see that simple functions with support in balls are dense in the space, therefore it is enough to see that

\[
|T_G(\sum_{i=1}^{n} b_i \chi_{E_i})| \leq C |G|_{s,q} \|\sum_{i=1}^{n} b_i \chi_{E_i}\|_{H^1_p}
\]

To see (3.11) we first invoke Lemma 3.1 to find a sequence of measures \(G_n\) in \(V^\infty(\Omega, X^*)\), that according to (3.2) verifies \(\lim_{n \to \infty} T_{G_n}(s) = T_G(s)\) for all simple function supported in a ball.

Secondly we use Lemma 3.2, together with (3.1) to get

\[
|T_G(s)| \leq \lim_{n \to \infty} |T_{G_n}(s)| \leq C \lim_{n \to \infty} |G_n|_{s,q} \|s\|_{H^1_p} \leq C \lim_{n \to \infty} C_n \|s\|_{H^1_p} \leq C |G|_{s,q} \|s\|_{H^1_p}.
\]

For the converse we shall deal first with the case \(m(\Omega) < \infty\). Let us take now a functional \(T\) in \((H^1_p(\Omega, X))^*\). Since constant functions are also considered as \(X\)-atoms in the case of finite measure we have that \(a \chi_E \in H^1_p(\Omega, X)\), what allows us to define the following \(X^*\) valued measure.

\[
< G(E), a > = T(a \chi_E) \quad (a \in X)
\]
Given a ball $B$ and a partition of $B$, say $\{E_i\}$, of pairwise disjoint sets, using the duality $(l^p(X))^* = l^q(X^*)$, we have

$$\left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^{q} \cdot \frac{m(E_i)}{m(B)} \right)^{1/q} =$$

$$\left(\sum_{i=1}^n \left\| \left( \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right) \left( \frac{m(E_i)}{m(B)} \right)^{1/q} \right\|_{X^*}^{q} \right)^{1/q} =$$

$$\sup \left\{ \sum_{i=1}^n \left( \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right) \left( \frac{m(E_i)}{m(B)} \right)^{1/q} \right| : \sum_{i=1}^n \left\| b_i \right\|_{X}^{p} = 1 \right\}. $$

On the other hand we have

$$\left| \sum_{i=1}^n \left( \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right) \left( \frac{m(E_i)}{m(B)} \right)^{1/q} \right| =$$

$$\frac{1}{m(B)^{1/q}} \left| \sum_{i=1}^n \left( \frac{G(E_i)}{m(E_i)} \right)^{1/p} b_i - \frac{G(B)}{m(B)} \sum_{i=1}^n m(E_i) b_i \right| =$$

$$= \frac{1}{m(B)^{1/q}} \left| T \left( \sum_{i=1}^n \frac{m(E_i)^{-1/p}}{b_i} \chi_{E_i} \right) - T(b \chi_B) \right|$$

where $b = \frac{1}{m(B)} \left( \sum_{i=1}^n m(E_i) b_i \right)$.

Denote by $a = \frac{1}{2 m(B)^{1/q}} \left( \sum_{i=1}^n m(E_i)^{-1/p} \chi_{E_i} - b \chi_B \right)$. It is elementary to show that if $\sum_{i=1}^n \left\| b_i \right\|_{X}^{p} = 1$ then $a$ is a $(X,p)$-atom.

Therefore we obtain

$$\left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^{q} \cdot \frac{m(E_i)}{m(B)} \right)^{1/q} \leq 2 \left| T(a) \right| \leq 2 \left\| T \right\|$$

This shows $\left\| G \right\|_{*q} \leq 2 \left\| T \right\|$. Since $T$ and $T_G$ coincide over simple atoms, we have $T = T_G$. On the other hand

$$\left\| G(\Omega) \right\| \leq \sup \left\{ \left| T(b \chi_B) \right| : \left\| b \right\| \leq 1 \right\} \leq m(\Omega) \left\| T \right\|$$

and this finishes the proof for the finite measure case.

Let us deal now with the case of $m(\Omega) = \infty$. Take a functional $T$ in $(H^1_0(\Omega, X))^*$ and a ball $B$ in $\Omega$. Let us consider the following space

$$L^p_0(B, X) = \{ f \in L_p(\Omega, X) : \text{supp} f \subset B \text{ and } \int_B f(x) \, dm(x) = 0 \}$$
The following function is an \((X,p)\)-atom
\[
a(x) = \frac{f(x)}{m(B)^{1/q} \|f\|_p} \quad \text{for } f \in L^p_0(B, X).
\]

hence
\[
\|f\|_{H^1_p} \leq m(B)^{1/q} \|f\|_p
\]
and therefore
\[
\|Tf\| \leq \|T\| m(B)^{1/q} \|f\|_p
\]
This shows that \(T\) defines a bounded functional on \(L^p_0(B, X)\) and hence from the Hahn-Banach extension theorem, we get an element in the dual of \(L^p_p(B, X)\). The characterization of the dual space \((L^p_p(B, X))^*\) in terms of \(X^*\)-valued measures of bounded \(q\)-variation allows us to find a measure \(G_B\) with values in \(X^*\) verifying
\[
(3.13) \quad T(f) = \int_B f \, dG_B \quad f \in L^p_0(B, X)
\]
(Note that this measure is uniquely determined up to a measure \(F(E) = \xi m(E \cap B)\) for some \(\xi \in X^*\)). Now if we take an increasing sequence of balls converging to \(\Omega\), say \(B_n\), and we determine \(G_{B_n}\) by the assumption \(G_{B_n}(B_1) = 0\), then we can construct a vector-valued measure on \(\Sigma_0\), given by \(G(E) = G_{B_n}(E)\) for \(E \subset B_n\). It is clear that \(G_{B_n}\) are absolutely continuous and hence the same is true for \(G\). Now from remark 1.2 we get an extension to \(\Sigma\).

\[
(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*} q \frac{m(E_i)}{m(B)} \right)^{1/q} = \sup_{\|f\|_p = 1} \left| \frac{1}{m(B)^{1/q}} \int_B f \, d(G - \frac{G(B)}{m(B)} m) \right|
\]
For each \(f \in L_p(B, X)\), consider \(a = \frac{1}{2m(B)^{1/q}} (f - f_B) \chi_B\) and therefore
\[
(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*} q \frac{m(E_i)}{m(B)} \right)^{1/q} = \sup_a 2 |T(a)| \leq 2 \|T\|
\]
This completes the proof.\(\diamondsuit\)

**Remark 3.1.** For \(1 < p, r < \infty\),

\[
MBMO_q(\Omega, X) = MBMO_r(\Omega, X) \quad \text{with equivalent norms}
\]
For dual spaces follows from the theorem and (1.1), and the general case is consequence of the embedding \(X \subset X^{**}\)

**REFERENCES**


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