The Poincaré and Geometrization Conjectures after R. Hamilton and G. Perelman

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Classification of surfaces

M^2, compact, connected, orientable

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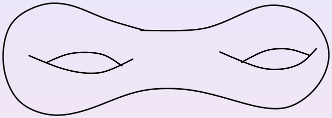
M^2, compact, connected, orientable $\rightsquigarrow$ boundary of a bretzel



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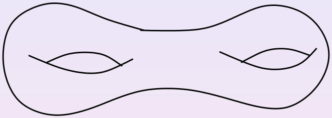

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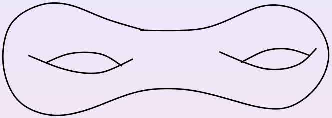

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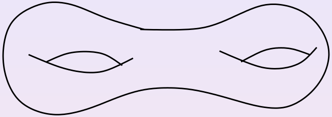

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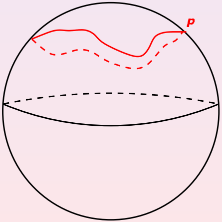

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Thurston puts Poincaré in a geometric setting.

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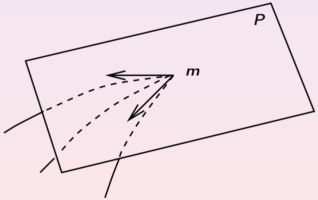

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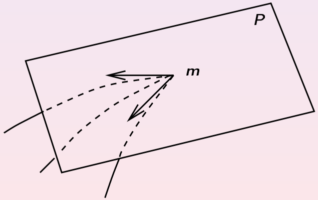

$K(P)$ = Gauß curvature at m of the sheet of geodesics.

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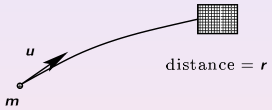

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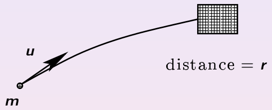

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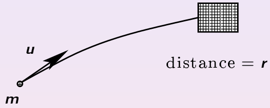

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Bilinear form on $T_m(M)$.

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(Almost) uniformization of surfaces.

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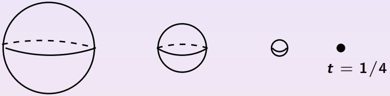

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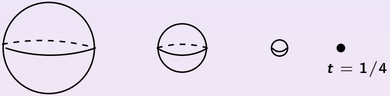

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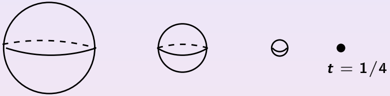

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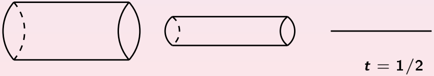
4. Cylinder $g(t) = (1 - 2t)g_{S^2} \oplus g_{\mathbf{R}}$.



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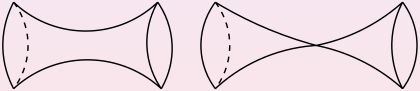
Example : the neckpinch



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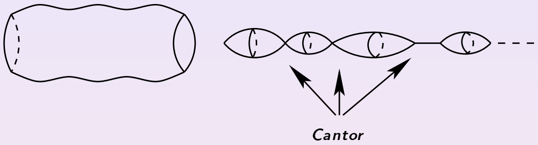

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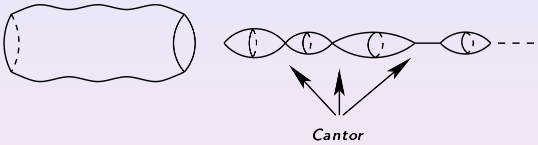

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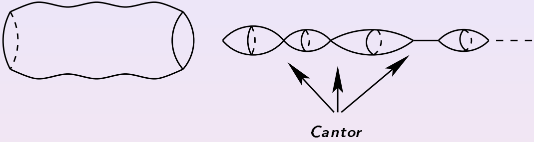

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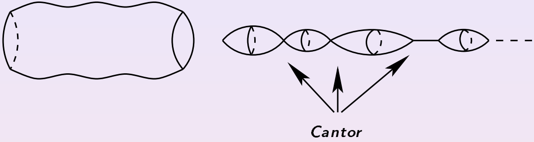

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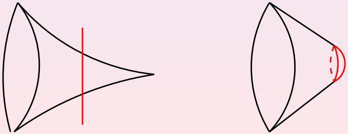

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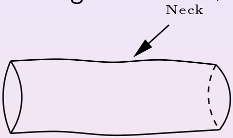

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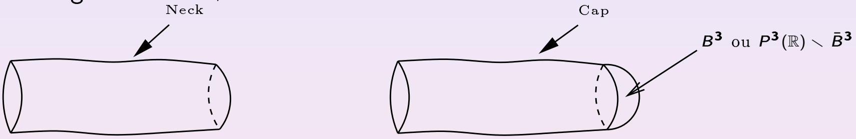

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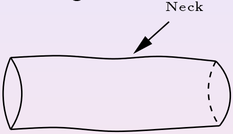

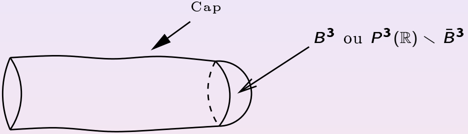

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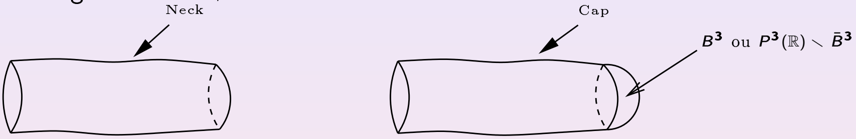

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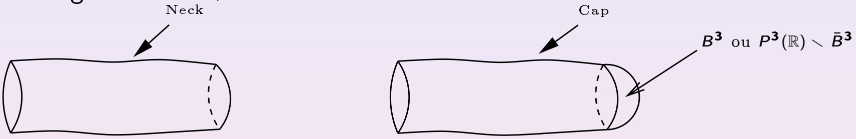

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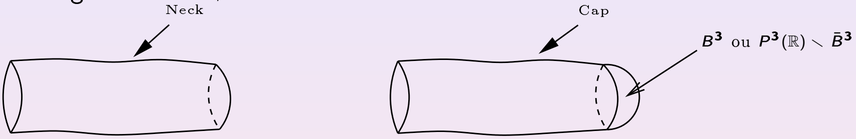

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2. g_0 *arbitrary* metric$\rightsquigarrow$ flow it up to first singular time :

The first singular time

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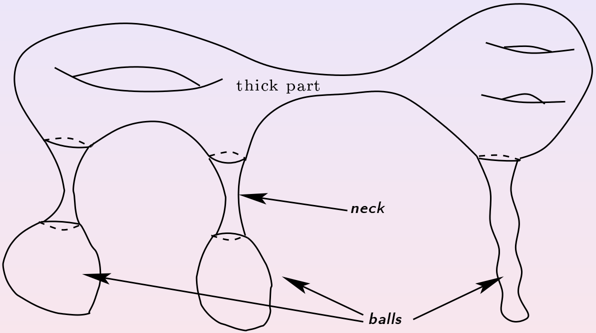
There are points with high curvature and points with "normal" curvature$\rightsquigarrow$ between there is a long neck.

surgery time



The surgery

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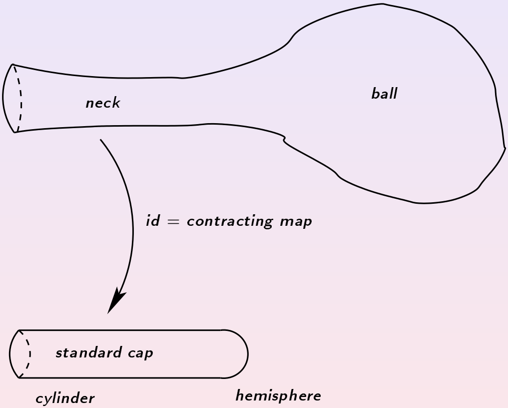
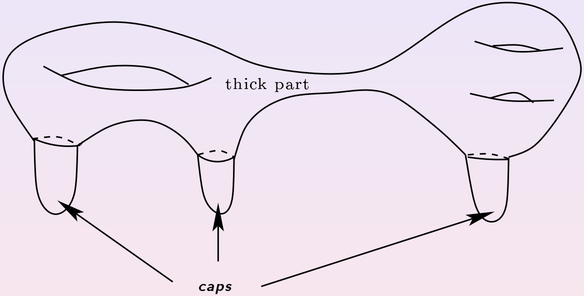


The result

The new metric is



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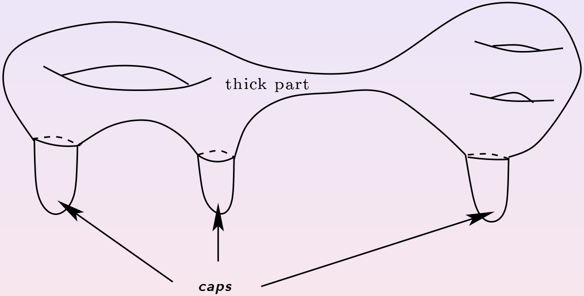

Conclusion : no topological surgery, just a discontinuity in the metric.

Claim (Perelman PII, 5) : Ricci-flow-with-surgery exists. There is no accumulation of surgeries $\rightsquigarrow$ finite number of surgeries in each finite interval.

The proof of the Poincaré conjecture

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If M is simply connected, then $M \simeq S^3$.

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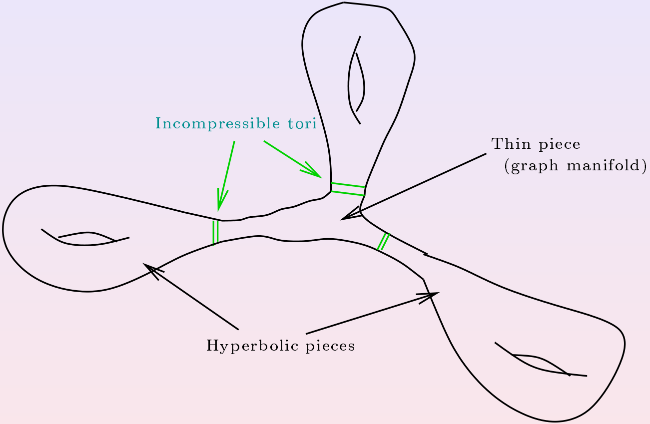
Claim (Perelman PII, 7,8) : For large t, M decomposes into thick and thin pieces (possibly empty) :

Thick-thin decomposition



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This would prove the Geometrization conjecture.

Previous result obtained by R. Hamilton in 1999 when there are no singularities and $t \sup_M |K(g(t))|$ is bounded.

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For both issue $\rightsquigarrow$ different approach by B3MP

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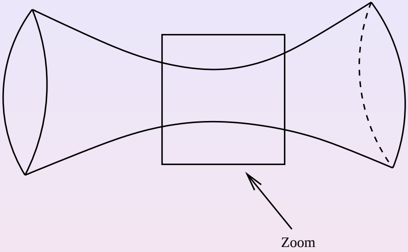
this is a **parabolic dilation**.

The ancient solutions



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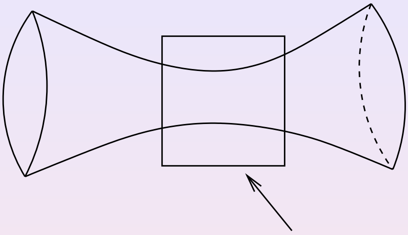

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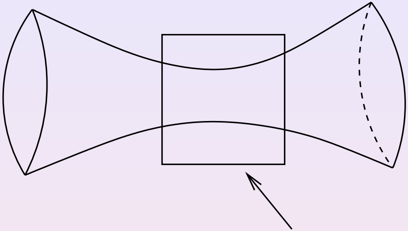

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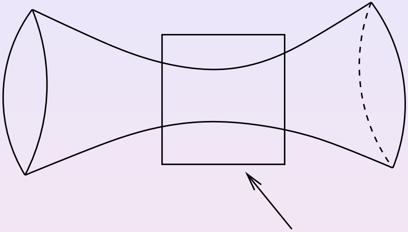

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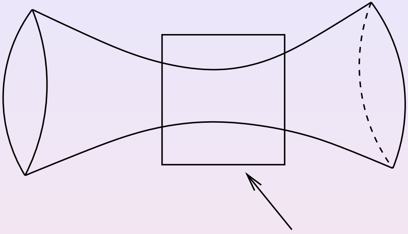

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It is an **ancient solution** $\rightsquigarrow$ infinitesimal models for singularities.

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For the ancient solutions,

- Curvature operator ≥ 0.
- Bounded sectional curvatures.

$+$ other properties $\Rightarrow$ classification.

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- B^3 and curvature > 0.