QUASI-LINEAR ELLIPTIC AND PARABOLIC EQUATIONS IN $L^1$ WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this paper we study the questions of existence and uniqueness of solutions for equations of the form

$$u - \text{div } a(x, Du) = f \quad \text{in } \Omega$$

$$-\frac{\partial u}{\partial \eta_a} \in \beta(u) \quad \text{on } \partial\Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $f \in L^1(\Omega)$. As a consequence, an $m$-completely accretive operator in $L^1(\Omega)$ can be associated to the corresponding parabolic equation, which permits to study this equation from the point of view of Nonlinear Semigroup Theory.

Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $1 < p < \infty$. Consider a vector valued function $a$ mapping $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^N$ and satisfying

(H$_1$) $a$ is a Carathéodory function (i.e., the map $\xi \to a(x, \xi)$ is continuous for almost all $x$ and the map $x \to a(x, \xi)$ is measurable for every $\xi$) and there exists $\lambda > 0$ such that

$$\langle a(x, \xi), \xi \rangle \geq \lambda |\xi|^p$$

holds for every $\xi$ and a.e. $x \in \Omega$, where $\langle , \rangle$ means scalar product in $\mathbb{R}^N$.

(H$_2$) For every $\xi$ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$ there holds

$$\langle a(x, \xi) - a(x, \eta), \xi - \eta \rangle > 0.$$
(H3) There exists $\Lambda \in \mathbb{R}$ such that

$$|a(x, \xi)| \leq \Lambda (j(x) + |\xi|^{p-1})$$

holds for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, with $j \in L^{p'}$, $p' = p/(p-1)$.

The hypotheses (H1), (H2) and (H3) are classical in the study of nonlinear operators in divergence form (see [LL]). The model example of function $a$ satisfying these hypotheses is $a(x, \xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \text{div}(|Du|^{p-2}Du)$.

The aim of this paper is to study existence and uniqueness of solutions for equations of the form

$$u - \text{div} \ a(x, Du) = f \quad \text{in} \ \Omega$$

($I$)

$$-\frac{\partial u}{\partial n_a} \in \beta(u) \quad \text{on} \ \partial \Omega$$

where $\partial/\partial n_a$ is the Neumann boundary operator associated to $a$, i.e.,

$$\frac{\partial u}{\partial n_a} := \langle a(x, Du), \eta \rangle$$

with $\eta$ the unit outward normal on $\partial \Omega$, $Du$ the gradient of $u$ and $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. These nonlinear flux on the boundary occurs in some problems in Mechanics and Physics [DL] (see also [Br2]). Observe also that the classical Neumann and Dirichlet boundary conditions correspond to $\beta = \mathbb{R} \times \{0\}$ and $\beta = \{0\} \times \mathbb{R}$, respectively.

We associate a completely accretive operator $A_0$ in $L^1(\Omega)$ with the formal differential expression

$$-\text{div} \ a(x, Du) + \text{nonlinear boundary conditions}.$$ 

Recently, in [B-V], a new concept of solution has been introduced for the elliptic equation

$$-\text{div} \ a(x, Du) = f(x) \quad \text{in} \ \Omega$$

$$u = 0 \quad \text{on} \ \partial \Omega,$$

namely entropy solution. We use the method developed in [B-V] to characterize and interpret the closure of the operator $A_0$.

The plan of the paper is as follows: Some preliminary results and notation are collected in Section 1. In the second section we study the problem (I) with variational methods. We introduce a completely accretive operator $A_0$ in $L^1(\Omega)$ which satisfies the range condition. Under certain restrictions another completely accretive operator $A$ satisfying the range condition and smaller than $A_0$ is also introduced. In order to characterize the closure of the operator $A$ we need to define the trace of functions which are not in the Sobolev spaces. This is the subject of Section 3. In the next section we characterize
the closure of the operator $A$. To do that, following [B-V], we introduce the concept of entropy solution for the elliptic problem. As a consequence of the m-accretivity of the closure of the operator $A$, we obtain a result of existence and uniqueness for the entropy solutions of the elliptic problem. Finally, in Section 5 we consider the evolution problem associated with the operator $A_0$.

1. Preliminaries

In this section we give some of the notation and definitions used later. If $\Omega \subset \mathbb{R}^N$ is a Lebesgue measurable set, $\lambda_N(\Omega)$ denotes its measure. The norm in $L^p(\Omega)$ is denoted by $\| \cdot \|_p$, $1 \leq p \leq \infty$. If $k \geq 0$ is an integer and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is the Sobolev space of functions $u$ on the open set $\Omega \subset \mathbb{R}^N$ for which $D^\alpha u$ belongs to $L^p(\Omega)$ when $|\alpha| \leq k$, with its usual norm $\| \cdot \|_{k,p}$. $W^{k,p}_0(\Omega)$ is the closure of $D(\Omega) = C^\infty_0(\Omega)$ in $W^{k,p}(\Omega)$. If $v \in L^1(\Omega)$ and $\lambda_N(\Omega) < \infty$, we denote by $\overline{v}$ the average of $v$, i.e.,

$$\overline{v} := \frac{1}{\lambda_N(\Omega)} \int_\Omega v(x) \, dx.$$ 

Given a finite measure space $(S, \nu)$, we denote by $M(S, \nu)$ the set of all measurable functions $u : S \to \mathbb{R}$ finite a.e., identifying the functions that are equal a.e. $M(S, \nu)$ is a metric space endowed with the distance function of the convergence in measure,

$$\rho(u, v) = \int_\Omega \frac{|u - v|}{1 + |u - v|}.$$ 

We recall, cf. [BBC], that for $0 < q < \infty$ the Marcinkiewicz space $M^q(\Omega)$ can be defined as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that the corresponding distribution function

$$\phi_f(k) = \lambda_N \{ x \in \Omega : |f(x)| > k \}$$

satisfies an estimate of the form

$$\phi_f(k) \leq C k^{-q}, \quad C < \infty.$$ 

For bounded $\Omega$’s, it is immediate that $M^q(\Omega) \subset M^\hat{q}(\Omega)$ if $\hat{q} \leq q$, also $L^q(\Omega) \subset M^q(\Omega) \subset L^r(\Omega)$ if $1 \leq r < q$.

We will use the following truncature operator: For a given constant $k > 0$ we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) := \begin{cases} 
    s & \text{if } |s| \leq k \\
    k \text{ sign } (s) & \text{if } |s| > k.
\end{cases}$$

For a function $u = u(x)$, $x \in \Omega$, we define the truncated function $T_k u$ pointwise, i.e., for every $x \in \Omega$ the value of $T_k u$ at $x$ is just $T_k(u(x))$. Observe that

$$\lim_{k \to 0} \frac{1}{k} T_k(s) = \text{sign}(s) := \begin{cases} 
    1 & \text{if } s > 0 \\
    0 & \text{if } s = 0 \\
    -1 & \text{if } s < 0.
\end{cases}$$
By the Stampacchia Theorem, cf. [KS], if $u \in W^{1,1}(\Omega)$, we have

$$DT_k(u) = 1_{\{ |u| < k \}} Du,$$

where $1_B$ denotes the characteristic function of a measurable set $B \subset \Omega$.

As we said in the introduction, our abstract framework is the Theory of Nonlinear Semigroups. We refer the reader to [Ba], [Be], [BCP] and [Cr] for background material on non-linear contraction semigroups.

In [BCr], Ph. Bénilan and M. Crandall introduce the concept of completely accretive operator, whose precedents are the results of Brezis-Strauss [BS] on semilinear elliptic equations (see also [Be]). To define this notion of operator we use the following notation. Let $u, v \in M(\Omega)$, we set $u \ll v$ if

$$\int_{\Omega} j(u) \leq \int_{\Omega} j(v) \quad \text{for} \quad j \in J_0,$$

where

$$J_0 = \{ \text{convex lower – semicontinuous maps } j : \mathbb{R} \to [0, \infty] \text{ satisfying } j(0) = 0 \}.$$

An operator $A$ in $M(\Omega)$, possibly multivaluated (i.e., $A \subset M(\Omega) \times M(\Omega)$), is said to be completely accretive if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad (u, v), (\hat{u}, \hat{v}) \in A.$$

Let

$$P_0 = \{ p \in C^\infty(\mathbb{R}) : 0 \leq p' \leq 1, \ \supp(p') \text{ is compact, and } 0 \notin \supp(p) \}.$$

By [BCr, Proposition 2.2], if $\Omega$ is bounded and $u, v \in L^1(\Omega)$, the following conditions are equivalent:

(1.1) \quad $u \ll u + \lambda v$ \quad for any $\lambda > 0$,

(1.2) \quad $\int_{\Omega} p(u)v \geq 0$ \quad for any $p \in P_0$.

Remark that if $A$ is a completely accretive operator in $L^1(\Omega)$ and $1 \leq q \leq \infty$, then the restriction $A_q$ of $A$ to $L^q(\Omega)$ is T-accretive in $L^q(\Omega)$, that is

$$\|(u - \hat{u} + \lambda(v - \hat{v}))^+\|_q \geq \|(u - \hat{u})^+\|_q \quad \text{for} \quad \lambda > 0, \quad (u, v), (\hat{u}, \hat{v}) \in A_q.$$

Consequently, its resolvent $J_\lambda = (I + \lambda A_q)^{-1}$ is an order-preserving contraction in $L^q(\Omega)$. If a completely accretive operator $A$ in $L^1(\Omega)$ satisfies the range condition:
"there exists $\lambda > 0$ such that $R(I + \lambda A)$ is dense in $L^1(\Omega)$", then the closure $\overline{A}$ of $A$ is an m-T-accretive operator in $L^1(\Omega)$, that is, $\overline{A}$ is T-accretive and there exists $\lambda > 0$ such that $R(I + \lambda \overline{A}) = L^1(\Omega)$. So, by Crandall-Liggett’s Theorem, the operator $\overline{A}$ generates, on the closure of its domain, a semigroup of order-preserving contractions given by the exponential formula

$$e^{-t\overline{A}} u = \lim_{n \to \infty} (I + \frac{t}{n} \overline{A})^{-n} u \quad \text{for} \quad u \in D(\overline{A}).$$

This semigroup solves the corresponding initial value problem for the operator $\overline{A}$

(1.3) \quad u' + \overline{A}u \geq 0, \quad u(0) = u_0.

The function $u(t) := e^{-t\overline{A}} u_0$ is called the mild-solution of problem (1.3).

2. Variational approach

From now on we assume $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$ of class $C^1$, $1 < p < \infty$, $a$ is a vector valued mapping from $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^N$ satisfying $(H_1) - (H_3)$ and $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$.

In this section we study the problem (I) with variational methods. We will introduce and study a nonlinear completely accretive operator $A_0$ in $L^1(\Omega)$ associated with the formal differential expression

$$-\text{div} \, a(x,Du) + \text{nonlinear boundary conditions}.$$ 

Since $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, there exists a convex lower semicontinuous (l.s.c.) function $j$ on $\mathbb{R}$, such that $\beta = \partial j$. Consider $\Phi : W^{1,p}(\Omega) \to [0, +\infty]$, defined by

$$\Phi(u) := \begin{cases} \int_{\partial \Omega} j(u) & \text{if } j(u) \in L^1(\partial \Omega) \\ +\infty & \text{if } j(u) \notin L^1(\partial \Omega). \end{cases}$$

It is well-known (cf. [Br2]) that $\Phi$ is a convex l.s.c. function in $W^{1,p}(\Omega)$. Now let us define the operator $A_0$ in $L^1(\Omega)$ by:

$$(u,v) \in A_0 \quad \text{if and only if} \quad u \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \ v \in L^1(\Omega) \ \text{and} \ \Phi(w) \geq \Phi(u) + \int_{\Omega} v(w-u) - \int_{\Omega} \langle a(x,Du), D(w-u) \rangle \ \text{for every } w \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Here and below the integrals over $\Omega$ are with respect to Lebesgue measure $\lambda_N$ and the integrals over $\partial \Omega$ are with respect to the area measure $\mu$ on $\partial \Omega$. Observe that taking $w = 0$ in the definition of $A_0$ we obtain that $D(A_0) \subset D(\Phi)$.

We can now formulate the main result of this section.
Theorem 2.1. The operator $A_0$ satisfies the following statements:

(i) $A_0$ is univalued, i.e., if $(u, v) \in A_0$, then

$$v = -\text{div} \, a(x, Du)$$

in the sense of distributions.

(ii) $A_0$ is completely accretive.

(iii) $L^\infty(\Omega) \subset R(I + A_0)$.

Proof. (i) : Let $(u, v) \in A_0$. Given $\phi \in D(\Omega)$, taking $w = u + \phi$ and $w = u - \phi$ as test functions in the definition of the operator $A_0$, we get

$$\int_\Omega v \phi = \int_\Omega \langle a(x, Du), D\phi \rangle,$$

and consequently $v = -\text{div} \, a(x, Du)$ in the sense of distributions.

(ii) : Let $p \in P_0$ and $(u, v), (\hat{u}, \hat{v}) \in A$. We will show that

$$\int_\Omega (v - \hat{v})p(u - \hat{u}) \geq 0.$$ 

Taking $w = u - p(u - \hat{u})$ and $w = \hat{u} + p(u - \hat{u})$ as test functions in the definition of the operator $A_0$, we get

$$\Phi(u - p(u - \hat{u})) \geq \Phi(u) - \int_\Omega vp(u - \hat{u}) + \int_\Omega \langle a(x, Du), Dp(u - \hat{u}) \rangle$$

and

$$\Phi(\hat{u} + p(u - \hat{u})) \geq \Phi(\hat{u}) + \int_\Omega \hat{v}p(u - \hat{u}) + \int_\Omega \langle a(x, D\hat{u}), Dp(u - \hat{u}) \rangle.$$

Adding (2.2) and (2.3) we obtain

$$\int_\Omega \langle a(x, Du) - a(x, D\hat{u}), Dp(u - \hat{u}) \rangle + \int_\Omega (\hat{v} - v)p(u - \hat{u}) \leq \Phi(u - p(u - \hat{u})) + \Phi(\hat{u} + p(u - \hat{u})) - \Phi(u) - \Phi(\hat{u}).$$

By (H_2), the first integral in the above inequality is positive. We thus get

$$\int_\Omega (\hat{v} - v)p(u - \hat{u}) \leq \Phi(u - p(u - \hat{u})) + \Phi(\hat{u} + p(u - \hat{u})) - \Phi(u) - \Phi(\hat{u}).$$

On the other hand, by the convexity of $j$, it is easy to see that

$$\Phi(u - p(u - \hat{u})) + \Phi(\hat{u} + p(u - \hat{u})) \leq \Phi(u) + \Phi(\hat{u}).$$

We conclude from (2.4) and (2.5) that (2.1) holds. Therefore, by the equivalence between (1.1) and (1.2), $A_0$ is a completely accretive operator in $L^1(\Omega)$. 
The proof of (iii) will involve several steps. The basic idea, however, is classical and consists in approximating the equation

\[ u - \text{div} \, a(x, Du) = v, \quad v \in L^\infty(\Omega), \]

by problems of the form

\[ \gamma_n(u_n) - \text{div} \, a(x, Du_n) = v, \quad \text{with} \quad \gamma_n(\xi) := T_n(\xi) + \frac{1}{n}|\xi|^{p-2}\xi. \]

**Step 1.** For every \( n \in \mathbb{N} \) we consider the operators \( A_n : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^* \), defined by

\[ \langle A_n u, w \rangle := \int_\Omega \langle a(x, Du), Dw \rangle + \int_\Omega \gamma_n(u)w. \]

It is easy to see that \( A_n \) is monotone and hemicontinuous. Hence, \( A_n \) is pseudomonotone.

Moreover, by (H3), we have

\[ \frac{\langle A_n u, u \rangle + \Phi(u)}{\|u\|_{1,p}} \geq \frac{\int_\Omega \langle a(x, Du), Du \rangle + \int_\Omega \gamma_n(u)u + \Phi(u)}{\|u\|_{1,p}} \geq \frac{\lambda \int_\Omega |Du|^p + 1/n \int_\Omega |u|^p}{\|u\|_{1,p}} \geq \min\{\lambda, 1/n\} \|u\|_{1,p}^{p-1} \]

and thus

\[ \lim_{\|u\|_{1,p} \to \infty} \frac{\langle A_n u, u \rangle + \Phi(u)}{\|u\|_{1,p}} = +\infty. \]

Then, given \( v \in L^\infty(\Omega) \), by [Br1, Corollary 30], there exists \( u_n \in W^{1,p}(\Omega) \), such that

\[ (v, w - u_n) - \langle A_n u_n, w - u_n \rangle \leq \Phi(w) - \Phi(u_n) \quad \text{for all} \ w \in W^{1,p}(\Omega). \]

That is

\[ (2.6) \quad \int_{\partial \Omega} j(w) \geq \int_{\partial \Omega} j(u_n) + \int_{\Omega} v(w-u_n) - \int_{\Omega} \gamma_n(u_n)(w-u_n) - \int_{\Omega} \langle a(x, Du_n), D(w-u_n) \rangle. \]

for all \( w \in W^{1,p}(\Omega) \).

**Step 2. A priori estimates.** Given \( p \in P_0 \), if we take \( w = u_n - p(T_n u_n) \) in (2.6), we get

\[ \int_{\partial \Omega} j(u_n - p(T_n u_n)) \geq (2.7) \quad \int_{\partial \Omega} j(u_n) - \int_{\Omega} vp(T_n u_n) + \int_{\Omega} \gamma_n(u_n)p(T_n u_n) + \int_{\Omega} \langle a(x, Du_n), Dp(T_n u_n) \rangle. \]

By the convexity of \( j \), it is easy to see that

\[ j(u_n - p(T_n u_n)) \leq j(u_n). \]
From here and (2.7) it follows that
\[ \int_\Omega T_n(u_n)p(T_n(u_n)) \leq \int_\Omega v p(T_n(u_n)). \]
This gives \( T_n(u_n) \ll v \). Thus, \( \| T_n(u_n) \|_\infty \leq \| v \|_\infty \) for all \( n \in \mathbb{N} \). In particular,
\[ \| u_n \|_\infty \leq \| v \|_\infty \quad \text{for all } n \geq \| v \|_\infty. \]

On the other hand, taking \( w = 0 \) as test function in (2.6) and using (H₁), we obtain
\[ \lambda \int_\Omega |Du_n|^p \leq \int_\Omega \langle a(x,Du_n),Du_n \rangle \leq \int_\Omega vu_n. \]

As a consequence of (2.8) and (2.9), \( \{ u_n \}_{n \in \mathbb{N}} \) is bounded in \( W^{1,p}(\Omega) \). Hence there exists a subsequence, still denoted \( u_n \), such that \( u_n \to u \) weakly in \( W^{1,p}(\Omega) \). Moreover, by the Rellich-Kondrachov Theorem, \( u_n \to u \) in \( L^p(\Omega) \), and by [M, Theorem 3.4.5], \( u_n \to u \) in \( L^p(\partial \Omega) \). After passing to a suitable subsequence, we can assume that \( u_n \to u \) a.e. in \( \Omega \). So, by (2.8), \( \| u \|_\infty \leq \| v \|_\infty \).

**Step 3. Convergence.** We now prove that \( Du_n \) converges to \( Du \) in measure, to do this we follow the same technique used in [BG] (see also [BW]). Since \( Du_n \) converges to \( Du \) weakly in \( L^p(\Omega) \), it is enough to show that \( \{ Du_n \} \) is a Cauchy sequence in measure. Let \( t \) and \( \epsilon > 0 \). For some \( A > 1 \) we set
\[ C(x,A,t) := \inf \{ \langle a(x,\xi) - a(x,\eta),\xi - \eta \rangle : \| \xi \| \leq A, \| \eta \| \leq A, \| \xi - \eta \| \geq t \}. \]

Having in mind that the function \( \psi \to a(x,\psi) \) is continuous for almost all \( x \in \Omega \) and the set \( \{ (\xi,\eta) : \| \xi \| \leq A, \| \eta \| \leq A, \| \xi - \eta \| \geq t \} \) is compact, the infimum in the definition of \( C(x,A,t) \) is a minimum. Hence, by (H₂), it follows that
\[ C(x,A,t) > 0 \quad \text{for almost all } x \in \Omega. \]

Now, for \( n,m \in \mathbb{N} \) and any \( k > 0 \), the following inclusions hold
\[ \{ |Du_n - Du_m| > t \} \subset \]
\[ \subset \{ |Du_n| \geq A \} \cup \{ |Du_m| \geq A \} \cup \{ |u_n - u_m| \geq k^2 \} \cup \{ C(x,A,t) \leq k \} \cup \]
\[ \cup \{ |u_n - u_m| \leq k^2, C(x,A,t) \geq k, |Du_n| \leq A, |Du_m| \leq A, |Du_n - Du_m| > t \}. \]

Since \( \{ Du_n \}_{n \in \mathbb{N}} \) is bounded in \( L^p(\Omega) \) we can choose \( A \) large enough in order to have
\[ \lambda_N(\{ |Du_n| \geq A \} \cup \{ |Du_m| \geq A \}) \leq \frac{\epsilon}{4} \quad \text{for all } n,m \in \mathbb{N}. \]

By (2.10) we can choose \( k \) small enough in order to have
\[ \lambda_N(\{ C(x,A,t) \leq k \}) \leq \frac{\epsilon}{4}. \]
On the other hand, if we use \( u_n - T_k(u_n - u_m) \) and \( u_m + T_k(u_n - u_m) \) as test functions in (2.6), we obtain

\[
- \int_{\Omega} v T_k(u_n - u_m) + \int_{\Omega} \gamma_n(u_n) T_k(u_n - u_m) + \int_{\Omega} \langle a(x, Du_n), DT_k(u_n - u_m) \rangle \leq \int_{\partial \Omega} j(u_n - T_k(u_n - u_m)) - \int_{\partial \Omega} j(u_n).
\]

Adding (2.14) and (2.15), we get

\[
\int_{\Omega} \langle a(x, Du_n) - a(x, Du_m), DT_k(u_n - u_m) \rangle + \int_{\Omega} (\gamma_n(u_n) - \gamma_m(u_m)) T_k(u_n - u_m) \leq \leq \int_{\partial \Omega} j(u_n - T_k(u_n - u_m)) + j(u_m + T_k(u_n - u_m)) - \int_{\partial \Omega} j(u_n) - \int_{\partial \Omega} j(u_m) \leq 0.
\]

Consequently,

\[
\int_{\Omega} \langle a(x, Du_n) - a(x, Du_m), DT_k(u_n - u_m) \rangle \leq k \int_{\Omega} |\gamma_n(u_n)| + |\gamma_m(u_m)| \leq k M.
\]

Hence

\[
\lambda_N (\{|u_n - u_m| \leq k^2, C(x, A, t) \geq k, |Du_n| \leq A, |Du_m| \leq A, |Du_n - Du_m| > t\}) \leq \lambda_N (\{|u_n - u_m| \leq k^2, \langle a(x, Du_n) - a(x, Du_m), D(u_n - u_m) \rangle \geq k\}) \leq \leq \frac{1}{k} \int_{\{|u_n - u_m| < k^2\}} \langle a(x, Du_n) - a(x, Du_m), D(u_n - u_m) \rangle \leq \frac{1}{k} k^2 M \leq \frac{\epsilon}{4},
\]

for \( k \) small enough.

Since \( A \) and \( k \) have been already choosen, if \( n_0 \) is large enough we have for \( n, m \geq n_0 \) the estimate \( \lambda_N (\{|u_n - u_m| \geq k^2\}) \leq \frac{\epsilon}{4} \). From here, using (2.11), (2.12), (2.13) and (2.16), we can conclude that

\[
\lambda_N (\{|Du_n - Du_m| \geq t\}) \leq \epsilon \quad \text{for} \quad m, n \geq n_0.
\]

According to Nemytskii’s Theorem [K, Lemma I.2.2.1] the convergence of \( Du_n \) to \( Du \) in measure implies that \( a(x, Du_n) \) converges in measure to \( a(x, Du) \), and a.e. (up to extraction of a subsequence, if necessary). Moreover, since \( \{Du_n\}_{n \in \mathbb{N}} \) is bounded in \( L^p(\Omega) \), from (H3) it follows that \( \{a(x, Du_n)\}_{n \in \mathbb{N}} \) is bounded in \( L^p(\Omega) \). Therefore,

\[
a(x, Du_n) \rightharpoonup a(x, Du) \quad \text{weakly in} \quad L^p(\Omega).
\]
Step 4. To complete the proof it remains to show that

\[(2.18) \quad \int_{\partial \Omega} j(w) \geq \int_{\partial \Omega} j(u) + \int_{\Omega} (v - u)(w - u) - \int_{\Omega} \langle a(x, Du), D(w - u) \rangle.\]

for all \( w \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \).

Let \( w \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \), from (2.17), it follows that

\[(2.19) \quad \int_{\Omega} \langle a(x, Du), Dw \rangle = \lim_{n \to \infty} \int_{\Omega} \langle a(x, Du_n), Dw \rangle.\]

By Fatou’s Lemma, we have

\[(2.20) \quad \int_{\Omega} \langle a(x, Du), Du \rangle \leq \liminf_{n \to \infty} \int_{\Omega} \langle a(x, Du_n), Du_n \rangle.\]

and

\[(2.21) \quad \int_{\partial \Omega} j(u) \leq \liminf_{n \to \infty} \int_{\partial \Omega} j(u_n).\]

Since \( u_n \to u \) in \( L^p(\Omega) \) we have

\[(2.22) \quad \lim_{n \to \infty} \int_{\Omega} (v - \gamma_n(u_n))(w - u_n) = \int_{\Omega} (v - u)(w - u).\]

From (2.19), (2.20), (2.21) and (2.22), passing to the limit in (2.6), we get (2.18), and the proof is concluded.

Next we are going to introduce another operator in \( L^1(\Omega) \), smaller than \( A_0 \), which will be used later to characterize the variational operator \( A_0 \). To define the new operator we need introduce the following subset of \( W^{1,p}(\Omega) \): Given \( \beta \) a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) with \( 0 \in \beta(0) \), we set

\[ W^{1,p}_\beta(\Omega) := \{ u \in W^{1,p}(\Omega) : u(x) \in \mathcal{D}(\beta) \text{ a.e. } x \in \partial \Omega \}. \]

The above definition uses the fact that the trace of \( u \in W^{1,1}(\Omega) \) on \( \partial \Omega \) is well defined in \( L^1(\partial \Omega) \) [N, Theorem 4.2]. Observe that we use the same notation \( u \) for \( u \) and its trace when convenient.

Remark that \( W^{1,p}_\beta(\Omega) \) is a closed convex subset of \( W^{1,p}(\Omega) \). In case \( \beta \) corresponds to the Dirichlet boundary condition, \( W^{1,p}_\beta(\Omega) = W^{1,p}_0(\Omega) \), and in case \( \beta \) corresponds to the Neumann boundary condition, \( W^{1,p}_\beta(\Omega) = W^{1,p}(\Omega) \).

We define the operator \( A \) in \( L^1(\Omega) \) by the rule:

\[(u, v) \in A \text{ if and only if } u \in W^{1,p}(\Omega) \cap L^\infty(\Omega), v \in L^1(\Omega), \text{ there exists } w \in L^1(\partial \Omega) \text{ with } -w(x) \in \beta(u(x)) \text{ a.e. on } \partial \Omega \text{ and} \]

\[ \int_{\Omega} \langle a(x, Du), D(u - \phi) \rangle \leq \int_{\Omega} v(u - \phi) + \int_{\partial \Omega} w(u - \phi) \text{ for every } \phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega). \]

Since \( \partial j = \beta \), it is easy to see that \( A \subset A_0 \). Consequently, \( A \) is a completely accretive operator. Let us see now, that under certain conditions, the operator \( A \) satisfies the range condition.
Theorem 2.2. If $\mathcal{D}(\beta)$ is closed, then the operator $A$ satisfies the range condition. More precisely,

$$L^\infty(\Omega) \subset R(I + A).$$

Proof. For every $n \in \mathbb{N}$, let $\gamma_n(\xi) := T_n(\xi) + \frac{1}{n}|\xi|^{p-2}\xi$, and $\beta_n$ be the Yosida approximation of $\beta$, i.e., $\beta_n = n(I - (I + 1/n \beta)^{-1})$. Consider the operators $A_n : W^{1,p}_\beta(\Omega) \to (W^{1,p}_\beta(\Omega))^*$, defined by

$$\langle A_n u, v \rangle := \int_\Omega \langle a(x, Du), Dv \rangle + \int_\Omega \gamma_n(u)v + \int_{\partial\Omega} T_n\beta_n(u)v.$$

As a consequence of (H3) and having in mind that $W^{1,p}_\beta(\Omega)$ is continuously embedded in $L^p(\partial\Omega)$ (see e.g., [M, Theorem 3.4.5]), we have that the operators $A_n$ are well defined and map bounded subsets into bounded subsets. Proceeding as in the proof of the above theorem we have $A_n$ is a monotone, hemicontinuous and coercive operator. Given $v \in L^\infty(\Omega)$, by a classical result of Browder (cf. [KS, Chapter III]), there exists $u_n \in W^{1,p}_\beta(\Omega)$ such that

$$\int_\Omega \langle a(x, Du_n), D(u_n - \phi) \rangle + \int_\Omega \gamma_n(u_n)(u_n - \phi) + \int_{\partial\Omega} T_n\beta_n(u_n)(u_n - \phi) \leq \int_\Omega v(u_n - \phi), \text{ for every } \phi \in W^{1,p}_\beta(\Omega).$$

(2.23)

Now, as in the steps 2 and 3 of the proof of the above theorem, we get the same estimates and convergence results for $\{u_n\}$. So the proof is completed by showing that there exists $w \in L^1(\partial\Omega)$ with $-w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, such that

$$\lim_{n \to \infty} \int_{\partial\Omega} T_n\beta_n(u_n)(u_n - \phi) = \int_{\partial\Omega} w(u - \phi).$$

(2.24)

In fact, since $u_n \in \mathcal{D}(\beta)$, $|\beta_n(u_n(x))| \leq \inf \{|r| : r \in \beta(u_n(x))\}$. From here, if $\mathcal{D}(\beta)$ is bounded, $\{\beta_n(u_n(x))\}_{n \in \mathbb{N}}$ is bounded in $L^\infty(\partial\Omega)$. In case $\mathcal{D}(\beta)$ is unbounded, the boundedness of $\{\beta_n(u_n(x))\}$ in $L^\infty(\partial\Omega)$ follows from the boundedness of $\{u_n(x)\}$ in $L^\infty(\Omega)$. Consequently,

$$T_n\beta_n(u_n) \to w \in L^1(\partial\Omega) \text{ weakly in } L^1(\partial\Omega).$$

(2.25)

Then, since $\{u_n - \phi : n \in \mathbb{N}\}$ is bounded in $L^\infty(\Omega)$, $u_n - \phi \to u - \phi$ a.e. in $\partial\Omega$ and $\mu(\partial\Omega) < \infty$, (2.25) yields (2.24).

Finally, since $u_n \to u$ in $L^1(\partial\Omega)$ and (2.25), from [BCS, Lemma G] it follows that $-w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and the proof is concluded.

The assumption $\mathcal{D}(\beta)$ closed can be removed when the function $a$ is “smooth enough”. More precisely, we give the following definition due to Ph. Bénilan (personal communication). We say that $a$ is smooth if for every $f \in L^\infty(\Omega)$ there exists $g \in L^1(\partial\Omega)$ such that the solution $u$ of the Dirichlet problem

$$-\text{div } a(x, Du) = f \text{ in } \Omega$$

\[ u = 0 \text{ on } \partial \Omega \]
is solution of the Neumann problem
\[ -\text{div } \mathbf{a}(x, Du) = f \text{ in } \Omega \]
\[ \frac{\partial u}{\partial \eta_a} = g \text{ on } \partial \Omega. \]

For \( \mathbf{a} \) smooth we have the following result.

**Theorem 2.3.** If \( \mathbf{a} \) is smooth, then the operator \( \mathcal{A} \) satisfies
\[ L^\infty(\Omega) \subset \mathcal{R}(I + \mathcal{A}). \]

**Proof.** Fix \( v \in L^\infty(\Omega) \). Let \( \beta_n \) be the Yosida approximations of \( \beta \). Since \( W^{1,p}_\beta(\Omega) = W^{1,p}(\Omega) \), by Theorem 2.2, there exists \( u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \) such that \( \beta_n(u_n) \in L^1(\partial \Omega) \) and
\[ (2.26) \quad \int_\Omega \langle \mathbf{a}(x, Du_n), D\phi \rangle + \int_\Omega u_n \phi = \int_\Omega v \phi - \int_{\partial \Omega} \beta_n(u_n) \phi, \]
for every \( \phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \).

We can now proceed analogously to the proof of Theorem 2.2. So the proof is completed by showing that the sequence \( \{ \beta_n(u_n) : n \in \mathbb{N} \} \) is a relatively weakly compact subset of \( L^1(\partial \Omega) \). To see this, let \( \hat{u} \) be the solution of the Dirichlet problem
\[ \hat{u} - \text{div } \mathbf{a}(x, D\hat{u}) = v \text{ in } \Omega \]
\[ \hat{u} = 0 \text{ on } \partial \Omega. \]

Since \( \mathbf{a} \) is smooth, it follows that
\[ (2.27) \quad \int_\Omega \hat{u} \phi + \int_\Omega \langle \mathbf{a}(x, D\hat{u}), D\phi \rangle = \int_\Omega v \phi + \int_{\partial \Omega} g \phi \]
for every \( \phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \).

Given \( p \in P_0 \), if we take \( \phi = p(\beta_n(u_n - \hat{u})) \) as test function in (2.26) and (2.27), substracting (2.27) from (2.26), we get
\[ \int_\Omega \langle \mathbf{a}(x, Du_n) - \mathbf{a}(x, D\hat{u}), Dp(\beta_n(u_n - \hat{u})) \rangle + \int_\Omega (u_n - \hat{u}) p(\beta_n(u_n - \hat{u})) = \]
\[ = - \int_{\partial \Omega} (g + \beta_n(u_n)) p(\beta_n(u_n - \hat{u})). \]
Since the left hand side is non negative and \( \hat{u} \in W^{1,p}_0(\Omega) \), it follows that
\[ \int_{\partial \Omega} (g + \beta_n(u_n)) p(\beta_n(u_n)) \leq 0. \]
Hence $\beta_n(u_n) \ll -g$ for all $n \in \mathbb{N}$. Then, since $g \in L^1(\partial \Omega)$, from [BCr, Proposition 2.11], it follows that $\{\beta_n(u_n) : n \in \mathbb{N}\}$ is a relatively weakly compact subset of $L^1(\partial \Omega)$ and the proof is completed.

**Remark 2.4.** The condition $a$ smooth is satisfied in some cases. For instance, in the linear case when the coefficients of $a$ are regular, cf. [Br2]. As a consequence of the regularity results given in [Li] it is also possible to show the smoothness of the $p$-Laplacian operator. We do not know if the assumptions (H1), (H2) and (H3) implies the smoothness of $a$.

An interesting problem is to characterize the operator $\overline{A}$. In the case of Dirichlet boundary condition, this has been done recently in [B-V], where a new concept of solution, named entropy solution, is defined for the elliptic problem

$$-\text{div } a(x, Du) = F(x, u) \quad \text{in } \Omega$$

$$u(x) = 0 \quad \text{on } \partial \Omega.$$  

For that, the authors need to introduce a new space $T^1_{loc}(\Omega)$, which is an extension of $W_0^{1,p}(\Omega)$. Here, in the same spirit, we need to extend the concept of trace. This is the subject of the next section.

**3. A generalization of the trace.**

Before to discuss the concept of trace we recall the following spaces introduced in [B-V]: $T^{1,1}_{loc}(\Omega)$ is defined as the set of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that for every $k > 0$ the truncated function $T_k(u)$ belongs to $W^{1,1}_{loc}(\Omega)$. For $1 < p < \infty$, $T^{1,p}_{loc}(\Omega)$ is the subset of $T^{1,1}_{loc}(\Omega)$ consisting of the functions $u$ such that $DT_k(u) \in L^p_{loc}(\Omega)$ for every $k > 0$. Likewise, $T^{1,p}(\Omega)$ is the subset of $T^{1,1}(\Omega)$ consisting of the functions $u$ such that $DT_k(u) \in L^p(\Omega)$ for every $k > 0$. Observe that in the definition of $T^{1,p}(\Omega)$ is not imposed the condition $T_k(u) \in L^p(\Omega)$. Of course, this condition follows immediately when $\Omega$ is bounded. So in this case we have

$$T^{1,p}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable } : T_k(u) \in W^{1,p}(\Omega) \text{ for all } k > 0\}.$$  

It is possible to give a sense to the derivative $Du$ of a function $u \in T^{1,1}_{loc}(\Omega)$, generalizing the usual concept of weak derivative in $W^{1,1}_{loc}(\Omega)$, thanks to the following result (see [B-V, Lemma 2.1]):

“For every $u \in T^{1,1}_{loc}(\Omega)$ there exists a unique measurable function $v : \Omega \rightarrow \mathbb{R}$ such that

$$DT_k(u) = v1_{\{|v| < k\}} \quad \text{a.e.}$$

Furthermore, if $u \in W^{1,1}_{loc}(\Omega)$, then $v = Du$ in the usual weak sense.”

The derivative $Du$ of a function $u \in T^{1,1}_{loc}(\Omega)$ is defined as the unique function $v$ satisfying (3.1). This notation will be used throughout in the sequel. We remark that this definition of derivative is not a definition in the sense of distributions.
Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ of class $C^1$ and $1 \leq p < \infty$. It is well-known (cf. [N] or [M]) that if $u \in W^{1,p}(\Omega)$, it is possible to define the trace of $u$ on $\partial \Omega$. More precisely, there is a bounded operator $\gamma$ from $W^{1,p}(\Omega)$ into $L^p(\partial \Omega)$ such that $\gamma(u) = u|_{\partial \Omega}$ whenever $u \in C(\overline{\Omega})$. Now, it is easy to see that, in general, it is not possible to define the trace of an element of $T^{1,p}(\Omega)$. In dimension one it is enough to consider the function $u(x) = 1/x$ for $x \in [0,1]$. Nevertheless, we are going to define the trace for the elements of a subset $T^{1,p}_t(\Omega)$ of $T^{1,p}(\Omega)$. $T^{1,p}_t(\Omega)$ will be the subset of $T^{1,p}(\Omega)$ consisting of the functions that can be approximated by functions of $W^{1,p}(\Omega)$ in the following sense: a function $u \in T^{1,p}(\Omega)$ belongs to $T^{1,p}_t(\Omega)$ if there exists a sequence $u_n \in W^{1,p}(\Omega)$ such that

\[(3.2)\quad W^{1,p}(\Omega) \subset T^{1,p}_t(\Omega) \subset T^{1,p}(\Omega).\]

In (3.2) the inclusions are strict. In fact: It is easy to see that the function $u(x) = 1/x$ for $x \in [0,1]$ is an element of $T^{1,1}([0,1]) \setminus T^{1,1}_t([0,1])$. Moreover the function $u$ defined by

$$u(x) := \begin{cases} 1/x & \text{if } x \in [0,1] \\ -1/x & \text{if } x \in [-1,0], \end{cases}$$

is an example of an element of $T^{1,1}_t([-1,1]) \setminus W^{1,1}([-1,1])$.

In the following result we obtain an extension of the trace defined in $W^{1,p}(\Omega)$.

**Theorem 3.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ of class $C^1$ and $1 \leq p < \infty$. Then, there exists a map $\tau : T^{1,p}_t(\Omega) \rightarrow M(\partial \Omega, \mu)$, such that

$$\tau(u) = \gamma(u) \quad \text{whenever } u \in W^{1,p}(\Omega).$$

Moreover,

(i) $\gamma(T_ku) = T_k(\tau u)$ for every $u \in T^{1,p}_t(\Omega)$ and $k > 0$.

(ii) If $u \in T^{1,p}_t(\Omega)$ and $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $u - \phi \in T^{1,p}_t(\Omega)$ and $\tau(u - \phi) = \tau(u) - \tau(\phi)$.

**Proof.** Given $u \in T^{1,p}_t(\Omega)$, let $u_n \in W^{1,p}(\Omega)$ be such that

(a) $u_n \rightarrow u$ a.e. in $\Omega$,

(b) $DT_k(u_n) \rightarrow DT_k(u)$ in $L^1(\Omega)$ for any $k > 0$,

(c) $\gamma(u_n) \rightarrow v \in M(\partial \Omega, \mu)$ a.e. in $\partial \Omega$.

Then, by the Dominated Convergence Theorem, $T_ku_n \rightarrow T_ku$ in $W^{1,1}(\Omega)$. Consequently, since the trace $\gamma$ from $W^{1,1}(\Omega)$ into $L^1(\partial \Omega)$ is a linear bounded operator, it follows that

$$\lim_{n \rightarrow \infty} \|\gamma(T_ku_n) - \gamma(T_ku)\|_{L^1(\partial \Omega)} = 0 \quad \text{for every } k \in \mathbb{N}.$$ 

Now, by (c), $T_k(\gamma(u_n)) \rightarrow T_k(v)$ a.e. in $\partial \Omega$. Consequently, $\gamma(T_ku) = T_k(v)$ a.e. in $\partial \Omega$ for any $k > 0$. Thus, $\gamma(T_k(u)) \rightarrow v$ a.e. in $\partial \Omega$. Therefore we can defined $\tau(u)$.
as the a.e. limit in $\partial \Omega$ of $\gamma(T_k(u))$ as $k \to \infty$. Obviously, $\tau(u) = \gamma(u)$ whenever $u \in W^{1,p}(\Omega)$. Also, by the definition of the map $\tau$, (i) holds.

(ii) Let $u \in T^{1,p}_{tr}(\Omega)$ and $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, there exists $u_n \in W^{1,p}(\Omega)$ satisfying (a), (b) and (c). Take $\psi_n := u_n - \phi \in W^{1,p}(\Omega)$. Then, $\psi_n \to u - \phi$ a.e. in $\Omega$ and $\gamma(\psi_n) \to \tau u - \tau \phi$ a.e. in $\partial \Omega$. Hence, to finish the proof we only need to prove that $DT_k(\psi_n) \to DT_k(u - \phi)$ a.e. On the other hand, if $M := k + \|\phi\|_\infty$, we have

$$|DT_k(\psi_n)| \leq |DT_M(u_n)| + |D\phi|.$$ 

Moreover, since $DT_M(u_n) \to DT_M(u)$ in $L^1(\Omega)$, there exists $g \in L^1(\Omega)$ such that $|DT_M(u_n)| \leq g$ a.e. Therefore, by the Dominated Convergence Theorem it follows that $DT_k(\psi_n) \to DT_k(u - \phi)$ in $L^1(\Omega)$.

To study the Dirichlet problem, in [B-V], it is introduced the subspace $T^{1,p}_{0}\Omega$ of $T^{1,p}(\Omega)$ consisting of the functions that can be approximated by smooth functions with compact support in $\Omega$ in the following sense: a function $u \in T^{1,p}(\Omega)$ belongs to $T^{1,p}_{0}\Omega$ if for every $k > 0$ there exists a sequence $\zeta_n \in C_0^\infty(\Omega)$ such that

$$\zeta_n \to T_k u \quad \text{in} \quad L^1_{loc}(\Omega),$$

$$D\zeta_n \to DT_k(u) \quad \text{in} \quad L^p(\Omega).$$

As a consequence of the characterizations of $T^{1,p}_{0}\Omega$ given in [B-V, Appendix II] we have

$$\ker(\tau) = T^{1,p}_{0}\Omega.$$ 

4. The closure of the operator $A$.

In this section we characterize the closure of the operator $A$ by means of a new concept of solution. As we mention in the introduction, a new concept of entropy solution for the Dirichlet problem

$$-\text{div} a(x, Du) = v(x) \quad \text{in} \quad \Omega$$

$$u = 0 \quad \text{on} \quad \partial \Omega.$$ 

is introduced in [B-V]. Following this idea we define the concept of entropy solution for the problem

$$(II)$$

$$-\text{div} a(x, Du) = v(x) \quad \text{in} \quad \Omega$$

$$-\frac{\partial u}{\partial n} = \beta(u) \quad \text{on} \quad \partial \Omega$$

where $v \in L^1(\Omega)$, in the following way: We say that $u \in T^{1,p}_{tr}(\Omega)$ is an entropy solution of (II) if there exists $w \in L^1(\partial \Omega)$, $-w(x) \in \beta(u(x))$ a.e. on $\partial \Omega$ such that

$$(4.1) \quad \int_{\Omega} \langle a(x, Du), DT_k(u - \phi) \rangle \leq \int_{\Omega} v T_k(u - \phi) + \int_{\partial \Omega} w T_k(u - \phi)$$
for every $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$ and $k > 0$.

Notice that the integrals in (4.1) are well defined. In general, the difference of two elements of $T^{1,p}(\Omega)$ is not an element of $T^{1,p}(\Omega)$ (see [B-V]), however, since $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have $u - \phi \in T^{1,p}(\Omega)$ (Theorem 3.1). Hence, $T_k(u - \phi) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and consequently the two first integrals in (4.1) are well defined. Moreover, in the last integral we can use the fact that the trace of $f \in W^{1,p}(\Omega)$ on $\partial \Omega$ is well defined in $L^p(\partial \Omega)$. Observe we use the same notation $f$ for $f$ and its trace when convenient.

Using the concept of entropy solution we define the operator $A$ in $L^1(\Omega)$ by the rule: $(u, v) \in A$ if and only if $u, v \in L^1(\Omega)$ and $u$ is an entropy solution of (II). Since $u \in T^{1,p}(\Omega)$, by Theorem 3.1, the trace of $u$ on $\partial \Omega$ is well defined as a function of $M(\partial \Omega, \mu)$.

In order to show that the operator $A$ is the closure of $A$ we need the following lemma.

**Lemma 4.1.** If $(u, v) \in A$ and $\alpha, k > 0$. Then the following inequality holds

$$\frac{1}{k} \int_{\{\alpha < |u| < \alpha + k\}} |Du|^p \leq \frac{1}{\lambda} \int_{\{|u| \geq \alpha\}} |v|.$$  

**Proof.** If we take $T_\alpha(u)$ as test function in (4.1) we have

$$\int_{\Omega} \langle a(x, Du), DT_k(u - T_\alpha(u)) \rangle \leq \int_{\Omega} vT_k(u - T_\alpha(u)) + \int_{\partial \Omega} wT_k(u - T_\alpha(u)).$$

Now, since the last integral is negative, using (H$_1$), it follows that

$$k \int_{\{|u| \geq \alpha\}} |v| \geq \int_{\Omega} \langle a(x, Du), DT_k(u - T_\alpha(u)) \rangle =$$

$$= \int_{\{\alpha < |u| < \alpha + k\}} \langle a(x, Du), Du \rangle \geq \lambda \int_{\{\alpha < |u| < \alpha + k\}} |Du|^p.$$

Now we come to the main result.

**Theorem 4.2.** Suppose $1 < p < N$ and assume that $D(\beta)$ is closed or $a$ is smooth. Then, the operator $A$ is the closure of $A$ in $L^1(\Omega)$. Consequently, $A$ is an $m$-$T$-accretive operator in $L^1(\Omega)$. Moreover, if $(u, v) \in A$, then

$$u \in M^{p_1}(\Omega), \quad |Du| \in M^{p_2}(\Omega)$$

where $p_1 = \frac{N(p-1)}{N-p}$ and $p_2 = \frac{N(p-1)}{N-1}$. In case $p > 2 - 1/N$, $u \in W^{1,q}(\Omega)$ for every $1 \leq q < p_2$.

**Proof.** By Theorem 2.2 or Theorem 2.3 and maximal accretivity, it is enough to show that $A$ is accretive in $L^1(\Omega)$ and $\overline{A} \subset A$.

**Step 1. Accretivity of $A$.** To prove the accretivity of $A$, we must show that

$$\int_{\Omega} |u - \tilde{u}| \leq \int_{\Omega} |f - \tilde{f}|$$  

(4.2)
whenever \( f \in u + A u \), \( \hat{f} \in \hat{u} + A \hat{u} \). In fact: Let \( w, \hat{w} \in L^1(\partial\Omega) \) with \(-w(x) \in \beta(u(x))\) and \(-\hat{w}(x) \in \beta(\hat{u}(x))\) a.e. on \( \partial\Omega \) such that for every \( h > 0 \),

\[
\int_{\Omega} \langle a(x, D u), D T_k(u - T_h(\hat{u})) \rangle \leq \int_{\Omega} (f - u) T_k(u - T_h(\hat{u})) + \int_{\partial\Omega} w T_k(u - T_h(\hat{u}))
\]
and

\[
\int_{\Omega} \langle a(x, D \hat{u}), D T_k(\hat{u} - T_h(u)) \rangle \leq \int_{\Omega} (\hat{f} - \hat{u}) T_k(\hat{u} - T_h(u)) + \int_{\partial\Omega} \hat{w} T_k(\hat{u} - T_h(u)).
\]

We write

\[
I_{h,k} := \int_{\Omega} \langle a(x, D u), D T_k(u - T_h(\hat{u})) \rangle + \int_{\Omega} \langle a(x, D \hat{u}), D T_k(\hat{u} - T_h(u)) \rangle,
\]

\[
J^1_h := \int_{\Omega} (f - u) T_k(u - T_h(\hat{u})) + \int_{\Omega} (\hat{f} - \hat{u}) T_k(\hat{u} - T_h(u))
\]
and

\[
J^2_h := \int_{\partial\Omega} w T_k(u - T_h(\hat{u})) + \int_{\partial\Omega} \hat{w} T_k(\hat{u} - T_h(u)).
\]

Adding up the above two inequalities we get

\[
(4.3) \quad I_{h,k} \leq J^1_h + J^2_h.
\]

By the Dominated Convergence Theorem it follows that

\[
(4.4) \quad \lim_{h \to \infty} J^1_h = \int_{\Omega} ((f - u) - (\hat{f} - \hat{u})) T_k(u - \hat{u}).
\]
and

\[
(4.5) \quad \lim_{h \to \infty} J^2_h = \int_{\partial\Omega} (w - \hat{w}) T_k(u - \hat{u}).
\]

Since the integral in (4.5) is negative, from (4.3) and (4.4) we have that

\[
(4.6) \quad \int_{\Omega} ((f - u) - (\hat{f} - \hat{u})) T_k(u - \hat{u}) \geq \liminf_{h \to \infty} I_{h,k}.
\]

Then, if we prove that,

\[
(4.7) \quad \liminf_{h \to \infty} I_{h,k} \geq 0 \quad \text{for any } k > 0,
\]

we get from (4.6)

\[
\frac{1}{k} \int_{\Omega} (u - \hat{u}) T_k(u - \hat{u}) \leq \frac{1}{k} \int_{\Omega} (f - \hat{f}) T_k(u - \hat{u}) \leq \int |f - \hat{f}|.
\]
From here, passing to the limit as \( k \to 0^+ \), it follows (4.2). To prove (4.7) we proceed by splitting the integral into different integration sets. We write

\[ I_{h,k} = I_{h,k}^1 + I_{h,k}^2 + I_{h,k}^3 + I_{h,k}^4, \]

where

\[ I_{h,k}^1 := \int_{\{ |u| < h, \ |\hat{u}| < h \}} \langle a(x, Du) - a(x, \hat{D}u), DT_k(u - \hat{u}) \rangle \geq 0. \]

\[ I_{h,k}^2 := \int_{\{ |u| < h, \ |\hat{u}| \geq h \}} \langle a(x, Du), DT_k(u - h\text{sign}(\hat{u})) \rangle + \int_{\{ |u| < h, \ |\hat{u}| \geq h \}} \langle a(x, \hat{D}u), DT_k(\hat{u} - u) \rangle \geq \int_{\{ |u| < h, \ |\hat{u}| \geq h \}} \langle a(x, Du), DT_k(u - \hat{u}) \rangle. \]

\[ I_{h,k}^3 := \int_{\{ |u| \geq h, \ |\hat{u}| < h \}} \langle a(x, Du), DT_k(u - \hat{u}) \rangle + \int_{\{ |u| \geq h, \ |\hat{u}| < h \}} \langle a(x, \hat{D}u), DT_k(\hat{u} - h\text{sign}(u)) \rangle \geq \int_{\{ |u| \geq h, \ |\hat{u}| < h \}} \langle a(x, Du), DT_k(u - \hat{u}) \rangle. \]

\[ I_{h,k}^4 := \int_{\{ |u| \geq h, \ |\hat{u}| \geq h \}} \langle a(x, Du), DT_k(u - h\text{sign}(\hat{u})) \rangle + \int_{\{ |u| \geq h, \ |\hat{u}| \geq h \}} \langle a(x, \hat{D}u), DT_k(\hat{u} - h\text{sign}(u)) \rangle \geq 0. \]

Combining the above estimates we get

\[ I_{h,k} \geq L_{h,k}^1 + L_{h,k}^2, \]

where

\[ L_{h,k}^1 := \int_{\{ |u| < h, \ |\hat{u}| \geq h \}} \langle a(x, D\hat{u}), DT_k(\hat{u} - u) \rangle \]

and

\[ L_{h,k}^2 := \int_{\{ |u| \geq h, \ |\hat{u}| < h \}} \langle a(x, Du), DT_k(u - \hat{u}) \rangle. \]

Now, if we put

\[ C(h, k) := \{ h < |u| < k + h \} \cap \{ h - k < |\hat{u}| < h \}, \]

we have

\[ |L_{h,k}^2| \leq \int_{\{ |u - \hat{u}| < k, \ |u| \geq h, \ |\hat{u}| < h \}} |\langle a(x, Du), Du - D\hat{u} \rangle| \leq \int_{C(h, k)} |\langle a(x, Du), Du \rangle| + \int_{C(h, k)} |\langle a(x, Du), D\hat{u} \rangle|. \]
Then, by Hölder’s inequality, we get
\[ |L^2_{h,k}| \leq \left( \int_{C(h,k)} |a(x, Du)|^{p'} \right)^{1/p'} \left( \left( \int_{C(h,k)} |Du|^p \right)^{1/p} + \left( \int_{C(h,k)} |\hat{D}u|^p \right)^{1/p} \right). \]

Now, by (H_3),
\[ \left( \int_{C(h,k)} |a(x, Du)|^{p'} \right)^{1/p'} \leq \Lambda \left( \|j\|^{p'}_{p'} + \int_{\{h<|u|<k+h\}} |Du|^p \right)^{1/p'}. \]

On the other hand, applying Lemma 4.1, we obtain
\[ \int_{\{h<|u|<k+h\}} |Du|^p \leq \frac{k}{\lambda} \int_{\{|u|\geq h\}} |f - u|. \]
and
\[ \int_{\{h-k<|\hat{u}|<h\}} |\hat{D}u|^p \leq \frac{k}{\lambda} \int_{\{|\hat{u}|\geq h-k\}} |\hat{f} - \hat{u}|. \]

From (4.8), (4.9) and (4.10), it follows that
\[ |L^2_{h,k}| \leq \Lambda 2^{1/2} \left( \|j\|^{p'}_{p'} + \int_{\{h<|u|<k+h\}} |Du|^p \right)^{1/p'}. \]

Then, since \( u, \hat{u}, f, \hat{f} \in L^1(\Omega) \), we have that
\[ \lim_{h \to \infty} L^2_{h,k} = 0. \]

Similarly,
\[ \lim_{h \to \infty} L^1_{h,k} = 0. \]

Therefore claim (4.7) holds and consequently the proof of step 1 is finished.

Step 2. \( \overline{A} \subset A \). Let \((u, v) \in \overline{A} \). Then, there exist \((u_n, v_n) \in A \) such that \( u_n \to u \) and \( v_n \to v \) in \( L^1(\Omega) \) and a.e. Also, there exists \( g \in L^1(\Omega) \) such that \( |u_n| \leq g, \ |v_n| \leq g \) a.e.
for all \( n \in \mathbb{N} \). Since \((u_n, v_n) \in A\) there exists \( w_n \in L^1(\partial \Omega)\) with \(-w_n(x) \in \beta(u_n(x))\) a.e. on \( \partial \Omega \) and

\[
\text{(4.11)} \quad \int_{\Omega} \langle a(x, Du_n), D(u_n - \phi) \rangle \leq \int_{\Omega} v_n(u_n - \phi) + \int_{\partial \Omega} w_n(u_n - \phi),
\]

for every \( \phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega) \). Taking \( u_n - T_k(u_n) \) as test function in (4.11), by (H1) we get

\[
\lambda \int_{\Omega} |DT_k(u_n)|^p \leq \int_{\{|u_n|<k\}} \langle a(x, Du_n), Du_n \rangle \leq \int_{\Omega} v_n T_k(u_n) + \int_{\partial \Omega} w_n T_k(u_n) \leq k \|g\|_1,
\]

i.e.,

\[
\text{(4.12)} \quad \int_{\Omega} |DT_k(u_n)|^p \leq \frac{k}{\lambda} \|g\|_1 \quad \text{for every } n \in \mathbb{N}.
\]

Moreover, from (4.12) we also get

\[
-\int_{\partial \Omega} w_n \frac{1}{k} T_k(u_n) \leq \int_{\Omega} v_n \frac{1}{k} T_k(u_n) \leq \|g\|_1.
\]

From here, letting \( k \to 0 \), we get the estimate

\[
\text{(4.13)} \quad \int_{\partial \Omega} |w_n| \leq \|g\|_1 \quad \text{for every } n \in \mathbb{N}.
\]

From (4.13) it follows that \( \{T_k(u_n) : n \in \mathbb{N}\} \) is a bounded subset of \( W^{1,p}(\Omega) \). Hence, after passing to a suitable subsequence, we have \( \{T_k(u_n)\}_{n \in \mathbb{N}} \) is weakly convergent in \( W^{1,p}(\Omega) \). Now, since \( T_k(u_n) \to T_k(u) \) in \( L^p(\Omega) \) as \( n \to \infty \), we have \( DT_k(u_n) \to DT_k(u) \) weakly in \( L^p(\Omega) \) as \( n \to \infty \). Therefore, \( T_k(u) \in W^{1,p}(\Omega) \) for every \( k > 0 \) and consequently, \( u \in T^{1,p}(\Omega) \).

In the next step we are going to see that \( \{Du_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in measure. Let \( t \) and \( \epsilon > 0 \). As in the proof of Proposition 2.3, if for some \( A > 1 \) we set

\[
C(x, A, t) := \inf \{ \langle a(x, \xi) - a(x, \eta), \xi - \eta \rangle : |\xi| \leq A, |\eta| \leq A, |\xi - \eta| \geq t \},
\]

we have that

\[
\text{(4.15)} \quad C(x, A, t) > 0 \quad \text{for almost all } x \in \Omega.
\]

For \( n, m \in \mathbb{N} \) and any \( k > 0 \), we have

\[
\{|Du_n - Du_m| > t\} \subset \{|DT_A u_n| \geq A\} \cup \{|DT_A u_m| \geq A\} \cup \{|u_n| > A\} \cup \left(\bigcup\{|u_m| > A\} \cup |u_n - u_m| \geq k^2\right) \cup \{C(x, A, t) \leq k\} \cup \{|u_n - u_m| \leq k^2\},
\]

\[
\text{(4.16)} \quad \text{for all } n, m \in \mathbb{N} \text{ and any } k > 0.
\]
\[ |u_n| \leq A, |u_m| \leq A, C(x, A, t) \geq k, \ |DT_A u_n| \leq A, \ |DT_A u_m| \leq A, \ |Du_n - Du_m| > t \].

Since \( \{u_n : n \in \mathbb{N}\} \) is bounded in \( L^1(\Omega) \) we can choose \( A \) large enough in order to have

\begin{equation}
\lambda_N(\{|u_n| \geq A\} \cup \{|u_m| \geq A\}) \leq \frac{\epsilon}{5} \quad \text{for all } n, m \in \mathbb{N}.
\end{equation}

Similarly, by (4.13), we can choose \( A \) large enough in order to have

\begin{equation}
\lambda_N(\{|DT_A u_n| \geq A\} \cup \{|DT_A u_m| \geq A\}) \leq \frac{\epsilon}{5} \quad \text{for all } n, m \in \mathbb{N}.
\end{equation}

By (4.15), taking \( k \) small enough we have

\begin{equation}
\lambda_N(\{|C(x, A, t) \leq k\}) \leq \frac{\epsilon}{5}.
\end{equation}

On the other hand, if

\[ S := \{|u_n - u_m| \leq k^2, \ |u_n| \leq A, \ |u_m| \leq A, \ C(x, A, t) \geq k, \ |DT_A u_n| \leq A, \ |DT_A u_m| \leq A, \ |Du_n - Du_m| > t \}, \]

since \( DT_A u_n = Du_n \) a.e. in \( S \), arguing as in the proof of (2.13) we get

\begin{equation}
\lambda_N(S) \leq \frac{\epsilon}{5} \quad \text{for } k \text{ small enough}.
\end{equation}

Since \( A \) and \( k \) have been already chosen, if \( n_0 \) is large enough we have for \( n, m \geq n_0 \) the estimate \( \lambda_N(\{|u_n - u_m| \geq k^2\}) \leq \frac{\epsilon}{5} \). From here, using (4.16) - (4.20), it follows that

\[ \lambda_N(\{|Du_n - Du_m| \geq t\}) \leq \epsilon \quad \text{for } m, n \geq n_0. \]

Consequently, \( \{Du_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in measure. So, there exists \( \varphi \in M(\Omega, \lambda_N) \) such that \( \{Du_n\}_{n \in \mathbb{N}} \) converges to \( \varphi \) in measure. Now, the above argument also shows that \( \{DT_k u_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in measure for every \( k > 0 \). Hence, since \( \{DT_k u_n\}_{n \in \mathbb{N}} \) weakly converges to \( DT_k (u) \) in \( L^p(\Omega) \), we have \( \{DT_k u_n\}_{n \in \mathbb{N}} \) converges in measure to \( DT_k (u) \). Thus, there exist \( n_1 < n_2 < \cdots < n_k < \cdots \), such that \( \lim_{k \to \infty} \varrho(DT_k u_{n_k}, DT_k u) = 0 \). Hence, \( \lim_{k \to \infty} \varrho(DT_k u_{n_k}, Du) = 0 \). Then, up to extraction of a subsequence, we have convergence a.e., and we can say that

\begin{equation}
\{Du_n\}_{n \in \mathbb{N}} \text{ converges to } Du \text{ in measure.}
\end{equation}

We claim

\begin{equation}
\{u_n\}_{n \in \mathbb{N}} \text{ is bounded in } M^{p_1}(\Omega), \quad p_1 = \frac{N(p - 1)}{N - p}.
\end{equation}
Fixed $n_0, k_0 \in \mathbb{N}$ such that
\[ |\bar{u}| + 1 < \frac{k_0}{2}, \quad |\bar{u}_n - \bar{u}| < 1/3 \quad \text{for all } n \geq n_0, \]
and
\[ \frac{2}{\lambda_N(\Omega)} \int_{\{g > k\}} g < 1/3 \quad \text{for all } k \geq k_0. \]
For $n \geq n_0$ and $k \geq k_0$ fixed, consider the sets
\[ C := \{|T_k(u_n)| > k/2 + |\bar{u}| + 1\}, \quad D := \{|T_k(u_n) - T_k(u_n)| > k/2\}. \]
If $x \in C \sim D$, we have
\[ \frac{k}{2} + |\bar{u}| + 1 < |T_k(u_n(x))| \leq |T_k(u_n(x)) - T_k(u_n)| \leq \]
\[ \leq k/2 + |T_k(u_n)| \leq k/2 + |T_k(u_n) - \bar{u}_n| + |\bar{u}| + 1/3 \leq \]
\[ \leq k/2 + |\bar{u}| + 1/3 + \frac{2}{\lambda_N(\Omega)} \int_{\{g > k\}} g \leq k/2 + |\bar{u}| + 2/3, \]
which is a contradiction. Thus, $C \subset D$. Hence, if $p^* = \frac{pN}{N-p}$,
\[ \lambda_N(\{|n| > k\}) \leq \lambda_N(\{|T_k(u_n)| > k/2 + |\bar{u}| + 1\}) = \lambda_N(C) \leq \]
\[ \leq \lambda_N(D) \leq \frac{1}{(k/2)^{p^*}}\|T_k(u_n) - T_k(u_n)\|_{p^*}. \]
Then, by Poincaré’s inequality (cf. [Zi, Cap. 4]) and (4.13), there exist constants $C, M > 0$ such that
\[ \lambda_N(\{|n| > k\}) \leq C \frac{1}{(k/2)^{p^*}}\|DT_k(u_n)\|_{p^*} \leq M \frac{k^{p^*/p}}{k^{p^*}} = Mk^{-p_1} \]
for every $n \geq n_0$ and $k \geq k_0$. Consequently, the claim (4.22) holds.

Next, we claim that
\[ (4.23) \quad \{Du_n\}_{n \in \mathbb{N}} \text{ is bounded in } M^{p_2}(\Omega), \quad p_2 = \frac{N(p-1)}{N-1}. \]
Let $t > 0$. By (4.13), there exists a constant $Q_1 > 0$, such that for every $k > 0$ and $n \in \mathbb{N}$,
\[ (4.24) \quad \lambda_N(\{|DT_k(u_n)| > t/2\}) \leq \frac{Q_1 k}{t^{p^*}}. \]
On the other hand, by (4.22), there exists a constant $Q_2 > 0$, such that
\[ (4.25) \quad \lambda_N(\{|n| \geq k\}) \leq \frac{Q_2}{k^{p_1}} \quad \text{for every } k > 0 \text{ and } n \in \mathbb{N}. \]
From (4.24) and (4.25), it follows that
\[ \lambda_N(\{|Du_n| > t\}) \leq \lambda_N(\{|Du_n - DT_k(u_n)| > t/2\}) + \lambda_N(\{|DT_k(u_n)| > t/2\}) \leq \lambda_N(\{|u_n| \geq k\}) + \lambda_N(\{|DT_k(u_n)| > t/2\}) \leq Q_2 \frac{k}{p_1} + \frac{Q_1 k}{tp}.
\]

Then, taking \( k := t^{p_2/p_1} \), we have
\[ \lambda_N(\{|Du_n| > t\}) \leq Qt^{-p_2} \quad \text{for every} \quad n \in \mathbb{N}. \]

Consequently, the claim (4.23) holds.

From (4.21), (4.22) and (4.23) we can state that
\[ u \in M^{p_1}(\Omega), \quad |Du| \in M^{p_2}(\Omega) \]
where \( p_1 = \frac{N(p-1)}{N-p} \) and \( p_2 = \frac{N(p-1)}{N-1} \). Suppose we are in the case \( p > 2 - 1/N \). Then, \( p_2 > 1 \). Hence, if \( 1 \leq q < p_2 \), we have that \( \{u_n : n \in \mathbb{N}\} \) is bounded in \( W^{1,q}(\Omega) \). Consequently, \( u \in W^{1,q}(\Omega) \).

Let us see now that \( u \in T_{tr}^{1,p}(\Omega) \). Indeed: Obviously, \( u_n \rightarrow u \) a.e. in \( \Omega \). Since \( \{DT_k(u_n) : n \in \mathbb{N}\} \) is bounded in \( L^p(\Omega) \) and \( DT_k(u_n) \rightarrow DT_k(u) \) in measure, it follows from [B-V, Lemma 6.1] that \( DT_k(u_n) \rightarrow DT_k(u) \) in \( L^1(\Omega) \). Finally, let us see that \( \{\gamma(u_n)\}_{n \in \mathbb{N}} \) converges a.e. in \( \partial\Omega \). For every \( k > 0 \), let
\[ A_k := \{x \in \partial\Omega : |T_k u(x)| < k\} \quad \text{and} \quad C := \partial\Omega \sim \cup_{k>0} A_k. \]

Then,
\[ \mu(C) = \frac{1}{k} \int_C |T_k u| \leq \frac{1}{k} \int_{\partial\Omega} |T_k u| \leq \frac{C_1}{k} \|T_k u\|_{W^{1,1}(\Omega)} \leq \frac{C_1}{k} \|T_k u\|_{L^1(\Omega)} + \frac{C_2}{k} \|DT_k u\|_{L^p(\Omega)}. \]

Now, by (4.13) and the boundedness of \( \{\|T_k u\|_{L^1(\Omega)} : k > 0\} \),
\[ \|DT_k u\|_{L^p(\Omega)} \leq \left( \frac{k}{\lambda \|g\|_1} \right)^{1/p} \quad \text{for any} \quad k > 0. \]

Hence,
\[ \mu(C) \leq \frac{C_3}{k} + \frac{C_4}{k^{1-1/p}} \quad \text{for any} \quad k > 0. \]

Taking limit as \( k \rightarrow \infty \) we have \( \mu(C) = 0 \). Moreover, \( A_k \subset A_r \) if \( k \leq r \). Thus, if we define in \( \partial\Omega \) the function \( v \) by
\[ v(x) := (T_k u)(x) \quad \text{if} \quad x \in A_k, \]
it is easy to see that \( u_n \rightarrow v \) a.e. in \( \partial\Omega \). Therefore, \( u \in T_{tr}^{1,p}(\Omega) \).
To complete the proof it remains to show that there exists \( w \in L^1(\partial\Omega) \) with \(-w(x) \in \beta(u(x))\) a.e. in \( \partial\Omega \), such that

\[
\int_\Omega \langle a(x, Du), DT_k(u - \phi) \rangle \leq \int_\Omega vT_k(u - \phi) + \int_{\partial\Omega} wT_k(u - \phi),
\]

for every \( \phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega) \). For this, let us see first that \( \{w_n : n \in \mathbb{N}\} \) is a Cauchy sequence in \( L^1(\partial\Omega) \). Indeed: Taking \( u_n - T_k(u_n - u_m) \) and \( u_m + T_k(u_n - u_m) \) as test functions in (4.11), we have

\[
-\int_{\partial\Omega} (w_n - w_m) \frac{1}{k} T_k(u_n - u_m) \leq \int_\Omega (v_n - v_m) \frac{1}{k} T_k(u_n - u_m) - \\
-\int_\Omega \langle a(x, Du_n) - a(x, Du_m), \frac{1}{k} D(T_k(u_n - u_m)) \rangle \leq \int_\Omega |v_n - v_m|.
\]

Letting \( k \to 0 \), we get

\[
\int_{\partial\Omega} |w_n - w_m| \leq \int_\Omega |v_n - v_m|.
\]

Now, since \( \{v_n\}_{n \in \mathbb{N}} \) is convergent in \( L^1(\Omega) \), by (4.27), \( \{w_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L^1(\partial\Omega) \).

We introduce the class \( \mathcal{F} \) of functions \( S \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) satisfying:

\[
S(0) = 0, \ 0 \leq S' \leq 1, \ S'(s) = 0 \ \text{for} \ s \ \text{large enough}, \\
S(-s) = -S(s), \ \text{and} \ S''(s) \leq 0 \ \text{for} \ s \geq 0.
\]

Let \( \phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega) \) and \( S \in \mathcal{F} \). Taking \( u_n - S(u_n - \phi) \) as test function in (4.11) we get

\[
\int_\Omega \langle a(x, Du_n), DS(u_n - \phi) \rangle \leq \int_\Omega v_n S(u_n - \phi) + \int_{\partial\Omega} w_n S(u_n - \phi).
\]

We can write the first member of (4.28) as

\[
\int_\Omega \langle a(x, Du_n), D(u_n - \phi) \rangle S'(u_n - \phi) - \int_\Omega \langle a(x, Du_n), D\phi \rangle S'(u_n - \phi).
\]

Since \( u_n \to u \) and \( Du_n \to Du \) a.e., Fatou’s Lemma yields

\[
\int_\Omega \langle a(x, Du), Du \rangle S'(u - \phi) \leq \liminf_{n \to \infty} \int_\Omega \langle a(x, Du_n), Du_n \rangle S'(u_n - \phi).
\]

The second term of (4.29) is estimated as follows. Let \( r := \|\phi\|_\infty + \|S\|_\infty \). By (4.13) and \((H_3)\), it follows that

\[
\{a(x, DT_ru_n) : n \in \mathbb{N}\} \ \text{is bounded in} \ L^{p'}(\Omega).
\]
Then, since $D_T u_n \to D_T u$ in measure, it follows that
\begin{equation}
\text{(4.30)} \quad a(x, D_T u_n) \to a(x, D_T u) \quad \text{weakly in } L^{p'}(\Omega).
\end{equation}
On the other hand,
$$|D\phi S'(u_n - \phi)| \leq |D\phi| \in L^p(\Omega).$$
Then, by the Dominated Convergence Theorem, we have
\begin{equation}
\text{(4.31)} \quad D\phi S'(u_n - \phi) \to D\phi S'(u - \phi) \quad \text{in } L^p(\Omega)^N.
\end{equation}
Hence, by (4.30) and (4.31), it follows that
\begin{equation}
\lim_{n \to \infty} \int_\Omega \langle a(x, D u_n), D\phi S'(u_n - \phi) \rangle = \int_\Omega \langle a(x, D u), D\phi S'(u - \phi) \rangle.
\end{equation}
Therefore, applying again the Dominated Convergence Theorem in the second member of (4.28), we obtain
$$\int_\Omega \langle a(x, D u), D S(u - \phi) \rangle \leq \int_\Omega v S(u - \phi) + \int_{\partial \Omega} w S(u - \phi).$$
From here, to get (4.26) we only need to apply the technique used in the proof of [B-V, Lemma 3.2].

Finally, since $w_n \to w$ in $L^1(\partial \Omega)$ and $-w_n \in \beta(u_n)$ a.e. in $\partial \Omega$, from the maximal monotonicity of $\beta$ it follows that $-w \in \beta(u)$ a.e. in $\partial \Omega$ and the proof concludes.

As a consequence of the m-accretivity of $A$ in $L^1(\Omega)$, we have the following result of existence and uniqueness for the entropy solutions of the nonlinear elliptic problem.

**Corollary 4.3.** Under the assumptions of Theorem 4.2, given $v \in L^1(\Omega)$ the problem
\begin{align*}
& u - \text{div } a(x, D u) = v \quad \text{in } \Omega \\
& -\frac{\partial u}{\partial \eta} = \beta(u) \quad \text{on } \partial \Omega.
\end{align*}
has a unique entropy solution. Moreover, if $2 - \frac{1}{N} < p < N$, the solution belongs to $W^{1,q}(\Omega)$ for every $1 \leq q < p_2$.

**Corollary 4.4.** Under the assumptions of Theorem 4.2, we have $A = A_0$ and $\overline{A}_0 = A$.

**Proof.** Since $A \subset A_0$, $A = \overline{A} \subset \overline{A}_0$. Therefore, by maximal accretivity we have $\overline{A}_0 = A$. Let $(u, v) \in A$. Then, $(u, v) \in A$ and $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Hence, there exists $w \in L^1(\partial \Omega)$ with $-w(x) \in \beta(u(x))$ a.e. on $\partial \Omega$ such that
$$\int_\Omega \langle a(x, D u), D T_k(u - \phi) \rangle \leq \int_\Omega v T_k(u - \phi) + \int_{\partial \Omega} w T_k(u - \phi)$$
for every $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$ and $k > 0$. From here, since $u - \phi \in L^\infty(\Omega)$, it follows that $(u, v) \in A$. 
5. The evolution problem

In this section we study in $L^1(\Omega)$, from the point of view of Nonlinear Semigroup Theory, the quasi-linear parabolic equation with nonlinear boundary condition

$$u_t = \text{div} \ a(x, Du) \text{ in } \Omega \times (0, \infty)$$

(III) $$-\frac{\partial u}{\partial \eta} \in \beta(u) \text{ on } \partial \Omega \times (0, \infty)$$

$$u(x,0) = u_0(x) \text{ in } \Omega.$$ 

We transcribe (III) as the evolution problem in $L^1(\Omega)$

(IV) $$\frac{du}{dt} + A_0 u = 0, \quad u(0) = u_0$$

In order to get solution of (IV) for every initial data $u_0 \in L^1(\Omega)$ we need the following result.

**Proposition 5.1.** The domain of the operator $A_0$ is dense in $L^1(\Omega)$.

**Proof.** We are going to prove that $L^\infty(\Omega) \subset D(A_0)$. Given $v \in L^\infty(\Omega)$, if we set

$$u_n := (I + 1/n A_0)^{-1} v,$$

then $(u_n, n(v - u_n)) \in A_0$, so taking $w = 0$ as test function in the definition of the operator $A_0$ we get

$$-n \int \Omega (v - u_n)u_n + \int \Omega \langle a(x, Du_n), Du_n \rangle \leq -\int \partial \Omega j(u_n) \leq 0.$$

From where it follows that

$$\frac{1}{n} \int \Omega \langle a(x, Du_n), Du_n \rangle + \int \Omega u_n^2 \leq \int \Omega vu_n.$$

Thus,

$$\int \Omega \langle a(x, Du_n), Du_n \rangle \leq n \int \Omega vu_n \leq n \|v\|_\infty^2 \lambda_N(\Omega) = nM.$$ 

Now, using (H_1) and (H_3), we have

$$\int \Omega |a(x, Du_n)|^{p'} \leq \int \Omega \Lambda^{p'} (j(x) + |Du_n|^{p-1})^{p'} \leq$$

$$\leq \Lambda' 2^{p'-1} \left( \int \Omega j(x)^{p'} + \int \Omega |Du_n|^p \right) \leq$$
\[ \leq \Lambda^{p'} 2^{p'-1} \left( \int_{\Omega} j(x)^{p'} + \frac{1}{\lambda} \int_{\Omega} \langle a(x, Du_n), Du_n \rangle \right) \leq \alpha_1 + n \alpha_2, \]

with

\[ \alpha_1 = \Lambda^{p'} 2^{p'-1} \int_{\Omega} j(x)^{p'} \quad \text{and} \quad \alpha_2 = \frac{M \Lambda^{p'} 2^{p'-1}}{\lambda}. \]

Consequently,

\[ (5.1) \int_{\Omega} | \frac{1}{n} a(x, Du_n) |^{p'} \leq \frac{\alpha_1}{n^{p'}} + \frac{n \alpha_2}{n^{p'}}. \]

On the other hand, if \( \phi \in D(\Omega) \), taking \( u_n - \phi \) and \( u_n + \phi \) as test functions in the definition of the operator \( A_0 \) we have that

\[ \frac{1}{n} \int_{\Omega} \langle a(x, Du_n), D\phi \rangle + \int_{\Omega} u_n \phi = \int_{\Omega} v \phi, \]

so, by (5.1), we get

\[ (5.2) \lim_{n \to \infty} \int_{\Omega} u_n \phi = \int_{\Omega} v \phi \quad \text{for every} \quad \phi \in D(\Omega). \]

Now, since \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( L^\infty(\Omega) \), there exists a subsequence \( \{u_{n_k}\}_{k \in \mathbb{N}} \) such that \( u_{n_k} \to u \) weakly in \( L^p(\Omega) \), which implies, by (5.2), that \( u = v \).

Finally, since \( A_0 \) is completely accretive, we have \( \|u_n\|_p \leq \|v\|_p \). Hence, \( u_{n_k} \to v \) in \( L^p(\Omega) \), and we can conclude that \( v \in \overline{D(A_0)} \).

Since the closure of the operator \( A_0 \) is an \( m\)-T-accretive operator in \( L^1(\Omega) \) with dense domain, according to Crandall-Liggett’s Semigroup Generation Theorem (cf. [Cr]) the operator \( \overline{A}_0 \) generates a semigroup of order-preserving contractions \( S(t) \) in \( L^1(\Omega) \) which solves, in a generalized sense called usually mild sense, the evolution problem (IV).

The mild-solution of problem (III) is given by \( u(\cdot, t) = S(t) u_0 \).

To finish this section we study the continuous dependence of \( u \) on \( a \) and \( \beta \) where \( u \) is the solution of the elliptic problem

\[ u - \text{div} \ a(x, Du) = v \quad \text{in} \quad \Omega \]

\[ -\frac{\partial u}{\partial \eta_a} \in \beta(u) \quad \text{on} \quad \partial \Omega. \]

Using nonlinear semigroup theory, this result implies the continuous dependence of solutions of the associated nonlinear parabolic problem on the nonlinearities in the problem.

As in the above section we assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \) of class \( C^1 \), \( 1 < p < N \). Also we assume that \( a_1, a_2, \ldots, a_\infty \) are Caratheodory functions from \( \Omega \times \mathbb{R}^N \) into \( \mathbb{R}^N \) satisfying

\[ (G_1) \quad \text{there exist} \quad \lambda_n > 0 \quad \text{with} \quad \inf_{n \in \mathbb{N}} \lambda_n = \lambda > 0 \quad \text{such that} \]

\[ \langle a_n(x, \xi), \xi \rangle \geq \lambda_n |\xi|^p \]
holds for every $\xi$ and a.e. $x \in \Omega$.

(G2) For every $\xi$ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$ there holds
\[
\langle a_n(x, \xi) - a_n(x, \eta), \xi - \eta \rangle > 0.
\]

(G3) There exist $\Lambda_n \in \mathbb{R}$ with $\sup_{n \in \mathbb{N}} \Lambda_n < \infty$ and $j_n \in L^{p'}$ with $\sup_{n \in \mathbb{N}} \|j_n\|_{p'} < \infty$, such that
\[
|a_n(x, \xi)| \leq \Lambda_n (j_n(x) + |\xi|^{p-1})
\]
holds for every $\xi \in \mathbb{R}^N$.

**Theorem 5.2.** Let $a_n$ be satisfying (G1), (G2), (G3) and for almost all $x \in \Omega$ $a_n(x, \xi) \to a_\infty(x, \xi)$ uniformly in $\xi$ on compact subsets. Let $\beta_n$ be maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta_n(0)$, $\beta_n \rightharpoonup \beta_\infty$ in the sense of maximal monotone graphs and $D(\beta_n) = D(\beta_\infty)$ for all $n \in \mathbb{N}$. Suppose $D(\beta_\infty)$ is closed. Let $A_n$ be the operator associated to $a_n$ and $\beta_n$. Then
\[
(I + A_n)^{-1}v \to (I + A_\infty)^{-1}v \quad \text{in} \quad L^1(\Omega)
\]
for every $v \in L^1(\Omega)$.

**Proof.** Since $(I + A_n)^{-1}$ are contractions in $L^1(\Omega)$ and $L^\infty(\Omega)$ is dense in $L^1(\Omega)$, we can suppose that $v \in L^\infty(\Omega)$ and work with the operators $A_n$. By Theorem 2.2, for every $n \in \mathbb{N}$ there exist $u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $w_n \in L^1(\partial \Omega)$ with $-w_n(x) \in \beta_n(u_n(x))$ a.e. in $\partial \Omega$ such that
\[
(5.3) \quad \int_{\Omega} \langle a_n(x, Du_n), D(u_n - \phi) \rangle + \int_{\Omega} u_n(u_n - \phi) \leq \int_{\Omega} v(u_n - \phi) + \int_{\partial \Omega} w_n(u_n - \phi),
\]
for every $\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)$. Moreover, the complete accretiveness of $A_n$ implies
\[
(5.4) \quad \|u_n\|_\infty \leq \|v\|_\infty \quad \text{for every} \quad n \in \mathbb{N}.
\]

Taking $\phi = 0$ as test function in (5.3) we obtain
\[
(5.5) \quad \|Du_n\|_p \leq \left( \frac{2}{\lambda_n} \|v\|^2_\infty \lambda_N(\Omega) \right)^{1/p} \quad \text{for every} \quad n \in \mathbb{N}.
\]

As a consequence of (5.4), (5.5), Rellich-Kondrachov’s Theorem and [M, Theorem 3.4.5] we can establish, after passing to a subsequence, the following fact:

\[
(5.6) \quad u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(\Omega).
\]

\[
(5.7) \quad u_n \to u \quad \text{in} \quad L^1(\Omega) \quad \text{and a.e.}
\]
With a similar argument than the one used in the proof of Theorem 2.2 we get, up to extraction of a subsequence if necessary, \( w_n \to w \in L^1(\partial \Omega) \) weakly in \( L^1(\partial \Omega) \). Moreover, by [BCS, Lemma G], \( -w(x) \in \beta_\infty(u(x)) \) a.e. in \( \partial \Omega \).

Now we prove that \( D u_n \) converges to \( Du \) in measure. Since \( D u_n \) converges to \( Du \) weakly in \( L^p(\Omega) \), it is enough to show that \( \{ D u_n \} \) is a Cauchy sequence in measure. Let \( t > 0 \). For some \( A > 1 \) and \( n, m \in \mathbb{N} \) we set

\[
C_{n,m}(x, A, t) := \inf \{ \langle a_n(x, \xi) - a_m(x, \eta), \xi - \eta \rangle : |\xi| \leq A, |\eta| \leq A, |\xi - \eta| \geq t \}.
\]

As in the proof of Theorem 2.1 we have that the infimum in the definition of \( C_{n,m}(x, A, t) \) is a minimum for almost all \( x \in \Omega \). Moreover, by (G2), and since \( a_n(x, \xi) \to a_\infty(x, \xi) \) uniformly in \( \xi \) on compact subsets, it follows that there exists \( n_0 \in \mathbb{N} \), such that for every \( n, m \geq n_0 \), we have

\[
(5.9) \quad C_{n,m}(x, A, t) > 0 \quad \text{for almost all } x \in \Omega.
\]

For \( n, m \in \mathbb{N} \) and \( k > 0 \), we have

\[
\{|Du_n - Du_m| > t\} \subset
\]

\[
(5.10) \quad \subset \{|Du_n| \geq A\} \cup \{|Du_m| \geq A\} \cup \{|u_n - u_m| \geq k^2\} \cup \{C_{n,m}(x, A, t) \leq k\} \cup
\]

\[
\cup \{|u_n - u_m| \leq k^2, C_{n,m}(x, A, t) \geq k, |Du_n| \leq A, |Du_m| \leq A, |Du_n - Du_m| > t\}.
\]

Since \( \{D u_n\}_{n \in \mathbb{N}} \) is bounded in \( L^p(\Omega) \) we can choose \( A \) large enough in order to have

\[
(5.11) \quad \lambda_N(\{|Du_n| \geq A\} \cup \{|Du_m| \geq A\}) \leq \frac{\epsilon}{4} \quad \text{for all } n, m \in \mathbb{N}.
\]

By (5.9) we can choose \( k \) small enough in order to have

\[
(5.12) \quad \lambda_N(\{C_{n,m}(x, A, t) \leq k\}) \leq \frac{\epsilon}{4} \quad \text{for } n, m \geq n_0.
\]

On the other hand, if we use \( u_n - T_{k^2}(u_n - u_m) \) and \( u_m + T_{k^2}(u_n - u_m) \) as test functions in (5.3), we obtain

\[
\int_{\{|u_n - u_m| \leq k^2\}} \langle a_n(x, Du_n) - a_m(x, Du_m), D(u_n - u_m) \rangle =
\]

\[
= \int_{\Omega} \langle a_n(x, Du_n) - a_m(x, Du_m), DT_{k^2}(u_n - u_m) \rangle \leq
\]

\[
\leq -\int_{\Omega} (u_n - u_m)T_{k^2}(u_n - u_m) + \int_{\partial \Omega} (w_n - w_m)T_{k^2}(u_n - u_m).
\]
Then, by (5.4),
\[
\int_{\{|u_n - u_m| < k^2\}} \langle a_n(x, Du_n) - a_m(x, Du_m), D(u_n - u_m) \rangle \leq k^2 Q.
\]
Hence
\[
\lambda_N(\{|u_n - u_m| \leq k^2; |Du_n| \leq A, |Du_m| \leq A, |Du_n - Du_m| > t\}) \leq \lambda_N(\{|u_n - u_m| \leq k^2; \langle a_n(x, Du_n) - a_m(x, Du_m), D(u_n - u_m) \rangle \geq k\}) \leq \frac{1}{k} \int_{\{|u_n - u_m| < k^2\}} \langle a_n(x, Du_n) - a_m(x, Du_m), D(u_n - u_m) \rangle \leq \frac{1}{k} k^2 Q \leq \frac{\epsilon}{4},
\]
for \(k\) small enough.

Since \(A\) and \(k\) have been already chosen, if \(n_0\) is large enough we have for \(n, m \geq n_0\) the estimate \(\lambda_N(\{|u_n - u_m| \geq k^2\}) \leq \frac{\epsilon}{4}\). From here, using (5.10), (5.11), (5.12) and (5.13), it follows that
\[
\lambda_N(\{|Du_n - Du_m| \geq t\}) \leq \epsilon \quad \text{for} \quad m, n \geq n_0.
\]
Consequently, \(\{Du_n\}\) is a Cauchy sequence in measure. Therefore, we can assume, after passing to a suitable subsequence, the a.e. convergence of \(Du_n\) to \(Du\), and since \(a_n(x, \xi) \rightarrow a_\infty(x, \xi)\) uniformly in \(\xi\) on compact subset, that \(a_n(x, Du_n) \rightarrow a_\infty(x, Du)\) a.e. in \(\Omega\). On the other hand, since \(\{Du_n\}_{n \in \mathbb{N}}\) is bounded in \(L^p(\Omega)\), it follows from (G3) that \(\{a_n(x, Du_n)\}_{n \in \mathbb{N}}\) is bounded in \(L^{p'}(\Omega)\). Therefore,
\[
a_n(x, Du_n) \rightarrow a_\infty(x, Du) \quad \text{weakly in} \quad L^{p'}(\Omega).
\]
In particular,
\[
\int_\Omega \langle a_n(x, Du_n), D\phi \rangle \rightarrow \int_\Omega \langle a_\infty(x, Du), D\phi \rangle,
\]
for every \(\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)\). From here, (5.5) and (5.8), passing to the limit in (5.3) we get
\[
\int_\Omega \langle a_\infty(x, Du), D(u - \phi) \rangle + \int_\Omega u(u - \phi) \leq \int_\Omega v(u - \phi) + \int_{\partial\Omega} w(u - \phi),
\]
for every \(\phi \in W^{1,p}_\beta(\Omega) \cap L^\infty(\Omega)\). Therefore, \(u = (I + A_\infty)^{-1}v\) and the proof is completed.

Remark 5.3. The above Theorem jointly with [Cr, Theorem 6] implies the continuous dependence of solutions of the associated nonlinear parabolic problem on the nonlinearities in the problem. More concretely, suppose \(a_1, a_2, \ldots, a_\infty\) and \(\beta_n\) satisfy the assumptions of Theorem 5.2. Let \(A_n\) be the operator associated to \(a_n\) and \(\beta_n\). Then, if \(u_n\) are the mild-solutions of the problems
\[
u_n' + A_n u_n \geq 0, \quad u_n(0) = f_n
\]
and \(f_n \rightarrow f_\infty\) in \(L^1(\Omega)\), we have that \(u_n\) converges in \(C([0, \infty[; L^1(\Omega))\) to the mild-solution \(u_\infty\) of the problem
\[
u_\infty' + A_\infty u_\infty \geq 0, \quad u_\infty(0) = f_\infty.
\]

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References


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