QUASI-LINEAR ELLIPTIC AND PARABOLIC EQUATIONS IN L^1 WITH NONLINEAR BOUNDARY CONDITIONS

F. Andreu, J. M. Mazón, S. Segura de León, J. Toledo[1]

Abstract. In this paper we study the questions of existence and uniqueness of solutions for equations of the form

$$u - \mathrm{div}\ \mathbf{a}(x, Du) = f \quad \text{in} \ \ \Omega$$

$$-\frac{\partial u}{\partial \eta_a} \in \beta(u) \quad \text{on} \ \ \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $f \in L^1(\Omega)$. As a consequence, an m-completely accretive operator in $L^1(\Omega)$ can be associated to the corresponding parabolic equation, which permits to study this equation from the point of view of Nonlinear Semigroup Theory.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $1 < p < \infty$. Consider a vector valued function $\mathbf{a}$ mapping $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^N$ and satisfying

(H_1) $\mathbf{a}$ is a Carathéodory function (i.e., the map $\xi \to \mathbf{a}(x, \xi)$ is continuous for almost all x and the map $x \to \mathbf{a}(x, \xi)$ is measurable for every ξ) and there exists $\lambda > 0$ such that

$$\langle \mathbf{a}(x, \xi), \xi \rangle \geq \lambda |\xi|^p$$

holds for every ξ and a.e. $x \in \Omega$, where $\langle , \rangle$ means scalar product in $\mathbb{R}^N$.

(H_2) For every ξ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$ there holds

$$\langle \mathbf{a}(x, \xi) - \mathbf{a}(x, \eta), \xi - \eta \rangle > 0.$$

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(H_3) There exists $\Lambda \in \mathbb{R}$ such that

$$|\mathbf{a}(x,\xi)| \leq \Lambda(j(x) + |\xi|^{p-1})$$

holds for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, with $j \in L^{p'}$, $p' = p/(p-1)$.

The hypotheses (H_1), (H_2) and (H_3) are classical in the study of nonlinear operators in divergence form (see [LL]). The model example of function $\mathbf{a}$ satisfying these hypotheses is $\mathbf{a}(x,\xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \mathrm{div}\big(|Du|^{p-2}\, Du\big)$.

The aim of this paper is to study existence and uniqueness of solutions for equations of the form

$$(I) \qquad \begin{aligned} & u - \mathrm{div}\ \mathbf{a}(x, Du) = f \quad \text{in } \Omega \\ & -\tfrac{\partial u}{\partial \eta_a} \in \beta(u) \quad \text{on } \partial\Omega \end{aligned}$$

where $\partial/\partial\eta_a$ is the Neumann boundary operator associated to $\mathbf{a}$, i.e.,

$$\frac{\partial u}{\partial \eta_a} := \langle \mathbf{a}(x, Du), \eta \rangle$$

with η the unit outward normal on $\partial\Omega$, Du the gradient of u and β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. These nonlinear flux on the boundary occurs in some problems in Mechanics and Physics [DL] (see also [Br_2]). Observe also that the classical Neumann and Dirichlet boundary conditions correspond to $\beta = \mathbb{R} \times \{0\}$ and $\beta = \{0\} \times \mathbb{R}$, respectively.

We associate a completely accretive operator A_0 in $L^1(\Omega)$ with the formal differential expression

$$-\mathrm{div}\ \mathbf{a}(x, Du) \quad + \quad \text{nonlinear boundary conditions.}$$

Recently, in [B-V], a new concept of solution has been introduced for the elliptic equation

$$-\mathrm{div}\ \mathbf{a}(x, Du) = f(x) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega,$$

namely *entropy solution.* We use the method developed in [B-V] to characterize and interpret the closure of the operator A_0.

The plan of the paper is as follows: Some preliminary results and notation are collected in Section 1. In the second section we study the problem (I) with variational methods. We introduce a completely accretive operator A_0 in $L^1(\Omega)$ which satisfies the range condition. Under certain restrictions another completely accretive operator A satisfying the range condition and smaller than A_0 is also introduced. In order to characterize the closure of the operator A we need to define the trace of functions which are not in the Sobolev spaces. This is the subject of Section 3. In the next section we characterize the closure of the operator A. To do that, following [B-V], we introduce the concept of entropy solution for the elliptic problem. As a consequence of the m-accretivity of the closure of the operator A, we obtain a result of existence and uniqueness for the entropy solutions of the elliptic problem. Finally, in Section 5 we consider the evolution problem associated with the operator A_0.

1. Preliminaries

In this section we give some of the notation and definitions used later. If $\Omega \subset \mathbb{R}^N$ is a Lebesgue measurable set, $\lambda_N(\Omega)$ denotes its measure. The norm in $L^p(\Omega)$ is denoted by $\|.\|_p$, $1 \le p \le \infty$. If $k \ge 0$ is an integer and $1 \le p \le \infty$, $W^{k,p}(\Omega)$ is the Sobolev space of functions u on the open set $\Omega \subset \mathbb{R}^N$ for which $D^\alpha u$ belongs to $L^p(\Omega)$ when $|\alpha| \le k$, with its usual norm $\|.\|_{k,p}$. $W_0^{k,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. If $v \in L^1(\Omega)$ and $\lambda_N(\Omega) < \infty$, we denote by $\overline{v}$ the average of v, i.e.,

$$\overline{v} := \frac{1}{\lambda_N(\Omega)} \int_\Omega v(x)\, dx.$$

Given a finite measure space (S, ν), we denote by $M(S, \nu)$ the set of all measurable functions $u : S \to \mathbb{R}$ finite a.e., identifying the functions that are equal a.e. $M(S, \nu)$ is a metric space endowed with the distance function of the convergence in measure,

$$\varrho(u, v) = \int_\Omega \frac{|u - v|}{1 + |u - v|}.$$

We recall, cf. [BBC], that for $0 < q < \infty$ the Marcinkiewicz space $M^q(\Omega)$ can be defined as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that the corresponding distribution function

$$\phi_f(k) = \lambda_N\{x \in \Omega \ : \ |f(x)| > k\}$$

satisfies an estimate of the form

$$\phi_f(k) \le Ck^{-q}, \qquad C < \infty.$$

For bounded Ω's, it is immediate that $M^q(\Omega) \subset M^{\hat{q}}(\Omega)$ if $\hat{q} \le q$, also $L^q(\Omega) \subset M^q(\Omega) \subset L^r(\Omega)$ if $1 \le r < q$.

We will use the following truncature operator: For a given constant $k > 0$ we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) := \begin{cases} s & \text{if } |s| \le k \\ k \text{ sign } (s) & \text{if } |s| > k. \end{cases}$$

For a function $u = u(x)$, $x \in \Omega$, we define the truncated function $T_k u$ pointwise, i.e., for every $x \in \Omega$ the value of $T_k u$ at x is just $T_k\big(u(x)\big)$. Observe that

$$\lim_{k \to 0} \frac{1}{k} T_k(s) = \text{sign}(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

By the Stampacchia Theorem, cf. [KS], if $u \in W^{1,1}(\Omega)$, we have

$$DT_k(u) = 1_{\{|u|<k\}}\ Du,$$

where 1_B denotes the characteristic function of a measurable set $B \subset \Omega$.

As we said in the introduction, our abstract framework is the Theory of Nonlinear Semigroups. We refer the reader to [Ba], [Be], [BCP] and [Cr] for background material on non-linear contraction semigroups.

In [BCr], Ph. Bénilan and M. Crandall introduce the concept of completely accretive operator, whose precedents are the results of Brezis-Strauss [BS] on semilinear elliptic equations (see also [Be]). To define this notion of operator we use the following notation. Let $u, v \in M(\Omega)$, we set $u \ll v$ if

$$\int_\Omega j(u) \leq \int_\Omega j(v) \quad \text{for} \quad j \in J_0,$$

where

$$J_0 = \{ \text{ convex lower} - \text{semicontinuous maps } \ j : \mathbb{R} \to [0, \infty] \ \text{ satisfying } j(0) = 0 \ \}.$$

An operator A in $M(\Omega)$, possibly multivaluated (i.e., $A \subset M(\Omega) \times M(\Omega)$), is said to be *completely accretive* if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \quad \text{for} \ \ \lambda > 0 \ \text{ and } \ (u, v), (\hat{u}, \hat{v}) \in A.$$

Let

$$P_0 = \{ \ p \in C^\infty(\mathbb{R}) \ : \ 0 \leq p' \leq 1, \ \ \text{supp}(p') \text{ is compact, and } 0 \notin \text{supp}(p) \ \}.$$

By [BCr, Proposition 2.2], if Ω is bounded and $u, v \in L^1(\Omega)$, the following conditions are equivalent:

$$u \ll u + \lambda v \quad \text{for any} \ \ \lambda > 0, \tag{1.1}$$

$$\int_\Omega p(u) v \geq 0 \quad \text{for any} \ \ p \in P_0. \tag{1.2}$$

Remark that if A is a completely accretive operator in $L^1(\Omega)$ and $1 \leq q \leq \infty$, then the restriction A_q of A to $L^q(\Omega)$ is T-accretive in $L^q(\Omega)$, that is

$$\| \big(u - \hat{u} + \lambda(v - \hat{v})\big)^+ \|_q \geq \| (u - \hat{u})^+ \|_q \qquad \text{for } \lambda > 0, \quad (u, v), (\hat{u}, \hat{v}) \in A_q.$$

Consequently, its resolvent $J_\lambda = (I + \lambda A_q)^{-1}$ is an order-preserving contraction in $L^q(\Omega)$. If a completely accretive operator A in $L^1(\Omega)$ satisfies the *range condition*:

"there exists $\lambda > 0$ such that $R(I + \lambda A)$ is dense in $L^1(\Omega)$",

then the closure $\overline{A}$ of A is an m-T-accretive operator in $L^1(\Omega)$, that is, $\overline{A}$ is T-accretive and there exists $\lambda > 0$ such that $R(I+\lambda\overline{A}) = L^1(\Omega)$. So, by Crandall-Liggett's Theorem, the operator $\overline{A}$ generates, on the closure of its domain, a semigroup of order-preserving contractions given by the exponential formula

$$e^{-t\overline{A}}\, u = \lim_{n\to\infty} \left(I + \frac{t}{n}\overline{A}\right)^{-n} u \quad \text{for } u \in \overline{D(\overline{A})}.$$

This semigroup solves the corresponding initial value problem for the operator $\overline{A}$

$$u' + \overline{A}u \ni 0, \quad u(0) = u_0. \tag{1.3}$$

The function $u(t) := e^{-t\overline{A}}\, u_0$ is called the *mild-solution* of problem (1.3).

2. Variational approach

From now on we assume Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ of class C^1, $1 < p < \infty$, $\mathbf{a}$ is a vector valued mapping from $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^N$ satisfying (H_1) - (H_3) and β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$.

In this section we study the problem (I) with variational methods. We will introduce and study a nonlinear completely accretive operator A_0 in $L^1(\Omega)$ associated with the formal differential expression

$$-\mathrm{div}\,\mathbf{a}(x, Du) \quad + \quad \text{nonlinear boundary conditions.}$$

Since β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$, there exists a convex lower semicontinuous (l.s.c.) function j on $\mathbb{R}$, such that $\beta = \partial j$. Consider $\Phi : W^{1,p}(\Omega) \to [0, +\infty]$, defined by

$$\Phi(u) := \begin{cases} \int_{\partial\Omega} j(u) & \text{if } j(u) \in L^1(\partial\Omega) \\ +\infty & \text{if } j(u) \notin L^1(\partial\Omega). \end{cases}$$

It is well-known (cf. [Br$_2$]) that Φ is a convex l.s.c. function in $W^{1,p}(\Omega)$. Now let us define the operator A_0 in $L^1(\Omega)$ by:

$(u, v) \in A_0$ if and only if $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v \in L^1(\Omega)$ and

$$\Phi(w) \geq \Phi(u) + \int_\Omega v(w-u) - \int_\Omega \langle \mathbf{a}(x, Du), D(w-u)\rangle \quad \text{for every } w \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Here and below the integrals over Ω are with respect to Lebesgue measure λ_N and the integrals over $\partial\Omega$ are with respect to the area measure μ on $\partial\Omega$. Observe that taking $w = 0$ in the definition of A_0 we obtain that $\mathcal{D}(A_0) \subset \mathcal{D}(\Phi)$.

We can now formulate the main result of this section.

Theorem 2.1. *The operator* A_0 *satisfies the following statements:*
(i) A_0 *is univalued, i.e., if* $(u,v) \in A_0$, *then*

$$v = -div\, \mathbf{a}(x, Du) \quad \textit{in the sense of distributions.}$$

(ii) A_0 *is completely accretive.*
(iii) $L^\infty(\Omega) \subset R(I + A_0)$.

Proof. (i) : Let $(u,v) \in A_0$. Given $\phi \in \mathcal{D}(\Omega)$, taking $w = u + \phi$ and $w = u - \phi$ as test functions in the definition of the operator A_0, we get

$$\int_\Omega v\phi = \int_\Omega \langle \mathbf{a}(x, Du), D\phi \rangle,$$

and consequently $v = -\text{div}\, \mathbf{a}(x, Du)$ in the sense of distributions.
(ii) : Let $p \in P_0$ and (u,v), $(\hat{u}, \hat{v}) \in A$. We will show that

$$\int_\Omega (v - \hat{v}) p(u - \hat{u}) \geq 0. \tag{2.1}$$

Taking $w = u - p(u - \hat{u})$ and $w = \hat{u} + p(u - \hat{u})$ as test functions in the definition of the operator A_0, we get

$$\Phi(u - p(u - \hat{u})) \geq \Phi(u) - \int_\Omega v p(u - \hat{u}) + \int_\Omega \langle \mathbf{a}(x, Du), Dp(u - \hat{u}) \rangle \tag{2.2}$$

and

$$\Phi(\hat{u} + p(u - \hat{u})) \geq \Phi(\hat{u}) + \int_\Omega \hat{v} p(u - \hat{u}) + \int_\Omega \langle \mathbf{a}(x, D\hat{u}), Dp(u - \hat{u}) \rangle. \tag{2.3}$$

Adding (2.2) and (2.3) we obtain

$$\int_\Omega \langle \mathbf{a}(x, Du) - \mathbf{a}(x, D\hat{u}), Dp(u - \hat{u}) \rangle + \int_\Omega (\hat{v} - v) p(u - \hat{u}) \leq$$

$$\leq \Phi(u - p(u - \hat{u})) + \Phi(\hat{u} + p(u - \hat{u})) - \Phi(u) - \Phi(\hat{u}).$$

By (H_2), the first integral in the above inequality is positive. We thus get

$$\int_\Omega (\hat{v} - v) p(u - \hat{u}) \leq \Phi(u - p(u - \hat{u})) + \Phi(\hat{u} + p(u - \hat{u})) - \Phi(u) - \Phi(\hat{u}). \tag{2.4}$$

On the other hand, by the convexity of j, it is easy to see that

$$\Phi(u - p(u - \hat{u})) + \Phi(\hat{u} + p(u - \hat{u})) \leq \Phi(u) + \Phi(\hat{u}). \tag{2.5}$$

We conclude from (2.4) and (2.5) that (2.1) holds. Therefore, by the equivalence between (1.1) and (1.2), A_0 is a completely accretive operator in $L^1(\Omega)$.

The proof of (iii) will involve several steps. The basic idea, however, is classical and consists in approximating the equation

$$u - \mathrm{div}\ \mathbf{a}(x, Du) = v, \qquad v \in L^\infty(\Omega),$$

by problems of the form

$$\gamma_n(u_n) - \mathrm{div}\ \mathbf{a}(x, Du_n) = v, \quad \text{with} \quad \gamma_n(\xi) := T_n(\xi) + \frac{1}{n}|\xi|^{p-2}\xi.$$

Step 1. For every $n \in \mathbb{N}$ we consider the operators $A_n : W^{1,p}(\Omega) \to \big(W^{1,p}(\Omega)\big)^\star$, defined by

$$\langle A_n u, w\rangle := \int_\Omega \langle \mathbf{a}(x, Du), Dw\rangle + \int_\Omega \gamma_n(u) w.$$

It is easy to see that A_n is monotone and hemicontinuous. Hence, A_n is pseudomonotone. Moreover, by (H_3), we have

$$\frac{\langle A_n u, u\rangle + \Phi(u)}{\|u\|_{1,p}} = \frac{\int_\Omega \langle \mathbf{a}(x, Du), Du\rangle + \int_\Omega \gamma_n(u)u + \Phi(u)}{\|u\|_{1,p}} \geq$$

$$\geq \frac{\lambda\ \int_\Omega |Du|^p + 1/n \int_\Omega |u|^p}{\|u\|_{1,p}} \geq \min\{\lambda, 1/n\}\ \|u\|_{1,p}^{p-1}$$

and thus

$$\lim_{\|u\|_{1,p}\to\infty} \frac{\langle A_n u, u\rangle + \Phi(u)}{\|u\|_{1,p}} = +\infty.$$

Then, given $v \in L^\infty(\Omega)$, by [Br$_1$, Corollary 30], there exists $u_n \in W^{1,p}(\Omega)$, such that

$$(v, w - u_n) - \langle A_n u_n, w - u_n\rangle \leq \Phi(w) - \Phi(u_n) \qquad \text{for all } w \in W^{1,p}(\Omega).$$

That is

$$\text{(2.6)} \quad \int_{\partial\Omega} j(w) \geq \int_{\partial\Omega} j(u_n) + \int_\Omega v(w-u_n) - \int_\Omega \gamma_n(u_n)(w-u_n) - \int_\Omega \langle \mathbf{a}(x, Du_n), D(w-u_n)\rangle.$$

for all $w \in W^{1,p}(\Omega)$.

Step 2. A priori estimates. Given $p \in P_0$, if we take $w = u_n - p(T_n u_n)$ in (2.6), we get

$$\text{(2.7)} \quad \begin{aligned} &\textstyle\int_{\partial\Omega} j(u_n - p(T_n u_n)) \geq \\ &\geq \textstyle\int_{\partial\Omega} j(u_n) - \int_\Omega v p(T_n u_n) + \int_\Omega \gamma_n(u_n) p(T_n u_n) + \int_\Omega \langle \mathbf{a}(x, Du_n), Dp(T_n u_n)\rangle. \end{aligned}$$

By the convexity of j, it is easy to see that

$$j(u_n - p(T_n u_n)) \leq j(u_n).$$

From here and (2.7) it follows that

$$\int_\Omega T_n(u_n)p(T_n(u_n)) \le \int_\Omega vp(T_n(u_n)).$$

This gives $T_n(u_n) \ll v$. Thus, $\|T_n(u_n)\|_\infty \le \|v\|_\infty$ for all $n \in \mathbb{N}$. In particular,

$$\|u_n\|_\infty \le \|v\|_\infty \quad \text{for all } n \ge \|v\|_\infty. \tag{2.8}$$

On the other hand, taking $w = 0$ as test function in (2.6) and using (H_1), we obtain

$$\lambda \int_\Omega |Du_n|^p \le \int_\Omega \langle \mathbf{a}(x, Du_n), Du_n\rangle \le \int_\Omega vu_n. \tag{2.9}$$

As a consequence of (2.8) and (2.9), $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. Hence there exists a subsequence, still denoted u_n, such that $u_n \to u \in W^{1,p}(\Omega)$ weakly in $W^{1,p}(\Omega)$. Moreover, by the Rellich-Kondrachov Theorem, $u_n \to u$ in $L^p(\Omega)$, and by [M, Theorem 3.4.5], $u_n \to u$ in $L^p(\partial\Omega)$. After passing to a suitable subsequence, we can assume that $u_n \to u$ a.e. in Ω. So, by (2.8), $\|u\|_\infty \le \|v\|_\infty$.

Step 3. Convergence. We now prove that Du_n converges to Du in measure, to do this we follow the same technique used in [BG] (see also [BW]). Since Du_n converges to Du weakly in $L^p(\Omega)$, it is enough to show that $\{Du_n\}$ is a Cauchy sequence in measure. Let t and $\epsilon > 0$. For some $A > 1$ we set

$$C(x, A, t) := \inf\{\langle \mathbf{a}(x, \xi) - \mathbf{a}(x, \eta), \xi - \eta\rangle \ : \ |\xi| \le A, \ |\eta| \le A, \ |\xi - \eta| \ge t \}.$$

Having in mind that the function $\psi \to \mathbf{a}(x, \psi)$ is continuous for almost all $x \in \Omega$ and the set $\{(\xi, \eta) \ : \ |\xi| \le A, \ |\eta| \le A, \ |\xi - \eta| \ge t \}$ is compact, the infimum in the definition of $C(x, A, t)$ is a minimum. Hence, by (H_2), it follows that

$$C(x, A, t) > 0 \quad \text{for almost all } x \in \Omega. \tag{2.10}$$

Now, for $n, m \in \mathbb{N}$ and any $k > 0$, the following inclusions hold

$$\{|Du_n - Du_m| > t\} \subset$$

$$\subset \{|Du_n| \ge A\} \cup \{|Du_m| \ge A\} \cup \{|u_n - u_m| \ge k^2\} \cup \{C(x, A, t) \le k\} \cup \tag{2.11}$$

$$\cup \{|u_n - u_m| \le k^2, \ C(x, A, t) \ge k, \ |Du_n| \le A, \ |Du_m| \le A, \ |Du_n - Du_m| > t\}.$$

Since $\{Du_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$ we can choose A large enough in order to have

$$\lambda_N(\{|Du_n| \ge A\} \ \cup \ \{|Du_m| \ge A\}) \le \frac{\epsilon}{4} \quad \text{for all } n, m \in \mathbb{N}. \tag{2.12}$$

By (2.10) we can choose k small enough in order to have

$$\lambda_N(\{C(x, A, t) \le k\}) \le \frac{\epsilon}{4}. \tag{2.13}$$

On the other hand, if we use $u_n - T_k(u_n - u_m)$ and $u_m + T_k(u_n - u_m)$ as test functions in (2.6), we obtain

$$(2.14)\qquad \begin{aligned} -\textstyle\int_\Omega vT_k(u_n-u_m) + \int_\Omega \gamma_n(u_n)T_k(u_n-u_m) + \int_\Omega \langle \mathbf{a}(x,Du_n), DT_k(u_n-u_m)\rangle \le \\ \le \textstyle\int_{\partial\Omega} j(u_n - T_k(u_n-u_m)) - \int_{\partial\Omega} j(u_n). \end{aligned}$$

and

$$(2.15)\qquad \begin{aligned} \textstyle\int_\Omega vT_k(u_n-u_m) - \int_\Omega \gamma_m(u_m)T_k(u_n-u_m) - \int_\Omega \langle \mathbf{a}(x,Du_m), DT_k(u_n-u_m)\rangle \le \\ \le \textstyle\int_{\partial\Omega} j(u_m + T_k(u_n-u_m)) - \int_{\partial\Omega} j(u_m). \end{aligned}$$

Adding (2.14) and (2.15), we get

$$\int_\Omega \langle \mathbf{a}(x,Du_n) - \mathbf{a}(x,Du_m), DT_k(u_n-u_m)\rangle + \int_\Omega \big(\gamma_n(u_n) - \gamma_m(u_m)\big)T_k(u_n-u_m) \le$$

$$\le \int_{\partial\Omega} j(u_n - T_k(u_n-u_m)) + j(u_m + T_k(u_n-u_m)) - \int_{\partial\Omega} j(u_n) - \int_{\partial\Omega} j(u_n) \le 0.$$

Consequently,

$$\int_\Omega \langle \mathbf{a}(x,Du_n) - \mathbf{a}(x,Du_m), DT_k(u_n-u_m)\rangle \le k\int_\Omega |\gamma_n(u_n)| + |\gamma_m(u_m)| \le kM.$$

Hence

$$\lambda_N(\{|u_n-u_m|\le k^2,\ C(x,A,t)\ge k,\ |Du_n|\le A,\ |Du_m|\le A,\ |Du_n - Du_m|>t\}) \le$$

$$(2.16)\qquad \le \lambda_N(\{|u_n-u_m|\le k^2,\ \langle \mathbf{a}(x,Du_n) - \mathbf{a}(x,Du_m), D(u_n-u_m)\rangle \ge k\}) \le$$

$$\le \frac{1}{k}\int_{\{|u_n-u_m|<k^2\}} \langle \mathbf{a}(x,Du_n) - \mathbf{a}(x,Du_m), D(u_n-u_m)\rangle \le \frac{1}{k}\,k^2M \le \frac{\epsilon}{4},$$

for k small enough.

Since A and k have been already choosen, if n_0 is large enough we have for $n,m\ge n_0$ the estimate $\lambda_N(\{|u_n-u_m|\ge k^2\}) \le \frac{\epsilon}{4}$. From here, using (2.11), (2.12), (2.13) and (2.16), we can conclude that

$$\lambda_N(\{|Du_n - Du_m|\ge t\}) \le \epsilon \quad \text{for} \quad m,n\ge n_0.$$

According to Nemytskii's Theorem [K, Lemma I.2.2.1] the convergence of Du_n to Du in measure implies that $\mathbf{a}(x,Du_n)$ converges in measure to $\mathbf{a}(x,Du)$, and a.e. (up to extraction of a subsequence, if necessary). Moreover, since $\{Du_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, from (H_3) it follows that $\{\mathbf{a}(x,Du_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(\Omega)$. Therefore,

$$(2.17)\qquad \mathbf{a}(x,Du_n) \to \mathbf{a}(x,Du) \quad \text{weakly in} \quad L^{p'}(\Omega).$$

Step 4. To complete the proof it remains to show that

$$\int_{\partial\Omega} j(w) \geq \int_{\partial\Omega} j(u) + \int_{\Omega} (v-u)(w-u) - \int_{\Omega} \langle \mathbf{a}(x, Du), D(w-u)\rangle. \tag{2.18}$$

for all $w \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$.
Let $w \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, from (2.17), it follows that

$$\int_{\Omega} \langle \mathbf{a}(x, Du), Dw\rangle = \lim_{n\to\infty} \int_{\Omega} \langle \mathbf{a}(x, Du_n), Dw\rangle. \tag{2.19}$$

By Fatou's Lemma, we have

$$\int_{\Omega} \langle \mathbf{a}(x, Du), Du\rangle \leq \liminf_{n\to\infty} \int_{\Omega} \langle \mathbf{a}(x, Du_n), Du_n\rangle. \tag{2.20}$$

and

$$\int_{\partial\Omega} j(u) \leq \liminf_{n\to\infty} \int_{\partial\Omega} j(u_n). \tag{2.21}$$

Since $u_n \to u$ in $L^p(\Omega)$ we have

$$\lim_{n\to\infty} \int_{\Omega} (v - \gamma_n(u_n))(w-u_n) = \int_{\Omega} (v-u)(w-u). \tag{2.22}$$

From (2.19), (2.20), (2.21) and (2.22), passing to the limit in (2.6), we get (2.18), and the proof is concluded.

Next we are going to introduce another operator in $L^1(\Omega)$, smaller than A_0, which will be used later to characterize the variational operator A_0. To define the new operator we need introduce the following subset of $W^{1,p}(\Omega)$: Given β a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$, we set

$$W^{1,p}_{\beta}(\Omega) := \{u \in W^{1,p}(\Omega) \ : \ u(x) \in \mathcal{D}(\beta) \ \text{ a.e. } x \in \partial\Omega\}.$$

The above definition uses the fact that the trace of $u \in W^{1,1}(\Omega)$ on $\partial\Omega$ is well defined in $L^1(\partial\Omega)$ [N, Theorem 4.2]. Observe that we use the same notation u for u and its trace when convenient.

Remark that $W^{1,p}_{\beta}(\Omega)$ is a closed convex subset of $W^{1,p}(\Omega)$. In case β corresponds to the Dirichlet boundary condition, $W^{1,p}_{\beta}(\Omega) = W^{1,p}_{0}(\Omega)$, and in case β corresponds to the Neumann boundary condition, $W^{1,p}_{\beta}(\Omega) = W^{1,p}(\Omega)$.

We define the operator A in $L^1(\Omega)$ by the rule:

$(u,v) \in A$ if and only if $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, $v \in L^1(\Omega)$, there exists $w \in L^1(\partial\Omega)$ with $-w(x) \in \beta(u(x))$ a.e. on $\partial\Omega$ and

$$\int_{\Omega} \langle \mathbf{a}(x, Du), D(u-\phi)\rangle \leq \int_{\Omega} v(u-\phi) + \int_{\partial\Omega} w(u-\phi) \quad \text{for every } \ \phi \in W^{1,p}_{\beta}(\Omega) \cap L^{\infty}(\Omega).$$

Since $\partial j = \beta$, it is easy to see that $A \subset A_0$. Consequently, A is a completely accretive operator. Let us see now, that under certain conditions, the operator A satisfies the range condition.

Theorem 2.2. *If* $\mathcal{D}(\beta)$ *is closed, then the operator* A *satisfies the range condition. More precisely,*

$$L^\infty(\Omega) \subset R(I+A).$$

Proof. For every $n \in \mathbb{N}$, let $\gamma_n(\xi) := T_n(\xi) + \frac{1}{n}|\xi|^{p-2}\xi$, and β_n be the Yosida approximation of β, i.e., $\beta_n = n\big(I - (I + 1/n\ \beta)^{-1}\big)$. Consider the operators $A_n : W^{1,p}_\beta(\Omega) \to \big(W^{1,p}(\Omega)\big)^\star$, defined by

$$\langle A_n u, v\rangle := \int_\Omega \langle \mathbf{a}(x, Du), Dv\rangle + \int_\Omega \gamma_n(u)v + \int_{\partial\Omega} T_n\beta_n(u)v.$$

As a consequence of (H_3) and having in mind that $W^{1,p}(\Omega)$ is continuously embedded in $L^p(\partial\Omega)$ (see e.g., [M, Theorem 3.4.5]), we have that the operators A_n are well defined and map bounded subsets into bounded subsets. Proceeding as in the proof of the above theorem we have A_n is a monotone, hemicontinuous and coercive operator. Given $v \in L^\infty(\Omega)$, by a classical result of Browder (cf. [KS, Chapter III]), there exists $u_n \in W^{1,p}_\beta(\Omega)$ such that

$$\begin{aligned} &\textstyle\int_\Omega \langle \mathbf{a}(x, Du_n), D(u_n - \phi)\rangle + \int_\Omega \gamma_n(u_n)(u_n - \phi) + \int_{\partial\Omega} T_n\beta_n(u_n)(u_n - \phi) \le \\ &\qquad \le \textstyle\int_\Omega v(u_n - \phi), \quad \text{for every } \phi \in W^{1,p}_\beta(\Omega). \end{aligned} \tag{2.23}$$

Now, as in the the steps 2 and 3 of the proof of the above theorem, we get the same estimates and convergence results for $\{u_n\}$. So the proof is completed by showing that there exists $w \in L^1(\partial\Omega)$ with $-w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, such that

$$\lim_{n\to\infty} \int_{\partial\Omega} T_n\beta_n(u_n)(u_n - \phi) = \int_{\partial\Omega} w(u - \phi). \tag{2.24}$$

In fact, since $u_n \in \mathcal{D}(\beta)$, $|\beta_n(u_n(x))| \le \inf\{|r| \ : \ r \in \beta(u_n(x))\}$. From here, if $\mathcal{D}(\beta)$ is bounded, $\{\beta_n(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\partial\Omega)$. In case $\mathcal{D}(\beta)$ is unbounded, the boundedness of $\{\beta_n(u_n(x))\}$ in $L^\infty(\partial\Omega)$ follows from the boundedness of $\{u_n(x)\}$ in $L^\infty(\Omega)$. Consequently,

$$T_n\beta_n(u_n) \to w \in L^1(\partial\Omega) \quad \text{weakly in } L^1(\partial\Omega). \tag{2.25}$$

Then, since $\{u_n - \phi \ : \ n \in \mathbb{N}\}$ is bounded in $L^\infty(\Omega)$, $u_n - \phi \to u - \phi$ a.e. in $\partial\Omega$ and $\mu(\partial\Omega) < \infty$, (2.25) yields (2.24).

Finally, since $u_n \to u$ in $L^1(\partial\Omega)$ and (2.25), from [BCS, Lemma G] it follows that $-w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and the proof is concluded.

The assumption $\mathcal{D}(\beta)$ closed can be removed when the function $\mathbf{a}$ is "smooth enough". More precisely, we give the following definition due to Ph. Bénilan (personal communication). We say that $\mathbf{a}$ is *smooth* if for every $f \in L^\infty(\Omega)$ there exists $g \in L^1(\partial\Omega)$ such that the solution u of the Dirichlet problem

$$-\text{div}\ \mathbf{a}(x, Du) = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on} \quad \partial\Omega$$

is solution of the Neumann problem

$$-\text{div } \mathbf{a}(x, Du) = f \quad \text{in} \quad \Omega$$

$$\frac{\partial u}{\partial \eta_a} = g \quad \text{on} \quad \partial\Omega.$$

For $\mathbf{a}$ smooth we have the following result.

Theorem 2.3. *If* $\mathbf{a}$ *is smooth, then the operator* A *satisfies*

$$L^\infty(\Omega) \subset R(I + A).$$

Proof. Fix $v \in L^\infty(\Omega)$. Let β_n be the Yosida approximations of β. Since $W^{1,p}_{\beta_n}(\Omega) = W^{1,p}(\Omega)$, by Theorem 2.2, there exists $u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\beta_n(u_n) \in L^1(\partial\Omega)$ and

$$\int_\Omega \langle \mathbf{a}(x, Du_n), D\phi\rangle + \int_\Omega u_n \phi = \int_\Omega v\phi - \int_{\partial\Omega} \beta_n(u_n)\phi, \tag{2.26}$$

for every $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We can now proceed analogously to the proof of Theorem 2.2. So the proof is completed by showing that the sequence $\{\beta_n(u_n) : n \in \mathbb{N}\}$ is a relatively weakly compact subset of $L^1(\partial\Omega)$. To see this, let $\hat{u}$ be the solution of the Dirichlet problem

$$\hat{u} - \text{div } \mathbf{a}(x, D\hat{u}) = v \quad \text{in} \quad \Omega$$

$$\hat{u} = 0 \quad \text{on} \quad \partial\Omega.$$

Since $\mathbf{a}$ is smooth, it follows that

$$\int_\Omega \hat{u}\phi + \int_\Omega \langle \mathbf{a}(x, D\hat{u}), D\phi\rangle = \int_\Omega v\phi + \int_{\partial\Omega} g\phi \tag{2.27}$$

for every $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.
Given $p \in P_0$, if we take $\phi = p(\beta_n(u_n - \hat{u}))$ as test function in (2.26) and (2.27), substracting (2.27) from (2.26), we get

$$\int_\Omega \langle \mathbf{a}(x, Du_n) - \mathbf{a}(x, D\hat{u}), Dp(\beta_n(u_n - \hat{u}))\rangle + \int_\Omega (u_n - \hat{u})p(\beta_n(u_n - \hat{u})) =$$

$$= -\int_{\partial\Omega} \big(g + \beta_n(u_n)\big)p(\beta_n(u_n - \hat{u})).$$

Since the left hand side is non negative and $\hat{u} \in W^{1,p}_0(\Omega)$, it follows that

$$\int_{\partial\Omega} \big(g + \beta_n(u_n)\big)p(\beta_n(u_n)) \le 0.$$

Hence $\beta_n(u_n) \ll -g$ for all $n \in \mathbb{N}$. Then, since $g \in L^1(\partial\Omega)$, from [BCr, Proposition 2.11], it follows that $\{\beta_n(u_n) : n \in \mathbb{N}\}$ is a relatively weakly compact subset of $L^1(\partial\Omega)$ and the proof is completed.

Remark 2.4. The condition **a** smooth is satisfied in some cases. For instance, in the linear case when the coeficients of **a** are regular, cf. [Br$_2$]. As a consequence of the regularity results given in [Li] it is also possible to show the smoothness of the p-Laplacian operator. We do not know if the assumptions (H$_1$), (H$_2$) and (H$_3$) implies the smoothness of **a**.

An interesting problem is to characterize the operator $\overline{A}$. In the case of Dirichlet boundary condition, this has been done recently in [B-V], where a new concept of solution, named *entropy solution*, is defined for the elliptic problem

$$-\text{div } \mathbf{a}(x, Du) = F(x, u) \quad \text{in } \Omega$$

$$u(x) = 0 \quad \text{on } \partial\Omega.$$

For that, the authors need to introduce a new space $\mathcal{T}_0^{1,p}(\Omega)$, which is an extension of $W_0^{1,p}(\Omega)$. Here, in the same spirit, we need to extend the concept of trace. This is the subject of the next section.

3. A generalization of the trace.

Before to discuss the concept of trace we recall the following spaces introduced in [B-V]: $\mathcal{T}_{loc}^{1,1}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for every $k > 0$ the truncated function $T_k(u)$ belongs to $W_{loc}^{1,1}(\Omega)$. For $1 < p < \infty$, $\mathcal{T}_{loc}^{1,p}(\Omega)$ is the subset of $\mathcal{T}_{loc}^{1,1}(\Omega)$ consisting of the functions u such that $DT_k(u) \in L_{loc}^p(\Omega)$ for every $k > 0$. Likewise, $\mathcal{T}^{1,p}(\Omega)$ is the subset of $\mathcal{T}_{loc}^{1,1}(\Omega)$ consisting of the functions u such that $DT_k(u) \in L^p(\Omega)$ for every $k > 0$. Observe that in the definition of $\mathcal{T}^{1,p}(\Omega)$ is not imposed the condition $T_k(u) \in L^p(\Omega)$. Of course, this condition follows immediately when Ω is bounded. So in this case we have

$$\mathcal{T}^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable } : T_k(u) \in W^{1,p}(\Omega) \quad \text{for all } k > 0\}.$$

It is possible to give a sense to the derivative Du of a function $u \in \mathcal{T}_{loc}^{1,1}(\Omega)$, generalizing the usual concept of weak derivative in $W_{loc}^{1,1}(\Omega)$, thanks to the following result (see [B-V, Lemma 2.1]):

"For every $u \in \mathcal{T}_{loc}^{1,1}(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}$ such that

$$DT_k(u) = v 1_{\{|v|<k\}} \qquad a.e. \tag{3.1}$$

Furthermore, if $u \in W_{loc}^{1,1}(\Omega)$, then $v = Du$ in the usual weak sense."

The derivative Du of a function $u \in \mathcal{T}_{loc}^{1,1}(\Omega)$ is defined as the unique function v satisfying (3.1). This notation will be used throughout in the sequel. We remark that this definition of derivative is not a definition in the sense of distributions.

Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^1 and $1 \leq p < \infty$. It is well-known (cf. [N] or [M]) that if $u \in W^{1,p}(\Omega)$, it is possible to define the trace of u on $\partial\Omega$. More precisely, there is a bounded operator γ from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$ such that $\gamma(u) = u_{|\partial\Omega}$ whenever $u \in C(\overline{\Omega})$. Now, it is easy to see that, in general, it is not possible to define the trace of an element of $\mathcal{T}^{1,p}(\Omega)$. In dimension one it is enough to consider the function $u(x) = 1/x$ for $x \in]0,1[$. Nevertheless, we are going to define the trace for the elements of a subset $\mathcal{T}^{1,p}_{tr}(\Omega)$ of $\mathcal{T}^{1,p}(\Omega)$. $\mathcal{T}^{1,p}_{tr}(\Omega)$ will be the subset of $\mathcal{T}^{1,p}(\Omega)$ consisting of the functions that can be approximated by functions of $W^{1,p}(\Omega)$ in the following sense: a function $u \in \mathcal{T}^{1,p}(\Omega)$ belongs to $\mathcal{T}^{1,p}_{tr}(\Omega)$ if there exists a sequence $u_n \in W^{1,p}(\Omega)$ such that

(a) $u_n \to u$ a.e. in Ω,

(b) $DT_k(u_n) \to DT_k(u)$ in $L^1(\Omega)$ for any $k > 0$,

(c) the sequence $\{\gamma(u_n)\}$ converges a.e. in $\partial\Omega$.

Obviously, we have

$$W^{1,p}(\Omega) \subset \mathcal{T}^{1,p}_{tr}(\Omega) \subset \mathcal{T}^{1,p}(\Omega). \tag{3.2}$$

In (3.2) the inclusions are strict. In fact: It is easy to see that the function $u(x) = 1/x$ for $x \in]0,1[$ is an element of $\mathcal{T}^{1,1}(]0,1[) \setminus \mathcal{T}^{1,1}_{tr}(]0,1[)$. Moreover the function u defined by

$$u(x) := \begin{cases} 1/x & \text{if } x \in]0,1[\\ -1/x & \text{if } x \in]-1,0[, \end{cases}$$

is an example of an element of $\mathcal{T}^{1,1}_{tr}(]-1,1[) \setminus W^{1,1}(]-1,1[)$.

In the following result we obtain an extension of the trace defined in $W^{1,p}(\Omega)$.

Theorem 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^1 and $1 \leq p < \infty$. Then, there exists a map $\tau : \mathcal{T}^{1,p}_{tr}(\Omega) \to M(\partial\Omega, \mu)$, such that*

$$\tau(u) = \gamma(u) \quad \textit{whenever } u \in W^{1,p}(\Omega).$$

Moreover,

(i) $\gamma(T_k u) = T_k(\tau u)$ for every $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and $k > 0$.

(ii) If $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $u - \phi \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and $\tau(u - \phi) = \tau(u) - \tau(\phi)$.

Proof. Given $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$, let $u_n \in W^{1,p}(\Omega)$ be such that

(a) $u_n \to u$ a.e. in Ω,

(b) $DT_k(u_n) \to DT_k(u)$ in $L^1(\Omega)$ for any $k > 0$,

(c) $\gamma(u_n) \to v \in M(\partial\Omega, \mu)$ a.e. in $\partial\Omega$.

Then, by the Dominated Convergence Theorem, $T_k u_n \to T_k u$ in $W^{1,1}(\Omega)$. Consequently, since the trace γ from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$ is a linear bounded operator, it follows that

$$\lim_{n\to\infty} \|\gamma(T_k u_n) - \gamma(T_k u)\|_{L^1(\partial\Omega)} = 0 \qquad \text{for every } k \in \mathbb{N}.$$

Now, by (c), $T_k(\gamma(u_n)) \to T_k(v)$ a.e. in $\partial\Omega$. Consequently, $\gamma(T_k(u)) = T_k(v)$ a.e. in $\partial\Omega$ for any $k > 0$. Thus, $\gamma(T_k(u)) \to v$ a.e. in $\partial\Omega$. Therefore we can defined $\tau(u)$

as the a.e. limit in $\partial\Omega$ of $\gamma(T_k(u))$ as $k \to \infty$. Obviously, $\tau(u) = \gamma(u)$ whenever $u \in W^{1,p}(\Omega)$. Also, by the definition of the map τ, (i) holds.
(ii) Let $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ and $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, there exists $u_n \in W^{1,p}(\Omega)$ satisfying (a), (b) and (c). Take $\psi_n := u_n - \phi \in W^{1,p}(\Omega)$. Then, $\psi_n \to u - \phi$ a.e. in Ω and $\gamma(\psi_n) \to \tau u - \tau\phi$ a.e. in $\partial\Omega$. Hence, to finish the proof we only need to prove that $DT_k(\psi_n) \to DT_k(u-\phi)$ in $L^1(\Omega)$. Indeed: it is easy to see that $DT_k(\psi_n) \to DT_k(u-\phi)$ a.e. On the other hand, if $M := k + \|\phi\|_\infty$, we have

$$|DT_k(\psi_n)| \leq |DT_M(u_n)| + |D\phi|.$$

Moreover, since $DT_M(u_n) \to DT_M(u)$ in $L^1(\Omega)$, there exists $g \in L^1(\Omega)$ such that $|DT_M(u_n)| \leq g$ a.e. Therefore, by the Dominated Convergence Theorem it follows that $DT_k(\psi_n) \to DT_k(u-\phi)$ in $L^1(\Omega)$.

To study the Dirichlet problem, in [B-V], it is introduced the subspace $\mathcal{T}_0^{1,p}(\Omega)$ of $\mathcal{T}^{1,p}(\Omega)$ consisting of the functions that can be approximated by smooth functions with compact support in Ω in the following sense: a function $u \in \mathcal{T}^{1,p}(\Omega)$ belongs to $\mathcal{T}_0^{1,p}(\Omega)$ if for every $k > 0$ there exists a sequence $\zeta_n \in C_0^\infty(\Omega)$ such that

$$\zeta_n \to T_k u \quad \text{in } L^1_{loc}(\Omega),$$

$$D\zeta_n \to DT_k(u) \quad \text{in } L^p(\Omega).$$

As a consequence of the characterizations of $\mathcal{T}_0^{1,p}(\Omega)$ given in [B-V, Appendix II] we have

$$\mathrm{Ker}(\tau) = \mathcal{T}_0^{1,p}(\Omega).$$

4. The closure of the operator A.

In this section we characterize the closure of the operator A by means of a new concept of solution. As we mention in the introduction, a new concept of entropy solution for the Dirichlet problem

$$-\mathrm{div}\ \mathbf{a}(x, Du) = v(x) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega.$$

is introduced in [B-V]. Following this idea we define the concept of entropy solution for the problem

$$(II) \qquad \begin{cases} -\mathrm{div}\ \mathbf{a}(x, Du) = v(x) & \text{in } \Omega \\ -\frac{\partial u}{\partial \eta_a} = \beta(u) & \text{on } \partial\Omega \end{cases}$$

where $v \in L^1(\Omega)$, in the following way: We say that $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ is an *entropy solution* of (II) if there exists $w \in L^1(\partial\Omega)$, $-w(x) \in \beta(u(x))$ a.e. on $\partial\Omega$ such that

$$(4.1) \qquad \int_\Omega \langle \mathbf{a}(x, Du), DT_k(u-\phi)\rangle \leq \int_\Omega vT_k(u-\phi) + \int_{\partial\Omega} wT_k(u-\phi)$$

for every $\phi \in W_\beta^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$.

Notice that the integrals in (4.1) are well defined. In general, the diference of two elements of $\mathcal{T}^{1,p}(\Omega)$ is not an element of $\mathcal{T}^{1,p}(\Omega)$ (see [B-V]), however, since $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have $u - \phi \in \mathcal{T}_{tr}^{1,p}(\Omega)$ (Theorem 3.1). Hence, $T_k(u-\phi) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and consequently the two first integrals in (4.1) are well defined. Moreover, in the last integral we can use the fact that the trace of $f \in W^{1,p}(\Omega)$ on $\partial\Omega$ is well defined in $L^p(\partial\Omega)$. Observe we use the same notation f for f and its trace when convenient.

Using the concept of entropy solution we define the operator $\mathcal{A}$ in $L^1(\Omega)$ by the rule: $(u,v) \in \mathcal{A}$ if and only if $u, v \in L^1(\Omega)$ and u is an entropy solution of (II). Since $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$, by Theorem 3.1, the trace of u on $\partial\Omega$ is well defined as a function of $M(\partial\Omega, \mu)$.

In order to show that the operator $\mathcal{A}$ is the closure of A we need the following lemma.

Lemma 4.1. *If* $(u,v) \in \mathcal{A}$ *and* $\alpha, k > 0$*. Then the following inequality holds*

$$\frac{1}{k} \int_{\{\alpha < |u| < \alpha + k\}} |Du|^p \leq \frac{1}{\lambda} \int_{\{|u| \geq \alpha\}} |v|.$$

Proof. If we take $T_\alpha(u)$ as test function in (4.1) we have

$$\int_\Omega \langle \mathbf{a}(x, Du), DT_k(u - T_\alpha(u)) \rangle \leq \int_\Omega v T_k(u - T_\alpha(u)) + \int_{\partial\Omega} w T_k(u - T_\alpha(u)).$$

Now, since the last integral is negative, using (H_1), it follows that

$$k \int_{\{|u| \geq \alpha\}} |v| \geq \int_\Omega \langle \mathbf{a}(x, Du), DT_k(u - T_\alpha(u)) \rangle =$$

$$= \int_{\{\alpha < |u| < k + \alpha\}} \langle \mathbf{a}(x, Du), Du \rangle \geq \lambda \int_{\{\alpha < |u| < \alpha + k\}} |Du|^p.$$

Now we come to the main result.

Theorem 4.2. *Suppose* $1 < p < N$ *and assume that* $\mathcal{D}(\beta)$ *is closed or* $\mathbf{a}$ *is smooth. Then, the operator* $\mathcal{A}$ *is the closure of* A *in* $L^1(\Omega)$*. Consequently,* $\mathcal{A}$ *is an m-T-accretive operator in* $L^1(\Omega)$*. Moreover, if* $(u,v) \in \mathcal{A}$*, then*

$$u \in M^{p_1}(\Omega), \qquad |Du| \in M^{p_2}(\Omega)$$

where $p_1 = \frac{N(p-1)}{N-p}$ *and* $p_2 = \frac{N(p-1)}{N-1}$*. In case* $p > 2 - 1/N$*,* $u \in W^{1,q}(\Omega)$ *for every* $1 \leq q < p_2$*.*

Proof. By Theorem 2.2 or Theorem 2.3 and maximal accretivity, it is enough to show that $\mathcal{A}$ is accretive in $L^1(\Omega)$ and $\overline{A} \subset \mathcal{A}$.

Step 1. Accretivity of $\mathcal{A}$. To prove the accretivity of $\mathcal{A}$, we must show that

$$\int_\Omega |u - \hat{u}| \leq \int_\Omega |f - \hat{f}| \tag{4.2}$$

whenever $f \in u + \mathcal{A}u$, $\hat{f} \in \hat{u} + \mathcal{A}\hat{u}$. In fact: Let $w, \hat{w} \in L^1(\partial\Omega)$ with $-w(x) \in \beta(u(x))$ and $-\hat{w}(x) \in \beta(\hat{u}(x))$ a.e. on $\partial\Omega$ such that for every $h > 0$,

$$\int_\Omega \langle \mathbf{a}(x, Du), DT_k(u - T_h(\hat{u}))\rangle \leq \int_\Omega (f - u)T_k(u - T_h(\hat{u})) + \int_{\partial\Omega} wT_k(u - T_h(\hat{u}))$$

and

$$\int_\Omega \langle \mathbf{a}(x, D\hat{u}), DT_k(\hat{u} - T_h(u))\rangle \leq \int_\Omega (\hat{f} - \hat{u})T_k(\hat{u} - T_h(u)) + \int_{\partial\Omega} \hat{w}T_k(\hat{u} - T_h(u)).$$

We write

$$I_{h,k} := \int_\Omega \langle \mathbf{a}(x, Du), DT_k(u - T_h(\hat{u}))\rangle + \int_\Omega \langle \mathbf{a}(x, D\hat{u}), DT_k(\hat{u} - T_h(u))\rangle,$$

$$J_h^1 := \int_\Omega (f - u)T_k(u - T_h(\hat{u})) + \int_\Omega (\hat{f} - \hat{u})T_k(\hat{u} - T_h(u))$$

and

$$J_h^2 := \int_{\partial\Omega} wT_k(u - T_h(\hat{u})) + \int_{\partial\Omega} \hat{w}T_k(\hat{u} - T_h(u)).$$

Adding up the above two inequalities we get

$$I_{h,k} \leq J_h^1 + J_h^2. \tag{4.3}$$

By the Dominated Convergence Theorem it follows that

$$\lim_{h\to\infty} J_h^1 = \int_\Omega \big((f - u) - (\hat{f} - \hat{u})\big)T_k(u - \hat{u}). \tag{4.4}$$

and

$$\lim_{h\to\infty} J_h^2 = \int_{\partial\Omega} (w - \hat{w})T_k(u - \hat{u}). \tag{4.5}$$

Since the integral in (4.5) is negative, from (4.3) and (4.4) we have that

$$\int_\Omega \big((f - u) - (\hat{f} - \hat{u})\big)T_k(u - \hat{u}) \geq \liminf_{h\to\infty} I_{h,k}. \tag{4.6}$$

Then, if we prove that,

$$\liminf_{h\to\infty} I_{h,k} \geq 0 \quad \text{for any} \quad k > 0, \tag{4.7}$$

we get from (4.6)

$$\frac{1}{k}\int_\Omega (u - \hat{u})T_k(u - \hat{u}) \leq \frac{1}{k}\int_\Omega (f - \hat{f})T_k(u - \hat{u}) \leq \int_\Omega |f - \hat{f}|.$$

From here, passing to the limit as $k \to 0^+$, it follows (4.2). To prove (4.7) we proceed by splitting the integral into different integration sets. We write

$$I_{h,k} = I^1_{h,k} + I^2_{h,k} + I^3_{h,k} + I^4_{h,k},$$

where

$$I^1_{h,k} := \int_{\{|u|<h,\ |\hat{u}|<h\}} \langle \mathbf{a}(x, Du) - \mathbf{a}(x, D\hat{u}), DT_k(u - \hat{u})\rangle \geq 0.$$

$$I^2_{h,k} := \int_{\{|u|<h,\ |\hat{u}|\geq h\}} \langle \mathbf{a}(x, Du), DT_k(u - h\mathrm{sign}(\hat{u}))\rangle + \int_{\{|u|<h,\ |\hat{u}|\geq h\}} \langle \mathbf{a}(x, D\hat{u}), DT_k(\hat{u} - u)\rangle \geq$$

$$\geq \int_{\{|u|<h,\ |\hat{u}|\geq h\}} \langle \mathbf{a}(x, D\hat{u}), DT_k(\hat{u} - u)\rangle.$$

$$I^3_{h,k} := \int_{\{|u|\geq h,\ |\hat{u}|<h\}} \langle \mathbf{a}(x, Du), DT_k(u - \hat{u})\rangle + \int_{\{|u|\geq h,\ |\hat{u}|<h\}} \langle \mathbf{a}(x, D\hat{u}), DT_k(\hat{u} - h\mathrm{sign}(u))\rangle \geq$$

$$\geq \int_{\{|u|\geq h,\ |\hat{u}|<h\}} \langle \mathbf{a}(x, Du), DT_k(u - \hat{u})\rangle.$$

$$I^4_{h,k} := \int_{\{|u|\geq h,\ |\hat{u}|\geq h\}} \langle \mathbf{a}(x, Du), DT_k(u - h\mathrm{sign}(\hat{u}))\rangle +$$

$$+ \int_{\{|u|\geq h,\ |\hat{u}|\geq h\}} \langle \mathbf{a}(x, D\hat{u}), DT_k(\hat{u} - h\mathrm{sign}(u))\rangle \geq 0.$$

Combining the above estimates we get

$$I_{h,k} \geq L^1_{h,k} + L^2_{h,k},$$

where

$$L^1_{h,k} := \int_{\{|u|<h,\ |\hat{u}|\geq h\}} \langle \mathbf{a}(x, D\hat{u}), DT_k(\hat{u} - u)\rangle$$

and

$$L^2_{h,k} := \int_{\{|u|\geq h,\ |\hat{u}|<h\}} \langle \mathbf{a}(x, Du), DT_k(u - \hat{u})\rangle.$$

Now, if we put

$$C(h,k) := \{h < |u| < k + h\} \cap \{h - k < |\hat{u}| < h\},$$

we have

$$|L^2_{h,k}| \leq \int_{\{|u-\hat{u}|<k,\ |u|\geq h,\ |\hat{u}|<h\}} |\langle \mathbf{a}(x, Du), Du - D\hat{u}\rangle| \leq$$

$$\leq \int_{C(h,k)} |\langle \mathbf{a}(x, Du), Du\rangle| + \int_{C(h,k)} |\langle \mathbf{a}(x, Du), D\hat{u}\rangle|.$$

Then, by Hölder's inequality, we get

$$|L^2_{h,k}| \leq \left(\int_{C(h,k)} |\mathbf{a}(x,Du)|^{p'} \right)^{1/p'} \left(\left(\int_{C(h,k)} |Du|^p \right)^{1/p} + \left(\int_{C(h,k)} |D\hat{u}|^p \right)^{1/p} \right).$$

Now, by (H_3),

$$\left(\int_{C(h,k)} |\mathbf{a}(x,Du)|^{p'} \right)^{1/p'} \leq \left(\int_{C(h,k)} \left(\Lambda (j(x) + |Du|^{p-1}) \right)^{p'} \right)^{1/p'} \leq$$

$$\leq \Lambda 2^{\frac{1}{p}} \left(\|j\|_{p'}^{p'} + \int_{\{h<|u|<k+h\}} |Du|^p \right)^{1/p'}. \tag{4.8}$$

On the other hand, applying Lemma 4.1, we obtain

$$\int_{\{h<|u|<k+h\}} |Du|^p \leq \frac{k}{\lambda} \int_{\{|u|\geq h\}} |f-u|. \tag{4.9}$$

and

$$\int_{\{h-k<|\hat{u}|<h\}} |D\hat{u}|^p \leq \frac{k}{\lambda} \int_{\{|\hat{u}|\geq h-k\}} |\hat{f}-\hat{u}|. \tag{4.10}$$

From (4.8), (4.9) and (4.10), it follows that

$$|L^2_{h,k}| \leq \Lambda 2^{\frac{1}{p}} \left(\|j\|_{p'}^{p'} + \frac{k}{\lambda} \int_{\{|u|\geq h\}} |f-u| \right)^{1/p'} \cdot$$

$$\left(\left(\frac{k}{\lambda} \int_{\{|u|\geq h\}} |f-u| \right)^{1/p} + \left(\frac{k}{\lambda} \int_{\{|\hat{u}|\geq h-k\}} |\hat{f}-\hat{u}| \right)^{1/p} \right).$$

Then, since $u, \hat{u}, f, \hat{f} \in L^1(\Omega)$, we have that

$$\lim_{h\to\infty} L^2_{h,k} = 0.$$

Similarly,

$$\lim_{h\to\infty} L^1_{h,k} = 0.$$

Therefore claim (4.7) holds and consequently the proof of step 1 is finished.

Step 2. $\overline{A} \subset \mathcal{A}$. Let $(u,v) \in \overline{A}$. Then, there exist $(u_n, v_n) \in A$ such that $u_n \to u$ and $v_n \to v$ in $L^1(\Omega)$ and a.e. Also, there exists $g \in L^1(\Omega)$ such that $|u_n| \leq g$, $|v_n| \leq g$ a.e.

for all $n \in \mathbb{N}$. Since $(u_n, v_n) \in A$ there exists $w_n \in L^1(\partial\Omega)$ with $-w_n(x) \in \beta(u_n(x))$ a.e. on $\partial\Omega$ and

$$\int_\Omega \langle \mathbf{a}(x, Du_n), D(u_n - \phi)\rangle \leq \int_\Omega v_n(u_n - \phi) + \int_{\partial\Omega} w_n(u_n - \phi), \tag{4.11}$$

for every $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$. Taking $u_n - T_k(u_n)$ as test function in (4.11), by (H_1) we get

$$\begin{aligned} \lambda \textstyle\int_\Omega |DT_k(u_n)|^p \leq \int_{\{|u_n|<k\}} \langle \mathbf{a}(x, Du_n), Du_n\rangle \leq \\ = \textstyle\int_\Omega v_n T_k(u_n) + \int_{\partial\Omega} w_n T_k(u_n) \leq k\|g\|_1, \end{aligned} \tag{4.12}$$

i.e.,

$$\int_\Omega |DT_k(u_n)|^p \leq \frac{k}{\lambda}\|g\|_1 \quad \text{for every } \ n \in \mathbb{N}. \tag{4.13}$$

Moreover, from (4.12) we also get

$$-\int_{\partial\Omega} w_n \frac{1}{k} T_k(u_n) \leq \int_\Omega v_n \frac{1}{k} T_k(u_n) \leq \|g\|_1.$$

From here, letting $k \to 0$, we get the estimate

$$\int_{\partial\Omega} |w_n| \leq \|g\|_1 \quad \text{for every } \ n \in \mathbb{N}. \tag{4.14}$$

From (4.13) it follows that $\{T_k(u_n) \ : \ n \in \mathbb{N}\}$ is a bounded subset of $W^{1,p}(\Omega)$. Hence, after passing to a suitable subsequence, we have $\{T_k(u_n)\}_{n\in\mathbb{N}}$ is weakly convergent in $W^{1,p}(\Omega)$. Now, since $T_k(u_n) \to T_k(u)$ in $L^p(\Omega)$ as $n \to \infty$, we have $DT_k(u_n) \to DT_k(u)$ weakly in $L^p(\Omega)$ as $n \to \infty$. Therefore, $T_k(u) \in W^{1,p}(\Omega)$ for every $k > 0$ and consequently, $u \in \mathcal{T}^{1,p}(\Omega)$.

In the next step we are going to see that $\{Du_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Let t and $\epsilon > 0$. As in the proof of Proposition 2.3, if for some $A > 1$ we set

$$C(x, A, t) := \inf\{\langle \mathbf{a}(x,\xi) - \mathbf{a}(x,\eta), \xi - \eta\rangle \ : \ |\xi| \leq A, \ |\eta| \leq A, \ |\xi - \eta| \geq t \ \},$$

we have that

$$C(x, A, t) > 0 \quad \text{for almost all } \ x \in \Omega. \tag{4.15}$$

For $n, m \in \mathbb{N}$ and any $k > 0$, we have

$$\begin{aligned} \{|Du_n - Du_m| > t\} \subset \{|DT_A u_n| \geq A\} \cup \{|DT_A u_m| \geq A\} \cup \{|u_n| > A\} \cup \\ \cup \{|u_m| > A\} \cup \{|u_n - u_m| \geq k^2\} \cup \{C(x, A, t) \leq k\} \cup \{|u_n - u_m| \leq k^2, \end{aligned} \tag{4.16}$$

$$|u_n| \le A,\ |u_m| \le A,\ C(x,A,t) \ge k,\ |DT_A u_n| \le A,\ |DT_A u_m| \le A,\ |Du_n - Du_m| > t\}.$$

Since $\{u_n : n \in \mathbb{N}\}$ is bounded in $L^1(\Omega)$ we can choose A large enough in order to have

$$\lambda_N(\{|u_n| \ge A\} \cup \{|u_m| \ge A\}) \le \frac{\epsilon}{5} \quad \text{for all } n,m \in \mathbb{N}. \tag{4.17}$$

Similarly, by (4.13), we can choose A large enough in order to have

$$\lambda_N(\{|DT_A u_n| \ge A\} \cup \{|DT_A u_m| \ge A\}) \le \frac{\epsilon}{5} \quad \text{for all } n,m \in \mathbb{N}. \tag{4.18}$$

By (4.15), taking k small enough we have

$$\lambda_N(\{C(x,A,t) \le k\}) \le \frac{\epsilon}{5}. \tag{4.19}$$

On the other hand, if

$$S := \{|u_n - u_m| \le k^2,\ |u_n| \le A,\ |u_m| \le A,\ C(x,A,t) \ge k,\ |DT_A u_n| \le A,\ |DT_A u_m| \le A,$$
$$|Du_n - Du_m| > t\},$$

since $DT_A u_n = Du_n$ a.e. in S, arguing as in the proof of (2.13) we get

$$\lambda_N(S) \le \frac{\epsilon}{5} \quad \text{for } k \text{ small enough.} \tag{4.20}$$

Since A and k have been already choosen, if n_0 is large enough we have for $n,m \ge n_0$ the estimate $\lambda_N(\{|u_n - u_m| \ge k^2\}) \le \frac{\epsilon}{5}$. From here, using (4.16) - (4.20), it follows that

$$\lambda_N(\{|Du_n - Du_m| \ge t\}) \le \epsilon \quad \text{for } m,n \ge n_0.$$

Consequently, $\{Du_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. So, there exists $\varphi \in M(\Omega, \lambda_N)$ such that $\{Du_n\}_{n\in\mathbb{N}}$ converges to φ in measure. Now, the above argument also shows that $\{DT_k u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in measure for every $k > 0$. Hence, since $\{DT_k u_n\}_{n\in\mathbb{N}}$ weakly converges to $DT_k(u)$ in $L^p(\Omega)$, we have $\{DT_k u_n\}_{n\in\mathbb{N}}$ converges in measure to $DT_k(u)$. Thus, there exist $n_1 < n_2 < \cdots < n_k < \cdots$, such that $\lim_{k\to\infty} \varrho(DT_k u_{n_k}, DT_k u) = 0$. Hence, $\lim_{k\to\infty} \varrho(DT_k u_{n_k}, Du) = 0$. Then, up to extraction of a subsequence, we have convergence a.e., and we can say that

$$\{Du_n\}_{n\in\mathbb{N}} \quad \text{converges to } Du \text{ in measure.} \tag{4.21}$$

We claim

$$\{u_n\}_{n\in\mathbb{N}} \quad \text{is bounded in } M^{p_1}(\Omega), \quad p_1 = \frac{N(p-1)}{N-p}. \tag{4.22}$$

Fixed $n_0, k_0 \in \mathbb{N}$ such that

$$|\overline{u}| + 1 < \frac{k_0}{2}, \quad |\overline{u_n} - \overline{u}| < 1/3 \quad \text{for all } n \geq n_0,$$

and

$$\frac{2}{\lambda_N(\Omega)} \int_{\{g>k\}} g < 1/3 \qquad \text{for all } k \geq k_0.$$

For $n \geq n_0$ and $k \geq k_0$ fixed, consider the sets

$$C := \{|T_k(u_n)| > k/2 + |\overline{u}| + 1\}, \qquad D := \{|T_k(u_n) - \overline{T_k(u_n)}| > k/2\}.$$

If $x \in C \sim D$, we have

$$k/2 + |\overline{u}| + 1 < |T_k(u_n(x))| \leq |T_k(u_n(x)) - \overline{T_k(u_n)}| + |\overline{T_k(u_n)}| \leq$$

$$\leq k/2 + |\overline{T_k(u_n)}| \leq k/2 + |\overline{T_k(u_n)} - \overline{u_n}| + |\overline{u}| + 1/3 \leq$$

$$\leq k/2 + |\overline{u}| + 1/3 + \frac{2}{\lambda_N(\Omega)} \int_{\{g>k\}} g \leq k/2 + |\overline{u}| + 2/3,$$

which is a contradiction. Thus, $C \subset D$. Hence, if $p^\star = \frac{pN}{N-p}$,

$$\lambda_N(\{|u_n| > k\}) \leq \lambda_N(\{|T_k(u_n)| > k/2 + |\overline{u}| + 1\}) = \lambda_N(C) \leq$$

$$\leq \lambda_N(D) \leq \frac{1}{(k/2)^{p^\star}} \|T_k(u_n) - \overline{T_k(u_n)}\|_{p^\star}^{p^\star}.$$

Then, by Poincaré's inequality (cf. [Zi, Cap. 4]) and (4.13), there exist constants $C, M > 0$ such that

$$\lambda_N(\{|u_n| > k\}) \leq C \frac{1}{(k/2)^{p^\star}} \|DT_k(u_n)\|_p^{p^\star} \leq M \frac{k^{p^\star/p}}{k^{p^\star}} = M k^{-p_1}$$

for every $n \geq n_0$ and $k \geq k_0$. Consequently, the claim (4.22) holds.

Next, we claim that

$$\text{(4.23)} \qquad \{Du_n\}_{n\in\mathbb{N}} \quad \text{is bounded in } M^{p_2}(\Omega), \quad p_2 = \frac{N(p-1)}{N-1}.$$

Let $t > 0$. By (4.13), there exists a constant $Q_1 > 0$, such that for every $k > 0$ and $n \in \mathbb{N}$,

$$\text{(4.24)} \qquad \lambda_N(\{|DT_k(u_n)| > t/2\}) \leq \frac{Q_1 k}{t^p}.$$

On the other hand, by (4.22), there exists a constant $Q_2 > 0$, such that

$$\text{(4.25)} \qquad \lambda_N(\{|u_n| \geq k\}) \leq \frac{Q_2}{k^{p_1}} \quad \text{for every } k > 0 \text{ and } n \in \mathbb{N}.$$

From (4.24) and (4.25), it follows that

$$\lambda_N(\{|Du_n| > t\}) \le \lambda_N(\{|Du_n - DT_k(u_n)| > t/2\}) + \lambda_N(\{|DT_k(u_n)| > t/2\}) \le$$

$$\le \lambda_N(\{|u_n| \ge k\}) + \lambda_N(\{|DT_k(u_n)| > t/2\}) \le \frac{Q_2}{k^{p_1}} + \frac{Q_1 k}{t^p}.$$

Then, taking $k := t^{p_2/p_1}$, we have

$$\lambda_N(\{|Du_n| > t\}) \le Q t^{-p_2} \quad \text{for every } n \in \mathbb{N}.$$

Consequently, the claim (4.23) holds.

From (4.21), (4.22) and (4.23) we can state that

$$u \in M^{p_1}(\Omega), \quad |Du| \in M^{p_2}(\Omega)$$

where $p_1 = \frac{N(p-1)}{N-p}$ and $p_2 = \frac{N(p-1)}{N-1}$. Suppose we are in the case $p > 2 - 1/N$. Then, $p_2 > 1$. Hence, if $1 \le q < p_2$, we have that $\{u_n \ : \ n \in \mathbb{N}\}$ is bounded in $W^{1,q}(\Omega)$. Consequently, $u \in W^{1,q}(\Omega)$.

Let us see now that $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$. Indeed: Obviously, $u_n \to u$ a.e. in Ω. Since $\{DT_k(u_n) \ : \ n \in \mathbb{N}\}$ is bounded in $L^p(\Omega)$ and $DT_k(u_n) \to DT_k(u)$ in measure, it follows from [B-V, Lemma 6.1] that $DT_k(u_n) \to DT_k(u)$ in $L^1(\Omega)$. Finally, let us see that $\{\gamma(u_n)\}_{n\in\mathbb{N}}$ converges a.e. in $\partial\Omega$. For every $k > 0$, let

$$A_k := \{x \in \partial\Omega \ : \ |T_k u(x)| < k\} \quad \text{and} \quad C := \partial\Omega \sim \cup_{k>0} A_k.$$

Then,

$$\mu(C) = \frac{1}{k}\int_C |T_k u| \le \frac{1}{k}\int_{\partial\Omega} |T_k u| \le \frac{C_1}{k}\|T_k u\|_{W^{1,1}(\Omega)} \le$$

$$\le \frac{C_1}{k}\|T_k u\|_{L^1(\Omega)} + \frac{C_2}{k}\|DT_k u\|_{L^p(\Omega)}.$$

Now, by (4.13) and the boundedness of $\{\|T_k u\|_{L^1(\Omega)} \ : \ k > 0\}$,

$$\|DT_k u\|_{L^p(\Omega)} \le \left(\frac{k}{\lambda}\|g\|_1\right)^{1/p} \quad \text{for any } k > 0.$$

Hence,

$$\mu(C) \le \frac{C_3}{k} + \frac{C_4}{k^{1-1/p}} \quad \text{for any } k > 0.$$

Taking limit as $k \to \infty$ we have $\mu(C) = 0$. Moreover, $A_k \subset A_r$ if $k \le r$. Thus, if we define in $\partial\Omega$ the function v by

$$v(x) := (T_k u)(x) \quad \text{if } x \in A_k,$$

it is easy to see that $u_n \to v$ a.e. in $\partial\Omega$. Therefore, $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$.

To complete the proof it remains to show that there exists $w \in L^1(\partial\Omega)$ with $-w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, such that

$$\int_\Omega \langle \mathbf{a}(x, Du), DT_k(u-\phi)\rangle \le \int_\Omega vT_k(u-\phi) + \int_{\partial\Omega} wT_k(u-\phi), \tag{4.26}$$

for every $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$. For this, let us see first that $\{w_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^1(\partial\Omega)$. Indeed: Taking $u_n - T_k(u_n - u_m)$ and $u_m + T_k(u_n - u_m)$ as test functions in (4.11), we have

$$-\int_{\partial\Omega} (w_n - w_m)\frac{1}{k}T_k(u_n - u_m) \le \int_\Omega (v_n - v_m)\frac{1}{k}T_k(u_n - u_m) -$$

$$-\int_\Omega \langle \mathbf{a}(x, Du_n) - \mathbf{a}(x, Du_m), \frac{1}{k}D(T_k(u_n - u_m))\rangle \le \int_\Omega |v_n - v_m|.$$

Letting $k \to 0$, we get

$$\int_{\partial\Omega} |w_n - w_m| \le \int_\Omega |v_n - v_m|. \tag{4.27}$$

Now, since $\{v_n\}_{n\in\mathbb{N}}$ is convergent in $L^1(\Omega)$, by (4.27), $\{w_n\}_{\ n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\partial\Omega)$.

We introduce the class $\mathcal{F}$ of functions $S \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying:

$$S(0) = 0,\ 0 \le S' \le 1,\ S'(s) = 0 \ \text{ for } \ \text{s} \ \text{ large enough},$$

$$S(-s) = -S(s), \ \text{ and } S''(s) \le 0 \ \text{ for } \ s \ge 0.$$

Let $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$ and $S \in \mathcal{F}$. Taking $u_n - S(u_n - \phi)$ as test function in (4.11) we get

$$\int_\Omega \langle \mathbf{a}(x, Du_n), DS(u_n - \phi)\rangle \le \int_\Omega v_n S(u_n - \phi) + \int_{\partial\Omega} w_n S(u_n - \phi). \tag{4.28}$$

We can write the first member of (4.28) as

$$\int_\Omega \langle \mathbf{a}(x, Du_n), Du_n\rangle S'(u-\phi) - \int_\Omega \langle \mathbf{a}(x, Du_n), D\phi\rangle S'(u_n - \phi). \tag{4.29}$$

Since $u_n \to u$ and $Du_n \to Du$ a.e., Fatou's Lemma yields

$$\int_\Omega \langle \mathbf{a}(x, Du), Du\rangle S'(u-\phi) \le \liminf_{n\to\infty} \int_\Omega \langle \mathbf{a}(x, Du_n), Du_n\rangle S'(u_n - \phi).$$

The second term of (4.29) is estimated as follows. Let $r := \|\phi\|_\infty + \|S\|_\infty$. By (4.13) and ($\mathrm{H}_3$), it follows that

$$\{\mathbf{a}(x, DT_r u_n) \ : \ n \in \mathbb{N}\} \quad \text{is bounded in} \quad L^{p'}(\Omega).$$

Then, since $DT_r u_n \to DT_r u$ in measure, it follows that

$$\mathbf{a}(x, DT_r u_n) \to \mathbf{a}(x, DT_r u) \quad \text{weakly in } L^{p'}(\Omega). \tag{4.30}$$

On the other hand,

$$|D\phi S'(u_n - \phi)| \leq |D\phi| \in L^p(\Omega).$$

Then, by the Dominated Convergence Theorem, we have

$$D\phi S'(u_n - \phi) \to D\phi S'(u - \phi) \quad \text{in } L^p(\Omega)^N. \tag{4.31}$$

Hence, by (4.30) and (4.31), it follows that

$$\lim_{n\to\infty} \int_\Omega \langle \mathbf{a}(x, Du_n), D\phi \rangle S'(u_n - \phi) = \int_\Omega \langle \mathbf{a}(x, Du), D\phi \rangle S'(u - \phi).$$

Therefore, applying again the Dominated Convergence Theorem in the second member of (4.28), we obtain

$$\int_\Omega \langle \mathbf{a}(x, Du), DS(u - \phi) \rangle \leq \int_\Omega vS(u - \phi) + \int_{\partial\Omega} wS(u - \phi).$$

From here, to get (4.26) we only need to apply the technique used in the proof of [B-V, Lemma 3.2].

Finally, since $w_n \to w$ in $L^1(\partial\Omega)$ and $-w_n \in \beta(u_n)$ a.e. in $\partial\Omega$, from the maximal monotonicity of β it follows that $-w \in \beta(u)$ a.e. in $\partial\Omega$ and the proof concludes.

As a consequence of the m-accretivity of $\mathcal{A}$ in $L^1(\Omega)$, we have the following result of existence and uniqueness for the entropy solutions of the nonlinear elliptic problem.

Corollary 4.3. *Under the assumptions of Theorem 4.2, given* $v \in L^1(\Omega)$ *the problem*

$$u - \operatorname{div} \mathbf{a}(x, Du) = v \quad \text{in } \Omega$$

$$-\tfrac{\partial u}{\partial \eta_a} = \beta(u) \quad \text{on } \partial\Omega.$$

has a unique entropy solution. Moreover, if $2 - \frac{1}{N} < p < N$*, the solution belongs to* $W^{1,q}(\Omega)$ *for every* $1 \leq q < p_2$*.*

Corollary 4.4. *Under the assumptions of Theorem 4.2, we have* $A = A_0$ *and* $\overline{A}_0 = \mathcal{A}$*.*

Proof. Since $A \subset A_0$, $\mathcal{A} = \overline{A} \subset \overline{A}_0$. Therefore, by maximal accretivity we have $\overline{A}_0 = \mathcal{A}$. Let $(u, v) \in A_0$. Then, $(u, v) \in \mathcal{A}$ and $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Hence, there exists $w \in L^1(\partial\Omega)$ with $-w(x) \in \beta(u(x))$ a.e. on $\partial\Omega$ such that

$$\int_\Omega \langle \mathbf{a}(x, Du), DT_k(u - \phi) \rangle \leq \int_\Omega vT_k(u - \phi) + \int_{\partial\Omega} wT_k(u - \phi)$$

for every $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$ and $k > 0$. From here, since $u - \phi \in L^\infty(\Omega)$, it follows that $(u, v) \in A$.

5. The evolution problem

In this section we study in $L^1(\Omega)$, from the point of view of Nonlinear Semigroup Theory, the quasi-linear parabolic equation with nonlinear boundary condition

$$u_t = \operatorname{div} \mathbf{a}(x, Du) \quad \text{in} \ \ \Omega \times (0, \infty)$$

$$(III) \qquad -\frac{\partial u}{\partial \eta_a} \in \beta(u) \quad \text{on} \ \ \partial\Omega \times (0, \infty)$$

$$u(x, 0) = u_0(x) \quad \text{in} \ \ \Omega.$$

We transcribe (III) as the evolution problem in $L^1(\Omega)$

$$(IV). \qquad \frac{du}{dt} + A_0 u = 0, \qquad u(0) = u_0$$

In order to get solution of (IV) for every initial data $u_0 \in L^1(\Omega)$ we need the following result.

Proposition 5.1. *The domain of the operator* A_0 *is dense in* $L^1(\Omega)$.

Proof. We are going to prove that $L^\infty(\Omega) \subset \overline{\mathcal{D}(A_0)}$. Given $v \in L^\infty(\Omega)$, if we set

$$u_n := (I + 1/nA_0)^{-1} v,$$

then $(u_n, n(v - u_n)) \in A_0$, so taking $w = 0$ as test function in the definition of the operator A_0 we get

$$-n \int_\Omega (v - u_n) u_n + \int_\Omega \langle \mathbf{a}(x, Du_n), Du_n \rangle \leq - \int_{\partial\Omega} j(u_n) \leq 0.$$

From where it follows that

$$\frac{1}{n} \int_\Omega \langle \mathbf{a}(x, Du_n), Du_n \rangle + \int_\Omega u_n^2 \leq \int_\Omega v u_n.$$

Thus,

$$\int_\Omega \langle \mathbf{a}(x, Du_n), Du_n \rangle \leq n \int_\Omega v u_n \leq n \|v\|_\infty^2 \ \lambda_N(\Omega) = nM.$$

Now, using (H_1) and (H_3), we have

$$\int_\Omega |\mathbf{a}(x, Du_n)|^{p'} \leq \int_\Omega \Lambda^{p'} \left(j(x) + |Du_n|^{p-1} \right)^{p'} \leq$$

$$\leq \Lambda^{p'} 2^{p'-1} \left(\int_\Omega j(x)^{p'} + \int_\Omega |Du_n|^p \right) \leq$$

$$\leq \Lambda^{p'} 2^{p'-1} \left(\int_\Omega j(x)^{p'} + \frac{1}{\lambda} \int_\Omega \langle \mathbf{a}(x, Du_n), Du_n \rangle \right) \leq \alpha_1 + n\alpha_2,$$

with

$$\alpha_1 = \Lambda^{p'} 2^{p'-1} \int_\Omega j(x)^{p'} \quad \text{and} \quad \alpha_2 = \frac{M\Lambda^{p'} 2^{p'-1}}{\lambda}.$$

Consequently,

$$\int_\Omega |\frac{1}{n}\, \mathbf{a}(x, Du_n)|^{p'} \leq \frac{\alpha_1}{n^{p'}} + \frac{n\alpha_2}{n^{p'}}. \tag{5.1}$$

On the other hand, if $\phi \in \mathcal{D}(\Omega)$, taking $u_n - \phi$ and $u_n + \phi$ as test functions in the definition of the operator A_0 we have that

$$\frac{1}{n} \int_\Omega \langle \mathbf{a}(x, Du_n), D\phi \rangle + \int_\Omega u_n \phi = \int_\Omega v\phi,$$

so, by (5.1), we get

$$\lim_{n\to\infty} \int_\Omega u_n \phi = \int_\Omega v\phi \quad \text{for every} \quad \phi \in \mathcal{D}(\Omega). \tag{5.2}$$

Now, since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ weakly in $L^p(\Omega)$, which implies, by (5.2), that $u = v$.

Finally, since A_0 is completely accretive, we have $\|u_n\|_p \leq \|v\|_p$. Hence, $u_{n_k} \to v$ in $L^p(\Omega)$, and we can conclude that $v \in \overline{\mathcal{D}(A_0)}$.

Since the closure of the operator A_0 is an m-T-accretive operator in $L^1(\Omega)$ with dense domain, according to Crandall-Liggett's Semigroup Generation Theorem (cf. [Cr]) the operator $\overline{A}_0$ generates a semigroup of order-preserving contractions $S(t)$ in $L^1(\Omega)$ which solves, in a generalized sense called usually *mild sense*, the evolution problem (IV). The mild-solution of problem (III) is given by $u(.,t) = S(t)u_0$.

To finish this section we study the continuous dependence of u on $\mathbf{a}$ and β where u is the solution of the elliptic problem

$$u - \operatorname{div} \mathbf{a}(x, Du) = v \quad \text{in} \ \Omega$$

$$-\frac{\partial u}{\partial \eta_a} \in \beta(u) \quad \text{on} \ \partial\Omega.$$

Using nonlinear semigroup theory, this result implies the continuous dependence of solutions of the asociated nonlinear parabolic problem on the nonlinearities in the problem.

As in the above section we assume that Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ of class C^1, $1 < p < N$. Also we assume that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_\infty$ are Caratheodory functions from $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^N$ satisfying

(G_1) there exist $\lambda_n > 0$ with $\inf_{n\in\mathbb{N}} \lambda_n = \lambda > 0$ such that

$$\langle \mathbf{a}_n(x, \xi), \xi \rangle \geq \lambda_n |\xi|^p$$

holds for every ξ and a.e. $x \in \Omega$.

(G_2) For every ξ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$ there holds

$$\langle \mathbf{a}_n(x,\xi) - \mathbf{a}_n(x,\eta), \xi - \eta \rangle > 0.$$

(G_3) There exist $\Lambda_n \in \mathbb{R}$ with $\sup_{n\in\mathbb{N}} \Lambda_n < \infty$ and $j_n \in L^{p'}$ with $\sup_{n\in\mathbb{N}} \|j_n\|_{p'} < \infty$, such that

$$|\mathbf{a}_n(x,\xi)| \leq \Lambda_n(j_n(x) + |\xi|^{p-1})$$

holds for every $\xi \in \mathbb{R}^N$.

Theorem 5.2. *Let* $\mathbf{a}_n$ *be satisfying* (G_1), (G_2), (G_3) *and for almost all* $x \in \Omega$ $\mathbf{a}_n(x,\xi) \to \mathbf{a}_\infty(x,\xi)$ *uniformly in* ξ *on compact subsets. Let* β_n *be maximal monotone graphs in* $\mathbb{R} \times \mathbb{R}$ *with* $0 \in \beta_n(0)$, $\beta_n \to \beta_\infty$ *in the sense of maximal monotone graphs and* $\mathcal{D}(\beta_n) = \mathcal{D}(\beta_\infty)$ *for all* $n \in \mathbb{N}$. *Suppose* $\mathcal{D}(\beta_\infty)$ *is closed. Let* $\mathcal{A}_n$ *be the operator associated to* $\mathbf{a}_n$ *and* β_n. *Then*

$$(I + \mathcal{A}_n)^{-1} v \to (I + \mathcal{A}_\infty)^{-1} v \quad \text{in} \;\; L^1(\Omega)$$

for every $v \in L^1(\Omega)$.

Proof. Since $(I + \mathcal{A}_n)^{-1}$ are contractions in $L^1(\Omega)$ and $L^\infty(\Omega)$ is dense in $L^1(\Omega)$, we can suppose that $v \in L^\infty(\Omega)$ and work with the operators A_n. By Theorem 2.2, for every $n \in \mathbb{N}$ there exist $u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $w_n \in L^1(\partial\Omega)$ with $-w_n(x) \in \beta_n(u_n(x))$ a.e. in $\partial\Omega$ such that

$$\int_\Omega \langle \mathbf{a}_n(x, Du_n), D(u_n - \phi) \rangle + \int_\Omega u_n(u_n - \phi) \leq \int_\Omega v(u_n - \phi) + \int_{\partial\Omega} w_n(u_n - \phi), \tag{5.3}$$

for every $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$. Moreover, the complete accretiveness of A_n implies

$$\|u_n\|_\infty \leq \|v\|_\infty \quad \text{for every} \;\; n \in \mathbb{N}. \tag{5.4}$$

Taking $\phi = 0$ as test function in (5.3) we obtain

$$\|Du_n\|_p \leq \left(\frac{2}{\lambda_n} \, \|v\|_\infty^2 \lambda_N(\Omega) \right)^{1/p} \quad \text{for every} \;\; n \in \mathbb{N}. \tag{5.5}$$

As a consequence of (5.4), (5.5), Rellich-Kondrachov's Theorem and [M, Theorem 3.4.5] we can establish, after passing to a subsequence,the following fact:

$$u_n \to u \quad \text{weakly in} \;\; W^{1,p}(\Omega). \tag{5.6}$$

$$u_n \to u \quad \text{in} \;\; L^1(\Omega) \;\; \text{and a.e.} \tag{5.7}$$

$$(5.8) \qquad u_n \to u \quad \text{in} \ L^p(\partial\Omega).$$

With a similar argument than the one used in the proof of Theorem 2.2 we get, up to extraction of a subsequence if necessary, $w_n \to w \in L^1(\partial\Omega)$ weakly in $L^1(\partial\Omega)$. Moreover, by [BCS, Lemma G], $-w(x) \in \beta_\infty(u(x))$ a.e. in $\partial\Omega$.

Now we prove that Du_n converges to Du in measure. Since Du_n converges to Du weakly in $L^p(\Omega)$, it is enough to show that $\{Du_n\}$ is a Cauchy sequence in measure. Let t and $\epsilon > 0$. For some $A > 1$ and $n, m \in \mathbb{N}$ we set

$$C_{n,m}(x,A,t) := \inf\{\langle \mathbf{a}_n(x,\xi) - \mathbf{a}_m(x,\eta), \xi - \eta\rangle \ : \ |\xi| \le A, \ |\eta| \le A, \ |\xi - \eta| \ge t \ \}.$$

As in the proof of Theorem 2.1 we have that the infimum in the definition of $C_{n,m}(x,A,t)$ is a minimum for almost all $x \in \Omega$. Moreover, by (G_2), and since $\mathbf{a}_n(x,\xi) \to \mathbf{a}_\infty(x,\xi)$ uniformly in ξ on compact subsets, it follows that there exists $n_0 \in \mathbb{N}$, such that for every $n, m \ge n_0$, we have

$$(5.9) \qquad C_{n,m}(x,A,t) > 0 \quad \text{for almost all} \ x \in \Omega.$$

For $n, m \in \mathbb{N}$ and $k > 0$, we have

$$\{|Du_n - Du_m| > t\} \subset$$

$$(5.10) \qquad \subset \{|Du_n| \ge A\} \cup \{|Du_m| \ge A\} \cup \{|u_n - u_m| \ge k^2\} \cup \{C_{n,m}(x,A,t) \le k\} \cup$$

$$\cup \{|u_n - u_m| \le k^2, \ C_{n,m}(x,A,t) \ge k, \ |Du_n| \le A, \ |Du_m| \le A, \ |Du_n - Du_m| > t\}.$$

Since $\{Du_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$ we can choose A large enough in order to have

$$(5.11) \qquad \lambda_N(\{|Du_n| \ge A\} \ \cup \ \{|Du_m| \ge A\}) \le \frac{\epsilon}{4} \quad \text{for all} \ n, m \in \mathbb{N}.$$

By (5.9) we can choose k small enough in order to have

$$(5.12) \qquad \lambda_N(\{C_{n,m}(x,A,t) \le k\}) \le \frac{\epsilon}{4} \quad \text{for} \ n, m \ge n_0.$$

On the other hand, if we use $u_n - T_{k^2}(u_n - u_m)$ and $u_m + T_{k^2}(u_n - u_m)$ as test functions in (5.3), we obtain

$$\int_{\{|u_n - u_m| \le k^2\}} \langle \mathbf{a}_n(x, Du_n) - \mathbf{a}_m(x, Du_m), D(u_n - u_m)\rangle =$$

$$= \int_\Omega \langle \mathbf{a}_n(x, Du_n) - \mathbf{a}_m(x, Du_m), DT_{k^2}(u_n - u_m)\rangle \le$$

$$\le -\int_\Omega (u_n - u_m) T_{k^2}(u_n - u_m) + \int_{\partial\Omega} (w_n - w_m) T_{k^2}(u_n - u_m).$$

Then, by (5.4),

$$\int_{\{|u_n-u_m|<k^2\}} \langle \mathbf{a}_n(x,Du_n)-\mathbf{a}_m(x,Du_m), D(u_n-u_m)\rangle \le k^2 Q.$$

Hence

$$\lambda_N(\{|u_n-u_m|\le k^2,\ C_{n,m}(x,A,t)\ge k,\ |Du_n|\le A,\ |Du_m|\le A,\ |Du_n-Du_m|>t\})\le$$

$$\le \lambda_N(\{|u_n-u_m|\le k^2,\ \langle \mathbf{a}_n(x,Du_n)-\mathbf{a}_m(x,Du_m), D(u_n-u_m)\rangle \ge k\})\le \tag{5.13}$$

$$\le \frac{1}{k}\int_{\{|u_n-u_m|<k^2\}} \langle \mathbf{a}_n(x,Du_n)-\mathbf{a}_m(x,Du_m), D(u_n-u_m)\rangle \le \frac{1}{k}\ k^2 Q \le \frac{\epsilon}{4},$$

for k small enough.

Since A and k have been already choosen, if n_0 is large enough we have for $n,m\ge n_0$ the estimate $\lambda_N(\{|u_n-u_m|\ge k^2\})\le \frac{\epsilon}{4}$. From here, using (5.10), (5.11), (5.12) and (5.13), it follows that

$$\lambda_N(\{|Du_n-Du_m|\ge t\})\le \epsilon \quad \text{for} \quad m,n\ge n_0.$$

Consequently, $\{Du_n\}$ is a Cauchy sequence in measure. Therefore, we can assume, after passing to a suitable subsequence, the a.e. convergence of Du_n to Du, and since $\mathbf{a}_n(x,\xi)\to \mathbf{a}_\infty(x,\xi)$ uniformly in ξ on compact subset, that $\mathbf{a}_n(x,Du_n)\to \mathbf{a}_\infty(x,Du)$ a.e. in Ω. On the other hand, since $\{Du_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$, it follows from (G_3) that $\{\mathbf{a}_n(x,Du_n)\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(\Omega)$. Therefore,

$$\mathbf{a}_n(x,Du_n)\to \mathbf{a}_\infty(x,Du) \quad \text{weakly in} \quad L^{p'}(\Omega).$$

In particular,

$$\int_\Omega \langle \mathbf{a}_n(x,Du_n), D\phi\rangle \to \int_\Omega \langle \mathbf{a}_\infty(x,Du), D\phi\rangle,$$

for every $\phi\in W^{1,p}_\beta(\Omega)\cap L^\infty(\Omega)$. From here, (5.5) and (5.8), passing to the limit in (5.3) we get

$$\int_\Omega \langle \mathbf{a}_\infty(x,Du), D(u-\phi)\rangle + \int_\Omega u(u-\phi) \le \int_\Omega v(u-\phi) + \int_{\partial\Omega} w(u-\phi),$$

for every $\phi\in W^{1,p}_\beta(\Omega)\cap L^\infty(\Omega)$. Therefore, $u=(I+A_\infty)^{-1}v$ and the proof is completed.

Remark 5.3. The above Theorem jointly with [Cr, Theorem 6] implies the continuous dependence of solutions of the associated nonlinear parabolic problem on the nonlinearities in the problem. More concretely, suppose $\mathbf{a}_1,\mathbf{a}_2,\ldots \mathbf{a}_\infty$ and β_n satisfy the assumptions of Theorem 5.2. Let $\mathcal{A}_n$ be the operator associated to $\mathbf{a}_n$ and β_n. Then, if u_n are the mild-solutions of the problems

$$u_n' + \mathcal{A}_n u_n \ni 0, \quad u_n(0)=f_n$$

and $f_n\to f_\infty$ in $L^1(\Omega)$, we have that u_n converges in $C([0,\infty[;L^1(\Omega))$ to the mild-solution u_∞ of the problem

$$u_\infty' + \mathcal{A}_\infty u_\infty \ni 0, \quad u_\infty(0)=f_\infty.$$

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Departamento de Análisis Matemático
Universitat de València
46100 Burjassot (Valencia)
SPAIN