Appendix online

A Proof of Result 1

Optimization problem (1) in the main text may have three types of solutions: two corner solutions and one interior. Here we investigate under which conditions the solution to the problem is given by $q_t = 1$ for all t and all countries which implies that the total emissions are equal to N. According to the Kuhn-Tucker conditions the maximization of the r.h.s. of the Bellman equation yields this type of corner solution when

$$1 + \delta U'(\rho z_t + Q_t + q_t) \geq 0. \tag{A.1}$$

Then if this condition is satisfied, the Bellman equation, after the substitution of the optimal emissions, can be rewritten as

$$U(z_t) = 1 - \gamma z_t^2 + \delta U(\rho z_t + N).$$

Thus, if there exists a value function that satisfies this equation along with (A.1), we know that at least for certain values for the parameters of the model, the solution implies that emissions are equal to the unity for all z_t. We find that this is true for a quadratic value function $U(z_t) = a - bz_t - cz_t^2$ if

$$\gamma \leq \frac{(1-\delta\rho)(1-\rho)}{2\delta N}. \tag{A.2}$$

To show this we first calculate the parameters of the quadratic value function by equating coefficients

$$
\begin{aligned}
a &= \frac{1}{1-\delta} - \frac{\delta\gamma N^2(1+\delta\rho)}{(1-\delta\rho)(1-\delta\rho^2)(1-\delta)}, \\
b &= \frac{2\delta\rho\gamma N}{(1-\delta\rho)(1-\delta\rho^2)}, \ c = \frac{\gamma}{1-\delta\rho^2},
\end{aligned}
$$

so that condition (A.1) now can be written as $1 \geq \delta[b + 2c(\rho z_t + N)]$ which yields after substituting b and c

$$
1 \geq \delta \left[\frac{2\delta\rho\gamma N}{(1-\delta\rho)(1-\delta\rho^2)} + \frac{2\gamma(\rho z_t + N)}{1-\delta\rho^2} \right].
$$

Rearranging terms gives the following condition for γ

$$
\frac{(1-\delta\rho^2)(1-\delta\rho)}{2\delta(\delta\rho N + (1-\delta\rho)(\rho z_t + N))} \geq \gamma. \tag{A.3}
$$

This condition depends inversely on z_t so that we need to establish the dynamics of the cumulative stock of emissions to obtain a condition that applies for all periods. When total emissions are equal to N, the dynamics of z_t is given by the difference equation $z_{t+1} = \rho z_t + N$ whose solution obtained by iteration is $z_t = (1 + \rho + \rho^2 + ... + \rho^{t-1})N$ and whose steady-state value of the stock is $N/(1-\rho)$. This implies that the stock increases from zero until it reaches the steady-state value that is the maximum value that z_t can take. For this reason, if we substitute z_t in (A.3) by $N/(1-\rho)$ we obtain a condition based only on the parameters of the model that if it is satisfied guarantees that the optimal emissions for each period are the maximum emissions. Doing this substitution we obtain the right-hand side of (A.2) that we define as critical value $\bar{\gamma}$. See (2) in the main text. Finally, for $z_0 = 0$ the quadratic value function yields $U(0) = a$ which is the expression reported in Result 1 for the present value of net benefits. This concludes the proof.

B Proof of Result 2

First we investigate under which conditions the solution to optimization problem (3)-(4) in the main text is an interior solution for all t and all countries. For an interior solution the following first order conditions must be satisfied

$$1 + \delta V' \left(\rho z_t + \sum_{i=1}^{N} q_{it} \right) = 0, \quad \text{for all } i,$$

which yields the following condition if we look for a symmetric solution

$$1 + \delta V' \left(\rho z_t + N q_t \right) = 0. \tag{B.1}$$

where q_t stands for the emissions of the representative country. If we guess a quadratic value function $V(z_t) = \alpha - \beta z_t - \chi z_t^2$, this condition defines the following optimal policy or rule

$$q_t(z_t) = \frac{1}{N} \left(\frac{1 - \beta\delta}{2\delta\chi} - \rho z_t \right). \tag{B.2}$$

Then the Bellman equation can be written as

$$V(z_t) = N q_t(z_t) - N\gamma z_t^2 + \delta V(\rho z_t + N q_t(z_t)),$$

which (from (B.2)) yields

$$V(z_t) = \frac{1 - \beta\delta}{2\delta\chi} - \rho z_t - N\gamma z_t^2 + \delta \left(\alpha - \beta \frac{1 - \beta\delta}{2\delta\chi} - \chi \left(\frac{1 - \beta\delta}{2\delta\chi} \right)^2 \right).$$

Equating coefficients we obtain

$$\alpha = \frac{(1 - \delta\rho)^2}{4\delta\gamma N(1 - \delta)}, \; \beta = \rho \text{ and } \chi = N\gamma.$$

Thus, the optimal policy (B.2) in terms of the parameters of the model is

$$q_t(z_t) = \frac{1}{N} \left(\frac{1 - \rho\delta}{2\delta N\gamma} - \rho z_t \right).$$

For this rule, the dynamics of the stock is given by

$$z_{t+1} = \rho z_t + N q_t(z_t) = \rho z_t + \frac{1-\rho\delta}{2\delta N\gamma} - \rho z_t = \frac{1-\rho\delta}{2\delta N\gamma}.$$

This means that the steady-state stock is reached in one period and it is equal to $\hat{z} = (1-\rho\delta)/2\delta N\gamma$. Then the emissions of the representative country are

$$q_0 = \frac{1-\rho\delta}{2\delta N^2\gamma} \text{ for } z_0 = 0 \text{ and } q_t = \hat{q} = \frac{(1-\rho\delta)(1-\rho)}{2\delta N^2\gamma} = (1-\rho)q_0 \text{ for } \hat{z} \text{ and for } t > 0.$$

In both cases emissions are bigger than zero independently of the value for γ. However, q_t will be lower than the unity for all t if $q_0 = (1-\rho\delta)/2\delta N^2\gamma < 1$ since $\hat{q}$ is lower than q_0. From this condition we obtain critical value $\underline{\gamma} = (1-\rho\delta)/2\delta N^2$ that appears in the main text. See (5). This allows us to rewrite q_0 as $\underline{\gamma}/\gamma$, $\hat{z}$ as $N\underline{\gamma}/\gamma$ and $\hat{q}$ as $(1-\rho)\underline{\gamma}/\gamma$. Finally, for $z_0 = 0$ the quadratic value functions yields $V(0) = \alpha$ so that for the representative country the present value of net benefits is given by α/N which is the expression reported in Result 2. This concludes the proof.

C Proof of Corollary 1

The comparison of emissions is straightforward. However, the comparison between the stocks of the pollutant is not so straightforward. If we focus on the steady-state values we have that[1]

$$\hat{z}^n - \hat{z}^c = \frac{N}{1-\rho} - \frac{1-\delta\rho}{2\delta\gamma N} = \frac{2\delta\gamma N^2 - (1-\delta\rho)(1-\rho)}{2(1-\rho)\delta\gamma N}.$$

[1] The superscript n stands for the non-cooperative equilibrium whereas c stands for the cooperative one.

This difference is positive only if $\gamma > (1-\delta\rho)(1-\rho)/2\delta N^2 = (1-\rho)\underline{\gamma}$. But if γ is greater than $\underline{\gamma}$ as required by Result 2, then it is also greater than $(1-\rho)\underline{\gamma}$ and we can conclude that the steady-state stock in the cooperative equilibrium is less than the steady-state stock in the non-cooperative equilibrium. It is a little more complex to compare the discounted present values of the net benefits for the representative country. According to Results 1 and 2 the difference between the discounted present values is given by

$$\begin{aligned}\frac{V(0)}{N} - U(0) &= \frac{(1-\delta\rho)^2}{4\delta\gamma N^2(1-\delta)} - \frac{1}{1-\delta} + \frac{\delta\gamma N^2(1+\delta\rho)}{(1-\delta\rho)(1-\delta\rho^2)(1-\delta)} \\ &= \frac{1}{1-\delta}\left(\frac{f(\gamma)}{4\delta\gamma(1-\delta\rho)(1-\delta\rho^2)N^2}\right), \qquad \text{(C.1)}\end{aligned}$$

where

$$f(\gamma) = 4\delta^2(1+\delta\rho)N^4\gamma^2 - 4\delta(1-\delta\rho)(1-\delta\rho^2)N^2\gamma + (1-\delta\rho)^3(1-\delta\rho^2).$$

This is a strictly convex function with an unique minimum for $\gamma^* = (1-\delta\rho)(1-\delta\rho^2)/2\delta(1+\delta\rho)N^2$ and a minimum value for the function strictly positive $f(\gamma^*) = \delta\rho^2(1-\delta)(1-\delta\rho)^2(1-\delta\rho^2)/(1+\delta\rho)$ which implies that the function is strictly positive for all γ. This establishes that the difference (C.1) is positive for $\gamma \geq 0$ so that the discounted present value of the net benefits for the representative country is bigger in the cooperative equilibrium than in the non-cooperative one for $\gamma \in (\underline{\gamma}, \bar{\gamma}]$.□

D Step iii) of the Algorithm

If condition (16) is not satisfied by the proposed values of B and C we proceed as follows:

1) We calculate using the value of C a new value for B according to expression

$$\hat{B} = \frac{1}{\delta} - \frac{2N}{1-\rho}C. \qquad \text{(D.1)}$$

2) Then if $\hat{B}$ is greater than $1/\delta N$ so that the r.h.s. of (16) holds, we continue with the algorithm using as values $\hat{B}$ and C.

3) If $\hat{B}$ is lower or equal to $1/\delta N$ we select for C a value that is something lower than the critical value given by[2]

$$\frac{1}{\delta N} = \frac{1}{\delta} - \frac{2N}{1-\rho}\tilde{C}.$$

In particular, we select the value given by

$$\frac{1}{\delta N} + \frac{1}{10}\left(\frac{1}{\delta} - \frac{1}{\delta N}\right) = \frac{1}{\delta} - \frac{2N}{1-\rho}C,$$

which yields

$$\hat{C} = \frac{9(N-1)(1-\rho)}{20\delta N^2} < \tilde{C} \tag{D.2}$$

and the corresponding value for B given by (D.1)

$$\hat{B} = \frac{N+9}{10\delta N}, \tag{D.3}$$

and we continue with the algorithm using as values for B and C these two values.

[2] This condition defines a value for C, that we denote by $\tilde{C}$, such that if C is lower than $\tilde{C}$ then the value of B defined by (D.1) is greater than $1/\delta N$ so that the r.h.s. of (16) holds and also the l.h.s.