Birkhoff-Frink representations as functors

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In an earlier article we characterized, from the viewpoint of set theory, those closure operators for which the classical result of Birkhoff and Frink, stating the equivalence between algebraic closure spaces, subalgebra lattices and algebraic lattices, holds in a many-sorted setting. In the present article we investigate, from the standpoint of category theory, the form these equivalences take when the adequate morphisms of the several different species of structures implicated in them are also taken into account. Specifically, our main aim is to provide a functorial rendering of the Birkhoff-Frink representation theorems for both single-sorted algebras and many-sorted algebras, by defining the appropriate categories and functors, covariant and contravariant, involved in the process.

1 Introduction.

As it is well-known Birkhoff and Frink, in [5], proved, among other interesting results, the following representation theorems:

1. Let (A, J) be an algebraic closure space, i.e., a set A together with an algebraic closure operator J on A. Then there exists a single-sorted signature $\Sigma^{(A,J)}$ and a $\Sigma^{(A,J)}$-algebra structure $F^{(A,J)}$ on A such that (A, J) is identical with $(A, \mathrm{Sg}_{(A,F^{(A,J)})})$, where $\mathrm{Sg}_{(A,F^{(A,J)})}$ is the subalgebra generating operator on A induced by the $\Sigma^{(A,J)}$-algebra $(A, F^{(A,J)})$ (see [5], p. 300). In other words, the algebraic (*alias* inductive) closure spaces, or, what is equivalent, the algebraic (*alias* inductive) closure systems, are precisely the subalgebra systems of finitary algebras.

2. Let $\mathbf{L} = (L, \leq)$ be a lattice. Then $\mathbf{L}$ is algebraic if and only if there exists a single-sorted signature Σ and a Σ-algebra $\mathbf{A} = (A, F)$ such that $\mathbf{L}$ is isomorphic to the algebraic lattice $\mathbf{Fix}(\mathrm{Sg}_{\mathbf{A}})$ determined by the fixed points of the algebraic closure operator $\mathrm{Sg}_{\mathbf{A}}$ (see [5], p. 302). In other words, the algebraic (i.e., compactly generated complete) lattices are, up to isomorphism, the algebraic closure spaces.

The first representation theorem can be interpreted as meaning that there is a system $\mathrm{Alg}(1)$, of single-sorted algebras, i.e., pairs $(\Sigma, \mathbf{A})$, where Σ is a single-sorted signature and $\mathbf{A}$ a Σ-algebra, and a construction Sg from $\mathrm{Alg}(1)$ to the system AClSp, of algebraic closure spaces, which sends a single-sorted algebra $(\Sigma, \mathbf{A})$ to the algebraic closure space $\mathrm{Sg}(\Sigma, \mathbf{A}) = (A, \mathrm{Sg}_{\mathbf{A}})$, where $\mathrm{Sg}_{\mathbf{A}}$ is the subalgebra generating operator on A induced by the Σ-algebra $\mathbf{A}$, and this in such a way that it is surjective.

Before explaining the meaning of the second representation theorem we let $\mathbf{Fix}$ stand for the construction from AClSp to the system ALat, of algebraic lattices, which sends an algebraic closure space (A, J) to the algebraic lattice $\mathbf{Fix}(J)$ determined by the fixed points of the algebraic closure operator J.

The second representation theorem can then be partially interpreted as saying that there is another construction $\mathbf{Fix} \circ \mathrm{Sg}$ from $\mathrm{Alg}(1)$ to ALat which sends a single-sorted algebra $(\Sigma, \mathbf{A})$ to the algebraic lattice $\mathbf{Fix}(\mathrm{Sg}_{\mathbf{A}})$,

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but, in contrast to the previous representation theorem, only in an essentially surjective way. We notice that the expression "essentially surjective", as applied to a construction and after having transformed it into a suitable functor, should be understood as meaning that it is "surjective up to isomorphism".

We can summarize the just stated interpretations of the Birkhoff-Frink representation theorems for single-sorted algebras by stating that in the following diagram

$$\mathrm{Alg}(1) \xrightarrow{\ \mathrm{Sg}\ } \mathrm{AClSp} \xrightarrow{\ \mathbf{Fix}\ } \mathrm{ALat},$$

the construction Sg is surjective and the construction $\mathbf{Fix}$ is essentially surjective (from which it follows that the composition of both constructions is another essentially surjective construction).

With regard to the many-sorted case, Matthiessen, in [24], proved that there are sets of sorts S, with at least two elements, and algebraic S-closure spaces (the definition of the concept of algebraic S-closure space is analogous to that of algebraic closure space and is given in the third section) which cannot be concretely represented as the set of all subalgebras of some many-sorted algebra. In other words, it is not generally true that, for every set of sorts S, each algebraic S-closure space (A, J) in the system $\mathrm{AClSp}(S)$, of algebraic S-closure spaces, has the form $(A, \mathrm{Sg}_{\mathbf{A}})$, for some S-sorted algebra $(\Sigma, \mathbf{A})$ in the system $\mathrm{Alg}(S)$ (the definition of $\mathrm{Alg}(S)$ is analogous to that of $\mathrm{Alg}(1)$ and is given in the second section), of S-sorted algebras. Or, what is equivalent, that the construction

$$\mathrm{Alg}(S) \xrightarrow{\ \mathrm{Sg}^S\ } \mathrm{AClSp}(S),$$

which sends an S-sorted algebra $(\Sigma, \mathbf{A})$ to the algebraic S-closure space $(A, \mathrm{Sg}^S_{\mathbf{A}})$, where $\mathrm{Sg}^S_{\mathbf{A}}$ is the subalgebra generating operator on the S-sorted set A induced by the $\mathbf{\Sigma} = (S, \Sigma)$-algebra $\mathbf{A}$, is not surjective. However, the construction

$$\mathrm{AClSp}(S) \xrightarrow{\ \mathbf{Fix}^S\ } \mathrm{ALat},$$

which sends an algebraic S-closure space (A, J) to the algebraic lattice $\mathbf{Fix}^S(J)$ determined by the fixed points of the algebraic S-closure operator J is, as for the single-sorted case (and as we will prove later on), also essentially surjective.

Related to the work by Matthiessen mentioned above, we proved, in [8], that, for an algebraic S-closure operator J on an S-sorted set A, it happens that $J = \mathrm{Sg}_{\mathbf{A}}$, for some many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$ and some $\mathbf{\Sigma}$-algebra $\mathbf{A}$, if and only if J is uniform (i.e., such that, for every sub-S-sorted sets $X = (X_s)_{s\in S}$, $Y = (Y_s)_{s\in S}$ of A, if $\{s \in S \mid X_s \neq \varnothing\} = \{s \in S \mid Y_s \neq \varnothing\}$, then $\{s \in S \mid J(X)_s \neq \varnothing\} = \{s \in S \mid J(X)_s \neq \varnothing\}$, where, for a sub-$S$-sorted set $Z = (Z_s)_{s\in S}$ of A, $J(Z) = (J(Z)_s)_{s\in S}$). Therefore, by co-restricting the construction Sg^S to the subsystem $\mathrm{UAClSp}(S)$ (defined in the third section), of uniform algebraic S-closure spaces, of the system $\mathrm{AClSp}(S)$, we restored the surjective character of the construction

$$\mathrm{Alg}(S) \xrightarrow{\ \mathrm{Sg}^S\ } \mathrm{UAClSp}(S).$$

Our main aim in this article is to complete from the standpoint of category theory the Birkhoff-Frink representation theorems and their corresponding generalizations to the many-sorted case by giving a functorial version of them. Hence in making so we stay one step ahead of the foregoing purely set-theoretical treatment of such representation theorems, providing in this way a new perspective on the classical representation theorems. In this respect we think that the functorial rendering of the many-sorted case of the Birkhoff-Frink representation theorems, in particular, is fundamental and interesting since many-sorted algebras have a significant role in recent universal algebra, because they generalize single-sorted algebras to new and useful situations. For example, a typical higher order programming language has several data types, so a program written in the program language can be modeled as a many-sorted algebra. In fact, very general first order structures can be completely described as many-sorted algebras. Connected with the aforementioned generalization we claim that it is impossible to regard the many-sorted categories as being essentially subcategories of the classical ones. Indeed, this follows, ultimately, from the fact that, for a set of sorts S with two or more elements, in $\mathbf{Set}^S$, the topos of all S-tuples of sets, there are objects $A = (A_s)_{s\in S}$ that are different from the initial object, $(\varnothing)_{s\in S}$, but such that they are globally empty, i.e., such that $\mathrm{Hom}((1)_{s\in S}, A) = \varnothing$.

We emphasize that the categorical treatment of some questions belonging to the fields of lattice theory and universal algebra is not entirely new as shown by the following examples. In lattice theory the categorical equivalence between algebraic lattices and algebraic closure spaces falls under this treatment and is only one instance of a whole spectrum of Stone type dualities, all of which follow the same pattern (see e.g. [12], [13], and [14]). In universal algebra Birkhoff's variety theorem and Birkhoff's completeness of many-sorted equational logic have been investigated by many authors also in a categorical setting (e.g., the first one in [1], [23], and [26], and the second one in [9]). However, in contrast to Birkhoff's variety theorem and Birkhoff's completeness of many-sorted equational logic, the Birkhoff-Frink representation theorems did not seem to have attracted to the same extent the attention of "categorically minded people". This article intends to fill this gap.

To attain the aforesaid aim we define for the objects belonging to each one of the different species of structures mentioned above the adequate morphisms between them. In this way each one of the systems of objects is transformed into the set of objects of a convenient category (written as their corresponding underlying system of objects, but in bold print). Moreover, we extend the definitions of the constructions Sg and Sg^S, which were, originally, restricted to the objects in $\mathrm{Alg}(1)$ and $\mathrm{Alg}(S)$, respectively, to the morphisms in the associated categories $\mathbf{Alg}(1)$ and $\mathbf{Alg}(S)$ to get covariant functors from them into $\mathbf{AClSp}$ and $\mathbf{UAClSp}(S)$, respectively. Additionally, we extend the definition of the construction $\mathbf{Fix}^S$, which was, at the beginning, restricted to the objects in $\mathrm{UAClSp}(S)$, to the morphisms in the associated category $\mathbf{UAClSp}(S)$ to get a contravariant functor from it into the category $\mathbf{ALat}$. It is worth noting that the functor $\mathbf{Fix}\colon \mathbf{AClSp} \longrightarrow \mathbf{ALat}$ is well-known, cf., [12] and [14] (however, since $\mathbf{AClSp} = \mathbf{UAClSp}(1)$, see Remark 3.14, we have that $\mathbf{Fix} = \mathbf{Fix}^1$).

In this article we make use of the concept of fibration and of the Ehresmann-Grothendieck construction (we write it EG-construction for short). With regard to the EG-construction there are, essentially, two reasons for using it. On the one hand, since it allows us to obtain, from a contravariant functor F from a category $\mathbf{C}$ to $\mathbf{Cat}$ (the definition of $\mathbf{Cat}$ is given below), a category $\int^{\mathbf{C}} F$ over $\mathbf{C}$, by "pasting" together the family of "summand" categories $(F(c))_{c\in\mathbf{C}}$ by means of the family of "translators" $F(f)\colon F(c') \longrightarrow F(c)$, parameterized by the morphisms $f\colon c \longrightarrow c'$ in $\mathbf{C}$. On the other hand, because a great deal of the properties of $\int^{\mathbf{C}} F$ can be obtained from the corresponding ones of the categories $F(c)$, taking into account the properties of the functors of the family of "translators". For the convenience of the reader we recall next the definitions of the aforementioned notions, thus making our exposition self-contained. Let $F\colon \mathbf{C} \longrightarrow \mathbf{B}$ be a functor. A morphism $f\colon x \longrightarrow y$ in $\mathbf{C}$ is *cartesian over* $u\colon a \longrightarrow b$ in $\mathbf{B}$ if $F(f) = u$ and for every $v\colon c \longrightarrow a$ in $\mathbf{B}$, and for every $h\colon z \longrightarrow y$ in $\mathbf{C}$ such that $F(h) = u \circ v$, there exists a unique $g\colon z \longrightarrow x$ in $\mathbf{C}$ such that $F(g) = v$ and $f \circ g = h$. The functor $F\colon \mathbf{C} \longrightarrow \mathbf{B}$ is a *fibration* if for every $y \in \mathbf{C}$ and every $u\colon a \longrightarrow F(y)$ in $\mathbf{B}$, there exists a cartesian morphism $f\colon x \longrightarrow y$ in $\mathbf{C}$ above u. A fibration $F\colon \mathbf{C} \longrightarrow \mathbf{B}$ is *split* if, for every morphism $u\colon a \longrightarrow b$ in $\mathbf{B}$ and every object x in the *fiber* $\mathbf{C}_b$ *over* (b, id_b), which is the inverse image under F of (b, id_b), it is possible to choose a distinguished cartesian morphism $u_x\colon x_u \longrightarrow x$ over u in such a way that all these morphisms u_x constitute a subcategory of $\mathbf{C}$. For an exhaustive treatment of fibrations see [17]. The EG-*construction* establishes a passage from a contravariant functor $F\colon \mathbf{C} \longrightarrow \mathbf{Cat}$ to a pair $(\int^{\mathbf{C}} F, \pi_F)$, where $\int^{\mathbf{C}} F$ is the category with objects pairs (c, x), where c is an object of $\mathbf{C}$ and x an object of $F(c)$, and morphisms from (c, x) to (c', x') pairs (f, λ) consisting of a morphism $f\colon c \longrightarrow c'$ in $\mathbf{C}$ and a morphism $\lambda\colon x \longrightarrow F(f)(x')$ in $F(c)$, with the obvious composition, and $\pi_F\colon \int^{\mathbf{C}} F \longrightarrow \mathbf{C}$, the projection functor, a fibration. The earliest published papers we know of which deal with the EG-construction are [11], pp. 89–91 and [18], pp. (sub.) 175–177.

Next we proceed to, briefly, describe the contents of the subsequent sections of this article.

In the second section we define, by applying the EG-construction to convenient contravariant functors, the categories $\mathbf{MSet}$, of many-sorted sets, $\mathbf{Sig}$, of many-sorted signatures, and $\mathbf{Alg}$, of many-sorted algebras. Moreover, we state that there are two split bi-fibrations, one from $\mathbf{MSet}$ to $\mathbf{Set}$ and another from $\mathbf{Sig}$ to $\mathbf{Set}$, and a fibration from $\mathbf{Alg}$ to $\mathbf{Sig}$. Then, by composing this fibration and the second of the split bi-fibrations, we obtain a fibration from $\mathbf{Alg}$ to $\mathbf{Set}$ that, for every set of sorts S, allows us to get, by taking the inverse image of it at (S, id_S), the fiber $\mathbf{Alg}(S)$, which we call the category of S-algebras, and which has as underlying set of objects the system $\mathrm{Alg}(S)$.

In the third section we begin by associating with every set of sorts S the category $\mathbf{ClSp}(S)$, of S-closure spaces. After having done that we define, for every set of sorts S, two full subcategories of $\mathbf{ClSp}(S)$, the category $\mathbf{AClSp}(S)$, of algebraic S-closure spaces, and the category $\mathbf{UAClSp}(S)$, of uniform algebraic S-closure spaces, and state the existence of some adjoint situations between them. Moreover, we prove that every mapping $\varphi\colon S \longrightarrow T$ determines an adjunction $\coprod_{\varphi}^{\mathrm{cl}} \dashv \Delta_{\varphi}^{\mathrm{cl}}$ from $\mathbf{ClSp}(S)$ to $\mathbf{ClSp}(T)$. Then, since all these functors $\Delta_{\varphi}^{\mathrm{cl}}$ are the components of the morphism mapping of a contravariant functor Δ^{cl} from $\mathbf{Set}$ to $\mathbf{Cat}$, by applying the EG-construction to Δ^{cl}, we get the category $\mathbf{MClSp}$, of many-sorted closure spaces and continuous mappings, and a split fibration from $\mathbf{MClSp}$ to $\mathbf{Set}$.

In the fourth section, with the suitable categories at hand, we view the first classical Birkhoff-Frink representation as a functor Sg from $\mathbf{Alg}(1)$ to $\mathbf{AClSp}$ which is surjective on the objects (but, we remark, only surjective with regard to the injective continuous mappings between algebraic closure spaces), while we view the second one as a contravariant functor $\mathbf{Fix} \circ \mathrm{Sg}$ from $\mathbf{Alg}(1)$ to $\mathbf{ALat}$ which is essentially surjective. Then we extend both theorems to the many-sorted case and prove that they have similar features to the classical version, but with respect to the categories $\mathbf{UAClSp}(S)$ and the contravariant functors $\mathbf{Fix}^S$ from $\mathbf{UAClSp}(S)$ to $\mathbf{ALat}$. Actually, the Birkhoff-Frink representation theorems, as applied to the many-sorted case say, respectively, that, for a nonempty set of sorts S, there exists a functor Sg^S from $\mathbf{Alg}(S)$ to $\mathbf{UAClSp}(S)$ which is surjective on the objects, and that there exists a contravariant functor $\mathbf{Fix}^S \circ \mathrm{Sg}^S$ from $\mathbf{Alg}(S)$ to $\mathbf{ALat}$ which is essentially surjective.

In this way the functorial version of the Birkhoff-Frink representation theorems, both for single-sorted and for many-sorted algebras, has been reached.

In this article the foundational system underlying category theory is $\mathbf{ZFC} + \exists\, \mathrm{GU}(\boldsymbol{\mathcal{U}})$, i.e., Zermelo-Fraenkel set theory with the Axiom of Choice plus the existence of a Grothendieck universe $\boldsymbol{\mathcal{U}}$ fixed once and for all (for an explanation of the concept of *Grothendieck universe* see, e.g., [22], p. 22). Therefore every set we consider in this article will be either a $\boldsymbol{\mathcal{U}}$-small set, i.e., an element of $\boldsymbol{\mathcal{U}}$, or a $\boldsymbol{\mathcal{U}}$-large set, i.e., a subset of $\boldsymbol{\mathcal{U}}$, or a set which is neither $\boldsymbol{\mathcal{U}}$-small nor $\boldsymbol{\mathcal{U}}$-large. Besides, we let $\mathbf{Set}$ stand for the category with objects the $\boldsymbol{\mathcal{U}}$-small sets and morphisms the mappings between $\boldsymbol{\mathcal{U}}$-small sets, and $\mathbf{Cat}$ for the category of the $\boldsymbol{\mathcal{U}}$-categories (i.e., categories $\mathbf{C}$ such that the set of objects of $\mathbf{C}$ is a subset of $\boldsymbol{\mathcal{U}}$, and the hom-sets of $\mathbf{C}$ elements of $\boldsymbol{\mathcal{U}}$) and functors between $\boldsymbol{\mathcal{U}}$-categories.

In all that follows we use standard concepts and constructions from category theory, see e.g., [1], [17], [18], [19], and [22]; classical universal algebra, see e.g., [6], [10], [16], [21], and [25]; many-sorted algebra, see e.g., [2], [4], [20], [24], and [28]; and lattice theory, see e.g., [3], [6], and [25]. However, following the French mathematical tradition, we agree to call a functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ *essentially surjective* (instead of *representative* or *isomorphism-dense*) if for every object d in $\mathbf{D}$ there exists an object c in $\mathbf{C}$ such that $F(c)$ is isomorphic to d.

2 Many-sorted sets, signatures, and algebras.

In this section we provide accurate definitions of the concepts of: many-sorted set, many-sorted signature, and many-sorted algebra. More specifically, we define the category $\mathbf{MSet}$, of many-sorted sets, by applying the EG-construction to an appropriate contravariant functor MSet from $\mathbf{Set}$ to $\mathbf{Cat}$, and state that the projection functor π_{MSet} from $\mathbf{MSet}$ to $\mathbf{Set}$ is a split bi-fibration. Following this we define the category $\mathbf{Sig}$, of many-sorted signatures, by applying, again, the EG-construction to a suitable contravariant functor Sig from $\mathbf{Set}$ to $\mathbf{Cat}$, and state that the projection functor π_{Sig} from $\mathbf{Sig}$ to $\mathbf{Set}$ is also a split bi-fibration. Next we define the category $\mathbf{Alg}$, of many-sorted algebras (in which many-sorted algebras of different many-sorted signature are compared via suitable homomorphisms), by applying, once again, the EG-construction to an adequate contravariant functor Alg from $\mathbf{Sig}$ to $\mathbf{Cat}$, and state that the projection functor π_{Alg} from $\mathbf{Alg}$ to $\mathbf{Sig}$ is a fibration. Then, by composing the fibration π_{Alg} and the split bi-fibration π_{Sig}, we obtain the fibration $\pi_{\mathrm{Sig,Alg}}$ that allows us to get, for every set of sorts S, in a regular way, the corresponding fiber $\mathbf{Alg}(S)$, constituted by the S-sorted algebras.

Before stating the first proposition of this section, we agree upon calling, henceforth, for a set of sorts S in $\boldsymbol{\mathcal{U}}$, the objects of the category $\mathbf{Set}^S$ (i.e., the functions $A = (A_s)_{s\in S}$ from S to $\boldsymbol{\mathcal{U}}$) *S-sorted sets*, and the morphisms of the category $\mathbf{Set}^S$ from an S-sorted set A to another B (i.e., the ordered triples (A, f, B), written as $f\colon A \longrightarrow B$, where $f = (f_s)_{s\in S} \in \prod_{s\in S} \mathrm{Hom}(A_s, B_s)$) *$S$-sorted mappings from A to B*.

In the following proposition, which is basic for a great deal of what follows, for a mapping $\varphi\colon S \longrightarrow T$, we prove that there is an adjunction $\coprod_\varphi \dashv \Delta_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$, and state that there is a contravariant functor MSet from $\mathbf{Set}$ to $\mathbf{Cat}$.

Proposition 2.1 *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then there are functors Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ and $\coprod_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ such that $\coprod_\varphi \dashv \Delta_\varphi$. We let θ^φ stand for the natural isomorphism of the adjunction. Moreover, there exists a contravariant functor MSet from $\mathbf{Set}$ to $\mathbf{Cat}$ which sends a set S to the category $\mathrm{MSet}(S) = \mathbf{Set}^S$, and a mapping φ from S to T to the functor Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$.*

Proof. Let Δ_φ be the functor from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ defined as follows: its object mapping sends each T-sorted set A to the S-sorted set $A_\varphi = (A_{\varphi(s)})_{s\in S}$, i.e., the composite mapping $A \circ \varphi$; its arrow mapping sends each T-sorted mapping $f\colon A \longrightarrow B$ to the S-sorted mapping $f_\varphi = (f_{\varphi(s)})_{s\in S}\colon A_\varphi \longrightarrow B_\varphi$. Let $\coprod_\varphi$ be the functor from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ defined as follows: its object mapping sends each S-sorted set A to the T-sorted set $\coprod_\varphi A = (\coprod_{s\in\varphi^{-1}[t]} A_s)_{t\in T}$; its arrow mapping sends each S-sorted mapping $f\colon A \longrightarrow B$ to the T-sorted mapping $\coprod_\varphi f = (\coprod_{s\in\varphi^{-1}[t]} f_s)_{t\in T}\colon \coprod_\varphi A \longrightarrow \coprod_\varphi B$. Then the functor $\coprod_\varphi$ is a left adjoint for Δ_φ. This is proved by stating that, for every S-sorted set A, the pair $(\eta^\varphi_A, \coprod_\varphi A)$, where η^φ_A is the S-sorted mapping from A to $\Delta_\varphi(\coprod_\varphi A) = (\coprod_{x\in\varphi^{-1}[\varphi(s)]} A_x)_{s\in S}$ whose s-th coordinate, for every $s \in S$, is the canonical embedding of A_s into $\coprod_{x\in\varphi^{-1}[\varphi(s)]} A_x$, is a universal morphism from A to Δ_φ. Let B be a T-sorted set and let $f\colon A \longrightarrow B_\varphi$ be an S-sorted mapping. In the sequel, for every $t \in T$ and every $s \in \varphi^{-1}[t]$, in_s denotes the canonical embedding of A_s into $\coprod_{s\in\varphi^{-1}[t]} A_s$, and, for every $t \in T$, $f^\S_t$ denotes the unique mapping $[f_s]_{s\in\varphi^{-1}[t]}$ from $\coprod_{s\in\varphi^{-1}[t]} A_s$ to $B_t = B_{\varphi(s)}$ such that, for every $s \in \varphi^{-1}[t]$, $[f_s]_{s\in\varphi^{-1}[t]} \circ \mathrm{in}_s = f_s$. Then $f^\S = (f^\S_t)_{t\in T}$ is the unique T-sorted mapping from $\coprod_\varphi A$ to B such that $f = \Delta_\varphi(f^\S) \circ \eta^\varphi_A$. □

By applying the EG-construction to MSet we get the category of many-sorted sets as stated in the following

Definition 2.2 The category $\mathbf{MSet}$, of *many-sorted sets* and *many-sorted mappings*, is given by $\mathbf{MSet} = \int^{\mathbf{Set}} \mathrm{MSet}$. Therefore $\mathbf{MSet}$ has as objects the pairs (S, A), where S is a set and A an S-sorted set, and as morphisms from (S, A) to (T, B) the pairs (φ, f), where $\varphi\colon S \longrightarrow T$ and $f\colon A \longrightarrow B_\varphi$.

From the definition of $\mathbf{MSet}$ it follows that the projection functor π_{MSet} from $\mathbf{MSet}$ to $\mathbf{Set}$ is a split bi-fibration, i.e., a split fibration and a split op-fibration. Furthermore, for every set S, the fiber of π_{MSet} at (S, id_S) is, essentially, the category $\mathbf{Set}^S$ of S-sorted sets and S-sorted mappings.

Remark 2.3 The category $\mathbf{MSet}$ is complete and co-complete.

Remark 2.4 The EG-construction applied to the contravariant functor MSet generates, explicitly, the category $\mathbf{MSet}$ and, implicitly, a logic: the internal logic of $\mathbf{MSet}$ (which is the trivalent logic of Heyting) obtained by combining, by means of the logical morphisms between the fibers of π_{MSet}, the Boolean internal logics of the just named fibers. Loosely speaking, we can say that globally the category $\mathbf{MSet}$ has a nonclassical logic, but that locally (i.e., in its fibers) it is Boolean, although not well-pointed (in the same way as a manifold is a space which locally looks like $\mathbb{R}^n$ (or $\mathbb{C}^n$) but which globally is not necessarily like any of those local spaces). Thus, in this case, we see that the system of laws governing the world obtained by synthesizing a family of given interwoven worlds, each of them governed by its proper system of laws, is not necessarily identical to any of the local systems of laws.

Our next goal is to define the category $\mathbf{Sig}$. But before doing that we agree that, for a set of sorts S in $\boldsymbol{\mathcal{U}}$, $\mathbf{Sig}(S)$ denotes the category of S-sorted signatures and S-sorted signature morphisms, i.e., the category $\mathbf{Set}^{S^\star \times S}$, where $S^\star$ is the underlying set of the free monoid on S. Therefore an *S-sorted signature* is a function Σ from $S^\star \times S$ to $\boldsymbol{\mathcal{U}}$ which sends a pair $(w, s) \in S^\star \times S$ to the set $\Sigma_{w,s}$ of the *formal operations* of *arity* w, *sort* (or *coarity*) s, and *rank* (or *biarity*) (w, s); and an *S-sorted signature morphism* from Σ to Σ' is an ordered triple (Σ, d, Σ'), written as $d\colon \Sigma \longrightarrow \Sigma'$, where $d = (d_{w,s})_{(w,s)\in S^\star\times S} \in \prod_{(w,s)\in S^\star\times S} \mathrm{Hom}(\Sigma_{w,s}, \Sigma'_{w,s})$. Thus, for every $(w, s) \in S^\star \times S$, $d_{w,s}$ is a mapping from $\Sigma_{w,s}$ to $\Sigma'_{w,s}$ which sends a formal operation σ in $\Sigma_{w,s}$ to the formal operation $d_{w,s}(\sigma)$ ($d(\sigma)$ for short) in $\Sigma'_{w,s}$.

Proposition 2.5 *There exists a contravariant functor Sig from $\mathbf{Set}$ to $\mathbf{Cat}$ defined as follows:*

1. Sig *sends a set of sorts S to* $\mathrm{Sig}(S) = \mathbf{Sig}(S)$.

2. Sig *sends a mapping* φ *from* S *to* T *to the functor* $\mathrm{Sig}(\varphi) = \Delta_{\varphi^\star\times\varphi}$ *from* $\mathbf{Sig}(T)$ *to* $\mathbf{Sig}(S)$ *which relabels* T*-sorted signatures into* S*-sorted signatures, i.e.,* $\mathrm{Sig}(\varphi)$ *assigns to a* T*-sorted signature* Λ *the* S*-sorted signature* $\mathrm{Sig}(\varphi)(\Lambda) = \Lambda_{\varphi^\star\times\varphi}$*, and assigns to a morphism of* T*-sorted signatures* d *from* Λ *to* Λ' *the morphism of* S*-sorted signatures* $\mathrm{Sig}(\varphi)(d) = d_{\varphi^\star\times\varphi}$ *from* $\Lambda_{\varphi^\star\times\varphi}$ *to* $\Lambda'_{\varphi^\star\times\varphi}$.

By applying the EG-construction to Sig we get the category of many-sorted sets as stated in the following

Definition 2.6 The category $\mathbf{Sig}$, of *many-sorted signatures* and *many-sorted signature morphisms*, is given by $\mathbf{Sig} = \int^{\mathbf{Set}} \mathrm{Sig}$. Therefore the category $\mathbf{Sig}$ has as objects the pairs (S,Σ), where S is a set of sorts and Σ an S-sorted signature, and as many-sorted signature morphisms from (S,Σ) to (T,Λ) the pairs (φ,d), where $\varphi\colon S \longrightarrow T$ is a morphism in $\mathbf{Set}$ while $d\colon \Sigma \longrightarrow \Lambda_{\varphi^\star\times\varphi}$ is a morphism in $\mathbf{Sig}(S)$. The composition of $(\varphi,d)\colon (S,\Sigma) \longrightarrow (T,\Lambda)$ and $(\psi,e)\colon (T,\Lambda) \longrightarrow (U,\Omega)$, denoted by $(\psi,e)\circ(\varphi,d)$, is $(\psi\circ\varphi, e_{\varphi^\star\times\varphi}\circ d)$, where

$$e_{\varphi^\star\times\varphi}\colon \Lambda_{\varphi^\star\times\varphi} \longrightarrow (\Omega_{\psi^\star\times\psi})_{\varphi^\star\times\varphi} (= \Omega_{(\psi\circ\varphi)^\star\times(\psi\circ\varphi)}).$$

Henceforth, unless otherwise stated, we will write $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ instead of (S,Σ) and (T,Λ), respectively, and $\mathbf{d}$ instead of (φ,d). Furthermore, to shorten terminology, we will say *signature* and *signature morphism* instead of *many-sorted signature* and *many-sorted signature morphism*, respectively.

The above definition, obviously, does not exclude that there may be other interesting types of signatures and signature morphisms, but rather, it delimits the ones we use in this article. Examples of signatures and of signature morphisms which fall under Definition 2.6 can be found, e.g., in [15].

From the definition of $\mathbf{Sig}$ it follows that the projection functor π_{Sig} from $\mathbf{Sig}$ to $\mathbf{Set}$ is a split bi-fibration.

Remark 2.7 The category $\mathbf{Sig}$ is complete and co-complete.

Since it will be used afterwards we introduce, for a signature $\mathbf{\Sigma}$, an S-sorted set A, an S-sorted mapping f from A to B, and a word w on S, i.e., an element w of $S^\star$, the following notation and terminology. We write $|w|$ for the length of the word w, A_w for $\prod_{i\in|w|} A_{w_i}$, and f_w for the mapping $\prod_{i\in|w|} f_{w_i}$ from A_w to B_w which sends $(a_i)_{i\in|w|}$ in A_w to $(f_{w_i}(a_i))_{i\in|w|}$ in B_w. Moreover, we let $\mathcal{O}_S(A)$ stand for the $S^\star\times S$-sorted set $(\mathrm{Hom}(A_w,A_s))_{(w,s)\in S^\star\times S}$ and we call it the $S^\star\times S$-sorted set of the *finitary operations on* A.

We proceed next to define the category $\mathbf{Alg}$ of many-sorted algebras. But before doing that we agree that, for an arbitrary but fixed signature $\mathbf{\Sigma}$, $\mathbf{Alg}(\mathbf{\Sigma})$ denotes the category of $\mathbf{\Sigma}$-algebras (and $\mathbf{\Sigma}$-homomorphisms). By a $\mathbf{\Sigma}$*-algebra* is meant a pair $\mathbf{A} = (A,F)$, where A is an S-sorted set and F a $\mathbf{\Sigma}$*-algebra structure on* A, i.e., a morphism $F = (F_{w,s})_{(w,s)\in S^\star\times S}$ in $\mathbf{Sig}(S)$ from Σ to $\mathcal{O}_S(A)$ (for a pair $(w,s)\in S^\star\times S$ and a $\sigma\in\Sigma_{w,s}$, to simplify notation we let F_σ stand for $F_{w,s}(\sigma)$). A $\mathbf{\Sigma}$*-homomorphism* from a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ to another $\mathbf{B} = (B,G)$, is a triple $(\mathbf{A},f,\mathbf{B})$, written as $f\colon \mathbf{A}\longrightarrow\mathbf{B}$, where f is an S-sorted mapping from A to B that preserves the structure in the sense that, for every (w,s) in $S^\star\times S$, every σ in $\Sigma_{w,s}$, and every $(a_i)_{i\in|w|}$ in A_w, it happens that $f_s(F_\sigma((a_i)_{i\in|w|})) = G_\sigma(f_w((a_i)_{i\in|w|}))$.

Proposition 2.8 *There exists a contravariant functor* Alg *from* $\mathbf{Sig}$ *to* $\mathbf{Cat}$ *which sends a signature* $\mathbf{\Sigma}$ *to* $\mathrm{Alg}(\mathbf{\Sigma}) = \mathbf{Alg}(\mathbf{\Sigma})$*, the category of* $\mathbf{\Sigma}$*-algebras, and a signature morphism* $\mathbf{d}$ *from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ *to the functor* $\mathrm{Alg}(\mathbf{d}) = \mathbf{d}^*\colon \mathbf{Alg}(\mathbf{\Lambda})\longrightarrow\mathbf{Alg}(\mathbf{\Sigma})$ *defined as follows:*

1. $\mathbf{d}^*$ *assigns to a* $\mathbf{\Lambda}$*-algebra* $\mathbf{B} = (B,G)$ *the* $\mathbf{\Sigma}$*-algebra* $\mathbf{d}^*(\mathbf{B}) = (B_\varphi, G^{\mathbf{d}})$*, where* $G^{\mathbf{d}}$ *is the composition of the* $S^\star\times S$*-sorted mappings* $d\colon\Sigma\longrightarrow\Lambda_{\varphi^\star\times\varphi}$ *and* $G_{\varphi^\star\times\varphi}\colon \Lambda_{\varphi^\star\times\varphi}\longrightarrow\mathcal{O}_T(B)_{\varphi^\star\times\varphi}$*. We agree that, for* $\sigma\in\Sigma_{w,s}$*,* $G_{d(\sigma)}\colon B_{\varphi^\star(w)}\longrightarrow B_{\varphi(s)}$ *denotes the value of* $G^{\mathbf{d}}$ *at* σ*.*

2. $\mathbf{d}^*$ *assigns to a* $\mathbf{\Lambda}$*-homomorphism* f *from* $\mathbf{B}$ *to* $\mathbf{B}'$ *the* $\mathbf{\Sigma}$*-homomorphism* $\mathbf{d}^*(f) = f_\varphi$ *from* $\mathbf{d}^*(\mathbf{B})$ *to* $\mathbf{d}^*(\mathbf{B}')$.

Proof. For every $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B,G)$ it is the case that G is a morphism from Λ to $\mathcal{O}_T(B)$. Then, by composing $d\colon\Sigma\longrightarrow\Lambda_{\varphi^\star\times\varphi}$ and $G_{\varphi^\star\times\varphi}\colon\Lambda_{\varphi^\star\times\varphi}\longrightarrow\mathcal{O}_T(B)_{\varphi^\star\times\varphi}$, and taking into account that $\mathcal{O}_T(B)_{\varphi^\star\times\varphi}$ is identical with $\mathcal{O}_S(B_\varphi)$, we infer that $G^{\mathbf{d}} = G_{\varphi^\star\times\varphi}\circ d$ is a $\mathbf{\Sigma}$-algebra structure on the S-sorted set B_φ.

On the other hand, for every $(w,s)\in S^\star\times S$ and every $\sigma\in\Sigma_{w,s}$, it happens that $d(\sigma)\in\Lambda_{\varphi^\star(w),\varphi(s)}$. Thus, f being a $\mathbf{\Lambda}$-homomorphism from (B,G) to (B',G'), we infer that $f_{\varphi(s)}\circ G_{d(\sigma)} = G'_{d(\sigma)}\circ f_{\varphi^\star(w)}$. Hence, since $G^{\mathbf{d}}_\sigma = G_{d(\sigma)}$ and $G'^{\mathbf{d}}_\sigma = G'_{d(\sigma)}$, we can assert that $(f_\varphi)_s\circ G^{\mathbf{d}}_\sigma = G'^{\mathbf{d}}_\sigma\circ(f_\varphi)_w$. Therefore f_φ is a $\mathbf{\Sigma}$-homomorphism from $(B_\varphi, G^{\mathbf{d}})$ to $(B'_\varphi, G'^{\mathbf{d}})$.

Since the identities and the composites are, obviously, preserved by $\mathbf{d}^*$, it follows that $\mathbf{d}^*$ is a functor from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$. □

By applying the EG-construction to Alg we get the category of many-sorted sets as stated in the following

Definition 2.9 The category $\mathbf{Alg}$, of *many-sorted algebras* and *many-sorted algebra homomorphisms*, is given by $\mathbf{Alg} = \int^{\mathbf{Sig}} \mathrm{Alg}$. Therefore the category $\mathbf{Alg}$ has as objects the pairs $(\mathbf{\Sigma}, \mathbf{A})$, where $\mathbf{\Sigma}$ is a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, and as morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the pairs $(\mathbf{d}, f)$, with $\mathbf{d}$ a signature morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*(\mathbf{B})$. Henceforth, to shorten terminology, we will say *algebra* and *algebra homomorphism*, or, simply, *homomorphism*, instead of *many-sorted algebra* and *many-sorted algebra homomorphism*, respectively.

From the definition of $\mathbf{Alg}$ it follows that the projection functor π_{Alg} from $\mathbf{Alg}$ to $\mathbf{Sig}$ is a fibration. Moreover, for every set of sorts S, the fiber of $\pi_{\mathrm{Sig,Alg}} = \pi_{\mathrm{Sig}} \circ \pi_{\mathrm{Alg}}$ at (S, id_S) is, essentially, the category $\mathbf{Alg}(S)$ with objects the pairs $(\Sigma, \mathbf{A})$, where Σ is an S-sorted signature and $\mathbf{A} = (A, F)$ a Σ-algebra, and morphisms from $(\Sigma, \mathbf{A})$ to $(\Lambda, \mathbf{B})$, where $\mathbf{B} = (B, G)$, the pairs (d, f), where d is an S-sorted signature morphism from Σ to Λ and f a Σ-homomorphism from $\mathbf{A}$ to $\mathbf{B}^d = (B, G \circ d)$.

Remark 2.10 The category $\mathbf{Alg}$ is concrete and univocally transportable relative to a "forgetful" G from $\mathbf{Alg}$ to the fibered product $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. In addition, the functor G has a left adjoint $\mathbf{T}\colon \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$, obtained from the family $(\mathbf{T}_{\mathbf{\Sigma}})_{\mathbf{\Sigma} \in \mathbf{Sig}}$, where, for a signature $\mathbf{\Sigma}$ in $\mathbf{Sig}$, the functor $\mathbf{T}_{\mathbf{\Sigma}}$ from $\mathbf{Set}^S$ to $\mathbf{Alg}(\mathbf{\Sigma})$ is the left adjoint to the forgetful functor $\mathrm{G}_{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Set}^S$. It is worth pointing out that the functor $\mathbf{T}$ transforms objects of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into labeled term algebras in $\mathbf{Alg}$ and morphisms of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into translators between the associated labeled term algebras in $\mathbf{Alg}$. Moreover, $\mathbf{Alg}$ is complete and, since, for every signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the functor $\mathbf{d}^*$, defined in Proposition 2.8, has a left adjoint $\mathbf{d}_*$, it is also co-complete. On the other hand, for every set of sorts S, the category $\mathbf{Alg}(S)$ is concrete and univocally transportable relative to the functor G_S from $\mathbf{Alg}(S)$ to $\mathbf{Sig}(S) \times \mathbf{Set}^S$, obtained, by the universal property of the product, from the forgetful functors $G_{\mathbf{Sig}(S)}$ and $G_{\mathbf{Set}^S}$ from the category $\mathbf{Alg}(S)$ to the categories $\mathbf{Sig}(S)$ and $\mathbf{Set}^S$, respectively. Besides, the functor G_S has a left adjoint.

3 Many-sorted closure spaces.

In this section we begin by associating with every set of sorts S the categories $\mathbf{ClSySp}(S)$, of S-closure system spaces, and $\mathbf{ClOpSp}(S)$, of S-closure operator spaces, and proving that both these categories are concretely isomorphic. This is why we will refer to them, simply, as the category $\mathbf{ClSp}(S)$, of S-closure spaces. Following this we define, for every set of sorts S, two full subcategories of $\mathbf{ClSp}(S)$, the category $\mathbf{AClSp}(S)$, of algebraic S-closure spaces, and the category $\mathbf{UAClSp}(S)$, of uniform algebraic S-closure spaces, which will be fundamental to state, in the fourth section, the Birkhoff-Frink representation theorems. Moreover, we state that $\mathbf{AClSp}(S)$ is a co-reflective subcategory of $\mathbf{ClSp}(S)$ and that $\mathbf{UAClSp}(S)$ is a co-reflective subcategory of $\mathbf{AClSp}(S)$. Afterwards, we prove that every mapping $\varphi\colon S \longrightarrow T$ determines an adjunction $\coprod_{\varphi}^{\mathrm{cl}} \dashv \Delta_{\varphi}^{\mathrm{cl}}$ from $\mathbf{ClSp}(S)$ to $\mathbf{ClSp}(T)$ and that it can be restricted both to one from $\mathbf{AClSp}(S)$ to $\mathbf{AClSp}(T)$ and to one from $\mathbf{UAClSp}(S)$ to $\mathbf{UAClSp}(T)$. Then, since the functors $\Delta_{\varphi}^{\mathrm{cl}}$ from $\mathbf{ClSp}(T)$ to $\mathbf{ClSp}(S)$, parameterized by the mappings φ in $\mathbf{Set}$, are the components of the morphism mapping of a contravariant functor Δ^{cl} from $\mathbf{Set}$ to $\mathbf{Cat}$, by applying the EG-construction to the contravariant functor Δ^{cl}, we get the category $\mathbf{MClSp}$, of many-sorted closure spaces and continuous mappings. Finally, from the definition of the category $\mathbf{MClSp}$ we state that the projection functor π_{MClSp} for $\mathbf{MClSp}$ is a split fibration and that, for every set S, the fiber of π_{MClSp} at (S, id_S) is, essentially, the category $\mathbf{ClSp}(S)$ of S-closure spaces and S-continuous mappings.

Definition 3.1 Let A be an S-sorted set and let $\mathrm{Sub}(A) = \{\, X \in \boldsymbol{\mathcal{U}}^S \mid X \subseteq A \,\}$ be the set of all *sub-S-sorted sets* of A, where $X \subseteq A$ means, in this context, that, for all $s \in S$, $X_s \subseteq A_s$.

1. An *S-closure system on* A is a subset $\mathcal{C}$ of $\mathrm{Sub}(A)$ such that $A \in \mathcal{C}$ and, for any $\mathcal{D} \subseteq \mathcal{C}$, if $\mathcal{D} \neq \varnothing$, then $\bigcap \mathcal{D} = (\bigcap_{D \in \mathcal{D}} D_s)_{s \in S} \in \mathcal{C}$. We denote by $\mathrm{ClSy}(A)$ the set of all S-closure systems on A and by $\mathbf{ClSy}(A)$ the same set but partially ordered by inclusion. We call the pairs of the form $(A, \mathcal{C})$, with $\mathcal{C} \in \mathrm{ClSy}(A)$, *$S$-closure system spaces*.

2. An *S-closure operator on A* is an operator J on $\mathrm{Sub}(A)$, i.e., a mapping J of $\mathrm{Sub}(A)$ into itself, which sends $X = (X_s)_{s\in S} \subseteq A$ to $J(X) = (J(X)_s)_{s\in S} \subseteq A$, with the properties
   (a) $X \subseteq J(X)$, i.e., J is *extensive*,
   (b) If $X \subseteq Y$, then $J(X) \subseteq J(Y)$, i.e., J is *isotone*,
   (c) $J(J(X)) = J(X)$, i.e., J is *idempotent*,

   for all $X, Y \subseteq A$. We denote by $\mathrm{ClOp}(A)$ the set of all S-closure operators on A and by $\mathbf{ClOp}(A)$ the same set but partially ordered by declaring $J \leq K$ to mean that, for every $X \subseteq A$, $J(X) \subseteq K(X)$. We call the pairs of the form (A, J), with $J \in \mathrm{ClOp}(A)$, *S-closure operator spaces*.

Example 3.2 For a $\mathbf{\Sigma}$-algebra $\mathbf{A}$, the set $\mathrm{Sub}(\mathbf{A})$, of all subalgebras of $\mathbf{A}$, is an S-closure system on the S-sorted set A, and the subalgebra generating operator $\mathrm{Sg}^S_{\mathbf{A}}$ on A induced by $\mathbf{A}$ is an S-closure operator on A.

Example 3.3 For a $\mathbf{\Sigma}$-algebra $\mathbf{A}$, the set $\mathrm{Cgr}(\mathbf{A})$, of all congruences on $\mathbf{A}$, is an S-closure system on the S-sorted set $A \times A = (A_s \times A_s)_{s\in S}$, and the congruence generating operator $\mathrm{Cg}^S_{\mathbf{A}}$ on $A \times A$ induced by $\mathbf{A}$ is an S-closure operator on $A \times A$.

As in the single-sorted case, also in the many-sorted case, for a set of sorts S, every S-closure system $\mathcal{C}$ on an S-sorted set A, when ordered by inclusion, determines a complete lattice $\boldsymbol{\mathcal{C}} = (\mathcal{C}, \subseteq)$. Moreover, the ordered sets $\mathbf{ClOp}(A)$ and $\mathbf{ClSy}(A)$ are complete lattices and dually isomorphic under the correspondence Fix^S from $\mathbf{ClOp}(A)$ to $\mathbf{ClSy}(A)$ which sends an S-closure operator J on A to $\mathrm{Fix}^S(J) = \{X \subseteq A \mid J(X) = X\}$, the S-closure system on A of all *fixed points of* J.

After having defined, for a set of sorts S, the S-closure system spaces and the S-closure operator spaces, we proceed next to define the suitable morphisms both between S-closure system spaces and between S-closure operator spaces.

Definition 3.4 Let S be a set of sorts, $(A, \mathcal{C})$, $(B, \mathcal{D})$ two S-closure system spaces, and (A, J), (B, K) two S-closure operator spaces.

1. An *S-continuous mapping* from $(A, \mathcal{C})$ to $(B, \mathcal{D})$ is a triple $((A, \mathcal{C}), f, (B, \mathcal{D}))$, which we shall write as $f\colon (A, \mathcal{C}) \longrightarrow (B, \mathcal{D})$, where f is an S-sorted mapping from A to B such that $f^{-1}[D] \in \mathcal{C}$, for all $D \in \mathcal{D}$.
2. An *S-continuous mapping* from (A, J) to (B, K) is a triple $((A, J), f, (B, K))$, which we shall write as $f\colon (A, J) \longrightarrow (B, K)$, where f is an S-sorted mapping from A to B such that $f[J(X)] \subseteq K(f[X])$, for all $X \subseteq A$.

Example 3.5 For a $\mathbf{\Sigma}$-homomorphism f from $\mathbf{A}$ to $\mathbf{B}$ and a sub-S-sorted X of A, it happens that $f[\mathrm{Sg}^S_{\mathbf{A}}(X)]$ is identical with $\mathrm{Sg}^S_{\mathbf{B}}(f[X])$. Therefore the $\mathbf{\Sigma}$-homomorphism f determines an S-continuous (and closed) mapping from $(A, \mathrm{Sg}^S_{\mathbf{A}})$ to $(B, \mathrm{Sg}^S_{\mathbf{B}})$.

Example 3.6 For a $\mathbf{\Sigma}$-homomorphism f from $\mathbf{A}$ to $\mathbf{B}$ and a congruence Ψ on $\mathbf{B}$, it happens that $(f\times f)^{-1}[\Psi]$ is a congruence on $\mathbf{A}$. Therefore from the $\mathbf{\Sigma}$-homomorphism f we get the S-continuous mapping $f \times f$ from $(A \times A, \mathrm{Cgr}(\mathbf{A}))$ to $(B \times B, \mathrm{Cgr}(\mathbf{B}))$.

Also as for the single-sorted case, for every set of sorts S, there exists, up to a concrete isomorphism, a category of S-closure spaces, with objects given by an S-sorted set and, alternatively, but equivalently, an S-closure system or an S-closure operator on it.

Proposition 3.7 *Let S be a set of sorts. Then*

1. *The S-closure system spaces together with the S-continuous mappings between them, as defined in the first part of Definition 3.4, constitute a category $\mathbf{ClSySp}(S)$. Furthermore, the forgetful functor from $\mathbf{ClSySp}(S)$ to $\mathbf{Set}^S$ which sends an S-continuous mapping f from $(A, \mathcal{C})$ to $(B, \mathcal{D})$ to the S-sorted mapping f from A to B is faithful. Therefore $\mathbf{ClSySp}(S)$ is a concrete category over $\mathbf{Set}^S$.*
2. *The S-closure operator spaces together with the S-continuous mappings between them, as defined in the second part of Definition 3.4, constitute a category $\mathbf{ClOpSp}(S)$. Furthermore, the forgetful functor from $\mathbf{ClOpSp}(S)$ to $\mathbf{Set}^S$ which sends an S-continuous mapping f from (A, J) to (B, K) to the S-sorted mapping f from A to B is faithful. Therefore $\mathbf{ClOpSp}(S)$ is a concrete category over $\mathbf{Set}^S$.*

3. *The categories* $\mathbf{ClOpSp}(S)$ *and* $\mathbf{ClSySp}(S)$ *are concretely isomorphic. The functor that associates with each* S*-continuous mapping* f *from* (A, J) *to* (B, K) *the* S*-continuous mapping* f *from* $(A, \mathrm{Fix}^S(J))$ *to* $(B, \mathrm{Fix}^S(K))$ *is a concrete isomorphism.*

Henceforth, in virtue of the third part of Proposition 3.7, for a set of sorts S, by the category of S-closure spaces, denoted by $\mathbf{ClSp}(S)$, we will refer, indistinctly, to any of the categories $\mathbf{ClSySp}(S)$ or $\mathbf{ClOpSp}(S)$. However, it is useful to keep in mind, for a set of sorts S, both categories, because some of the properties which can be attributed to an S-closure space, like that of uniformity (which was defined in [8] and recalled immediately below), are much more easily stated in terms of the objects of one of them than in terms of the objects of the other.

Remark 3.8 As for the single-sorted case, for every set of sorts S, the forgetful functor from the category $\mathbf{ClSp}(S)$ to the category $\mathbf{Set}^S$ has both a left and a right adjoint and constructs limits and colimits (see [23], p. 149, for the definition of when a functor *constructs limits* and *colimits*).

Next we define, for a set of sorts S and an S-sorted set A, the concepts of: algebraic S-closure operator on A, support of A, uniformity for an operator on A, uniform S-closure operator on A, and uniform algebraic S-closure operator on A.

Definition 3.9 Let A be an S-sorted set. An S-closure operator J on A is said to be *algebraic* if, for every $X \subseteq A$, $J(X) = \bigcup_{Z \in \mathrm{Sub}_{\mathrm{f}}(X)} J(Z)$, where $\mathrm{Sub}_{\mathrm{f}}(X)$ is the set of all sub-S-sorted sets Z of X which are *finite*, i.e., such that $\mathrm{card}(\coprod_{s \in S} Z_s) < \aleph_0$. We let $\mathbf{AClSp}(S)$ stand for the full subcategory of $\mathbf{ClSp}(S)$ determined by the S-closure spaces (A, J) for which J is algebraic.

Notice that, for an S-sorted set A, the following conditions are equivalent: (1) A is finite, (2) A is finitary, and (3) A is strongly finitary (see [24], Lemma 3.5, p. 26, for a proof, and [19], Exercise 22E, p. 155, for the definition of *finitary* and of *strongly finitary*).

The classical characterization of algebraic closure operators, due to Schmidt [27], is also valid for many-sorted algebraic closure operators. That is, for an S-closure operator J on an S-sorted set A, the following conditions are equivalent: (1) J is algebraic, (2) for every nonempty directed family $(X^i)_{i \in I}$ in $\mathrm{Sub}(A)$, it happens that $J(\bigcup_{i \in I} X^i) = \bigcup_{i \in I} J(X^i)$, where, for a family of S-sorted sets $(Z^i)_{i \in I}$, $\bigcup_{i \in I} Z^i = (\bigcup_{i \in I} Z^i_s)_{s \in S}$.

Example 3.10 The set $\mathrm{Sub}(\mathbf{A})$ is an algebraic S-closure system on the S-sorted set A, and $\mathrm{Sg}^S_{\mathbf{A}}$ an algebraic S-closure operator on A.

Example 3.11 The set $\mathrm{Cgr}(\mathbf{A})$ is an algebraic S-closure system on the S-sorted set $A \times A$, and $\mathrm{Cg}^S_{\mathbf{A}}$ an algebraic S-closure operator on $A \times A$.

Definition 3.12 Let A be an S-sorted set. The *support of* A, from now on denoted by $\mathrm{supp}(A)$, is the subset of S defined as $\mathrm{supp}(A) = \{s \in S \mid A_s \neq \varnothing\}$.

Definition 3.13 Let A be an S-sorted set. An operator J on $\mathrm{Sub}(A)$ is said to be *uniform* if, for all $X, Y \subseteq A$, if $\mathrm{supp}(X) = \mathrm{supp}(Y)$, then $\mathrm{supp}(J(X)) = \mathrm{supp}(J(Y))$. We let $\mathbf{UClSp}(S)$ stand for the full subcategory of $\mathbf{ClSp}(S)$ determined by the S-closure spaces (A, J) for which J is uniform. On the other hand, we say that J is a *uniform algebraic* S*-closure operator on* A if J is an algebraic S-closure operator on A and J is uniform. We let $\mathbf{UAClSp}(S)$ stand for the full subcategory of $\mathbf{AClSp}(S)$ determined by the algebraic S-closure spaces (A, J) for which J is uniform.

Remark 3.14 Unlike what it happens in the many-sorted case, the concept of uniformity does not play any role in the single-sorted one, since in it every operator on a set is always uniform, hence $\mathbf{UAClSp}(1) = \mathbf{AClSp}(1)$.

Remark 3.15 For a $\boldsymbol{\Sigma}$-algebra $\mathbf{A}$, as we will show in the fourth section, it happens that $\mathrm{Sg}^S_{\mathbf{A}}$ is a uniform algebraic S-closure operator on A.

By taking the dual of Exercise 5 in [10], pp. 46–47, which Cohn attributes to G. Higman, given a set A and a closure operator J on A, there exists a greatest algebraic closure operator J_{f} on A contained in J. We prove next the many-sorted counterpart of this result and, in addition, that if J is uniform, then J_{f} is also uniform.

Proposition 3.16 *Let* A *be an* S*-sorted set and* J *an* S*-closure operator on* A*. Then* J_{f}*, where, for every* $X \subseteq A$, $J_{\mathrm{f}}(X) = \bigcup_{Y \in \mathrm{Sub}_{\mathrm{f}}(X)} J(Y)$*, is the greatest algebraic* S*-closure operator on* A *such that* $J_{\mathrm{f}} \leq J$*. Moreover, if* J *is uniform, then* J_{f} *is the greatest uniform algebraic* S*-closure operator on* A *such that* $J_{\mathrm{f}} \leq J$*.*

P r o o f. We restrict ourselves to prove that J_{f} is uniform since the proof of the first part of the proposition is formally identical with that corresponding to the single-sorted case. Let $X, Y \subseteq A$ be such that $\mathrm{supp}(X) = \mathrm{supp}(Y)$ and $s \in \mathrm{supp}(J_{\mathrm{f}}(X))$. Then s belongs to the support of the union (and, therefore, to the union of the supports) of the closures under J of the finite sub-S-sorted sets of X. Hence there exists a $Z \in \mathrm{Sub}_{\mathrm{f}}(X)$ such that $s \in \mathrm{supp}(J(Z))$. But, because Z is finite, there exists a $Z' \in \mathrm{Sub}_{\mathrm{f}}(Y)$ such that $\mathrm{supp}(Z) = \mathrm{supp}(Z')$ and, since J is uniform, $s \in \mathrm{supp}(J(Z'))$, thus $s \in \mathrm{supp}(J_{\mathrm{f}}(Y))$. □

Following this we prove, for a set of sorts S, that there are adjoint situations between the categories $\mathbf{ClSp}(S)$, $\mathbf{AClSp}(S)$, and $\mathbf{UAClSp}(S)$.

Proposition 3.17 *Let S be a set of sorts. Then $\mathbf{AClSp}(S)$ is a co-reflective subcategory of $\mathbf{ClSp}(S)$, i.e., there exists a functor from $\mathbf{ClSp}(S)$ to $\mathbf{AClSp}(S)$ right adjoint for the canonical embedding of $\mathbf{AClSp}(S)$ in $\mathbf{ClSp}(S)$. Moreover, $\mathbf{UAClSp}(S)$ is a co-reflective subcategory of $\mathbf{AClSp}(S)$, and therefore also of $\mathbf{ClSp}(S)$.*

P r o o f. We restrict ourselves to prove the first part of the proposition, since the second one follows immediately from it. Let (A, J) be an S-closure space. Then the algebraic S-closure space (A, J_{f}), together with the morphism from (A, J_{f}) to (A, J) determined by id_A, has the property that, for every algebraic S-closure space (B, K) and every morphism $f\colon (B, K) \longrightarrow (A, J)$, there exists a unique morphism $f^\flat\colon (B, K) \longrightarrow (A, J_{\mathrm{f}})$ such that $\mathrm{id}_A \circ f^\flat = f$. In fact, if $f\colon (B, K) \longrightarrow (A, J)$ is a morphism, then $f^\flat = ((B, K), f, (A, J_{\mathrm{f}}))$ is a morphism, since, for every $X \subseteq B$, every $s \in S$, and every $a \in K(X)_s$, it happens that $a \in K(F)_s$, for some $F \in \mathrm{Sub}_{\mathrm{f}}(X)$, because K is algebraic, from which we infer that $f_s(a) \in J(f[F])_s$, thus $f_s(a) \in J_{\mathrm{f}}(f[X])_s$. □

Our next goal is to prove that every mapping $\varphi\colon S \longrightarrow T$ determines an adjunction $\coprod^{\mathrm{cl}}_\varphi \dashv \Delta^{\mathrm{cl}}_\varphi$ from $\mathbf{ClSp}(S)$ to $\mathbf{ClSp}(T)$. In order to get such a proof it is convenient to introduce the following notational conventions. For a T-sorted set B and a subset $\mathcal{D}$ of $\mathrm{Sub}(B)$, let $\Delta_\varphi[\mathcal{D}]$ denote the subset $\{\, D_\varphi \mid D \in \mathcal{D} \,\}$ of $\mathrm{Sub}(B_\varphi)$, and, for an S-sorted set A and a subset $\mathcal{C}$ of $\mathrm{Sub}(A)$, let $\coprod_\varphi[\mathcal{C}]$ denote the subset $\{\, \coprod_\varphi C \mid C \in \mathcal{C} \,\}$ of $\mathrm{Sub}(\coprod_\varphi A)$.

Proposition 3.18 *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then from $\mathbf{ClSp}(T)$ to $\mathbf{ClSp}(S)$ there exists a functor $\Delta^{\mathrm{cl}}_\varphi$ defined as follows:*

1. *$\Delta^{\mathrm{cl}}_\varphi$ sends $(B, \mathcal{D})$ in $\mathbf{ClSp}(T)$ to $(B_\varphi, \Delta_\varphi[\mathcal{D}])$ in $\mathbf{ClSp}(S)$.*
2. *$\Delta^{\mathrm{cl}}_\varphi$ sends a T-continuous mapping f from $(B, \mathcal{D})$ to $(B', \mathcal{D}')$ to the S-continuous mapping f_φ from $(B_\varphi, \Delta_\varphi[\mathcal{D}])$ to $(B'_\varphi, \Delta_\varphi[\mathcal{D}'])$.*

P r o o f. Let $\mathcal{D}$ be a T-closure system on B, then $\Delta_\varphi[\mathcal{D}]$ is an S-closure system on B_φ, since, for every nonempty family $(Y^i)_{i\in I}$ of T-sorted sets, it happens that $(\bigcap_{i\in I} Y^i)_\varphi = \bigcap_{i\in I} Y^i_\varphi$. Moreover, if f is a T-continuous mapping from $(B, \mathcal{D})$ to $(B', \mathcal{D}')$ and $Y'_\varphi \in \Delta_\varphi[\mathcal{D}']$, then $Y' \in \mathcal{D}'$ and $f^{-1}[Y'] \in \mathcal{D}$, thus $\Delta_\varphi(f^{-1}[Y']) \in \Delta_\varphi[\mathcal{D}]$. But $\Delta_\varphi(f^{-1}[Y'])$ is identical with $(\Delta_\varphi(f))^{-1}[Y'_\varphi]$, therefore f_φ is an S-continuous mapping. □

Proposition 3.19 *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then from $\mathbf{ClSp}(S)$ to $\mathbf{ClSp}(T)$ there exists a functor $\coprod^{\mathrm{cl}}_\varphi$ defined as follows:*

1. *$\coprod^{\mathrm{cl}}_\varphi$ sends $(A, \mathcal{C})$ in $\mathbf{ClSp}(S)$ to $(\coprod_\varphi A, \coprod_\varphi[\mathcal{C}])$ in $\mathbf{ClSp}(T)$.*
2. *$\coprod^{\mathrm{cl}}_\varphi$ sends an S-continuous mapping f from $(A, \mathcal{C})$ to $(A', \mathcal{C}')$ to the T-continuous mapping $\coprod_\varphi f$ from $(\coprod_\varphi A, \coprod_\varphi[\mathcal{C}])$ to $(\coprod_\varphi A', \coprod_\varphi[\mathcal{C}'])$.*

P r o o f. Let $\mathcal{C}$ be an S-closure system on A, then $\coprod_\varphi[\mathcal{C}]$ is a T-closure system on $\coprod_\varphi A$, since, for every nonempty family $(X^i)_{i\in I}$ of S-sorted sets, it happens that $\coprod_\varphi \bigcap_{i\in I} X^i = \bigcap_{i\in I} \coprod_\varphi X^i$. Moreover, if f is an S-continuous mapping from $(A, \mathcal{C})$ to $(A', \mathcal{C}')$ and $\coprod_\varphi X \in \coprod_\varphi[\mathcal{C}']$, then $X' \in \mathcal{C}'$ and $f^{-1}[X'] \in \mathcal{C}$, consequently $\coprod_\varphi f^{-1}[X'] \in \coprod_\varphi[\mathcal{C}]$. But $\coprod_\varphi f^{-1}[X']$ is identical with $(\coprod_\varphi f)^{-1}[\coprod_\varphi X']$, therefore $\coprod_\varphi f$ is a T-continuous mapping. □

Proposition 3.20 *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then the functor $\coprod^{\mathrm{cl}}_\varphi$ is a left adjoint for the functor $\Delta^{\mathrm{cl}}_\varphi$.*

P r o o f. The natural isomorphism θ^{φ} of the adjunction $\coprod_{\varphi} \dashv \Delta_{\varphi}$, stated in Proposition 2.1, also happens to be a natural isomorphism

$$\mathrm{Hom}((A,\mathcal{C}),(B_{\varphi},\Delta_{\varphi}[\mathcal{D}])) \cong \mathrm{Hom}((\coprod_{\varphi} A, \coprod_{\varphi}[\mathcal{C}]),(B,\mathcal{D})),$$

for every $(A,\mathcal{C})$ in $\mathbf{ClSp}(S)$ and every $(B,\mathcal{D})$ in $\mathbf{ClSp}(T)$.

Let f be an S-continuous mapping from $(A,\mathcal{C})$ to $(B_{\varphi},\Delta_{\varphi}[\mathcal{D}])$, and Y an element of $\mathcal{D}$. Since $Y_{\varphi} \in \Delta_{\varphi}[\mathcal{D}]$ and f is continuous, we infer that $f^{-1}[Y_{\varphi}] \in \mathcal{C}$ and $\coprod_{\varphi} f^{-1}[Y_{\varphi}] \in \coprod_{\varphi}[\mathcal{C}]$. But $\coprod_{\varphi} f^{-1}[Y_{\varphi}]$ is identical with $((\theta^{\varphi})^{-1}(f))^{-1}[Y]$, because

$$\begin{aligned} ((\theta^{\varphi})^{-1}(f))^{-1}[Y] &= (\{(a,s) \in (\textstyle\coprod_{\varphi} A)_t \mid a \in A_s,\ \varphi(s)=t,\ f_s(a) \in Y_t\})_{t\in T} \\ &= (\{(a,s) \in (\textstyle\coprod_{\varphi} A)_t \mid a \in f^{-1}[Y_{\varphi}]_s,\ \varphi(s)=t\})_{t\in T} \\ &= (\textstyle\coprod_{s\in\varphi^{-1}[t]} f^{-1}[Y_{\varphi}]_s)_{t\in T} \\ &= \textstyle\coprod_{\varphi} f^{-1}[Y_{\varphi}], \end{aligned}$$

where, we recall, for $t \in T$, $(\coprod_{\varphi} A)_t = \coprod_{s\in\varphi^{-1}[t]} A_s$. Therefore $(\theta^{\varphi})^{-1}(f)$ is a T-continuous mapping.

Reciprocally, let us suppose that g is a T-continuous mapping from $(\coprod_{\varphi} A, \coprod_{\varphi}[\mathcal{C}])$ to $(B,\mathcal{D})$. Let Y_{φ} be an element of $\Delta_{\varphi}[\mathcal{D}]$, then $Y \in \mathcal{D}$ and $g^{-1}[Y] \in \coprod_{\varphi}[\mathcal{C}]$. But it happens that

$$\begin{aligned} g^{-1}[Y] &= (\{(a,s) \in (\textstyle\coprod_{\varphi} A)_t \mid g_t(a,s) \in Y_t\})_{t\in T} \\ &= (\textstyle\coprod_{s\in\varphi^{-1}[t]}\{a \in A_s \mid g_{\varphi(s)}(a,s) \in Y_{\varphi(s)}\})_{t\in T} \\ &= \textstyle\coprod_{\varphi}((\{a \in A_s \mid g_{\varphi(s)}(a,s) \in Y_{\varphi(s)}\})_{s\in S}), \end{aligned}$$

and, additionally, that

$$\begin{aligned} (\{a \in A_s \mid g_{\varphi(s)}(a,s) \in Y_{\varphi(s)}\})_{s\in S} &= (\{a \in A_s \mid \theta^{\varphi}(g)_s(a) \in Y_{\varphi(s)}\})_{s\in S} \\ &= (\theta^{\varphi}(g))^{-1}[Y_{\varphi}], \end{aligned}$$

thus $g^{-1}[Y] = \coprod_{\varphi}(\theta^{\varphi}(g))^{-1}[Y_{\varphi}]$, therefore $(\theta^{\varphi}(g))^{-1}[Y_{\varphi}] \in \mathcal{C}$ and $\theta^{\varphi}(g)$ is an S-continuous mapping. □

Now we prove that, for a mapping $\varphi\colon S \longrightarrow T$, the adjunction $\coprod^{\mathrm{cl}}_{\varphi} \dashv \Delta^{\mathrm{cl}}_{\varphi}$ can be restricted both to one from $\mathbf{AClSp}(S)$ to $\mathbf{AClSp}(T)$ and to one from $\mathbf{UAClSp}(S)$ to $\mathbf{UAClSp}(T)$.

Proposition 3.21 *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then*

1. *If $(B,\mathcal{D})$ is an algebraic T-closure space, then $\Delta^{\mathrm{cl}}_{\varphi}(B,\mathcal{D})$ is an algebraic S-closure space and if $(A,\mathcal{C})$ is an algebraic S-closure space, then $\coprod^{\mathrm{cl}}_{\varphi}(A,\mathcal{C})$ is an algebraic T-closure space. We agree to denote by $\coprod^{\mathrm{acl}}_{\varphi} \dashv \Delta^{\mathrm{acl}}_{\varphi}$ the corresponding adjunction from $\mathbf{AClSp}(S)$ to $\mathbf{AClSp}(T)$.*

2. *If (B,K) is a uniform algebraic T-closure space, then $\Delta^{\mathrm{cl}}_{\varphi}(B,K)$ is a uniform algebraic S-closure space and if (A,J) is a uniform algebraic S-closure space and φ is injective, then $\coprod^{\mathrm{cl}}_{\varphi}(A,J)$ is a uniform algebraic T-closure space. We agree to denote by $\coprod^{\mathrm{uacl}}_{\varphi} \dashv \Delta^{\mathrm{uacl}}_{\varphi}$ the corresponding adjunction from $\mathbf{UAClSp}(S)$ to $\mathbf{UAClSp}(T)$.*

P r o o f. Let $(B,\mathcal{D})$ be an algebraic T-closure space. Then $\Delta^{\mathrm{cl}}_{\varphi}(B,\mathcal{D})$ is an algebraic S-closure space, since, for every nonempty family $(D^i_{\varphi})_{i\in I}$ in $\Delta_{\varphi}[\mathcal{D}]$, directed or not, it happens that $(\bigcup_{i\in I} D^i)_{\varphi} = \bigcup_{i\in I} D^i_{\varphi}$.

Let $(A,\mathcal{C})$ be an algebraic S-closure space. Then $\coprod^{\mathrm{cl}}_{\varphi}(A,\mathcal{C})$ is an algebraic T-closure space, since, for every nonempty family $(C^i_{\varphi})_{i\in I}$ in $\coprod_{\varphi}[\mathcal{C}]$, directed or not, it happens that $\bigcup_{i\in I}(\coprod_{\varphi}(D^i)) = \coprod_{\varphi}(\bigcup_{i\in I} D^i)$.

Let (B,K) be a uniform algebraic T-closure space. Then $\Delta^{\mathrm{cl}}_{\varphi}(B,K)$ is a uniform algebraic S-closure space. In fact, for every $Y, Z \subseteq B_{\varphi}$, if $\mathrm{supp}(Y) = \mathrm{supp}(Z)$, then $\mathrm{supp}(\coprod_{\varphi} Y) = \mathrm{supp}(\coprod_{\varphi} Z)$, hence, since K is uniform, we infer that $\mathrm{supp}(K(\coprod_{\varphi} Y)) = \mathrm{supp}(K(\coprod_{\varphi} Z))$, thus $\mathrm{supp}(K(\coprod_{\varphi} Y)_{\varphi}) = \mathrm{supp}(K(\coprod_{\varphi} Z)_{\varphi})$.

Let (A,J) be a uniform algebraic S-closure space. If $\varphi\colon S \longrightarrow T$ is injective, then $\coprod_{\varphi}^{\mathrm{cl}}(A,J)$ is a uniform algebraic T-closure space. In fact, for every $\coprod_{\varphi} Y, \coprod_{\varphi} Z \subseteq \coprod_{\varphi} A$, if $\mathrm{supp}(\coprod_{\varphi} Y) = \mathrm{supp}(\coprod_{\varphi} Z)$, then, by the injectivity of φ, $\mathrm{supp}(Y) = \mathrm{supp}(Z)$, hence, since J is uniform, $\mathrm{supp}(J(Y)) = \mathrm{supp}(J(Z))$, therefore $\mathrm{supp}(\coprod_{\varphi} J(Y)) = \mathrm{supp}(\coprod_{\varphi} J(Z))$, i.e., $\mathrm{supp}(J_{\varphi}(\coprod_{\varphi} Y)) = \mathrm{supp}(J_{\varphi}(\coprod_{\varphi} Z))$. □

The functors $\Delta_{\varphi}^{\mathrm{cl}}$ and $\coprod_{\varphi}^{\mathrm{cl}}$ can, evidently, also be defined for T-closure and S-closure operators, respectively. Actually, the definition of the functor $\coprod_{\varphi}^{\mathrm{cl}}$ for S-closure operators, as shown in the following proposition, is immediate.

Proposition 3.22 *Given a mapping $\varphi\colon S \longrightarrow T$ and an S-closure space (A,J), the pair $(\coprod_{\varphi} A, J_{\varphi})$ is a T-closure space, where the operator J_{φ} on $\coprod_{\varphi} A$ assigns to $\coprod_{\varphi} X$, for any $X \subseteq A$, the T-sorted set $\coprod_{\varphi} J(X)$.*

Proof. The definition of the operator J_{φ} is sound, since $\mathrm{Sub}(A) \cong \mathrm{Sub}(\coprod_{\varphi} A)$ and $\mathrm{Sub}(\coprod_{\varphi} A)$ is precisely $\coprod_{\varphi}[\mathrm{Sub}(A)]$. □

However, the corresponding definition of the functor $\Delta_{\varphi}^{\mathrm{cl}}$ for T-closure operators is more involved, since, for a T-sorted set B, we only have, in general, that $\Delta_{\varphi}[\mathrm{Sub}(B)] \subseteq \mathrm{Sub}(B_{\varphi})$.

Proposition 3.23 *Given a mapping $\varphi\colon S \longrightarrow T$ and a T-closure space (B,K), the pair $(B_{\varphi}, K_{\varphi})$ is an S-closure space, where the operator K_{φ} on B_{φ} is defined as follows:*

$$K_{\varphi}\begin{cases} \mathrm{Sub}(B_{\varphi}) \longrightarrow \mathrm{Sub}(B_{\varphi}) \\ \quad Y \longmapsto K((\bigcup_{s\in\varphi^{-1}[t]} Y_s)_{t\in T})_{\varphi} \end{cases}$$

Proof. The definition of the operator K_{φ} as the composite $\Delta_{\varphi,B} \circ K \circ \bigcup_{\varphi,B}$ is sound since the mapping $\bigcup_{\varphi,B}$ from $\mathrm{Sub}(B_{\varphi})$ to $\mathrm{Sub}(B)$, which sends a subset Y of B_{φ} to the subset $(\bigcup_{s\in\varphi^{-1}[t]} Y_s)_{t\in T}$ of B, is isotone and has the mapping $\Delta_{\varphi,B}$ from $\mathrm{Sub}(B)$ to $\mathrm{Sub}(B_{\varphi})$, which sends a subset X of B to the subset X_{φ} of B_{φ}, as a right adjoint. □

The functors $\Delta_{\varphi}^{\mathrm{cl}}\colon \mathbf{ClSp}(T) \longrightarrow \mathbf{ClSp}(S)$, parameterized by the mappings φ in $\mathbf{Set}$, are the components of the morphism mapping of a contravariant functor Δ^{cl} from $\mathbf{Set}$ to $\mathbf{Cat}$.

Proposition 3.24 *There exists a contravariant functor Δ^{cl} from $\mathbf{Set}$ to $\mathbf{Cat}$ which sends a set S to $\Delta^{\mathrm{cl}}(S) = \mathbf{ClSp}(S)$, the category of S-closure spaces, and a mapping $\varphi\colon S \longrightarrow T$ to the functor $\Delta_{\varphi}^{\mathrm{cl}}$ from $\mathbf{ClSp}(T)$ to $\mathbf{ClSp}(S)$ defined as follows:*

1. *$\Delta_{\varphi}^{\mathrm{cl}}$ assigns to a T-closure space $(B,\mathcal{D})$ the S-closure space $(B_{\varphi}, \Delta_{\varphi}[\mathcal{D}])$.*
2. *$\Delta_{\varphi}^{\mathrm{cl}}$ assigns to a T-continuous mapping f from $(B,\mathcal{D})$ to $(B',\mathcal{D}')$ the S-continuous mapping f_{φ} from $(B_{\varphi}, \Delta_{\varphi}[\mathcal{D}])$ to $(B'_{\varphi}, \Delta_{\varphi}[\mathcal{D}'])$.*

Remark 3.25 The functors $\coprod_{\varphi}^{\mathrm{cl}}\colon \mathbf{ClSp}(S) \longrightarrow \mathbf{ClSp}(T)$, parameterized by the mappings φ in $\mathbf{Set}$, are the components of the morphism mapping of a of a pseudo-functor $\coprod^{\mathrm{cl}}$ from $\mathbf{Set}$ to $\mathbf{Cat}$.

By applying the EG-construction to Δ^{cl} we get the category of many-sorted closure spaces as stated in the following

Definition 3.26 The category $\mathbf{MClSp}$, of *many-sorted closure spaces and continuous mappings*, is given by $\mathbf{MClSp} = \int^{\mathbf{Set}} \Delta^{\mathrm{cl}}$. Therefore $\mathbf{MClSp}$ has as objects the triples $(S,A,\mathcal{C})$, where S is a set and $(A,\mathcal{C})$ an S-closure space, and as morphisms from $(S,A,\mathcal{C})$ to $(T,B,\mathcal{D})$ the triples $((S,A,\mathcal{C}),(\varphi,f),(T,B,\mathcal{D}))$, which we agree to denote by $(\varphi,f)\colon (S,A,\mathcal{C}) \longrightarrow (T,B,\mathcal{D})$, where φ is a mapping from S to T and f an S-continuous mapping from $(A,\mathcal{C})$ to $(B_{\varphi}, \Delta_{\varphi}[\mathcal{D}])$. Henceforth, to shorten terminology, we will say *closure space* and *continuous mapping*, instead of *many-sorted closure space* and *many-sorted continuous mapping*, respectively, when this is unlikely to cause confusion.

From the definition of $\mathbf{MClSp}$ it follows that the projection functor π_{MClSp} for $\mathbf{MClSp}$ is a split fibration. Moreover, for every set S, the fiber of π_{MClSp} at (S, id_S) is, essentially, the category $\mathbf{ClSp}(S)$.

Remark 3.27 The forgetful functor from the category $\mathbf{MClSp}$ to the category $\mathbf{MSet}$ has left and right adjoints and constructs limits and colimits, exactly as it happens for the forgetful functor from the category $\mathbf{ClSp}(S)$ to the category $\mathbf{Set}^S$ (as indicated in Remark 3.8).

4 Functorization of the Birkhoff-Frink representation theorems.

Our main aim in this last section is to provide the functorial version of the Birkhoff-Frink representation theorems both for single-sorted algebras and for many-sorted algebras, by defining the appropriate categories and functors, covariant and contravariant, involved in the process. The categories which are necessary for it are of three types. On the one hand, we have the categories $\mathbf{Alg}(S)$, of S-algebras and S-homomorphisms, which are, essentially, the fibers, at the pairs (S, id_S), of the fibration $\pi_{\mathrm{Sig},\mathrm{Alg}}$ from $\mathbf{Alg}$ to $\mathbf{Set}$, as stated after Definition 2.9. In particular, the category $\mathbf{Alg}(1)$, i.e., the fiber of $\pi_{\mathrm{Sig},\mathrm{Alg}}$ at $(1, \mathrm{id}_1)$, where 1 is the standard final set, has

1. As objects, essentially, the pairs $(\Sigma, \mathbf{A})$, where $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ is a single-sorted signature, i.e., an object of $\mathbf{Set}^{\mathbb{N}}$, and $\mathbf{A} = (A, F)$ a Σ-algebra, i.e., an ordinary set A together with an $\mathbb{N}$-sorted mapping F from Σ to $(\mathrm{Hom}(A^n, A))_{n\in\mathbb{N}}$, and
2. As morphisms from $(\Sigma, \mathbf{A})$ to $(\Lambda, \mathbf{B})$, where $\mathbf{B} = (B, G)$, the pairs (d, f) with $d = (d_n)_{n\in\mathbb{N}}$ an $\mathbb{N}$-sorted mapping from Σ to Λ in $\mathbf{Set}^{\mathbb{N}}$, and f a Σ-homomorphism from $\mathbf{A} = (A, F)$ to $\mathbf{B}^d = (B, G \circ d)$.

On the other hand, we have the categories $\mathbf{AClSp}(S)$, of algebraic S-closure spaces and S-continuous mappings, and $\mathbf{UAClSp}(S)$, of uniform algebraic S-closure spaces and S-continuous mappings, which are subcategories of the fibers $\mathbf{ClSp}(S)$, at the pairs (S, id_S), of the split fibration π_{MClSp} from $\mathbf{MClSp}$ to $\mathbf{Set}$. In particular, the category $\mathbf{AClSp}(1)$, i.e., the fiber of π_{MClSp} at $(1, \mathrm{id}_1)$, is, essentially, the category $\mathbf{AClSp}$, of algebraic closure spaces and continuous mappings. And, finally, we have the category $\mathbf{ALat}$ which is the subcategory of the category $\mathbf{CLat}_{\bigwedge}$, of complete lattices and lattice morphisms which preserve arbitrary meets, determined by the algebraic lattices and the morphisms which, in addition, preserve directed joins.

The functors involved in the Birkhoff-Frink representation theorems are of two types, both parameterized by sets (of sorts) S in $\mathbf{Set}$. On the one hand, we have the covariant functors Sg^S from $\mathbf{Alg}(S)$ to $\mathbf{UAClSp}(S)$, which assign to an S-algebra $(\Sigma, \mathbf{A})$ the uniform algebraic S-closure space $(A, \mathrm{Sg}^S_{\mathbf{A}})$, where, we recall, $\mathrm{Sg}^S_{\mathbf{A}}$ is the subalgebra generating operator on A induced by $\mathbf{A}$. On the other hand, we have the contravariant functors $\mathbf{Fix}^S$ from $\mathbf{UAClSp}(S)$ to $\mathbf{ALat}$, which assign to a uniform algebraic S-closure space (A, J) the complete lattice $\mathbf{Fix}^S(J)$ of the fixed points of J.

We begin by stating the functorial version of the first classical Birkhoff-Frink representation theorem. But before doing that we introduce, for a single-sorted signature Σ and a Σ-algebra $\mathbf{A}$, the following notation. We will write $\mathrm{E}_{\mathbf{A}}$ to denote the operator on $\mathrm{Sub}(A)$ which sends a subset X of A precisely to $X \cup (\bigcup_{\substack{n\in\mathbb{N}\\ \sigma\in\Sigma_n}} F_\sigma[X^n])$. We will also use $(\mathrm{E}^n_{\mathbf{A}}(X))_{n\in\mathbb{N}}$ to denote the family in $\mathrm{Sub}(A)$ specified as: $\mathrm{E}^0_{\mathbf{A}}(X) = X$, and, for all $n \in \mathbb{N}$, $\mathrm{E}^{n+1}_{\mathbf{A}}(X) = \mathrm{E}_{\mathbf{A}}(\mathrm{E}^n_{\mathbf{A}}(X))$, and $\mathrm{E}^\omega_{\mathbf{A}}(X)$ as an alternate notation to $\bigcup_{n\in\mathbb{N}} \mathrm{E}^n_{\mathbf{A}}(X)$.

Proposition 4.1 *There exists a functor* Sg *from* $\mathbf{Alg}(1)$ *to* $\mathbf{AClSp}$ *which is surjective on the objects.*

P r o o f. It suffices to take as Sg the functor from $\mathbf{Alg}(1)$ to $\mathbf{AClSp}$ which sends an algebra $(\Sigma, \mathbf{A})$ to the algebraic closure space $(A, \mathrm{Sg}_{\mathbf{A}})$, where $\mathrm{Sg}_{\mathbf{A}}$ is the subalgebra generating operator on A induced by $\mathbf{A}$; and a morphism (d, f) from $(\Sigma, \mathbf{A})$ to $(\Lambda, \mathbf{B})$ in $\mathbf{Alg}(1)$ to the morphism f from $(A, \mathrm{Sg}_{\mathbf{A}})$ to $(B, \mathrm{Sg}_{\mathbf{B}})$ in $\mathbf{AClSp}$.

The action of Sg on the morphisms is well defined, i.e., for every $X \subseteq A$, $f[\mathrm{Sg}_{\mathbf{A}}(X)] \subseteq \mathrm{Sg}_{\mathbf{B}}(f[X])$. This follows, by induction, taking into account the following facts: (1) $\mathrm{Sg}_{\mathbf{A}}(X) = \mathrm{E}^\omega_{\mathbf{A}}(X)$, (2) $f[\mathrm{E}^\omega_{\mathbf{A}}(X)] = \bigcup_{n\in\mathbb{N}} f[\mathrm{E}^n_{\mathbf{A}}(X)]$, (3) $\mathrm{Sg}_{\mathbf{B}}(f[X])$ is a subalgebra of the Λ-algebra $\mathbf{B}$, and (4) f is a Σ-homomorphism from $\mathbf{A} = (A, F)$ to $\mathbf{B}^d = (B, G \circ d)$.

The proof of the fact that the functor Sg from $\mathbf{Alg}(1)$ to $\mathbf{AClSp}$ is surjective on the objects can be found, e.g., in [10], p. 80. □

Remark 4.2 Of course, as pointed out, e.g., by Cohn in [10], p. 81, there are many ways of choosing the structure of single-sorted algebra to produce the given algebraic closure space. However, if we take into account only the standard way in which it has been chosen in [10], p. 80, [16], p. 48, [21], p. 91, and [25], p. 183, then this allows us to assert that the morphism mapping of Sg is also surjective with regard to the injective continuous mappings between algebraic closure spaces. In fact, let f be an injective continuous mapping from an algebraic closure space (A, J) to another (B, K). Then the pair (d_f, f), where d_f is the morphism from $\Sigma^{(A,J)}$ to $\Sigma^{(B,K)}$ which to a pair (x, a), with $x \in A^n$, for some $n \in \mathbb{N}$, and $a \in J(\mathrm{Im}(x))$, assigns the pair $(f^n(x), f(a))$, is a homomorphism from the algebra $(\Sigma^{(A,J)}, (A, F^{(A,J)}))$ to the algebra $(\Sigma^{(B,K)}, (B, F^{(B,K)}))$, since the injectivity of f entails (taking into account the definition of the structural operations of the involved algebras given in any of the four items referred to above) that $F^{(B,K)}_{f^n(x),f(a)} \circ f^n = f \circ F^{(A,J)}_{x,a}$. Obviously, the functor Sg sends (d_f, f) to the morphism f. Therefore, the morphism mapping of the functor Sg is surjective, as asserted, for the injective continuous mappings between algebraic closure spaces.

To render functorial the second Birkhoff-Frink representation theorem it is required to work with the subcategory $\mathbf{ALat}$ of the category $\mathbf{CLat}_\wedge$.

Proposition 4.3 *There exists an essentially surjective contravariant functor* $\mathbf{Fix}$ *from* $\mathbf{AClSp}$ *to* $\mathbf{ALat}$.

P r o o f. Let $\mathbf{Fix}$ be the contravariant functor from $\mathbf{AClSp}$ to $\mathbf{ALat}$ which assigns to an algebraic closure space (A, J) the algebraic lattice $\mathbf{Fix}(J)$ of the fixed points of J, and to a morphism $f\colon (A, J) \longrightarrow (B, K)$ the morphism $\mathbf{Fix}(f) = f^{-1}[\cdot]$ from $\mathbf{Fix}(K)$ to $\mathbf{Fix}(J)$ which sends a fixed point $Y = K(Y)$ of K to $f^{-1}[Y]$, its inverse image under f, which is, obviously, a fixed point of J. The morphism mapping of $\mathbf{Fix}$ is well defined since, on the one hand, for every $\mathcal{Y} \subseteq \mathrm{Fix}(K)$, it happens that

$$f^{-1}[\bigwedge \mathcal{Y}] = f^{-1}[\bigcap_{Y\in\mathcal{Y}} Y] = \bigcap_{Y\in\mathcal{Y}} f^{-1}[Y] = \bigwedge\{f^{-1}[Y] \mid Y \in \mathcal{Y}\},$$

and, on the other hand, for every directed subset $\mathcal{Y}$ of $\mathrm{Fix}(K)$, we have that $f^{-1}[\bigvee \mathcal{Y}] = \bigvee\{f^{-1}[Y] \mid Y \in \mathcal{Y}\}$. Finally, on account of Theorem 5.8 in [6], we conclude that $\mathbf{Fix}$ is essentially surjective. □

From Proposition 4.1 and Proposition 4.3 we obtain the functorial version of the second classical Birkhoff-Frink representation theorem as stated in the following

Corollary 4.4 *The contravariant functor* $\mathbf{Fix} \circ \mathrm{Sg}$ *from* $\mathbf{Alg}(1)$ *to* $\mathbf{ALat}$ *is essentially surjective.*

P r o o f. Because the object mapping of the functor Sg is surjective and the contravariant functor $\mathbf{Fix}$ is essentially surjective. □

Our next goal is to extend Proposition 4.1 and Corollary 4.4 to the many-sorted case, i.e., to the case when the set of sorts S is such that $\mathrm{card}(S) \geq 2$. We begin by proving for the functor Sg^S from $\mathbf{Alg}(S)$ to $\mathbf{UAClSp}(S)$ (to be defined below) the many-sorted counterpart of the first classical Birkhoff-Frink representation theorem. However, to do this we need to state beforehand, for an S-sorted signature Σ and a Σ-algebra $\mathbf{A}$, a constructive characterization of the subalgebra generating operator $\mathrm{Sg}^S_{\mathbf{A}}$ on A induced by $\mathbf{A}$, and the uniformity of $\mathrm{Sg}^S_{\mathbf{A}}$.

Definition 4.5 Let Σ be an S-sorted signature and $\mathbf{A} = (A, F)$ a Σ-algebra.

1. The operator $\mathrm{E}_{\mathbf{A}}$ on $\mathrm{Sub}(A)$ is defined, for every $X \subseteq A$, by $\mathrm{E}_{\mathbf{A}}(X) = X \cup \left(\bigcup_{\sigma\in\Sigma_{\cdot,s}} F_\sigma[X_{\mathrm{ar}(\sigma)}]\right)_{s\in S}$, where, for $s \in S$, $\Sigma_{\cdot,s} = \bigcup_{w\in S^\star} \Sigma_{w,s}$ and, for $\mathrm{ar}(\sigma) = (s_j)_{j\in m} \in S^\star$, the arity of σ, $X_{\mathrm{ar}(\sigma)} = \prod_{j\in m} X_{s_j}$.
2. Let X be a sub-S-sorted set of A. Then the family $(\mathrm{E}^n_{\mathbf{A}}(X))_{n\in\mathbb{N}}$ in $\mathrm{Sub}(A)$ is defined by recursion as: $\mathrm{E}^0_{\mathbf{A}}(X) = X$, and, for all $n \in \mathbb{N}$, $\mathrm{E}^{n+1}_{\mathbf{A}}(X) = \mathrm{E}_{\mathbf{A}}(\mathrm{E}^n_{\mathbf{A}}(X))$.
3. The operator $\mathrm{E}^\omega_{\mathbf{A}}$ on $\mathrm{Sub}(A)$ is defined, for every $X \subseteq A$, by $\mathrm{E}^\omega_{\mathbf{A}}(X) = \bigcup_{n\in\mathbb{N}} \mathrm{E}^n_{\mathbf{A}}(X)$.

Proposition 4.6 *Let* Σ *be an* S*-sorted signature,* $\mathbf{A}$ *a* Σ*-algebra, and* $X \subseteq A$*. Then* $\mathrm{Sg}^S_{\mathbf{A}}(X) = \mathrm{E}^\omega_{\mathbf{A}}(X)$.

P r o o f. See [7]. □

Proposition 4.7 *Let* Σ *be an* S*-sorted signature,* $\mathbf{A}$ *a* Σ*-algebra, and* $X, Y \subseteq A$*. Then*

1. *If* $\mathrm{supp}(X) = \mathrm{supp}(Y)$*, then, for each* $n \in \mathbb{N}$, $\mathrm{supp}(\mathrm{E}^n_{\mathbf{A}}(X)) = \mathrm{supp}(\mathrm{E}^n_{\mathbf{A}}(Y))$.
2. $\mathrm{supp}(\mathrm{Sg}^S_{\mathbf{A}}(X)) = \bigcup_{n\in\mathbb{N}} \mathrm{supp}(\mathrm{E}^n_{\mathbf{A}}(X))$.

3. *If* $\operatorname{supp}(X) = \operatorname{supp}(Y)$*, then* $\operatorname{supp}(\mathrm{Sg}^S_{\mathbf{A}}(X)) = \operatorname{supp}(\mathrm{Sg}^S_{\mathbf{A}}(Y))$.

Therefore the many-sorted algebraic closure operator $\mathrm{Sg}^S_{\mathbf{A}}$ *is uniform.*

P r o o f. See [7] again. □

Proposition 4.8 *Let S be a nonempty set of sorts. Then there exists a functor Sg^S from $\mathbf{Alg}(S)$ to $\mathbf{UAClSp}(S)$ which is surjective on the objects.*

P r o o f. Let Sg^S be the functor from $\mathbf{Alg}(S)$ to $\mathbf{UAClSp}(S)$ which sends an S-algebra $(\Sigma, \mathbf{A})$ to the uniform algebraic S-closure space $(A, \mathrm{Sg}^S_{\mathbf{A}})$, where $\mathrm{Sg}^S_{\mathbf{A}}$ is the subalgebra generating operator on A induced by $\mathbf{A}$, and a morphism (d, f) from $(\Sigma, \mathbf{A})$ to $(\Lambda, \mathbf{B})$ in $\mathbf{Alg}(S)$ to the morphism f from $(A, \mathrm{Sg}^S_{\mathbf{A}})$ to $(B, \mathrm{Sg}^S_{\mathbf{B}})$ in $\mathbf{UAClSp}(S)$.

To prove that the action of Sg^S on the morphisms is well defined, i.e., that, for every $X \subseteq A$, $f[\mathrm{Sg}^S_{\mathbf{A}}(X)]$ is a sub-S-sorted set of $\mathrm{Sg}^S_{\mathbf{B}}(f[X])$, we can now proceed analogously to the proof of the corresponding fact in Proposition 4.1.

The proof of the fact that the functor Sg^S from $\mathbf{Alg}(S)$ to $\mathbf{UAClSp}(S)$ is surjective on the objects can be found in [8], and this is precisely the first Birkhoff-Frink representation theorem for many-sorted algebras. □

Remark 4.9 In what follows, to shorten notation, we continue to write, for a set of sorts U, Sg^U for the composite of $\mathrm{Sg}^U\colon \mathbf{Alg}(U) \longrightarrow \mathbf{UAClSp}(U)$ and of the canonical embedding of $\mathbf{UAClSp}(U)$ into $\mathbf{AClSp}(U)$. Let $\varphi\colon S \longrightarrow T$ be a mapping. Then the functor φ^* from $\mathbf{Alg}(T)$ to $\mathbf{Alg}(S)$ which sends a T-algebra $(\Lambda, \mathbf{B})$, with $\mathbf{B} = (B, G)$, to the S-algebra $\varphi^*(\Lambda, \mathbf{B}) = (\Lambda_{\varphi^\star \times \varphi}, (B_\varphi, G_{\varphi^\star \times \varphi}))$, and a morphism (e, g) from $(\Lambda, \mathbf{B})$ to $(\Lambda', \mathbf{B}')$ in $\mathbf{Alg}(T)$ to the morphism $\varphi^*(e, g) = (e_{\varphi^\star \times \varphi}, g_\varphi)$ from $\varphi^*(\Lambda, \mathbf{B})$ to $\varphi^*(\Lambda', \mathbf{B}')$ in $\mathbf{Alg}(S)$, has a left adjoint φ_*. Moreover, there are natural transformations β^φ from $\coprod^{\mathrm{acl}}_\varphi \circ \mathrm{Sg}^S$ to $\mathrm{Sg}^T \circ \varphi_*$ and α^φ from $\mathrm{Sg}^S \circ \varphi^*$ to $\Delta^{\mathrm{acl}}_\varphi \circ \mathrm{Sg}^T$ adjoint for the noncommutative square constructed from the adjunction $\varphi_* \dashv \varphi^*$, the adjunction $\coprod^{\mathrm{acl}}_\varphi \dashv \Delta^{\mathrm{acl}}_\varphi$ (stated in the first part of Proposition 3.21), and the functors Sg^S and Sg^T (for an explanation of the concept of *adjoint natural transformations* see, e.g., [29], p. 14). Finally, if φ is injective, then, considering the adjunction $\coprod^{\mathrm{uacl}}_\varphi \dashv \Delta^{\mathrm{uacl}}_\varphi$ (stated in the second part of Proposition 3.21), there are also, as above, natural transformations which are adjoint for the corresponding noncommutative square.

After having stated the functorial version of the first Birkhoff-Frink representation theorem for many-sorted algebras, we devote the remainder of this section to state the second one. To this end we need to prove, ultimately, that, for every set of sorts S, there exists an essentially surjective contravariant functor $\mathbf{Fix}^S$ from $\mathbf{UAClSp}(S)$ to $\mathbf{ALat}$ and for this we must state the following auxiliary results.

Proposition 4.10 *Let S be a set of sorts. Then there exists a contravariant functor $\mathbf{Fix}^S$ from $\mathbf{ClSp}(S)$ to $\mathbf{CLat}_\bigwedge$.*

P r o o f. Let $\mathbf{Fix}^S$ be the contravariant functor from $\mathbf{ClSp}(S)$ to $\mathbf{CLat}_\bigwedge$ which assigns to an S-closure space (A, J) the complete lattice $\mathbf{Fix}^S(J)$ of the fixed points of J, and to a morphism f from (A, J) to (B, K) the morphism $\mathbf{Fix}^S(f) = f^{-1}[\cdot]$ from $\mathbf{Fix}^S(K)$ to $\mathbf{Fix}^S(J)$ which sends a fixed point $Y = K(Y)$ of K to $f^{-1}[Y] = (f_s^{-1}[Y_s])_{s\in S}$, its inverse image under f, which is, obviously, a fixed point of J. What is left is to show that the morphism mapping of $\mathbf{Fix}^S$ is well defined, i.e., that, for every $\mathcal{Y} \subseteq \mathrm{Fix}^S(K)$, it happens that $f^{-1}[\bigwedge \mathcal{Y}] = \bigwedge\{f^{-1}[Y] \mid Y \in \mathcal{Y}\}$. But the same proof works for it as for the homologous fact in Proposition 4.3. □

The contravariant functor $\mathbf{Fix}^S$ from $\mathbf{ClSp}(S)$ to $\mathbf{CLat}_\bigwedge$ can actually be restricted to a contravariant functor from the subcategory $\mathbf{AClSp}(S)$ of $\mathbf{ClSp}(S)$ to the subcategory $\mathbf{ALat}$ of $\mathbf{CLat}_\bigwedge$, and to a contravariant functor from the subcategory $\mathbf{UAClSp}(S)$ of $\mathbf{AClSp}(S)$ to $\mathbf{ALat}$. That is, for every algebraic or uniform algebraic S-closure operator J on an S-sorted set A, it happens that $\mathbf{Fix}^S(J)$ is an algebraic lattice. However, in order to verify it we need to prove beforehand the following property relative to the finite sub-S-sorted sets of the union of a family of S-sorted sets.

Lemma 4.11 *Let $(A^i)_{i\in I}$ be a family of S-sorted sets and $X \in \mathrm{Sub}_{\mathrm{f}}(\bigcup_{i\in I} A^i)$. Then there exists a finite subset K of I such that $X \in \mathrm{Sub}_{\mathrm{f}}(\bigcup_{i\in K} A^i)$.*

P r o o f. For each $s \in S$ and each $x \in X_s$, define $I_{x,s} = \{i \in I \mid x \in A^i_s\}$. Then $(I_{x,s})_{(x,s)\in\coprod X}$, where $\coprod X = \bigcup_{s\in S}(X_s \times \{s\})$, is a family of nonempty sets, since, for every $s \in S$ and every $x \in X_s$, $x \in \bigcup_{i\in I} A^i_s$, therefore $\prod_{(x,s)\in\coprod X} I_{x,s} \neq \varnothing$. Let f be an arbitrary but fixed element of $\prod_{(x,s)\in\coprod X} I_{x,s}$ and $K = \mathrm{Im}(f)$. Then $X \subseteq \bigcup_{i\in K} A^i$ and, since $\mathrm{card}(K) \leq \mathrm{card}(\coprod X)$ and $\coprod X$ is finite, K is finite. □

Proposition 4.12 *Let J be an algebraic S-closure operator on an S-sorted set A. Then $\mathbf{Fix}^S(J)$ is an algebraic lattice and the compacts of $\mathbf{Fix}^S(J)$ are precisely the S-sorted sets of the form $J(X)$, where X is a finite sub-S-sorted set of A.*

P r o o f. We restrict ourselves to prove that if X is a finite sub-S-sorted set of A, then $J(X)$ is a compact of $\mathbf{Fix}^S(J)$ since the verification of the remaining parts is straightforward. Let us suppose that $J(X)$ is a sub-S-sorted of $\bigvee_{i\in I} J(X^i)$. Then, since $X \subseteq J(X)$ and J is algebraic, it happens that

$$X \subseteq \bigvee_{i\in I} J(X^i) = J(\bigcup_{i\in I} X^i) = \bigcup_{Z\in\mathrm{Sub_f}(\bigcup_{i\in I} X^i)} J(Z).$$

Therefore, for every $s \in S$ and every $x \in X_s$, there exists a $Z^{x,s} \in \mathrm{Sub_f}(\bigcup_{i\in I} X^i)$ such that $x \in J(Z^{x,s})_s$. Hence, by Lemma 4.11, there exists a $K^{x,s} \in \mathrm{Sub_f}(I)$ such that $Z^{x,s} \in \mathrm{Sub_f}(\bigcup_{i\in K^{x,s}} X^i)$. Let us denote by K the set $\bigcup_{(x,s)\in\coprod X} K^{x,s}$. Then, for every $s \in S$ and every $x \in X_s$, it happens that $Z^{x,s} \in \mathrm{Sub_f}(\bigcup_{i\in K} X^i)$, thus

$$J(Z^{x,s}) \subseteq J(\bigcup_{i\in K} X^i) = \bigcup_{Z\in\mathrm{Sub_f}(\bigcup_{i\in K} X^i)} J(Z).$$

From the above it follows that

$$X \subseteq \bigcup_{(x,s)\in\coprod X} J(Z^{x,s}) \subseteq \bigcup_{Z\in\mathrm{Sub_f}(\bigcup_{i\in K} X^i)} J(Z) \subseteq J(\bigcup_{i\in K} X^i) = \bigvee_{i\in K} J(X^i).$$

Consequently, since J is idempotent, $J(X) \subseteq \bigvee_{i\in K} J(X^i)$, which is the desired conclusion. □

Corollary 4.13 *The contravariant functor $\mathbf{Fix}^S$ from $\mathbf{ClSp}(S)$ to $\mathbf{CLat}_\bigwedge$ can be restricted to a contravariant functor from $\mathbf{AClSp}(S)$ to $\mathbf{ALat}$, as well as to a contravariant functor from $\mathbf{UAClSp}(S)$ to $\mathbf{ALat}$, both of them denoted by $\mathbf{Fix}^S$ for short.*

For a nonempty set of sorts S the contravariant functor $\mathbf{Fix}^S$ from $\mathbf{ClSp}(S)$ to $\mathbf{CLat}_\bigwedge$ is essentially surjective, as we prove in the following

Proposition 4.14 *Let S be a nonempty set of sorts. Then every complete lattice is isomorphic to the complete lattice of the fixed points of an S-closure operator on an S-sorted set, i.e., the contravariant functor $\mathbf{Fix}^S$ from $\mathbf{ClSp}(S)$ to $\mathbf{CLat}_\bigwedge$ is essentially surjective.*

P r o o f. Let $\mathbf{L}$ be a complete lattice. Since the set of sorts S is nonempty, for an arbitrary but fixed sort $t \in S$, let $\delta^{t,\mathbf{L}}$ be the S-sorted set whose s-th coordinate, for $s \in S$, is L, if $s = t$, and $\varnothing$, if $s \neq t$. Then for the S-closure system $\mathcal{C}_\mathbf{L}$ on $\delta^{t,\mathbf{L}}$ defined as $\mathcal{C}_\mathbf{L} = \{X \subseteq \delta^{t,\mathbf{L}} \mid \exists\, a \in L\,(X_t =\downarrow a)\}$, where, for every $a \in L$, $\downarrow a = \{b \in L \mid b \leq a\}$ is the principal ideal in $\mathbf{L}$ associated to a, or, what is equivalent, for the S-closure operator $J_\mathbf{L}$ on $\delta^{t,\mathbf{L}}$ defined, for every $X \subseteq \delta^{t,\mathbf{L}}$ and every $s \in S$, as $J_\mathbf{L}(X)_s = \{a \in L \mid a \leq \bigvee X_t\}$, if $s = t$, and as $J_\mathbf{L}(X)_s = \varnothing$, if $s \neq t$, it happens that $\boldsymbol{\mathcal{C}}_\mathbf{L} = (\mathcal{C}_\mathbf{L}, \subseteq) = \mathbf{Fix}^S(J_\mathbf{L})$ is isomorphic to $\mathbf{L}$. □

As above, also, for a nonempty set of sorts S, the contravariant functor $\mathbf{Fix}^S$ from $\mathbf{AClSp}(S)$ to $\mathbf{ALat}$ is essentially surjective, as we prove in the following

Proposition 4.15 *Let S be a nonempty set of sorts. Then every algebraic lattice is isomorphic to the algebraic lattice of the fixed points of an algebraic S-closure operator on an S-sorted set, i.e., the contravariant functor $\mathbf{Fix}^S$ from $\mathbf{AClSp}(S)$ to $\mathbf{ALat}$ is essentially surjective.*

P r o o f. Let $\mathbf{L}$ be an algebraic lattice, $\mathrm{K}(\mathbf{L})$ the set of all compact elements of $\mathbf{L}$, and $\delta^{t,\mathrm{K}(\mathbf{L})}$ the S-sorted set whose s-th coordinate, for $s \in S$, is $\mathrm{K}(\mathbf{L})$, if $s = t$, and $\varnothing$, if $s \neq t$. Then for the S-closure operator $J_\mathbf{L}$ on $\delta^{t,\mathrm{K}(\mathbf{L})}$ defined, for every $X \subseteq \delta^{t,\mathrm{K}(\mathbf{L})}$ and every $s \in S$, as $J_\mathbf{L}(X)_s = \{a \in \mathrm{K}(\mathbf{L}) \mid a \leq \bigvee X_t\}$, if $s = t$, and as $J_\mathbf{L}(X)_s = \varnothing$, if $s \neq t$, we infer that $\mathbf{Fix}^S(J_\mathbf{L})$ is isomorphic to $\mathbf{L}$. □

Finally, since the algebraic S-closure operator $J_{\mathbf{L}}$ defined in the proof of the above proposition is, in addition, uniform, we have proved all auxiliary results needed to show that there exists an essentially surjective contravariant functor $\mathbf{Fix}^{S}$ from $\mathbf{UAClSp}(S)$ to $\mathbf{ALat}$.

Corollary 4.16 *Let S be a nonempty set of sorts. Then every algebraic lattice is isomorphic to the algebraic lattice of the fixed points of a uniform algebraic S-closure operator on an S-sorted set, i.e., the contravariant functor $\mathbf{Fix}^{S}$ from $\mathbf{UAClSp}(S)$ to $\mathbf{ALat}$ is essentially surjective.*

From Proposition 4.8 and Corollary 4.16 we obtain the many-sorted functorial version of the second Birkhoff-Frink representation theorem as stated in the following

Corollary 4.17 *Let S be a nonempty set of sorts. Then the contravariant functor $\mathbf{Fix}^{S} \circ \mathrm{Sg}^{S}$ from $\mathbf{Alg}(S)$ to $\mathbf{ALat}$, obtained by composing the functor Sg^{S} from $\mathbf{Alg}(S)$ to $\mathbf{UAClSp}(S)$ and the contravariant functor $\mathbf{Fix}^{S}$ from $\mathbf{UAClSp}(S)$ to $\mathbf{ALat}$, is essentially surjective.*

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