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Attractor for a Degenerate Nonlinear Diffusion Problem with Nonlinear Boundary Condition

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In this paper we prove the existence of a compact attractor in $L^{\infty}(\Omega)$ for a degenerate nonlinear diffusion problem with nonlinear flux on the boundary. In order to formulate the equation as a dynamical system, some existence and uniqueness results for weak solutions are proved.

KEY WORDS: Nonlinear parabolic equations; attractors; asymptotic behavior; flows in porous media; diffusion of biological populations.

AMS SUBJECT CLASSIFICATIONS: 35K65, 35B40.

1. INTRODUCTION

Attractors play an important role in the study of the asymptotic behavior of the solutions of partial differential equations. The existence of a maximal attractor, i.e., a compact set that attracts all solutions as time goes to infinity, has been derived for a large class of PDEs (see for instance Babin and Vishik, 1992; Teman, 1988; Haraux, 1991). In this paper we study the following initial boundary value problem

$$\begin{cases} u_t = \Delta\varphi(u) + f(u) & \text{in } Q = \Omega \times (0, \infty) \\ -\dfrac{\partial\varphi(u)}{\partial\eta} = g(u) & \text{on } S = \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{I}$$

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in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, where $\varphi: \mathbb{R} \to \mathbb{R}$ is a continuous increasing function which includes the case of pure powers $\varphi(r) = |r|^m \operatorname{sign}(r)$, $m \geqslant 1$, and $f, g \in C(\mathbb{R})$ satisfy some growth conditions which will be precised later.

The aim of this paper is to prove the existence of a maximal compact attractor in $L^\infty(\Omega)$. The choice of the space $L^\infty(\Omega)$ is motivated by the fact that the solutions of (I) are bounded for bounded initial data and the compactness of the trajectories is given by classical results (DiBenedetto, 1983). Remark that if $m > 1$ the problem is not semilinear and the "natural" spaces to consider the structure of the attractor for this type of quasilinear equations are not necessary Hilbert spaces (see Feireisl *et al.*, 1995, Bénilan and Labani, 1989).

Problems of this form arise in a number of areas of science, for instance, in models for gas or fluid flow in porous media (Bear, 1972; Aronson, 1986) and for the spread of certain biological populations (Gurtin and MacCamy, 1977; Okubo, 1980). There is an extensive literature about the large time behavior of solutions of problems of this type (see for instance Aronson *et al.*, 1982; Langlais and Phillips, 1985; Bertsch *et al.*, 1982; Alikakos and Rostamian, 1981; Andreu *et al.*, 1995; Filo and Mottoni, 1992; Eden *et al.*, 1991). In some of these papers, it is shown that the solutions stabilize as time goes to infinity by converging to a function. In our case the dynamic is more complicated due to the relative generality of f and g and we are not able to have a so precise result. Note that the existence of a global attractor for a similar problem with other growth conditions on f and different boundary conditions has been considered in Eden *et al.*, 1991.

In order to formulate the equation as a dynamical system, we establish the existence and uniqueness of a global weak solution of problem (I) when the initial datum $u_0 \in L^\infty(\Omega)$.

We consider the following assumptions on the data. We always assume Ω to be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Respect to φ, f and g, we will assume the following hypotheses, which we shall refer to collectively as (H):

(H_1) $\varphi \in C^1(\mathbb{R})$, $\varphi(0) = 0$, $\varphi'(r) > 0$ for all $r \neq 0$ and

(i) there exist $r_0 > 0$, $\alpha_0 > 0$, $\alpha_1 > 0$, $0 \leqslant m_0 \leqslant m_1 < 1$ such that

$$\alpha_0 \, |\varphi(r)|^{m_0} \leqslant \varphi'(r) \leqslant \alpha_1 \, |\varphi(r)|^{m_1} \qquad \forall \, |r| \geqslant r_0$$

(ii) if $\varphi'(0) = 0$, there exists $\varepsilon > 0$ such that

$$\varphi' \text{ is increasing in }]0, \varepsilon], \qquad \varphi' \text{ is decreasing in } [-\varepsilon, 0[$$

and

$$\varphi'(\varepsilon) \leqslant \varphi'(s) \qquad \text{in} \quad [\varepsilon, +\infty[,$$

$$\varphi'(-\varepsilon) \leqslant \varphi'(s) \qquad \text{in} \quad]-\varepsilon, \varepsilon].$$

(H_2) $f \in C^1(\mathbb{R})$ and there exist $c_1 \geqslant 0$ and $0 \leqslant \alpha < 1$ such that

$$f(r)\,\text{sign}(r) \leqslant c_1\, |\varphi(r)|^\alpha \qquad \text{for all} \quad |r| \geqslant r_0$$

where sign is the function

$$\text{sign}(r) = \begin{cases} 1 & \text{if} \quad r > 0 \\ 0 & \text{if} \quad r = 0 \\ -1 & \text{if} \quad r < 0 \end{cases}$$

(H_3) $g \in C(\mathbb{R})$, $g \circ \varphi^{-1} \in C^1(\mathbb{R} \setminus \{0\})$, $\limsup_{r \to 0} |(g \circ \varphi^{-1})'(r)| < \infty$ and there exists $c_2 > 0$ such that

$$g(r)\,\text{sign}(r) \geqslant c_2\, |\varphi(r)| \qquad \text{for all} \quad |r| \geqslant r_0$$

Remark. Hypothese $(H_1(i))$ will be enough to prove the existence and uniqueness results but we need $(H_1(ii))$ to apply the continuity result of DiBenedetto, 1983.

Note that assumption (H_1) is satisfied in the case $\varphi(r) = |r|^m \,\text{sign}(r)$ is $m > 1$, that is, for the porous medium equation, and in the semilinear case $\varphi(r) = r$. Remark also that (H_1) is also satisfied by a Crandall and Pierre condition (See M. Crandall and M. Pierre, 1982):

$$0 < m_0 \leqslant \frac{\varphi \varphi''}{(\varphi')^2} \leqslant m_1 < 1$$

The plan of the paper is as follows. Some a priori estimates for smooth solutions are obtained in Section 2. In the third section we establish the existence and uniqueness of a global weak solution when the initial datum is a function in $L^\infty(\Omega)$. Finally, in Section 4 we prove that the semigroup on $L^\infty(\Omega)$ defined by the global solution obtained in the previous section has a compact global attractor.

2. A PRIORI ESTIMATES FOR SMOOTH SOLUTIONS

In this section we shall establish a priori estimates for the smooth solutions which will be fundamental for the rest of the paper. For $T > 0$, consider the problem

$$\begin{cases} u_t = \Delta\varphi(u) + f(u) & \text{in} \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial\varphi(u)}{\partial\eta} = g(u) & \text{on} \quad S_T = \partial\Omega \times (0, T) \end{cases} \tag{P}$$

Definition 1. By a smooth solution of problem (P) on Q_T we mean a function $u \in C^{2,1}(\overline{Q_T})$ such that $\nabla\varphi(u) \in C^{2,1}(Q_T)$, $u_t \in C^{2,1}(Q_T)$ and satisfies (P) in a classical sense. We shall say that u is a global smooth solution of problem (P) if u is a smooth solution on Q_T for all positive T.

For a function $u(x, t)$ we use the notation $u(t)$ to denote the function-valued map $t \to u(\cdot, t)$.

Proposition 1. *Assume (H) holds. Let u be a global smooth solution of problem (P) with initial datum $u(0) = u_0 \in L^\infty(\Omega)$. For any $1 \leqslant p < \infty$, there exists $C(\tau, \|u_0\|_{L^\infty(\Omega)}, p)$ with $\lim_{\tau\to\infty} C(\tau, \|u_0\|_{L^\infty(\Omega)}, p) = C(p)$ such that*

$$\|\varphi(u(t))\|_{L^p(\Omega)} \leqslant C(\tau, \|u_0\|_{L^\infty(\Omega)}, p) \qquad \textit{for every} \quad t \geqslant \tau \geqslant 0 \tag{2.1}$$

Proof. Multiplying the equation of (P) by $((\varphi(u) - \varphi(u_0))^+)^p$ and performing obvious manipulations it yields

$$\frac{d}{dt}\int_\Omega \Phi_p(u) + \frac{4p}{(p+1)^2}\int_\infty |\nabla((\varphi(u) - \varphi(r_0))^+)^{(p+1)/2}|^2$$
$$+ \int_{\partial\Omega} g(u)((\varphi(u) - \varphi(r_0))^+)^p$$
$$\leqslant \int_\Omega f(u)((\varphi(u) - \varphi(r_0))^+)^p$$

where

$$\Phi_p(r) = \int_0^r ((\varphi(s) - \varphi(r_0))^+)^p \, ds$$

From here, having in mind the assumptions (H_2) and (H_3), if $d=\min\{1, c_2\}$, we have

$$\frac{d}{dt}\int_\Omega \Phi_p(u)+\frac{4pd}{(p+1)^2}\left[\int_\Omega |\nabla((\varphi(u)-\varphi(r_0))^+)^{(p+1)/2}|^2 + \int_{\partial\Omega}((\varphi(u)-\varphi(r_0))^+)^{p+1}\right]$$
$$\leqslant c_1\int_\Omega((\varphi(u)-\varphi(r_0))^+)^{\alpha+p}+c_1'\int_\Omega((\varphi(u)-\varphi(r_0))^+)^p \qquad (2.2)$$

Now, by the generalized Poincaré inequality (see Teman, 1988), we have

$$\int_\Omega((\varphi(u)-\varphi(r_0))^+)^{p+1}$$
$$\leqslant C\left[\int_\Omega |\nabla((\varphi(u)-\varphi(r_0))^+)^{(p+1)/2}|^2+\int_{\partial\Omega}((\varphi(u)-\varphi(r_0))^+)^{p+1}\right]$$

Then, by (2.2), using Hölder and Young inequalities, we can find constants $\rho, \sigma>0$ such that

$$\frac{d}{dt}\int_\Omega \Phi_p(u)+\rho\int_\Omega((\varphi(u)-\varphi(r_0))^+)^{p+1}\leqslant\sigma \qquad (2.3)$$

On the other hand, the hypotheses on φ yield

$$\int_\Omega \Phi_p(u)\leqslant C(p)\int_\Omega((\varphi(u)-\varphi(r_0))^+)^{p+1-m_0}$$

Consequently, from (2.3) we get the following differential inequality, with constants depending on p,

$$\frac{d}{dt}\int_\Omega \Phi_p(u)+\delta\left(\int_\Omega \Phi_p(u)\right)^{(p+1)/(p+1-m_0)}\leqslant\sigma$$

which implies, from Gronwall lemma if $m_0=0$ or from a lemma of Ghidaglia (Lemma 5.1 of Teman, 1988) if $m_0>0$, that

$$\int_\Omega \Phi_p(u(x,t))\,dx\leqslant A(\tau, \|u_0\|_{L^\infty(\Omega)}, p) \qquad \forall 0\leqslant\tau\leqslant t \qquad (2.4)$$

with

$$A(\tau, \|u_0\|_{L^\infty(\Omega)}, p) = C_1(\|u_0\|_{L^\infty(\Omega)}, p)\, e^{-\delta\tau} + C_2(p) \qquad \text{if} \quad m_0 = 0$$

and

$$A(\tau, \|u_0\|_{L^\infty(\Omega)}, p)$$
$$= C_3(p) + \frac{1}{(C_4(\|u_0\|_{L^\infty(\Omega)}, p) + C_5(p)\,\tau)^{(p+1-m_0)/m_0}} \qquad \text{if} \quad m_0 > 0$$

with $C_4(\|u_0\|_{L^\infty(\Omega)}, p) > 0$.

Now, the hypotheses on φ also yield

$$((\varphi(r) - \varphi(r_0))^+)^{p+1-m_1} \leqslant c(p)\, \Phi_p(r) + d(p)\, \Phi_{p-m_1}(r) \tag{2.5}$$

Hence, using (2.4) and (2.5) we get

$$\int_\Omega ((\varphi(u(x,t)) - \varphi(r_0))^+)^p\, dx$$
$$\leqslant c(p)\, A(\tau, \|u_0\|_{L^\infty(\Omega)}, p + m_1 - 1)$$
$$+ d(p)\, A(\tau, \|u_0\|_{L^\infty(\Omega)}, p - 1), \qquad \forall 0 \leqslant \tau \leqslant t \tag{2.6}$$

Now, since $v = -u$ satisfies

$$\begin{cases} v_t = \Delta\tilde{\varphi}(v) + \tilde{f}(v) & \text{in } Q \\ -\dfrac{\partial\tilde{\varphi}(v)}{\partial\eta} = \tilde{g}(v) & \text{on } S \end{cases}$$

where $\tilde{\varphi}(r) = -\varphi(-r)$, $\tilde{f}(r) = -f(-r)$ and $\tilde{g}(r) = -g(-r)$, if we proceed as in the previous step, we obtain the similar estimate for v. Finally, since φ is bounded in $[-r_0, r_0]$, from (2.6) and the previous remark we get (2.1) and the proof concludes.

Our main goal now is to get uniform estimates for smooth solutions independent on time. To do this we apply Moser type techniques. We make essential use of the following result proved in Laurençot (1993).

Lemma 1. *Let* $a > 1$, $b \geqslant 0$, $c \in \mathbb{R}$, $C_0 \geqslant 1$, $C_1 \geqslant 1$ *and* p_0 *be given numbers such that*

$$p_0 + \frac{c}{a-1} > 0$$

Consider the sequence of real numbers (p_k) *defined by*

$$p_{k+1} = ap_k + c, \qquad k \in \mathbb{N}$$

Assume that (γ_k) *is a sequence of positive real numbers satisfying*

$$\gamma_0 \leqslant C_1^{p_0}$$
$$\gamma_{k+1} \leqslant C_0 p_{k+1}^b \max\{C_1^{p_{k+1}}, \gamma_k^a\}$$

Then, the sequence (γ_k^{1/p_k}) *is bounded. More concretely,*

$$\gamma_k^{1/p_k} \leqslant (C_0^{1/(a-1)} \aleph^{b/(a-1)})^{(a^k-1)/p_k} a^{\delta_k} C_1^{\aleph(a^k/p_k)}$$

where

$$\aleph = 2\left(p_0 + \left|\frac{c}{a-1}\right|\right)$$

and

$$\delta_k = \frac{b}{a(a-1)^2} \frac{a^{k+1} - (k+1)\,a + k}{p_k}$$

Proposition 2. *Assume that (H) holds. Let u be a global smooth solution of problem (P) with initial datum* $u(0) = u_0 \in L^\infty(\Omega)$*. Then, there exists a constant C, depending only on* $\|u_0\|_{L^\infty(\Omega)}$*, such that*

$$\|u\|_{L^\infty(Q)} \leqslant C \tag{2.7}$$

Proof. Let $1 \leqslant p < \infty$ and $\psi(r) = (\varphi(r) - \varphi(r_0))^+$. Then, if

$$\Phi_p(r) = \int_0^r \psi(s)^p\, ds$$

working as in the first part of the proof of Proposition 1, taking p large enough such that $((p+1)^2/4p)\, c_2 \geqslant 1$, we obtain

$$\frac{d}{dt} \int_\Omega \Phi_p(u) + \frac{4p}{(p+1)^2} \left[\int_\Omega |\nabla \psi(u)^{(p+1)/2}|^2 + \int_{\partial\Omega} \psi(u)^{p+1} \right]$$
$$\leqslant \int_\Omega f(u)\, \psi(u)^p \tag{2.8}$$

Now, by the generalized Poincaré inequality and the Sobolev embeddings, there are $q > 1$ and $a_1 > 0$ such that

$$\frac{4p}{(p+1)^2}\left[\int_\Omega |\nabla\psi(u)^{(p+1)/2}|^2 + \int_{\partial\Omega} \psi(u)^{p+1}\right] \geqslant \frac{4pa_1}{(p+1)^2}\left(\int_\Omega \psi(u)^{q(p+1)}\right)^{1/q}$$

Hence, from (2.8), it follows that

$$\frac{d}{dt}\int_\Omega \Phi_p(u) + \frac{4pa_1}{(p+1)^2}\left(\int_\Omega \psi(u)^{q(p+1)}\right)^{1/q} \leqslant \int_\Omega f(u)\,\psi(u)^p \tag{2.9}$$

Now, by (H_2), $f(s)\,\psi(s)^p \leqslant c_1(\psi(s)^\alpha + \varphi(r_0)^\alpha)\,\varphi(s)^p$ for any $s \in \mathbb{R}$. Then, taking $F(s) = c_1(\psi(s)^\alpha + \varphi(r_0)^\alpha)$ and applying Hölder inequality, we get for $r > 1$

$$\int_\Omega f(u)\,\psi(u)^p \leqslant a_2\,\|F(u)\|_{L^r(\Omega)}\left(\int_\Omega \psi(u)^{(p+1)\,r/r-1}\right)^{p(r-1)/(p+1)\,r}$$

For any $s > 1$, we can write the above inequality in the form

$$\int_\Omega f(u)\,\psi(u)^p \leqslant a_2\,\|F(u)\|_{L^r(\Omega)}\left(\int_\Omega \varphi(u)^{q(p+1)(r/(r-1)\,q)(s-1/s)}\right.$$
$$\left.\times\,\varphi(u)^{((p+1)\,r/(r-1))\,(1/s)}\right)^{p(r-1)/(p+1)\,r}$$

Now, if $r > q/(q-1)$, we can choose s large enough in order to apply again Hölder inequality and obtain

$$\int_\Omega f(u)\,\psi(u)^p \leqslant a_2\,\|F(u)\|_{L^r(\Omega)}\left(\int_\Omega \psi(u)^{q(p+1)}\right)^{(s-1)\,p/qs(p+1)}$$
$$\times\left(\int_\Omega \psi(u)^{(p+1)\,rq/[(r-1)\,sq-r(s-1)]}\right)^\alpha$$
$$=\left[\frac{2a_1 s}{(p+1)(s-1)}\left(\int_\Omega \psi(u)^{q(p+1)}\right)^{1/q}\right]^{(s-1)\,p/s(p+1)}$$
$$\times\left(\frac{(p+1)(s-1)}{2sa_1}\right)^{(s-1)\,p/s(p+1)}$$
$$\times\,a_2\,\|F(u)\|_{L^r(\Omega)}\left(\int_\Omega \psi(u)^{(p+1)\,rq/[(r-1)\,sq-r(s-1)]}\right)^\alpha \tag{2.10}$$

where

$$\alpha = \frac{(r-1)\,sq - r(s-1)}{rsq} \cdot \frac{p}{p+1}$$

Applying Young inequality in (2.10), (2.9) yields the following

$$\frac{d}{dt}\int_\Omega \Phi_p(u) + \frac{4pa_1}{(p+1)^2}\left(\int_\Omega \psi(u)^{q(p+1)}\right)^{1/q}$$

$$\leqslant \frac{2pa_1}{(p+1)^2}\left(\int_\Omega \psi(u)^{q(p+1)}\right)^{1/q} + \frac{s+p}{(p+1)\,s}$$

$$\times \left(\frac{(p+1)(s-1)}{2sa_1}\right)^{p(s-1)/(s+p)} (1+a_2)^{(p+1)\,s/(s+p)} \, \|F(u)\|_{L^r(\Omega)}^{(p+1)\,s/(s+p)}$$

$$\times \left(\max\left\{1, \int_\Omega \psi(u)^{(p+1)\,rq/[(r-1)\,sq - r(s-1)]}\right\}\right)^{[((r-1)\,sq - r(s-1))/rq]\cdot[p/(s+p)]}$$

Changing r_0 if necessary, we may suppose that $\|F(u)\|_{L^r(\Omega)} > 1$, consequently

$$\frac{d}{dt}\int_\Omega \Phi_p(u) \leqslant \left(\frac{1}{2a_1}\right)^{s-1} (p+1)^{s-1}\,(1+a_2)^s\,\|F(u)\|_{L^r(\Omega)}^s$$

$$\times \left(\max\left\{1, \int_\Omega \psi(u)^{(p+1)/\beta_s}\right\}\right)^{\beta_s}$$

where

$$\beta_s = \frac{(r-1)\,sq - r(s-1)}{rq}$$

Therefore, we can find a constant $b_1 > 0$ such that

$$\frac{d}{dt}\int_\Omega \Phi_p(u) \leqslant b_1^s (p+1)^{s-1}\,\|F(u)\|_{L^r(\Omega)}^s \left(\max\left\{1, \int_\Omega \varphi(u)^{(p+1)/\beta_s}\right\}\right)^{\beta_s}$$

Integrating this inequality from 0 to t and having in mind that

$$\int_\Omega \Phi_p(u(0)) \leqslant \mu(\Omega)\,\|u_0\|_{L^\infty(\Omega)}\,\psi(\|u_0\|_{L^\infty(\Omega)})^p \leqslant b_2^{p+1}$$

with the constant $b_2 > 1$ depending only on $\|u_0\|_{L^\infty(\Omega)}$, we obtain

$$\int_\Omega \Phi_p(u(t)) \leqslant b_2^{p+1} + b_1^s(p+1)^{s-1} \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \, d\tau$$
$$\times \sup_{\tau \in [0,t]} \left(\max \left\{ 1, \int_\Omega \psi(u(\tau))^{(p+1)/\beta_s} \right\} \right)^{\beta_s} \qquad (2.11)$$

Now, the hypotheses on φ yield

$$((\varphi(r) - \varphi(r_0))^+)^{p-m_1+1} \leqslant b_3(p-m_1+1)\, \Phi_p(r) + b_4(p-m_1+1)\, \Phi_{p-m_1}(r) \qquad (2.12)$$

Hence using (2.11) and (2.12) we get

$$\int_\Omega (\psi(u(t)))^{p-m_1+1}$$
$$\leqslant b_3(p-m_1+1)\, b_2^{p+1} + b_3(p-m_1+1)\, b_1^s(p+1)^{s-1}$$
$$\times \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \, d\tau \sup_{\tau \in [0,t]} \left(\max \left\{ 1, \int_\Omega \psi(u(\tau))^{(p+1)/\beta_s} \right\} \right)^{\beta_s}$$
$$+ b_4(p-m_1+1)\, b_5^{p-m_1+1} + b_4(p-m_1+1)\, b_6^s(p-m_1+1)^{s-1}$$
$$\times \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \, d\tau \sup_{\tau \in [0,t]} \left(\max \left\{ 1, \int_\Omega \psi(u(\tau))^{(p-m_1+1)/\beta_s} \right\} \right)^{\beta_s}$$

Now, applying Hölder inequality, it is easy to see that

$$\max \left\{ 1, \int_\Omega \psi(u(\tau))^{(p-m_1+1)/\beta_s} \right\} \leqslant c \max \left\{ 1, \int_\Omega \psi(u(\tau))^{(p+1)/\beta_s} \right\}$$

Thus, we can write

$$\int_\Omega \psi(u(t))^{p-m_1+1} \leqslant b_7^{p-m_1+1}(p-m_1+1)^s$$
$$+ b_8^s(p-m_1+1)^s \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \, d\tau$$
$$\times \sup_{\tau \in [0,t]} \left(\max \left\{ 1, \int_\Omega \psi(u(\tau))^{(p+1)/\beta_s} \right\} \right)^{\beta_s} \qquad (2.13)$$

Take p_0 large enough and define inductively the sequence

$$p_{k+1} = \beta_s p_k - m_1$$

Since $r > q/(q-1)$, we can choose s large enough such that $\beta_s > 1$, and consequently $p_k \to +\infty$ as $k \to +\infty$.

If we define

$$\gamma_k = \sup_{\tau \in [0, t]} \max \left\{ 1, \int_\Omega \psi(u(\tau))^{p_k} \right\}$$

taking $p = p_{k+1} + m_1 - 1$ in (2.13), as we can suppose that the right hand side of the above inequality is greater than 1, it follows that

$$\gamma_{k+1} \leqslant b_7^{p_{k+1}} p_{k+1}^s + b_8^s p_{k+1}^s \left(\int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \, d\tau \right) \gamma_k^{\beta_s}$$

Consequently,

$$\gamma_{k+1} \leqslant 2b_8^s \max \left\{ 1, \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \right\} p_{k+1}^s \max\{b_7^{p_{k+1}}, \gamma_k^{\beta_s}\}$$

Then, we are in the situation of Lemma 1, with

$$C_0 = 2b_8^s \max \left\{ 1, \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \right\}$$

$$b = s, \qquad C_1 = b_7, \qquad a = \beta_s, \qquad \text{and} \qquad c = -m_1$$

Obviously, we can choose s large enough, such that

$$p_0 - \frac{m_1}{\beta_s - 1} > 0$$

Moreover, by (2.1)

$$\int_\Omega \psi(u(\tau))^p \leqslant C(p, \|u_0\|_{L^\infty(\Omega)}), \qquad \text{for all} \quad \tau \geqslant 0$$

Hence, we can take b_7 large enough such that

$$\gamma_0 = \sup_{\tau \in [0, t]} \max \left\{ 1, \int_\Omega \psi(u(\tau))^{p_0} \right\} \leqslant C_1^{p_0}$$

Therefore, applying Lemma 1

$$(\gamma_k)^{1/p_k} = \left(\sup_{\tau \in [0,t]} \max \left\{ 1, \int_\Omega \psi(u(\tau))^{p_k} \right\} \right)^{1/p_k} \leqslant M \qquad \forall\, k \in \mathbb{N}$$

where

$$M \leqslant \left[\left(2b_8^s \max \left\{ 1, \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \right\} \right)^{1/s} \right.$$
$$\left. \times \left(2 \left(p_0 + \frac{m_1}{\beta_s - 1} \right) \right) \right]^{[(p_k - p_0)/p_k][s/(p_0(\beta_s - 1) - m_1)]}$$
$$\times \beta_s^{\{s/\beta_s(\beta_s - 1)\} \cdot \{[(p_{k+1} - m_1/(\beta_s - 1))/p_k][1/(p_0(\beta_s - 1) - m_1)] - [((k+1)\beta_s - k)/p_k(\beta_s - 1)]\}}$$
$$\times b_7^{2[p_0 + m_1/(\beta_s - 1)][(p_k - m_1/(\beta_s - 1))/p_k][1/(p_0 - m_1/(\beta_s - 1))]}$$

Therefore, taking limit as $k \to \infty$, we get

$$\|\psi(u)\|_{L^\infty(Q_t)}$$
$$\leqslant \left[\left(2b_8^s \max \left\{ 1, \int_0^t \|F(u(\tau))\|_{L^r(\Omega)}^s \right\} \right)^{1/s} \right.$$
$$\left. \times \left(2 \left(p_0 + \frac{m_1}{\beta_s - 1} \right) \right) \right]^{s/[p_0(\beta_s - 1) - m_1]}$$
$$\times \beta_s^{[s/\beta_s(\beta_s - 1)] \cdot [1/(p_0(\beta_s - 1) - m_1)]} b_7^{2[p_0 + m_1/(\beta_s - 1)][1/(p_0 - m_1/(\beta_s - 1))]}$$

From where it follows, taking $s \to \infty$, that

$$\|\psi(u)\|_{L^\infty(Q_t)} \leqslant [2p_0 b_8 \max\{1, \sup_{\tau \in [0,t]} \|F(u(\tau))\|_{L^r(\Omega)}\}]^{rq/[p_0((r-1)q - r)]} b_7^2 \tag{2.14}$$

Now, by (H_2) and (2.1) we have

$$\|F(u(\tau))\|_{L^r(\Omega)} \leqslant C(r, \|u_0\|_{L^\infty(\Omega)}), \qquad \forall\, \tau \geqslant 0 \tag{2.15}$$

Proceeding as at the end of the proof of Proposition 1, we obtain the same result for $\psi(u) = (\varphi(-r_0) - \varphi(u))^+$. Then, from (2.14) and (2.15) we obtain (2.7).

In the next result we obtain the following energy estimates.

Proposition 3. *Assume that (H) holds. Let u be a global smooth solution of problem (P) with initial datum* $u(0)=u_0 \in L^\infty(\Omega)$. *Then, given* $\tau>0$, *there exist constants depending only on* $\|u_0\|_{L^\infty(\Omega)}$ *and* τ, *such that*

$$\|\nabla\varphi(u(t))\|_{L^2(\Omega)} \leqslant C(\tau, \|u_0\|_{L^\infty(\Omega)}), \qquad \forall\, t \geqslant \tau \tag{2.16}$$

$$\int_\tau^\infty \int_\Omega u_t \varphi(u)_t \leqslant K(\tau, \|u_0\|_{L^\infty(\Omega)}) \tag{2.17}$$

Proof. Multiplying the equation of (P) by $\varphi(u)_t$ we get

$$\begin{aligned} 0 &\leqslant \int_\Omega u_t \varphi(u)_t \\ &= -\int_\Omega \nabla\varphi(u)\cdot\nabla(\varphi(u))_t - \int_{\partial\Omega} g(u)(\varphi(u))_t + \int_\Omega f(u)(\varphi(u))_t \end{aligned} \tag{2.18}$$

If we set

$$F(r)=\int_0^r f(s)\,\varphi'(s)\,ds, \qquad G(r)=\int_0^t g(s)\,\varphi'(s)\,ds$$

from (2.18) it follows that

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla\varphi(u)|^2 + \int_{\partial\Omega} G(u) - \int_\Omega F(u)\right) \leqslant 0 \tag{2.19}$$

Now, by the above Proposition, there are constants $M_1, M_2>0$ such that

$$-M_1 \leqslant \int_{\partial\Omega} G(u) \leqslant M_1, \qquad -M_2 \leqslant \int_\Omega F(u) \leqslant M_2$$

and consequently

$$\int_{\partial\Omega} G(u) + M_1 \geqslant 0, \qquad -\int_\Omega F(u) + M_2 \geqslant 0$$

Moreover, from (2.19), we can write

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla\varphi(u)|^2 + \int_{\partial\Omega} G(u) + M_1 - \int_\Omega F(u) + M_2\right) \leqslant 0$$

On the other hand, multiplying the equation of (P) by $\varphi(u)$, integrating over $]t, t+h[\, \times \Omega$ and using Proposition 2, it is easy to see that

$$\int_t^{t+h} \left(\frac{1}{2} \int_\Omega |\nabla\varphi(u)|^2 + \int_{\partial\Omega} G(u) - \int_\Omega F(u) \right) \leqslant C(h, \|u_0\|_{L^\infty(\Omega)}) \qquad \text{for} \quad t \geqslant \tau$$

Then, by the uniform Gronwall's Lemma (Teman, 1988), we obtain (2.16). Finally, integrating (2.18) over $]\tau, T[$, we get

$$\int_\tau^T \int_\Omega u_t \varphi(u)_t + \frac{1}{2} \int_\Omega (|\nabla\varphi(u(T))|^2 - |\nabla\varphi(u(\tau))|^2)$$
$$+ \int_{\partial\Omega} (G(u(T)) - G(u(\tau))) + \int_\Omega (F(u(\tau)) - F(u(T))) = 0$$

Hence, by (2.16), (2.17) holds.

We finish this section with the following result.

Lemma 2. *Let* $F, \hat{F} \in L^1(Q_t)$, $G, \hat{G} \in L^1(S_T)$ *and* $v_0, \hat{v}_0 \in L^\infty(\Omega)$. *Suppose* v *and* $\hat{v}$ *are smooth solutions of the problem*

$$\begin{cases} v_t = \Delta\varphi(v) + F & in \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial\varphi(v)}{\partial\eta} = G & on \quad S_T = \partial\Omega \times (0, T) \\ v(x, 0) = v_0(x) & in \ \Omega \end{cases}$$

and

$$\begin{cases} \hat{v}_t = \Delta\varphi(\hat{v}) + \hat{F} & in \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial\varphi(\hat{v})}{\partial\eta} = G & on \quad S_T = \partial\Omega \times (0, T) \\ \hat{v}(x, 0) = \hat{G} & in \ \Omega \end{cases}$$

respectively. Let $\psi \in C^2(\bar{\Omega})$, $\psi \geqslant 1$ *on* $\bar{\Omega}$ *such that*

$$\frac{\partial\psi}{\partial\eta} \geqslant L\psi \qquad on\ \partial\Omega \tag{2.20}$$

where $L > 0$ is a given constant. Then

$$\int_{\Omega} (v(t) - \hat{v}(t))^+ \psi + L \int_0^t \int_{\partial\Omega} (\varphi(v) - \varphi(\hat{v}))^+ \psi$$
$$\leqslant \int_{\Omega} (v_0 - \hat{v}_0)^+ \psi$$
$$+ \int_0^t \int_{\Omega} [(\varphi(v) - \varphi(\hat{v}))^+ |\Delta\psi| + (F - \hat{F}) \operatorname{sign}^+(v - \hat{v}) \psi]$$
$$- \int_0^t \int_{\partial\Omega} (G - \hat{G}) \operatorname{sign}^+(v - \hat{v}) \psi$$

for every $0 < t < T$.

Proof. Multiplying the difference of the two equations by $\operatorname{sign}^+(v - \hat{v}) \psi$ and integrating over Ω, we get

$$\int_{\Omega} (v - \hat{v})_t \operatorname{sign}^+(v - \hat{v}) \psi$$
$$= -\int_{\Omega} \nabla(\varphi(v) - \varphi(\hat{v}))^+ \cdot \nabla\psi + \int_{\partial\Omega} \frac{\partial}{\partial\eta} (\varphi(v) - \varphi(\hat{v})) \operatorname{sign}^+(v - \hat{v}) \psi$$
$$+ \int_{\Omega} (F - \hat{F}) \operatorname{sign}^+(v - \hat{v}) \psi$$

Hence

$$\frac{d}{ds} \int_{\Omega} (v - \hat{v})^+ \psi \leqslant \int_{\Omega} (\varphi(v) - \varphi(\hat{v}))^+ \Delta\psi - \int_{\partial\Omega} \frac{\partial\psi}{\partial\eta} (\varphi(v) - \varphi(\hat{v}))^+$$
$$- \int_{\partial\Omega} (G - \hat{G}) \operatorname{sign}^+(v - \hat{v}) \psi + \int_{\Omega} (F - \hat{F}) \operatorname{sign}^+(v - \hat{v}) \psi$$

Integrating the above inequality from 0 to t and using (2.20) the proof concludes.

3. EXISTENCE AND UNIQUENESS OF GLOBAL WEAK SOLUTIONS

In this section we prove the existence and uniqueness of a global weak solution of problem (I) when the initial datum is in $L^\infty(\Omega)$.

Definition 2. Given $u_0 \in L^\infty(\Omega)$, by a weak solution of problem (I) on Q_T we mean a function $u \in C([0, T]; L^1(\Omega)) \cap L^\infty(Q_T)$ such that $\varphi(u) \in L^2(0, T; H^1(\Omega))$ and satisfies the identity

$$\int_{Q_T} (\nabla\varphi(u) \cdot \nabla\phi - u\phi_t - f(u)\,\phi) + \int_{S_T} g(u)\,\phi = \int_\Omega u_0(x)\,\phi(x, 0)\,dx$$

for any function $\phi \in L^2(0, T; H^1(\Omega)) \cap W^{1,1}(0, T: L^1(\Omega))$ with $\phi(T) = 0$.

We shall say that u is a global weak solution of problem (I) if u is a weak solution on Q_T for all positive T.

Theorem 1. *Under the assumptions (H), for every $u_0 \in L^\infty(\Omega)$ there exists a unique global weak solution u of problem (I) satisfying*

$$\|u\|_{L^\infty(Q)} \leqslant C(\|u_0\|_{L^\infty(\Omega)}) \tag{3.1}$$

$$\|\nabla\varphi(u(t))\|_{L^2(\Omega)} \leqslant C(\tau, \|u_0\|_{L^\infty(\Omega)}), \qquad \forall t \geqslant \tau > 0 \tag{3.2}$$

$$\int_\tau^\infty \int_\Omega (\varphi(u)_t)^2 \leqslant K(\tau, \|u_0\|_{L^\infty(\Omega)}), \qquad \forall \tau > 0 \tag{3.3}$$

We are going to divide the proof into several steps.

Existence of Solution

To prove the existence of solution we will consider a sequence of approximated nondegenerate problems which can be solved in a classical sense. To do that we consider sequences of functions (φ_n), (f_n) and (g_n) satisfying:

$$\begin{cases} \varphi_n \in C^\infty(\mathbb{R}), \qquad \varphi_n' \geqslant \max\{\varphi', 1/n\}, \qquad \varphi_n(0) = 0, \\ \varphi_n(r) \text{ constant for } |r| \geqslant n, \\ \alpha_0\,|\varphi_n(r)|^{m_0} \leqslant \varphi_n'(r) \leqslant \alpha_1\,|\varphi_n(r)|^{m_1} \qquad \forall\,|r| \geqslant r_0, \\ \varphi_n \to \varphi, \qquad \varphi_n' \to \varphi' \text{ uniformly on compact subsets of } \mathbb{R}. \end{cases} \tag{G$_1$}$$

$$\begin{cases} f_n \in C^\infty(\mathbb{R}), \qquad f_n(r) \text{ constant for } |r| \geqslant n, \\ \sup_{[0, M]} |f_n'| \leqslant \sup_{[0, M]} |f'| \qquad \text{for all} \quad M > 0, \\ f_n(r)\,\text{sign}(r) \leqslant c_1\,|\varphi_n(r)|^\alpha \qquad \text{for all} \quad |r| \geqslant r_0, \\ t_n \to f \text{ uniformly on compact subsets of } \mathbb{R}. \end{cases} \tag{G$_2$}$$

$$\begin{cases} g_n \in C^\infty(\mathbb{R}), \quad g_n(r) \text{ constant for } |r| \geqslant n, \\ g_n(r)\,\mathrm{sign}(r) \geqslant c_2\, |\varphi_n(r)| \quad \text{for all} \quad |r| \geqslant r_0, \\ \text{for any } R > 0, \text{ there exists } c_3(R) > 0 \text{ such that} \\ (g_n(r) - g_n(\hat{r}))\,\mathrm{sign}^+(r - \hat{r}) \geqslant -c_3(R)(\varphi_n(r) - \varphi_n(\hat{r}))^+ \quad \forall r, \hat{r} \in [-R, R], \\ g_n \to g \text{ uniformly on compact subsets of } \mathbb{R}. \end{cases} \tag{G$_3$}$$

Using the same technique as in the proof of Proposition 3 of Filo and de Mottoni (1992), we can find functions $u_{0,n} \in C^3(\bar{\Omega})$, $\|u_{0,n}\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 1$, satisfying the compatibility condition

$$-\frac{\partial \varphi_n(u_{0,n})}{\partial \eta} = g_n(u_{0,n}) \qquad \text{on } \partial\Omega$$

and

$$\|u_{0,n} - u_0\|_{L^1(\Omega)} \to 0 \qquad \text{as} \quad n \to \infty.$$

Consider the approximated problems

$$\begin{cases} (u_n)_t = \Delta\varphi_n(u_n) + f_n(u_n) & \text{in} \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial \varphi_n(u_n)}{\partial \eta} = g_n(u_n) & \text{on} \quad S_T = \partial\Omega \times (0, T) \\ u_n(x, 0) = u_{0,n}(x) & \text{in } \Omega \end{cases} \tag{P$_n$}$$

By a classical result (Theorem 7.4 of Ladyzenskaja *et al.*, 1968), for any $T > 0$, (P_n) has a unique smooth solution u_n in Q_T. Moreover, as a consequence of Propositions 2 and 3, the following estimates hold:

$$\|u_n\|_{L^\infty(Q_T)} \leqslant D_1(\|u_0\|_{L^\infty(\Omega)}) \tag{3.4}$$

$$\|\nabla\varphi_n(u_n(t))\|_{L^2(\Omega)} \leqslant D_2(\tau, \|u_0\|_{L^\infty(\Omega)}), \qquad \forall 0 < \tau \leqslant t < T \tag{3.5}$$

$$\int_\tau^T \int_\Omega (u_n)_t\, \varphi_n(u_n)_t \leqslant D_3(\tau, \|u_0\|_{L^\infty(\Omega)}), \qquad \forall 0 < \tau < T \tag{3.6}$$

In the next step we are going to see, by compactness arguments, that it is possible to pass to the limit in (P_n) in order to get a weak solution of problem (I).

From (3.4) we can suppose (up to extraction of a subsequence, if necessary) that

$$u_n \to u \qquad \text{weakly in } L^2(Q_T) \tag{3.7}$$

On the other hand, (3.6) and (3.4) yields

$$\int_\tau^T \int_\Omega ((\varphi_n(u_n))_t)^2 \leqslant D_4(\tau, \|u_0\|_{L^\infty(\Omega)}) \tag{3.8}$$

Multiplying the equation of (P_n) by $\varphi_n(u_n)$ and integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega \psi_n(u_n) = \int_\Omega (u_n)_t\, \varphi_n(u_n)$$

$$= -\int_\Omega |\nabla\varphi_n(u_n)|^2 - \int_{\partial\Omega} g_n(u_n)\, \varphi_n(u_n) + \int_\Omega f_n(u_n)\, \varphi_n(u_n)$$

where

$$\psi_n(r) = \int_0^t \varphi_n(s)\, ds$$

Then, integrating over $[0, T]$ and using (3.4) we get

$$\int_{Q_T} |\nabla\varphi_n(u_n)|^2 \leqslant C, \qquad \forall n \in \mathbb{N} \tag{3.9}$$

Now, fixed $0 < \tau < T$, by (3.4), (3.8) and (3.9), we have that

$$\{\varphi_n(u_n)\colon n \in \mathbb{N}\} \text{ is a relatively compact subset of } L^2(]\tau, T[\times\Omega) \tag{3.10}$$

Since $\varphi_n \to \varphi$ uniformly on compact subsets of $\mathbb{R}$, by (3.4), (3.7) and (3.10), it is easy to see that

$$u_n \to u \qquad \text{in} \quad L^2(]\tau, T[\times\Omega) \qquad \text{for any} \quad 0 < \tau < T$$

Consequently, up to extraction of a subsequence, we can obtain

$$u_n \to u \qquad \text{a.e. in } Q_T \tag{3.11}$$

Next, we get by (3.9)

$$\nabla\varphi_n(u_n) \to \Psi \qquad \text{weakly in } L^2(Q_T)$$

New, by (3.11) and the Dominate Convergence Theorem, we get

$$\varphi_n(u_n) \to \varphi(u_n) \qquad L^2(Q_T) \tag{3.12}$$

As a consequence of (3.9) and (3.12), we obtain $\Psi = \nabla\varphi(u)$.

From (3.9), (3.12) and Theorem 3.4.5 of Morrey (1966), it is easy to see that

$$g_n(u_n) \to g(u) \qquad \text{in } L^2(0, T : L^2(\partial\Omega))$$

We now prove that $u \in C([0, T]; L^1(\Omega))$. Given $0 < \tau < T$, by (3.8) we can suppose that

$$(\varphi_n(u_n))_t \to (\varphi(u))_t \qquad \text{weakly in } L^2(\tau, T; L^2(\Omega))$$

From where it follows that $\varphi(u) \in C(]0, T]; L^1(\Omega))$ and consequently $u \in C(]0, T]; L^1(\Omega))$. To get the continuity of u at 0 we need the following result.

Lemma 3. *Given $u_0, \hat{u}_0 \in L^\infty(\Omega)$, let u and $\hat{u}$ be limit of the smooth solutions of the approximated problem*

$$\begin{cases} (u_n)_t = \Delta\varphi_n(u_n) + f_n(u_n) & \text{in} \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial\varphi_n(u_n)}{\partial\eta} = g_n(u_n) & \text{on} \quad S_T = \partial\Omega \times (0, T) \\ u_n(x, 0) = u_{0,n}(x) & \text{in } \Omega \end{cases} \qquad (P_n)$$

and

$$\begin{cases} (\hat{u}_n)_t = \Delta\varphi_n(\hat{u}_n) + f_n(\hat{u}_n) & \text{in} \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial\varphi_n(\hat{u}_n)}{\partial\eta} = g_n(\hat{u}_n) & \text{on} \quad S_T = \partial\Omega \times (0, T) \\ \hat{u}_n(x, 0) = \hat{u}_{0,n}(x) & \text{in } \Omega \end{cases} \qquad (\hat{P}_n)$$

respectively, where φ_n, f_n and g_n satisfy (G_1), (G_2) and G_3, $u_{0,n} \to u_n$, $\hat{u}_{0,n} \to \hat{u}_0$ in $L^1(\Omega)$ bounded in $L^\infty(\Omega)$ independently on n and verify the corresponding compatibility conditions. Then, there exists $C > 0$ such that

$$\|(u(t) - \hat{u}(t))^+\|_{L^1(\Omega)} \leqslant e^{CT} \|(u_0 - \hat{u}_0)^+\|_{L^1(\Omega)}, \qquad \forall 0 < t \leqslant T$$

In particular we also have

$$\|u(t) - \hat{u}(t)\|_{L^1(\Omega)} \leqslant e^{CT} \|u_0 - \hat{u}_0\|_{L^1(\Omega)}, \qquad \forall 0 < t \leqslant T$$

Proof. Applying Lemma 2 we have that

$$\int_\Omega (u_n(t)-\hat{u}_n(t))^+ \psi + L\int_0^t\int_{\partial\Omega} (\varphi_n(u_n)-\varphi_n(\hat{u}_n))^+ \psi$$

$$\leqslant \int_\Omega (u_{0,n}-\hat{u}_{0,n})^+ \psi + \int_0^t\int_\Omega [(\varphi_n(u_n)-\varphi_n(\hat{u}_n))^+ |\Delta\psi|$$

$$+ (f_n(u_n)-f_n(\hat{u}_n)) \operatorname{sign}^+(u_n-\hat{u}_n)\psi]$$

$$-\int_0^t\int_{\partial\Omega} (g_n(u_n)-g_n(\hat{u}_n)) \operatorname{sign}^+(u_n-\hat{u}_n)\psi$$

By (G_1), (G_2) and (3.4) we get

$$\|\varphi_n'(u_n)\|_{L^\infty(Q_T)} \leqslant C, \qquad \|f_n'(u_n)\|_{L^\infty(Q_T)} \leqslant C$$

Now, by (3.4) there exists $R_0>0$ such that $\|u_n\|_{L^\infty(\Omega)} \leqslant R_0$, $\|\hat{u}_n\|_{L^\infty(\Omega)} \leqslant R_0$. Then, taking $L=c_3(R_0)$ we get for any $t\in[0,T]$

$$\int_\Omega (u_n(t)-\hat{u}_n(t))^+ \leqslant M_1\int_\Omega (u_{0,n}-\hat{u}_{0,n})^+ + M_2\int_0^t\int_\Omega (u_n(s)-\hat{u}_n(s))^+\,ds$$

where $M_i=M_i(R_0)$, $i=1,2$. Hence, applying Gronwall Lemma and passing to the limit when $n\to\infty$ we obtain the desired conclusion with $C=C(R_0)$ for all $0<t\leqslant T$.

To finish the proof of the continuity of u at 0, firstly we assume $u_0\in C^1(\bar{\Omega})$. Then, in the construction of $u_{0,n}$ we can suppose that $\{|\nabla\varphi_n(u_{0,n})| : n\in\mathbb{N}\}$ is bounded in $L^2(\Omega)$. Multiplying the equation (P_n) by $\varphi_n(u_n)_t$, we have

$$\int_\Omega (u_n)_t\,(\varphi_n(u_n))_t$$

$$=-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi_n(u_n)|^2 - \int_{\partial\Omega} g_n(u_n)(\varphi_n(u_n))_t + \int_\Omega f_n(u_n)(\varphi_n(u_n))_t \qquad (3.13)$$

If we set

$$F_n(r)=\int_0^t f_n(s)\,\varphi_n'(s)\,ds, \qquad G_n(r)=\int_0^r g_n(s)\,\varphi_n'(s)\,ds$$

integrating (3.13) we obtain

$$\int_0^t \int_\Omega (u_n)_t (\varphi_n(u_n))_t + \tfrac{1}{2}\int_\Omega |\nabla\varphi_n(u_n(t))|^2$$

$$= \tfrac{1}{2}\int_\Omega |\nabla\varphi_n(u_{0,n})|^2 + \int_{\partial\Omega} (G_n(u_{0,n}) - G_n(u_n(t))) + \int_\Omega (F_n(u_n(t)) - F_n(u_{0,n}))$$

From where it follows that $\{(\varphi_n(u_n))_t : n \in \mathbb{N}\}$ is bounded in $L^2(Q_T)$. Then, proceeding as in the proof of the continuity in $]0, T]$, we conclude that $u \in C([0, T]; L^1(\Omega))$ and $u(0) = u_0$.

It remains to drop the condition u_0 smooth. If $u_0 \in L^\infty(\Omega)$, there exists a sequence of smooth functions $v_{0,n}$ such that $v_{0,n} \to u_0$ in $L^1(\Omega)$. The corresponding v_n, constructed as above, are continuous at 0, then as

$$\|u(t) - u_0\|_1 \leqslant \|u(t) - v_n(t)\|_1 + \|v_n(t) - v_{0,n}\|_1 + \|v_{0,n} - u_0\|_1 \qquad \forall t > 0$$

by Lemma 3, we obtain the continuity of u at 0.

Finally, since u_n is a smooth solution of (P_n), it clearly satisfies

$$\int_{Q_T} (\nabla\varphi_n(u_n) \cdot \nabla\phi - u_n\phi_t - f_n(u_n)\,\phi) + \int_{S_T} g_n(u_n)\,\phi$$

$$= \int_\Omega u_{0,n}(x)\,\phi(x, 0)\,dx$$

for any test function ϕ. From here, passing to the limit when $n \to \infty$ we obtain that u is a weak solution of problem (I).

Uniqueness

Definition 3. Let $F \in L^\infty(Q_T)$, $G \in L^\infty(S_T)$ and $v_0 \in L^\infty(\Omega)$, we say that v is a weak solution of problem

$$S(\varphi, F, G, v_0) \begin{cases} v_t = \Delta\varphi(v) + F & \text{in} \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial\varphi(v)}{\partial\eta} = G & \text{on} \quad S_T = \partial\Omega \times (0, T) \\ v(x, 0) = v_0(x) & \text{in } \Omega \end{cases}$$

if $v \in C([0, T], L^1(\Omega)) \cap L^\infty(Q_T)$, $\varphi(v) \in L^2(0, T; H^1(\Omega))$ and satisfies

$$\int_{Q_T} (\nabla\varphi(v) \cdot \nabla\phi - v\phi_t - F\phi) + \int_{S_T} G\phi = \int_\Omega v_0(x)\, \phi(x, 0)\, dx$$

for any function $\phi \in L^2(0, T; H^1(\Omega)) \cap W^{1,1}(0, T; L^1(\Omega))$ with $\phi(T) = 0$.

In order to prove the uniqueness we need the following lemma.

Lemma 4. *Let $F, \hat{F} \in L^\infty(Q_T)$, $G, \hat{G} \in L^\infty(S_T)$, $v_0, \hat{v}_0 \in L^\infty(\Omega)$ and φ, $\hat{\varphi}$ satisfying (H_1). If v and $\hat{v}$ are weak solutions of $S(\varphi, F, G, v_0)$ and $S(\hat{\varphi}, \hat{F}, \hat{G}, \hat{v}_0)$ respectively, then*

$$\begin{aligned}\int_{Q_T} &(v - \hat{v})(\varphi(v) - \hat{\varphi}(\hat{v})) \\ &\leqslant \int_{Q_T} (F - \hat{F}) \int_t^T (\varphi(v) - \hat{\varphi}(\hat{v})) \\ &\quad - \int_{S_T} (G - \hat{G}) \int_t^T (\varphi(v) - \hat{\varphi}(\hat{v})) + \int_\Omega (v_0 - \hat{v}_0) \int_0^T (\varphi(v) - \hat{\varphi}(\hat{v}))\end{aligned}$$

Proof. It is enough to take as test function

$$\eta(x, t) = \begin{cases} \int_t^T (\varphi(v(x, s)) - \varphi(\hat{v}(x, s)))\, ds, & \text{if } 0 \leqslant t < T \\ 0, & \text{if } t \geqslant T \end{cases}$$

and the result follows.

Proof of Uniqueness. Suppose that u and $\hat{u}$ are two weak solutions of problem (I) on Q_T with the same initial datum $u_0 \in L^\infty(\Omega)$. Let F_n, $\hat{F}_n$, G_n, $\hat{G}_n$ smooth functions, F_n, $\hat{F}_n$ bounded in $L^\infty(Q_T)$ and G_n, $\hat{G}_n$ bounded in $L^\infty(S_T)$ uniformly in n, such that

$$\begin{aligned} &F_n \to f(u), \quad \hat{F}_n \to f(\hat{u}) \quad \text{in } L^2(Q_T) \quad \text{and} \\ &G_n \to g(u), \quad \hat{G}_n \to g(\hat{u}) \quad \text{in } L^2(S_T) \end{aligned}$$

Let φ_n satisfying (G_1). Using again the same technique introduce in Proposition 3 of Filo and de Mottoni (1992), we can find functions $u_{0,n}, \hat{u}_{0,n} \in C^3(\bar{\Omega})$ bounded in $L^\infty(\Omega)$ uniformly in n satisfying the compatibility conditions

$$-\frac{\partial\varphi_n(u_{0,n})}{\partial\eta}=G_n(\cdot,0) \qquad \text{on } \partial\Omega$$

$$-\frac{\partial\varphi_n(\hat{u}_{0,n})}{\partial\eta}=\hat{G}_n(\cdot,0) \qquad \text{on } \partial\Omega$$

and

$$u_{0,n}\to u_0, \qquad \hat{u}_{0,n}\to u_0 \qquad \text{in } L^1(\Omega)$$

By classical results (see Theorem 7.4 of Ladyzenskaja *et al.*, 1968), there exist u_n and $\hat{u}_n$ smooth solutions of the problem $S(\varphi_n, F_n, G_n, u_{0,n})$ and $S(\varphi_n, \hat{F}_n, \hat{G}_n, \hat{u}_{0,n})$ respectively.

First, using the maximum principle, we prove that u_n are bounded in $L^\infty(Q_T)$ uniformly in n. Indeed, there exists $C>0$ such that $\|F_n\|_{L^\infty(Q_T)}\leqslant C$, $\|G_n\|_{L^\infty(S_T)}\leqslant C$ and $\|\varphi_n(u_{0,n})\|_{L^\infty(\Omega)}\leqslant C$. Consider $\psi\in C^2(\bar{\Omega})$ satisfying

$$\psi>C \qquad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial\psi}{\partial\eta}\geqslant\psi \qquad \text{on } \partial\Omega$$

Set $\xi_n=\varphi_n^{-1}(\psi+\gamma t)$ where γ is a positive constant. On the one hand, for C large enough, we have by (G_1)

$$(\xi_n)_t=\frac{\gamma}{\varphi_n'(\varphi_n^{-1}(\psi+\gamma t))}\geqslant\frac{\gamma}{\alpha_1(\|\psi\|_{L^\infty(\Omega)}+\gamma T)^{m_1}}$$

which implies

$$(u_n)_t-\Delta\varphi_n(u_n)-F_n=0\leqslant(\xi_n)_t-\Delta\varphi_n(\xi_n)-F_n \qquad \text{in } Q_T$$

for γ large enough. On the other hand, we have

$$\frac{\partial\varphi_n(\xi_n)}{\partial\eta}=\frac{\partial\psi}{\partial\eta}\geqslant C\geqslant\frac{\partial\varphi_n(u_n)}{\partial\eta} \qquad \text{in } S_T.$$

Consequently $u_n\leqslant\xi_n\leqslant C(\|\psi\|_{L^\infty(\Omega)},\gamma,T)$ on Q_T. Similarly a lower bound for u_n can be obtained.

Now, multiplying the equation by $\varphi_n(u_n)$ and integrating on Q_T it is easy to see that $\{|\nabla\varphi_n(u_n)|: n\in\mathbb{N}\}$ is bounded in $L^2(Q_T)$.

Using Lemma 4 for u_n, the solution of $S(\varphi_n, F_n, G_n, u_{0,n})$, and for u, the weak solution of $S(\varphi, f(u), g(u), u_0)$, we have

$$\int_{Q_T}(u_n-u)(\varphi_n(u_n)-\varphi(u))\leqslant a_n, \qquad \text{with} \quad \lim_{n\to\infty}a_n=0$$

Then, as

$$\int_{Q_T} (u_n - u)(\varphi(u_n) - \varphi(u))$$
$$= \int_{Q_T} (u_n - u)(\varphi(u_n) - \varphi_n(u_n)) + \int_{Q_T} (u_n - u)(\varphi_n(u_n) - \varphi(u))$$

and $\{u_n : n \in \mathbb{N}\}$ is bounded in $L^\infty(Q_T)$, we obtain

$$\int_{Q_T} (u_n - u)(\varphi(u_n) - \varphi(u)) \to 0$$

By the monotonicity of φ we conclude, up to extraction of a subsequence, that u_n converges almost everywhere to u. Consequently, $\varphi_n(u_n) \to \varphi(u)$ in $L^2(Q_T)$. Moreover, $\{|\nabla\varphi_n(u_n)| : n \in \mathbb{N}\}$ is bounded in $L^2(Q_T)$. Hence, by Theorem 3.4.5 of Morrey (1966), it is easy to see that

$$\varphi_n(u_n) \to \varphi(u) \qquad \text{in } L^2(0, T : L^2(\partial\Omega))$$

Similarly, we obtain the same for $\hat{u}_n$. Then, applying Lemma 2 and passing to the limit we obtain

$$\int_\Omega (u(t) - \hat{u}(t))^+ \psi + L \int_0^t \int_{\partial\Omega} (\varphi(u) - \varphi(\hat{u}))^+ \psi$$
$$\leqslant \int_0^t \int_\Omega (\varphi(u) - \varphi(\hat{u}))^+ |\Delta\varphi| + \int_0^t \int_\Omega (f(u) - f(\hat{u}))\, \mathrm{sign}^+(u - \hat{u})\, \psi$$
$$- \int_0^t \int_{\partial\Omega} (g(u) - g(\hat{u}))\, \mathrm{sign}^+(u - \hat{u})\, \psi$$

Proceeding as in the proof of Lemma 3 we conclude $u \leqslant \hat{u}$ a.e. on Q_T. Interchanging the role of u and $\hat{u}$ the proof of uniqueness is finished.

As a consequence of Lemma 3 and the uniqueness of weak solutions we have the following L^1-*Contraction Principle*, which will be useful in the next section.

Proposition 4. *Suppose $u_0, \hat{u}_0 \in L^\infty(\Omega)$ and let u, $\hat{u}$ be weak solutions of problem (I) on Q_T with initial date u_0, $\hat{u}_0$, respectively. Let R_0 be an upper bound of $\|u_0\|_{L^\infty(\Omega)}$, $\|\hat{u}_0\|_{L^\infty(\Omega)}$, then there exists a constant $C = C(R_0) > 0$, such that*

$$\|(u(t) - \hat{u}(t))^+\|_{L^1(\Omega)} \leqslant e^{CT} \|(u_0 - \hat{u}_0)^+\|_{L^1(\Omega)}, \qquad \forall t \in [0, T]$$

and

$$\|u(t) - \hat{u}(t)\|_{L^1(\Omega)} \leqslant e^{CT} \|u_0 - \hat{u}_0\|_{L^1(\Omega)}, \qquad \forall t \in [0, T]$$

4. EXISTENCE OF THE GLOBAL ATTRACTOR

By Theorem 1, we can define a semigroup $(S(t))_{t \geqslant 0}$ on $L^\infty(\Omega)$ by $S(t)\, u_0 = u(\cdot, t)$, where u is the unique global weak solution of problem (I) with initial datum u_0. In this section we establish the existence of a global compact attractor for $(S(t))_{t \geqslant 0}$ in this space (see Teman, 1988, for the definition of attractor and related concepts). We start proving that the operators $S(t)$ are uniformly compact for t large.

Lemma 5. *Given a bounded subset B of $L^\infty(\Omega)$ and $t_0 > 0$ the set*

$$\bigcup_{t \geqslant t_0} S(t)\, B$$

is relatively compact in $L^\infty(\Omega)$.

Proof. By (3.1) of Theorem 1, we have that

$$\bigcup_{t \geqslant 0} S(t)\, B$$

is bounded in $L^\infty(\Omega)$. Now, as a consequence of the corollary of Theorem 6.2 of DiBenedetto (1983), which is also true for the non-degenerate case (see also Ladyzenskaja *et al.*, 1968), we obtain that for any $t_0 > 0$ the set

$$\bigcup_{t \geqslant t_0} S(t)\, B$$

is equicontinuous. Then, from Ascoli's Theorem we conclude the proof.

Next we are going to prove the existence of a bounded absorbing set. For this we proceed in two steps. Firstly, we see that the set of stationary solutions of (I) is bounded in $L^\infty(\Omega)$. Then, to conclude, we use the fact that the problem is gradient-like (i.e., if $S(t_n)\, v \to w$ in $L^\infty(\Omega)$, where $t_n \to \infty$ and $v \in L^\infty(\Omega)$, then w is a stationary solution of (I)).

Lemma 6. *Let $S = \{w \in L^\infty(\Omega) \colon S(t)\, w = w, \forall t \geqslant 0\}$ be the set of the stationary solutions. Then, there exists $\rho_0 > 0$ such that $S \subset B(0, \rho_0)$, where $B(0, \rho_0)$ is the ball in $L^\infty(\Omega)$ centered at 0 of radius ρ_0.*

Proof. Let $w \in S$. Take $w_n \in C^3(\bar{\Omega})$ such that $w_n \to w$ in $L^1(\Omega)$, w_n bounded in $L^\infty(\Omega)$ uniformly in n and satisfies the compatibility condition. Let u_n be the smooth solution of problem

$$\begin{cases} (u_n)_t = \Delta \varphi_n(u_n) + f_n(u_n) & \text{in} \quad Q_T = \Omega \times (0, T) \\ -\dfrac{\partial \varphi_n(u_n)}{\partial \eta} = g_n(u_n) & \text{on} \quad S_T = \partial\Omega \times (0, T) \\ u_n(x, 0) = w_n(x) & \text{in } \Omega \end{cases} \tag{$\mathrm{S_n}$}$$

Then, if

$$\varphi_n(r) = (\varphi_n(r) - \varphi_n(r_0))^+, \qquad \Phi_{n,p} = \int_0^t \psi_n(s)^p \, ds, \qquad \text{and}$$

$$F_n(u_n) = c_1(\psi_n(u_n)^\alpha + \varphi_n(r_0)^\alpha)$$

proceeding as in the proof of Proposition 2, we get

$$\int_0^t \frac{d}{dt} \int_\Omega \Phi_{n,p}(u_n) + \int_0^t \frac{2pa_1}{(p+1)^2} \left(\int_\Omega \psi_n(u_n)^{(p+1)q} \right)^{1/q}$$
$$\leqslant \int_0^t b_1^s (p+1)^{s-1} \|F_n(u_n)\|_{L^r(\Omega)}^s \left(\max\left\{1, \int_\Omega \psi_n(u_n)^{(p+1)/\beta_s} \right\} \right)^{\beta_s}$$

Letting $n \to \infty$ in the last inequality, it follows that

$$\frac{2pa_1}{(p+1)^2} \left(\int_\Omega \psi(w)^{(p+1)q} \right)^{1/q}$$
$$\leqslant b_1^s (p+1)^{s-1} \|F(w)\|_{L^r(\Omega)}^s \left(\max\left\{1, \int_\Omega \psi(w)^{(p+1)/\beta_s} \right\} \right)^{\beta_s}$$

where $\psi(r) = (\varphi(r) - \varphi(r_0))^+$ and $F(w) = c_1(\psi(w)^\alpha + \varphi(r_0)^\alpha)$. Consequently,

$$\int_\Omega \psi(w)^{(p+1)} \leqslant b_2 b_1^s (p+1)^s \|F(w)\|_{L^r(\Omega)}^s \left(\max\left\{1, \int_\Omega \psi(w)^{(p+1)/\beta_s} \right\} \right)^{\beta_s} \tag{4.1}$$

Take $p_0 \geqslant 1$ and define inductively the sequence

$$p_{k+1} = \beta_s p_k$$

As in the proof of Proposition 2, $p_k \to \infty$ as $k \to \infty$. Taking $p = \beta_s p_k - 1$ in (4.1), we obtain

$$\max\left\{1, \int_\Omega \psi(w)^{p_{k+1}}\right\} \leqslant b_2 b_1^s p_{k+1}^s \|F(w)\|_{L^r(\Omega)}^s \left(\max\left\{1, \int_\Omega \varphi(w)^{p_k}\right\}\right)^{\beta_s} \tag{4.2}$$

If we define

$$\gamma_k = \max\left\{1, \int_\Omega \psi(w)^{p_k}\right\}$$

(4.2) can be rewritten as follows

$$\gamma_{k+1} \leqslant b_2 b_1^s p_{k+1}^s \|F(w)\|_{L^r(\Omega)}^s \gamma_k^{\beta_s} \tag{4.3}$$

New, using the fact that $\lim_{\tau \to \infty} C(\tau, \|u_0\|_{L^\infty(\Omega)}, p) = C(p)$ in Proposition 1, there exists $C_1 \geqslant 1$ (independent of $\|w\|_{L^\infty(\Omega)}$) such that $\gamma_0 \leqslant C_1^{p_0}$. Consequently, (4.3) yields

$$\gamma_{k+1} \leqslant b_2 b_1^s \|F(w)\|_{L^r(\Omega)}^s p_{k+1}^s \max\{C_1^{p_{k+1}}, \gamma_k^{\beta_s}\}$$

Therefore, applying Lemma 1

$$\gamma_k^{1/p_k} \leqslant C(b_1, b_2, \|F(w)\|_{L^r(\Omega)}, C_1, s, \beta_s)$$

Now, by (H_2) and Proposition 2.1, $\|F(w)\|_{L^r(\Omega)} \leqslant C(r)$. Then, taking $k \to \infty$ and proceeding as in the proof of Proposition 1, we conclude.

***Lemma** 7. The ball $B(0, 2\rho_0)$ is an absorbing set in $L^\infty(\Omega)$ for the semigroup $(S(t))_{t \geqslant 0}$.*

Proof. Let first see that the problem is gradient-like. Given $w \in \omega(v)$, $v \in L^\infty(\Omega)$, there exists $t_n \to \infty$ such that $S(t_n)\, v \to w$ in $L^\infty(\Omega)$. Fix $t > 0$. Then

$$\begin{aligned}\|\varphi(S(t)\, w) - \varphi(w)\|_{L^2(\Omega)} \leqslant{}& \|\varphi(S(t)\, w) - \varphi(S(t + t_n)\, v)\|_{L^2(\Omega)} \\ &+ \|\varphi(S(t + t_n)\, v) - \varphi(S(t_n)\, v)\|_{L^2(\Omega)} \\ &+ \|\varphi(S(t_n)\, v) - \varphi(w)\|_{L^2(\Omega)}\end{aligned} \tag{4.4}$$

By (3.3), there exists $\varepsilon(n)$, $\lim_{n\to\infty}\varepsilon(n)=0$ such that

$$\int_\Omega |\varphi(S(t+t_n)\,v)-\varphi(S(t_n)\,v)|^2 \leqslant t\varepsilon(n) \tag{4.5}$$

Now

$$\|\varphi(S(t_n)\,v)-\varphi(w)\|_{L^2(\Omega)} \to 0 \tag{4.6}$$

On the other hand, since $\varphi\in C^1(\mathbb{R})$ and by Proposition 4

$$\begin{aligned}\|\varphi(S(t)\,w)-\varphi(S(t+t_n)\,v)\|_{L^2(\Omega)} &\leqslant C(\|S(t)\,w-S(t+t_n)\,v\|_{L^1(\Omega)})^{1/2}\\ &\leqslant C'(\|w-S(t_n)\,v\|_{L^1(\Omega)})^{1/2}\end{aligned} \tag{4.7}$$

From (4.4), (4.5), (4.6) and (4.7), $\varphi(S(t)\,w)=\varphi(w)$ and consequently $S(t)\,w=w$.

Suppose now that $B(0,2\rho_0)$ is not an absorbing set. Then, there exists a bounded subset B_0 of $L^\infty(\Omega)$, $t_n\to\infty$ and $u_{0,n}\in B$, satisfying

$$S(t_n)\,u_{0,n}\notin B(0,2\rho_0), \qquad \forall n\in\mathbb{N} \tag{4.8}$$

By Lemma 5, we can suppose (up to extraction of a subsequence, if necessary) that

$$S(t_n)\,u_{0,n}\to w \qquad \text{in } L^\infty(\Omega) \tag{4.9}$$

Now, there exists a constant $k>0$ such that $-k\leqslant u_{0,n}\leqslant k$ a.e. in Ω. By Proposition 4, $S(t_n)(-k)\leqslant S(t_n)(u_{0,n})\leqslant S(t_n)(k)$ a.e. in Ω. Then, by the previous step we can suppose (up to extraction of a subsequence, if necessary) that $S(t_n)(-k)\to w_1$ and $S(t_n)(k)\to w_2$, where $w_1, w_2\in S$. Consequently $w\in B(0,\rho_0)$, which is a contradiction with (4.8) and (4.9).

With all these ingredients, if we could define a dynamical system (see Haraux, 1991) in $L^\infty(\Omega)$, the existence of a compact global attractor is well known (see Teman, 1988). But since we only have the continuity of $u\to S(t)\,u$ in L^1-norm and the continuity at 0 of $t\to S(t)\,u$ in the L^1-norm, this is not possible. Anyway, we can prove the existence of the attractor in $L^\infty(\Omega)$ as a consequence of the following lemma.

Lemma 8. *Let Ω be a bounded subset of $\mathbb{R}^N$ and $S(t)$: $L^\infty(\Omega)\to L^\infty(\Omega)$ a semigroup satisfying $S(t)\in C([0,T], L^1(\Omega))$, for all $T>0$ and*

$$\begin{aligned}&\|S(t)\,u_0-S(t)\,\hat{u}_0\|_{L^1(\Omega)}\leqslant e^{CT}\,\|u_0-\hat{u}_0\|_{L^1(\Omega)},\\ &\forall 0\leqslant t\leqslant T, \qquad u_0,\hat{u}_0\in L^\infty(\Omega)\end{aligned} \tag{4.10}$$

with $C = C(R_0)$ where R_0 is an upper bound of $\|u_0\|_{L^\infty(\Omega)}$, $\|\hat{u}_0\|_{L^\infty(\Omega)}$. Assume there exists an absorbing set B_0 in $L^\infty(\Omega)$ for the semigroup $(S(t))_{t\geq 0}$ and $\tau > 0$ such that $S(\tau)$: $L^\infty(\Omega) \to L^\infty(\Omega)$ is a compact operator. Then, there exists a compact attractor $\mathscr{A}$ in $L^\infty(\Omega)$ for $(S(t))_{t\geq 0}$.

Proof. We set

$$\mathscr{K} = \overline{S(\tau)\, B_0}^{L^\infty(\Omega)}$$

Then, $\mathscr{K}$ is a compact absorbing set in $L^\infty(\Omega)$. Let

$$\mathscr{A} = \omega(\mathscr{K}) = \{v \in L^\infty(\Omega) \colon \exists t_n \to \infty \text{ and } \exists u_n \in \mathscr{K} \text{ such that } S(t_n)\, u_n \to v\}$$

Let us see that $\mathscr{A}$ is attractive in $L^\infty(\Omega)$. Let B be a bounded subset of $L^\infty(\Omega)$. Since $\mathscr{K}$ is absorbing, there exists $t(B) > 0$ such that

$$S(t)\, v \in \mathscr{K} \qquad \text{for all} \quad t \geq t(B) \qquad \text{and} \qquad v \in B$$

Suppose that

$$\sup_{v \in B} \operatorname{dist}(S(t)\, v, \mathscr{A}) \not\to 0 \qquad \text{as} \quad t \to \infty$$

Then, there exist $t_n \to \infty$, $v_n \in B$ and $\varepsilon > 0$ such that

$$\operatorname{dist}(S(t_n)\, v_n, \mathscr{A}) \geq \varepsilon, \qquad \forall n \in \mathbb{N} \tag{4.11}$$

Since we can take $t_n > t(B)$, $S(t_n)\, v_n \in \mathscr{K}$ for all $n \in \mathbb{N}$. Consequently, we can assume that

$$S(t_n)\, v_n \to w \qquad \text{in} \quad L^\infty(\Omega)$$

Therefore $w \in \mathscr{A}$ which is a contradiction with (4.11).

Finally, let us see that $\mathscr{A}$ is an invariant set, i.e., $S(t)\, \mathscr{A} = \mathscr{A}$ for all $t \geq 0$. If $w \in S(t)\, \mathscr{A}$, there exists $v \in \mathscr{A}$ such that $w = S(t)\, v$. Since $v \in \mathscr{A}$, there exist $t_n \to \infty$ and $v_n \in \mathscr{K}$ such that $S(t_n)\, v_n \to v$ in $L^\infty(\Omega)$, consequently $S(t_n)\, v_n \to v$ in $L^1(\Omega)$. Hence, by (4.10)

$$S(t)(S(t_n)\, v_n) \to S(t)\, v = w \qquad \text{in } L^1(\Omega)$$

Now, for n large enough, $S(t + t_n)\, v_n \in \mathscr{K}$. So, we also have

$$S(t + t_n)\, v_n \to w \qquad \text{in } L^\infty(\Omega)$$

Therefore, $w \in \mathscr{A}$.

Reciprocally, given $w \in \mathscr{A}$, there exist $t_n \to \infty$ and $v_n \in \mathscr{K}$ such that

$$S(t_n)\, v_n \to w \qquad \text{in} \quad L^\infty(\Omega)$$

Since $\mathscr{K}$ is a compact absorbing set, we can suppose that $S(t_n - t)\, v_n \in \mathscr{K}$ and

$$S(t_n - t)\, v_n \to v \in \mathscr{A} \qquad \text{in } L^\infty(\Omega)$$

for some v. From where it follows that

$$S(t)(S(t_n - t)\, v_n) \to S(t)\, v \qquad \text{in } L^1(\Omega)$$

and we conclude that $w = S(t)\, v$ and $w \in S(t)\, \mathscr{A}$.

Finally, as a consequence of Proposition 4, Lemmas 5, 7 and 8, we can state the main result of this section.

Theorem 2. *The semigroup $(S(t))_{t \geqslant 0}$ admits a compact global attractor in $L^\infty(\Omega)$.*

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