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A degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions

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We prove existence and uniqueness of weak solutions for a general degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions. Particular instances of this problem appear in various phenomena with changes of phase like the multiphase Stefan problem and in the weak formulation of the mathematical model of the so called Hele–Shaw problem. Also, the problem with nonhomogeneous Neumann boundary conditions is included.

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1. Introduction

The purpose of this paper is to establish the existence and uniqueness of a weak solution for a degenerate elliptic-parabolic problem with nonlinear dynamical boundary condition of the form

$$P_{\gamma,\beta}(f, g, z_0, w_0) \quad \begin{cases} z_t - \operatorname{div} \mathbf{a}(x, Du) = f, \quad z \in \gamma(u), \quad \text{in } Q_T :=]0, T[\times \Omega, \\ w_t + \mathbf{a}(x, Du) \cdot \eta = g, \quad w \in \beta(u), \quad \text{on } S_T :=]0, T[\times \partial\Omega, \\ z(0) = z_0 \quad \text{in } \Omega, \quad w(0) = w_0 \quad \text{in } \partial\Omega, \end{cases}$$

where $T > 0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $z_0 \in L^1(\Omega)$, $w_0 \in L^1(\partial\Omega)$, $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$ and η is the unit outward normal on $\partial\Omega$. Here $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray–Lions conditions. The nonlinearities γ and β are maximal monotone graphs in $\mathbb{R}^2$ (see, e.g., [21]) such that $0 \in \gamma(0)$, $\mathrm{Dom}(\gamma) = \mathbb{R}$, and $0 \in \beta(0)$. In particular, γ and β may be multivalued. This allows one to include the homogeneous Dirichlet boundary condition (taking β to be the monotone graph $\mathsf{D} = \{0\} \times \mathbb{R}$), in which case we consider, in fact, the following problem with static boundary conditions:

$$DP_{\gamma}(f, z_0) \quad \begin{cases} z_t - \operatorname{div} \mathbf{a}(x, Du) = f, \quad z \in \gamma(u), \quad \text{in } Q_T, \\ u = 0 \quad \text{on } S_T, \\ z(0) = z_0 \quad \text{in } \Omega; \end{cases}$$

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and the nonhomogeneous Neumann boundary condition (taking β to be the monotone graph N defined by $\mathsf{N}(r) = 0$ for all $r \in \mathbb{R}$), in which case we consider the problem

$$NP_\gamma(f, g, z_0) \quad \begin{cases} z_t - \operatorname{div} \mathbf{a}(x, Du) = f, \quad z \in \gamma(u), \quad \text{in } Q_T, \\ \mathbf{a}(x, Du) \cdot \eta = g \quad \text{on } S_T, \\ z(0) = z_0 \quad \text{in } \Omega; \end{cases}$$

as well as many other nonlinear fluxes on the boundary that occur in mechanics and physics (see, e.g., [28] or [20]). Note also that, since γ may be multivalued, problems of type $P_{\gamma,\beta}(f, g, z_0, w_0)$ appear in various phenomena with changes of phase like the multiphase Stefan problem ([26]) and in the weak formulation of the mathematical model of the so called Hele–Shaw problem (see [27] and [29]). Moreover, if $\gamma = \mathsf{N}$, we consider the following elliptic problem with nonlinear dynamical boundary conditions:

$$BP_\beta(f, g, w_0) \quad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = f \quad \text{in } Q_T, \\ w_t + \mathbf{a}(x, Du) \cdot \eta = g, \quad w \in \beta(u), \quad \text{on } S_T, \\ w(0) = w_0 \quad \text{in } \partial\Omega. \end{cases}$$

The dynamical boundary conditions, although not too widely considered in the mathematical literature, are very natural in many mathematical models including heat transfer in a solid in contact with a moving fluid, thermoelasticity, diffusion phenomena, problems in fluid dynamics, etc. (see [9], [24], [30], [48] and the references therein). These dynamical boundary conditions also appear in the study of the Stefan problem when the boundary material has large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regarded as the boundary of the domain. For instance, this is the case if one considers an iron ball in which water and ice coexist. For more details about these physical considerations one can see for instance [1]. They also appear in the study of the Hele–Shaw problem. Recall that in [27] the authors give the weak formulation of the problem in the form of a nonlinear degenerate parabolic problem, governed by the Laplace operator and the multivalued Heaviside function, with static boundary conditions. From the physical point of view they assume that the prescribed value of the flux on the boundary is known. But, in some practical situations, it may not be possible to prescribe or to control the exact value of the flux on the boundary. In [47] (see also [49]), the authors consider the case of nonlocal dynamical boundary conditions and use variational methods to solve the problem.

In the present paper, we cover the case of general nonlinear diffusion and local dynamical boundary conditions. Notice that general nonlinear diffusion operators of Leray–Lions type, different from the Laplacian, appear when one deals with non-Newtonian fluids (see, e.g., [8], [42], [43] and the references therein for the case of the Hele–Shaw problem with non-Newtonian fluids). Another interesting application we have in mind concerns the filtration equation with dynamical boundary conditions (see, e.g., [50]), which appears for example in the study of rainfall infiltration through soil, when accumulation of water on the ground surfaces caused by saturation of the surface layer is taken into account. Observe that β may be such that $\operatorname{Ran}(\beta)$ is different from $\mathbb{R}$, so that we cover the case where the boundary conditions are either dynamical or static with respect to the values of w in the problem under consideration. This is the situation where the saturation happens only for values of w in a subinterval of $\mathbb{R}$.

In contrast to the case of Dirichlet boundary conditions (problem $DP_\gamma(f, z_0)$), which is well known (see [2], [4], [16], [18], [22], [39] and the references therein), to our knowledge there is little literature on problems $P_{\gamma,\beta}(f, g, z_0, w_0)$, and the results mostly concern particular nonlinearities and the Laplace operator. For instance, the problem $NP_\gamma(f, g, z_0)$ is treated by Hulshof in [33] in the particular case where γ is a uniformly Lipschitz continuous function, $\gamma(r) = 1$ for $r \in \mathbb{R}^+$, $\gamma \in C^1(\mathbb{R}^-)$, $\gamma' > 0$ on $\mathbb{R}^-$ and $\lim_{r\downarrow-\infty} \gamma(r) = 0$ and for some particular functions g. Kenmochi in [40] considers the same problem in the case $\gamma \in C(\mathbb{R})$ with $\mathrm{Ran}(\gamma)$ a closed bounded interval. The second author of the present paper studies the cases where γ is the Heaviside maximal monotone graph and the case where $\gamma(r) = \exp(r)$ in [34] and [36], respectively. In one space dimension, much more literature exists (see [17] and [51] and the references therein).

For elliptic-parabolic problems with dynamical boundary conditions, the cases in which γ and β are both linear are well known (see, e.g., [32], [30], [31], [44], [3] and the references therein). For the general nonlinear case, that is, when γ and β are maximal monotone graphs, most of the papers in the literature concern the Laplace operator and γ and β with range $\mathbb{R}$ (see [50], [1] and the references therein). The problem becomes more complicated if one of the ranges of γ and β may not be equal to $\mathbb{R}$, and there are few relevant results in the literature. In [35] the case where β is a continuous nondecreasing function (possibly depending on x) and γ is the Heaviside maximal monotone graph, which corresponds to the Hele–Shaw problem, is studied. In [38], the authors consider the homogeneous case, i.e., $f = 0$ and $g = 0$, with γ and β being maximal monotone graphs everywhere defined.

Roughly speaking, in contrast to the Dirichlet boundary condition, for the nonhomogeneous Neumann boundary condition and/or dynamical boundary conditions, the problem is noncoercive, and moreover, the conservation of mass exhibits a necessary condition for the existence of a solution related to the ranges of the nonlinearities γ and β (see (6)). Indeed, prescribing the value of u on some part of the lateral boundary, one can control the Sobolev norm of the solution in the interior of Ω by the L^p norm of the gradient in Ω. This is not possible in the case of purely Neumann boundary conditions or dynamical ones, and some substitute for this kind of argument has to be found. In the case where the nonlinearities have ranges equal to $\mathbb{R}$ and assuming additional assumptions on f and g one can obtain L^∞-estimates for the solutions (see, e.g., [33] and [38]). If one of the ranges is not equal to $\mathbb{R}$, the L^∞-estimates are lost and the existence of solutions becomes complicated.

Another main difficulty when dealing with doubly nonlinear parabolic problems is the uniqueness. For the Laplace operator, thanks to the linearity of the operator, the problem can be solved by using suitable test functions with respect to u (see, e.g., [38]). For nonlinear operators this kind of argument cannot be used. In [16], for an elliptic-parabolic problem with Dirichlet boundary conditions, it is shown that the notion of integral solution ([10]) is a very useful tool to prove uniqueness (see also [37] for nonhomogeneous and time dependent Neumann boundary conditions). For general nonlinearities, even for homogeneous Dirichlet boundary condition, the question of uniqueness is more difficult and most of the arguments used in the literature are based on doubling variable methods (see, e.g., [22], [23], [39], [18], [5] and the references therein). In this paper we show that applying the notion of integral solution simplifies the proof of uniqueness, which is obtained without using doubling variable methods. Moreover with this technique uniqueness is proved without any assumption on the jumps of γ and β.

We also make use of nonlinear semigroup theory ([14], [52]). So we need to consider the elliptic problem

$$(S^{\gamma,\beta}_{\phi,\psi}) \qquad \begin{cases} -\mathrm{div}\, \mathbf{a}(x, Du) + \gamma(u) \ni \phi & \text{in } \Omega, \\ \mathbf{a}(x, Du) \cdot \eta + \beta(u) \ni \psi & \text{on } \partial\Omega. \end{cases}$$

In [6], under rather general assumptions, existence of solutions and a contraction principle for problem $(S^{\gamma,\beta}_{\phi,\psi})$ are obtained. Using these results we prove the existence of mild solutions for the associated Cauchy problem. In principle it is not clear how these mild solutions have to be interpreted for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. Under some additional natural conditions (see Theorem 3.3), we show that mild solutions are weak solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. To get uniqueness, we show that weak solutions are integral solutions.

Let us briefly summarize the content of the paper. In Section 2 we fix the notation and give some preliminaries; we also introduce the concept of weak solution for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ and state the existence and uniqueness result for weak solutions of $P_{\gamma,\beta}(f, g, z_0, w_0)$ and a contraction principle satisfied by weak solutions. In Section 3 we study the problem from the point of view of nonlinear semigroup theory. In Section 4 we prove the existence of weak solutions and in Section 5 we obtain a contraction principle. Finally, in the appendix we give the proof of the characterization of the closure of the domain of the operator associated to our problem.

2. Preliminaries and main result

In this section, after some preliminaries, we give the concept of weak solution for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ and we state the existence and uniqueness result for this type of solution.

Throughout the paper, $\Omega \subset \mathbb{R}$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 1$, γ and β are maximal monotone graphs in $\mathbb{R}^2$ such that $\mathrm{Dom}(\gamma) = \mathbb{R}$, $0 \in \gamma(0) \cap \beta(0)$, and the Carathéodory function $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies

(H_1) there exists $\lambda > 0$ such that $\mathbf{a}(x, \xi) \cdot \xi \geqslant \lambda |\xi|^p$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$,

(H_2) there exist $c > 0$ and $\varrho \in L^{p'}(\Omega)$ such that $|\mathbf{a}(x, \xi)| \leqslant \sigma(\varrho(x) + |\xi|^{p-1})$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = p/(p-1)$,

(H_3) $(\mathbf{a}(x, \xi) - \mathbf{a}(x, \eta)) \cdot (\xi - \eta) > 0$ for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$.

The hypotheses (H_1)–(H_3) are classical in the study of nonlinear operators in divergence form (see, e.g., [46] or [11]). The model example of a function $\mathbf{a}$ satisfying these hypotheses is $\mathbf{a}(x, \xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \mathrm{div}(|Du|^{p-2}Du)$.

We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbb{R}^N$ or its $(N-1)$-Hausdorff measure. For $1 \leqslant q < +\infty$, $L^q(\Omega)$ and $W^{1,q}(\Omega)$ denote respectively the standard Lebesgue and Sobolev spaces, and $W^{1,q}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,q}(\Omega)$. For $u \in W^{1,q}(\Omega)$, we denote by u or $\mathtt{tr}(u)$ the trace of u on $\partial\Omega$ in the usual sense. Recall that $\mathtt{tr}(W^{1,q}(\Omega)) = W^{1/q',q}(\partial\Omega)$ and $\mathrm{Ker}(\mathtt{tr}) = W^{1,q}_0(\Omega)$.

We introduce the sets

$$V^{1,q}(\Omega) := \left\{\phi \in L^1(\Omega) : \exists M > 0 \text{ such that } \int_\Omega |\phi v| \leqslant M\|v\|_{W^{1,q}(\Omega)}\ \forall v \in W^{1,q}(\Omega)\right\}$$

and

$$V^{1,q}(\partial\Omega) := \left\{\psi \in L^1(\partial\Omega) : \exists M > 0 \text{ such that } \int_{\partial\Omega} |\psi v| \leqslant M\|v\|_{W^{1,q}(\Omega)}\ \forall v \in W^{1,q}(\Omega)\right\}.$$

$V^{1,q}(\Omega)$ is a Banach space endowed with the norm

$$\|\phi\|_{V^{1,q}(\Omega)} := \inf\left\{M > 0 : \int_\Omega |\phi v| \leqslant M\|v\|_{W^{1,q}(\Omega)}\ \forall v \in W^{1,q}(\Omega)\right\},$$

and $V^{1,q}(\partial\Omega)$ is a Banach space endowed with the norm

$$\|\psi\|_{V^{1,q}(\partial\Omega)} := \inf\left\{M > 0 : \int_{\partial\Omega} |\psi v| \leqslant M\|v\|_{W^{1,q}(\Omega)}\ \forall v \in W^{1,q}(\Omega)\right\}.$$

Observe that Sobolev embeddings and trace theorems imply, for $1 \leqslant q < N$,

$$L^{q'}(\Omega) \subset L^{(Nq/(N-q))'}(\Omega) \subset V^{1,q}(\Omega)$$

and

$$L^{q'}(\partial\Omega) \subset L^{((N-1)q/(N-q))'}(\partial\Omega) \subset V^{1,q}(\partial\Omega).$$

Also,

$$V^{1,q}(\Omega) = L^1(\Omega) \quad \text{and} \quad V^{1,q}(\partial\Omega) = L^1(\partial\Omega) \qquad \text{when } q > N,$$

$$L^q(\Omega) \subset V^{1,N}(\Omega) \quad \text{and} \quad L^q(\partial\Omega) \subset V^{1,N}(\partial\Omega) \qquad \text{for any } q > 1.$$

We say that $\mathbf{a}$ is *smooth* (see [7] and [6]) when, for any $\phi \in L^\infty(\Omega)$ such that there exists a bounded weak solution u of the homogeneous Dirichlet problem

$$(D) \qquad \begin{cases} -\operatorname{div}\mathbf{a}(x, Du) = \phi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

there exists $\psi \in L^1(\partial\Omega)$ such that u is also a weak solution of the Neumann problem

$$(N) \qquad \begin{cases} -\operatorname{div}\mathbf{a}(x, Du) = \phi & \text{in } \Omega, \\ \mathbf{a}(x, Du)\cdot\eta = \psi & \text{on } \partial\Omega. \end{cases}$$

Functions $\mathbf{a}$ corresponding to linear operators with smooth coefficients and p-Laplacian type operators are smooth (see [20] and [45]). In [6], we prove that $\mathbf{a}$ is smooth if and only if for any $\phi \in V^{1,p}(\Omega)$ there exists $\psi = T(\phi) \in V^{1,p}(\partial\Omega)$ such that the weak solution u of (D) is a weak solution of (N). Moreover,

$$\int_\Omega (T(\phi_1) - T(\phi_2))^+ \leqslant \int_\Omega (\phi_1 - \phi_2)^+ \quad \text{for all } \phi_1, \phi_2 \in V^{1,p}(\Omega).$$

For a maximal monotone graph ϑ in $\mathbb{R}\times\mathbb{R}$ the *main section* ϑ^0 of ϑ is defined by

$$\vartheta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \vartheta(s) & \text{if } \vartheta(s) \neq \emptyset, \\ +\infty \quad \text{if } [s, +\infty) \cap \operatorname{Dom}(\vartheta) = \emptyset, \\ -\infty \quad \text{if } (-\infty, s] \cap \operatorname{Dom}(\vartheta) = \emptyset. \end{cases}$$

We write $\vartheta_- := \inf \operatorname{Ran}(\vartheta)$ and $\vartheta_+ := \sup \operatorname{Ran}(\vartheta)$. If $0 \in \operatorname{Dom}(\vartheta)$, then $j_\vartheta(r) = \int_0^r \vartheta^0(s)\,\mathrm{d}s$ defines a convex l.s.c. function such that $\vartheta = \partial j_\vartheta$. If j_ϑ^* is the Legendre transform of j_ϑ then $\vartheta^{-1} = \partial j_\vartheta^*$.

In [13] the following relation for $u, v \in L^1(\Omega)$ is defined: $u \ll v$ if

$$\int_\Omega (u-k)^+ \leqslant \int_\Omega (v-k)^+ \quad \text{and} \quad \int_\Omega (u+k)^- \leqslant \int_\Omega (v+k)^- \quad \text{for any } k > 0,$$

and the following facts are proved.

PROPOSITION 2.1 Let Ω be a bounded domain in $\mathbb{R}^N$.

(i) If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_q \leqslant \|v\|_q$ for any $q \in [1, +\infty]$.
(ii) If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.

As said in the introduction, our aim is to study the existence and uniqueness of a weak solution for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. The concept of weak solution we have in mind is the following.

DEFINITION 2.2 Given $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$, a *weak solution* of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$ is a couple (z, w) such that $z \in C([0, T]; L^1(\Omega))$, $w \in C([0, T]; L^1(\partial\Omega))$, $z(0) = z_0$, $w(0) = w_0$ and there exists $u \in L^p(0, T; W^{1,p}(\Omega))$ such that $z \in \gamma(u)$ a.e. in Q_T, $w \in \beta(u)$ a.e. on S_T and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z(t)\xi + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} w(t)\xi + \int_\Omega \mathbf{a}(x, Du(t)) \cdot D\xi = \int_\Omega f(t)\xi + \int_{\partial\Omega} g(t)\xi \quad \text{in } \mathcal{D}'(]0, T[) \tag{1}$$

for any $\xi \in C^1(\overline{\Omega})$.

REMARK 2.3 Observe that taking $\xi = 1$ in the above definition, we get

$$\int_\Omega z(t) + \int_{\partial\Omega} w(t) = \int_\Omega z_0 + \int_{\partial\Omega} w_0 + \int_0^t \left(\int_\Omega f + \int_{\partial\Omega} g \right) \quad \forall t \in [0, T]. \tag{2}$$

Recall that in the case $\beta = 0$, for the Laplacian operator and γ the multivalued Heaviside function (i.e., for the Hele–Shaw problem), existence and uniqueness of weak solutions for this problem is known to hold only if

$$\int_\Omega z_0 + \int_0^t \left(\int_\Omega f + \int_{\partial\Omega} g \right) \in]0, |\Omega|[\quad \text{for any } t \in [0, T[$$

(see [34] or [40])). For the maximal monotone graphs γ and β, we set

$$\mathcal{R}^+_{\gamma,\beta} := \gamma_+|\Omega| + \beta_+|\partial\Omega|, \qquad \mathcal{R}^-_{\gamma,\beta} := \gamma_-|\Omega| + \beta_-|\partial\Omega|.$$

We suppose $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$ and we write $\mathcal{R}_{\gamma,\beta} :=]\mathcal{R}^-_{\gamma,\beta}, \mathcal{R}^+_{\gamma,\beta}[$.

The main results of this paper are the following contraction principle and the following existence and uniqueness theorem.

THEOREM 2.4 Let $T > 0$. For $i = 1, 2$, let $f_i \in L^1(0, T; L^1(\Omega))$, $g_i \in L^1(0, T; L^1(\partial\Omega))$, $z_{i0} \in L^1(\Omega)$ and $w_{i0} \in L^1(\partial\Omega)$; let (z_i, w_i) be a weak solution in $[0, T]$ of problem $P_{\gamma,\beta}(f_i, g_i, z_{i0}, w_{i0})$, $i = 1, 2$. Then

$$\begin{aligned}\int_\Omega (z_1(t) - z_2(t))^+ + \int_{\partial\Omega} (w_1(t) - w_2(t))^+ \leqslant \int_\Omega (z_{10} - z_{20})^+ + \int_{\partial\Omega} (w_{10} - w_{20})^+ \\ + \int_0^t \int_\Omega (f_1(\tau) - f_2(\tau))^+ \,\mathrm{d}\tau + \int_0^t \int_{\partial\Omega} (g_1(\tau) - g_2(\tau))^+ \,\mathrm{d}\tau\end{aligned} \tag{3}$$

for almost every $t \in]0, T[$.

THEOREM 2.5 Assume $\mathrm{Dom}(\gamma) = \mathbb{R}$, $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$, and $\mathrm{Dom}(\beta) = \mathbb{R}$ or **a** smooth. Let $T > 0$. Let $f \in L^{p'}(0, T; L^{p'}(\Omega))$, $g \in L^{p'}(0, T; L^{p'}(\partial\Omega))$, $z_0 \in L^{p'}(\Omega)$ and $w_0 \in L^{p'}(\partial\Omega)$ be such that

$$\gamma_- \leqslant z_0 \leqslant \gamma_+, \qquad \beta_- \leqslant w_0 \leqslant \beta_+, \tag{4}$$

$$\int_\Omega j^*_\gamma(z_0) + \int_\Gamma j^*_\beta(w_0) < +\infty, \tag{5}$$

and

$$\int_\Omega z_0 + \int_{\partial\Omega} w_0 + \int_0^t \left(\int_\Omega f + \int_{\partial\Omega} g \right) \in \mathcal{R}_{\gamma,\beta} \quad \forall t \in [0, T]. \tag{6}$$

Then there exists a unique weak solution (z, w) of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$.

The uniqueness part of Theorem 2.5 follows from Theorem 2.4. To prove Theorem 2.4 and the existence part of Theorem 2.5 we shall use the theory of nonlinear semigroups (see, e.g., [10], [14] or [25]). We will show the existence of a mild solution and we will prove that it is a weak solution of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. To prove the contraction principle we will show that weak solutions are integral solutions. To do this we need to rewrite $P_{\gamma,\beta}(f, g, z_0, w_0)$ as an abstract Cauchy problem and to use the results obtained in [6] for the associated elliptic problem.

3. Mild solutions

First let us recall some basic facts for the elliptic problem $(S^{\gamma,\beta}_{\phi,\psi})$ given in [6], which will be crucial for the proof of our main results. In [6] the following concept of solution for problem $(S^{\gamma,\beta}_{\phi,\psi})$ is introduced.

DEFINITION 3.1 Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$. A triple of functions $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a *weak solution* of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v \tag{7}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

Observe that if $(S^{\gamma,\beta}_{\phi,\psi})$ has a weak solution, then necessarily ϕ and ψ must satisfy

$$\mathcal{R}^-_{\gamma,\beta} \leqslant \int_{\partial\Omega} \psi + \int_\Omega \phi \leqslant \mathcal{R}^+_{\gamma,\beta}.$$

Indeed, by taking $v = 1$ as a test function in (7), we get

$$\int_\Omega z + \int_{\partial\Omega} w = \int_{\partial\Omega} \psi + \int_\Omega \phi.$$

Moreover the following existence and uniqueness results about weak solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$ have been obtained in [6].

THEOREM 3.2 (i) Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$, and let $[u_1, z_1, w_1]$ and $[u_2, z_2, w_2]$ be weak solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Then there exists a constant $c \in \mathbb{R}$ such that

$$\begin{aligned} u_1 - u_2 = c & \quad \text{a.e. in } \Omega, \\ z_1 - z_2 = 0 & \quad \text{a.e. in } \Omega, \\ w_1 - w_2 = 0 & \quad \text{a.e. in } \partial\Omega. \end{aligned}$$

Moreover, if $c \neq 0$, there exists a constant $k \in \mathbb{R}$ such that $z_1 = z_2 = k$.

(ii) For any weak solution $[u_1, z_1, w_1]$ of problem $(S^{\gamma,\beta}_{\phi_1,\psi_1})$, $\phi_1 \in L^1(\Omega)$, $\psi_1 \in L^1(\partial\Omega)$, and any weak solution $[u_2, z_2, w_2]$ of problem $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, $\phi_2 \in L^1(\Omega)$, $\psi_2 \in L^1(\partial\Omega)$, we have

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leqslant \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

Theorem 3.2(ii) is given in [6] in a different way. With the technique used in Section 5 we can get exactly the above result.

THEOREM 3.3 Assume $\mathrm{Dom}(\gamma) = \mathbb{R}$, and $\mathrm{Dom}(\beta) = \mathbb{R}$ or **a** smooth. For any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ with

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta}, \tag{8}$$

there exists a weak solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Moreover $z \in V^{1,p}(\Omega)$, $w \in V^{1,p}(\partial\Omega)$ and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v \quad \text{for any } v \in W^{1,p}(\Omega).$$

These results imply that the natural space to study problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ from the point of view of nonlinear semigroup theory is $X = L^1(\Omega) \times L^1(\partial\Omega)$ provided with the natural norm

$$\|(f, g)\| := \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$

In this space we define the following operator

$$\mathcal{B}^{\gamma,\beta} := \{((z, w), (\hat{z}, \hat{w})) \in X \times X : \exists u \in W^{1,p}(\Omega) \text{ such that } [u, z, w] \text{ is a weak solution of } (S^{\gamma,\beta}_{z+\hat{z}, w+\hat{w}})\},$$

in other words, $(\hat{z}, \hat{w}) \in \mathcal{B}^{\gamma,\beta}(z, w)$ if and only if there exists $u \in W^{1,p}(\Omega)$ such that $z(x) \in \gamma(u(x))$ a.e. in Ω, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv = \int_\Omega \hat{z} v + \int_{\partial\Omega} \hat{w} v \tag{9}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, which allows us to rewrite problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ as the following abstract Cauchy problem in X:

$$\begin{cases} V'(t) + \mathcal{B}^{\gamma,\beta}(V(t)) \ni (f, g), & t \in]0, T[, \\ V(0) = (z_0, w_0). \end{cases} \tag{10}$$

A direct consequence of Theorems 3.2 and 3.3 is the following result.

COROLLARY 3.4 The operator $\mathcal{B}^{\gamma,\beta}$ is T-accretive in X and, assuming $\mathrm{Dom}(\gamma) = \mathbb{R}$, and $\mathrm{Dom}(\beta) = \mathbb{R}$ or $\mathbf{a}$ smooth, it satisfies the following range condition:

$$\left\{(\phi, \psi) \in V^{1,p}(\Omega) \times V^{1,p}(\partial\Omega) : \int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta}\right\} \subset \mathrm{Ran}(I + \mathcal{B}^{\gamma,\beta}).$$

Moreover, we can characterize $\overline{D(\mathcal{B}^{\gamma,\beta})}^{L^1(\Omega)\times L^1(\partial\Omega)}$ as follows.

THEOREM 3.5 Under the hypothesis $\mathrm{Dom}(\gamma) = \mathbb{R}$, and $\mathrm{Dom}(\beta) = \mathbb{R}$ or $\mathbf{a}$ smooth, we have

$$\overline{D(\mathcal{B}^{\gamma,\beta})}^{L^1(\Omega)\times L^1(\partial\Omega)} = \{(z, w) \in L^1(\Omega) \times L^1(\partial\Omega) : \gamma_- \leqslant z \leqslant \gamma_+,\ \beta_- \leqslant w \leqslant \beta_+\}.$$

The proof of this theorem is quite technical and given in the Appendix.

The above results allow us to prove the following theorem concerning mild solutions.

THEOREM 3.6 Let $T > 0$. Under the hypothesis $\mathrm{Dom}(\gamma) = \mathbb{R}$, and $\mathrm{Dom}(\beta) = \mathbb{R}$ or $\mathbf{a}$ smooth, for any $z_0 \in L^1(\Omega)$, $w_0 \in L^1(\partial\Omega)$ and any $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$ satisfying (4) and (6), there exists a unique mild solution of (10).

Proof. For $n \in \mathbb{N}$, let $\epsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \epsilon$, $f_1^\epsilon, \ldots, f_n^\epsilon \in L^{p'}(\Omega)$, $g_1^\epsilon, \ldots, g_n^\epsilon \in L^{p'}(\partial\Omega)$, $w_0^\epsilon \in L^{p'}(\Omega)$, $z_0^\epsilon \in L^{p'}(\partial\Omega)$ with

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\|f(t) - f_i^\epsilon\|_{L^1(\Omega)} + \|g(t) - g_i^\epsilon\|_{L^1(\partial\Omega)})\, \mathrm{d}t \leqslant \epsilon$$

and

$$\|z_0^\epsilon - z_0\|_{L^1(\Omega)} + \|w_0^\epsilon - w_0\|_{L^1(\partial\Omega)} \leqslant \epsilon.$$

If we set

$$f_\epsilon(t) = f_i^\epsilon, \quad g_\epsilon(t) = g_i^\epsilon \quad \text{for } t \in\,]t_{i-1}, t_i],\ i = 1, \ldots, n,$$

it follows that

$$\int_0^T (\|f(t) - f_\epsilon(t)\|_{L^1(\Omega)} + \|g(t) - g_\epsilon(t)\|_{L^1(\partial\Omega)})\, \mathrm{d}t \leqslant \epsilon. \tag{11}$$

By Theorem 3.3 applied recursively, for n large enough, there exists a weak solution $[u_i^\epsilon, z_i^\epsilon, w_i^\epsilon]$ of

$$\begin{cases} \gamma(u_i^\epsilon) - \epsilon \operatorname{div} \mathbf{a}(x, Du_i^\epsilon) \ni \epsilon f_i^\epsilon + z_{i-1}^\epsilon, \\ \epsilon \mathbf{a}(x, Du_i^\epsilon) \cdot \eta + \beta(u_i^\epsilon) \ni \epsilon g_i^\epsilon + w_{i-1}^\epsilon, \end{cases} \tag{12}$$

for $i = 1, \ldots, n$. That is, there exists a unique solution $(z_i^\epsilon, w_i^\epsilon) \in X$ of the time discretized scheme associated with (10),

$$(z_i^\epsilon, w_i^\epsilon) + \epsilon \mathcal{B}^{\gamma,\beta}(z_i^\epsilon, w_i^\epsilon) \ni \epsilon(f_i^\epsilon, g_i^\epsilon) + (z_{i-1}^\epsilon, w_{i-1}^\epsilon) \quad \text{for } i = 1, \ldots, n. \tag{13}$$

Observe that to apply Theorem 3.3 we need to know that (8) holds in every step. Indeed, for the first step this is straightforward. For the other steps, we use the following argument. For each $i = 1, \ldots, n$, we have $[u_i^\epsilon, z_i^\epsilon, w_i^\epsilon] \in W^{1,p}(\Omega) \times V^{1,p}(\Omega) \times V^{1,p}(\partial\Omega)$ and

$$\int_\Omega \mathbf{a}(x, Du_i^\epsilon) \cdot Dv + \int_\Omega \frac{z_i^\epsilon - z_{i-1}^\epsilon}{\epsilon} v + \int_{\partial\Omega} \frac{w_i^\epsilon - w_{i-1}^\epsilon}{\epsilon} v = \int_\Omega f_i^\epsilon v + \int_{\partial\Omega} g_i^\epsilon v \tag{14}$$

for all $v \in W^{1,p}(\Omega)$. Therefore, taking $v = 1$ in (14), we have

$$\int_\Omega z_i^\epsilon + \int_{\partial\Omega} w_i^\epsilon = \epsilon\left(\int_\Omega f_i^\epsilon + \int_{\partial\Omega} g_i^\epsilon\right) + \int_\Omega z_{i-1}^\epsilon + \int_{\partial\Omega} w_{i-1}^\epsilon. \tag{15}$$

It follows that

$$\int_\Omega z_i^\epsilon + \int_{\partial\Omega} w_i^\epsilon = \epsilon \sum_{j=1}^{i}\left(\int_\Omega f_j^\epsilon + \int_{\partial\Omega} g_j^\epsilon\right) + \int_\Omega z_0^\epsilon + \int_{\partial\Omega} w_0^\epsilon,$$

and if we take n large enough, condition (8) is always satisfied.

Therefore, if we define $V_\epsilon(t) = (z_\epsilon(t), w_\epsilon(t))$ by

$$\begin{cases} z_\epsilon(0) = z_0^\epsilon, \quad w_\epsilon(0) = w_0^\epsilon, \\ z_\epsilon(t) = z_i^\epsilon, \quad w_\epsilon(t) = w_i^\epsilon \quad \text{for } t \in]t_{i-1}, t_i],\ i = 1, \ldots, n, \end{cases} \tag{16}$$

it is an ϵ-approximate solution of problem (10).

Now by nonlinear semigroup theory (see [10], [14], [25]), on account of Corollary 3.4 and Theorem 3.5, problem (10) has a unique mild solution $V(t) = (z(t), w(t)) \in C([0, T]; X)$, $z(t) = L^1(\Omega)\text{-}\lim_{\epsilon\to 0} z_\epsilon(t)$ and $w(t) = L^1(\partial\Omega)\text{-}\lim_{\epsilon\to 0} w_\epsilon(t)$ uniformly for $t \in [0, T]$. □

In principle, it is not clear how these mild solutions have to be interpreted for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. We will see that under the hypothesis of Theorem 2.5 they are weak solutions of $P_{\gamma,\beta}(f, g, z_0, w_0)$, which proves the existence part of that theorem.

4. Existence of weak solutions

As said in the previous section, the existence part of Theorem 2.5 is shown by proving that the mild solution of problem (10) is a weak solution of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ whenever the assumptions of Theorem 2.5 are fulfilled. Before giving the proof we need to prove some technical lemmas.

4.1 *Preparatory lemmas*

We shall use the following integration by parts lemma in the proof of the existence part and in the proof of the contraction principle. We denote by $(\cdot, \cdot)$ the pairing between $(W^{1,p}(\Omega))'$ and $W^{1,p}(\Omega)$.

LEMMA 4.1 Let $(z, w) \in C([0, T]; L^1(\Omega)) \times L^1(\partial\Omega))$ and $F \in L^{p'}(0, T; (W^{1,p}(\Omega))')$ be such that

$$\int_0^T \int_\Omega z(t)\psi_t + \int_0^T \int_{\partial\Omega} w(t)\psi_t = \int_0^T (F(t), \psi(t))\,\mathrm{d}t \tag{17}$$

for any $\psi \in W^{1,1}(0, T; W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$ with $\psi(0) = \psi(T) = 0$. Then

$$\int_0^T \int_\Omega \left(\int_0^{z(t)} H(\cdot, (\gamma^{-1})^0(s))\,\mathrm{d}s\right)\psi_t + \int_0^T \int_{\partial\Omega}\left(\int_0^{w(t)} H(\cdot, (\beta^{-1})^0(s))\,\mathrm{d}s\right)\psi_t$$
$$= \int_0^T (F(t), H(\cdot, u(t))\psi(t))\,\mathrm{d}t,$$

for any $u \in L^p(0,T;W^{1,p}(\Omega))$ with $z \in \gamma(u)$ a.e. in Q_T, $w \in \beta(u)$ a.e. in S_T, for any $\psi \in \mathcal{D}(]0,T[\times\mathbb{R}^N)$, and for any Carathéodory function $H : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $H(x,r)$ is nondecreasing in r, $H(\cdot,u) \in L^p(0,T;W^{1,p}(\Omega))$, $\int_0^z H(x,(\gamma^{-1})^0(s))\,\mathrm{d}s \in L^1(Q_T)$ and $\int_0^w H(x,(\beta^{-1})^0(s))\,\mathrm{d}s \in L^1(S_T)$.

Proof. The proof is similar to the one given in [23] for Dirichlet boundary conditions. We give it here for the sake of completeness.

Let $\psi \in \mathcal{D}(]0,T[\times\mathbb{R}^N)$, $\psi \geqslant 0$, and for $H_\tau = T_{1/\tau}H$, $\tau > 0$, let

$$\eta_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau} H_\tau(\cdot,u(s))\psi(s)\,\mathrm{d}s.$$

Then η_τ can be used as a test function in (17) and therefore

$$\begin{aligned}
\int_0^T (F(t),\eta_\tau(t))\,\mathrm{d}t &= \int_0^T\int_\Omega z(t)(\eta_\tau)_t + \int_0^T\int_{\partial\Omega} w(t)(\eta_\tau)_t \\
&= \int_0^T\int_\Omega z(t)\frac{H_\tau(\cdot,u(t+\tau))\psi(t+\tau) - H_\tau(\cdot,u(t))\psi(t)}{\tau} \\
&\quad + \int_0^T\int_{\partial\Omega} w(t)\frac{H_\tau(\cdot,u(t+\tau))\psi(t+\tau) - H_\tau(\cdot,u(t))\psi(t)}{\tau} \\
&= \int_0^T\int_\Omega \frac{z(t-\tau)-z(t)}{\tau}H_\tau(\cdot,u(t))\psi(t) \\
&\quad + \int_0^T\int_{\partial\Omega} \frac{w(t-\tau)-w(t)}{\tau}H_\tau(\cdot,u(t))\psi(t).
\end{aligned}$$

Now, since

$$\begin{aligned}
&H_\tau(\cdot,\gamma^{-1}(r)) \subset \partial\left(\int_0^r H_\tau(\cdot,(\gamma^{-1})^0(s))\,\mathrm{d}s\right),\\
&H_\tau(\cdot,\beta^{-1}(r)) \subset \partial\left(\int_0^r H_\tau(\cdot,(\beta^{-1})^0(s))\,\mathrm{d}s\right),\\
&H_\tau(\cdot,u(t)) \in H_\tau(\cdot,\gamma^{-1}(z(t))) \qquad \text{a.e. in } \Omega,\\
&H_\tau(\cdot,u(t)) \in H_\tau(\cdot,\beta^{-1}(w(t))) \qquad \text{a.e. on } \partial\Omega,
\end{aligned}$$

we have

$$\begin{aligned}
(z(t-\tau)-z(t))H_\tau(\cdot,u(t)) &\leqslant \int_{z(t)}^{z(t-\tau)} H_\tau(\cdot,(\gamma^{-1})^0(s))\,\mathrm{d}s \qquad \text{a.e. in } \Omega,\\
(w(t-\tau)-w(t))H_\tau(\cdot,u(t)) &\leqslant \int_{w(t)}^{w(t-\tau)} H_\tau(\cdot,(\beta^{-1})^0(s))\,\mathrm{d}s \qquad \text{a.e. on } \partial\Omega.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^T (F(t),\eta_\tau(t))\,\mathrm{d}t \leqslant{}& \int_0^T\int_\Omega \frac{1}{\tau}\int_{z(t)}^{z(t-\tau)} H_\tau(\cdot,(\gamma^{-1})^0(s))\,\mathrm{d}s\,\psi(t)\\
&+ \int_0^T\int_{\partial\Omega} \frac{1}{\tau}\int_{w(t)}^{w(t-\tau)} H_\tau(\cdot,(\beta^{-1})^0(s))\,\mathrm{d}s\,\psi(t)
\end{aligned}$$

$$= \int_0^T \int_\Omega \int_0^{z(t)} H_\tau(\cdot, (\gamma^{-1})^0(s))\,\mathrm{d}s\, \frac{\psi(t+\tau)-\psi(t)}{\tau} + \int_0^T \int_{\partial\Omega} \int_0^{w(t)} H_\tau(\cdot, (\beta^{-1})^0(s))\,\mathrm{d}s\, \frac{\psi(t+\tau)-\psi(t)}{\tau}.$$

Letting $\tau \to 0^+$ we get

$$\int_0^T (F(t), H(\cdot, u(t))\psi(t))\,\mathrm{d}t \leqslant \int_0^T \int_\Omega \left(\int_0^{z(t)} H(x, (\gamma^{-1})^0(s))\,\mathrm{d}s\right)\psi_t + \int_0^T \int_{\partial\Omega} \left(\int_0^{w(t)} H(x, (\beta^{-1})^0(s))\,\mathrm{d}s\right)\psi_t.$$

Taking now $\tilde{\eta}_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau} H_\tau(\cdot, u(s-\tau))\psi(s)\,\mathrm{d}s$, and arguing as above we get the other inequality. □

To prove the existence of weak solutions we shall also use the following result.

LEMMA 4.2 Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$, $\{z_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$, $\{w_n\}_{n\in\mathbb{N}} \subset L^1(\partial\Omega)$ be such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ a.e. in Ω and $w_n \in \beta(u_n)$ a.e. in $\partial\Omega$. Suppose that

(i) if $\mathcal{R}^+_{\gamma,\beta} = +\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^+ + \int_{\partial\Omega} w_n^+ < M \qquad \forall n \in \mathbb{N},$$

(ii) if $\mathcal{R}^+_{\gamma,\beta} < +\infty$, there exist $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n + \int_{\partial\Omega} w_n < M < \mathcal{R}^+_{\gamma,\beta} \qquad \forall n \in \mathbb{N}$$

and

$$\max\left\{\int_{\{x\in\Omega\,:\,z_n(x)<-h\}} |z_n|, \int_{\{x\in\partial\Omega\,:\,w_n(x)<-h\}} |w_n|\right\} < \frac{\mathcal{R}^+_{\gamma,\beta} - M}{8} \qquad \forall n \in \mathbb{N}.$$

Then there exists a constant $C = C(M)$ in case (i), and $C = C(M, h)$ in case (ii), such that

$$\|u_n^+\|_{L^p(\Omega)} \leqslant C(\|Du_n^+\|_{L^p(\Omega)} + 1) \qquad \forall n \in \mathbb{N}.$$

In order to prove Lemma 4.2, we use the following well known result (see [53]).

LEMMA 4.3 (i) There exists a constant $C(\Omega, N, p)$ such that, for any $K \subset \Omega$ with $|K| > 0$,

$$\|u\|_{L^p(\Omega)} \leqslant \frac{C(\Omega, N, p)}{|K|^{1/p}}(\|Du\|_{L^p(\Omega)} + \|u\|_{L^p(K)}) \qquad \forall u \in W^{1,p}(\Omega). \tag{18}$$

(ii) There exists a constant $\hat{C}(\Omega, N, p)$ such that, for any $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$,

$$\|u\|_{L^p(\Omega)} \leqslant \frac{\hat{C}(\Omega, N, p)}{|\Gamma|^{1/p}}(\|Du\|_{L^p(\Omega)} + \|u\|_{L^p(\Gamma)}) \qquad \forall u \in W^{1,p}(\Omega). \tag{19}$$

Proof of Lemma 4.2. Assume first that $\mathcal{R}^+_{\gamma,\beta} = +\infty$. Then $\gamma^+ = +\infty$ or $\beta^+ = +\infty$. Suppose first that $\gamma^+ = +\infty$. Then, by assumption, there exists $M > 0$ such that

$$\int_\Omega z_n^+ < M \qquad \forall n \in \mathbb{N}.$$

Let $K_n = \{x \in \Omega : z_n^+(x) < 2M/|\Omega|\}$ for every $n \in \mathbb{N}$. Then

$$0 \leqslant \int_{K_n} z_n^+ = \int_\Omega z_n^+ - \int_{\Omega\setminus K_n} z_n^+ \leqslant M - (|\Omega| - |K_n|)\frac{2M}{|\Omega|} = |K_n|\frac{2M}{|\Omega|} - M.$$

Therefore, $|K_n| \geqslant |\Omega|/2$, and

$$\|u_n^+\|_{L^p(K_n)} \leqslant |K_n|^{1/p} \sup \gamma^{-1}\left(\frac{2M}{|\Omega|}\right).$$

Then, by Lemma 4.3, for all $n \in \mathbb{N}$,

$$\|u_n^+\|_{L^p(\Omega)} \leqslant C(\Omega, N, p)\left(\left(\frac{2}{|\Omega|}\right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \gamma^{-1}\left(\frac{2M}{|\Omega|}\right)\right).$$

If $\beta^+ = +\infty$, we similarly get, for all $n \in \mathbb{N}$,

$$\|u_n^+\|_{L^p(\Omega)} \leqslant \hat{C}(\Omega, N, p)\left(\left(\frac{2}{|\partial\Omega|}\right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \beta^{-1}\left(\frac{2M}{|\partial\Omega|}\right)\right),$$

where $\hat{C}(\Omega, N, p)$ is given in Lemma 4.3.

Now assume that $\mathcal{R}^+_{\gamma,\beta} < +\infty$, and let $\delta = \mathcal{R}^+_{\gamma,\beta} - M$. Then, by assumption,

$$\int_\Omega z_n + \int_{\partial\Omega} w_n < \mathcal{R}^+_{\gamma,\beta} - \delta.$$

Consequently, for every $n \in \mathbb{N}$,

$$\int_\Omega z_n < \gamma^+|\Omega| - \frac{\delta}{2} \tag{20}$$

or

$$\int_{\partial\Omega} w_n < \beta^+|\partial\Omega| - \frac{\delta}{2}. \tag{21}$$

For $n \in \mathbb{N}$ such that (20) holds, let $K_n = \{x \in \Omega : z_n(x) < \gamma^+ - \delta/(4|\Omega|)\}$. Then, on the one hand,

$$\int_{K_n} z_n = \int_\Omega z_n - \int_{\Omega\setminus K_n} z_n < -\frac{\delta}{4} + |K_n|\left(\gamma^+ - \frac{\delta}{4|\Omega|}\right),$$

and, on the other hand,

$$\int_{K_n} z_n = -\int_{K_n\cap\{x\in\Omega : z_n<-h\}} |z_n| + \int_{K_n\cap\{x\in\Omega : z_n\geqslant -h\}} z_n \geqslant -\frac{\delta}{8} - h|K_n|.$$

Therefore,

$$|K_n|\left(h - \frac{\delta}{4|\Omega|} + \gamma^+\right) \geqslant \frac{\delta}{8}.$$

Hence $|K_n| > 0$, $h - \delta/(4|\Omega|) + \gamma^+ > 0$ and

$$|K_n| \geqslant \frac{\delta/8}{h - \delta/(4|\Omega|) + \gamma^+}.$$

Consequently,

$$\|u_n^+\|_{L^p(K_n)} \leqslant |K_n|^{1/p} \sup \gamma^{-1}\left(\gamma^+ - \frac{\delta}{4|\Omega|}\right).$$

Then, by Lemma 4.3,

$$\|u_n^+\|_{L^p(\Omega)} \leqslant C(\Omega, N, p)\left(\left(\frac{h - \delta/(4|\Omega|) + \gamma^+}{\delta/8}\right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \gamma^{-1}\left(\gamma^+ - \frac{\delta}{4|\Omega|}\right)\right).$$

Similarly, for $n \in \mathbb{N}$ such that (21) holds, we get $|\{x \in \partial\Omega : w_n(x) < \beta^+ - \delta/(4|\partial\Omega|)\}| > 0$, $h - \delta/(4|\partial\Omega|) + \beta^+ > 0$ and

$$\|u_n^+\|_{L^p(\Omega)} \leqslant \hat{C}(\Omega, N, p)\left(\left(\frac{h - \delta/(4|\partial\Omega|) + \beta^+}{\delta/8}\right)^{1/p} \|Du_n^+\|_{L^p(\Omega)} + \sup \beta^{-1}\left(\beta^+ - \frac{\delta}{4|\partial\Omega|}\right)\right),$$

where $\hat{C}(\Omega, N, p)$ is given in Lemma 4.3. □

4.2 *Proof of the existence part of Theorem 2.5*

Let $T > 0$. Let $f \in L^{p'}(0, T; L^{p'}(\Omega))$, $g \in L^{p'}(0, T; L^{p'}(\partial\Omega))$, $z_0 \in L^{p'}(\Omega)$ and $w_0 \in L^{p'}(\partial\Omega)$ satisfy (4)–(6). Let $V(t) = (z(t), w(t))$ be the mild solution of problem (10) given by Theorem 3.6. Our aim is to prove that (z, w) is a weak solution of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$.

Following the proof of the existence of the mild solution (Theorem 3.6), for $n \in \mathbb{N}$, let $\epsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \epsilon$ and $f_1, \ldots, f_n \in L^{p'}(\Omega)$, $g_1, \ldots, g_n \in L^{p'}(\partial\Omega)$ with

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\|f(t) - f_i\|_{L^{p'}(\Omega)}^{p'} + \|g(t) - g_i\|_{L^{p'}(\partial\Omega)}^{p'})\, dt \leqslant \epsilon.$$

Then

$$\begin{aligned} z(t) &= L^1(\Omega)\text{-}\lim_{\epsilon\to 0} z_\epsilon(t) \qquad \text{uniformly for } t \in [0, T], \\ w(t) &= L^1(\partial\Omega)\text{-}\lim_{\epsilon\to 0} w_\epsilon(t) \qquad \text{uniformly for } t \in [0, T], \end{aligned} \tag{22}$$

where $z_\epsilon(t)$ and $w_\epsilon(t)$ are given, for ϵ small enough, by

$$\begin{cases} z_\epsilon(t) = z_0, \quad w_\epsilon(t) = w_0 \quad \text{for } t \in\,]-\infty, 0], \\ z_\epsilon(t) = z_i, \quad w_\epsilon(t) = w_i \quad \text{for } t \in\,]t_{i-1}, t_i],\ i = 1, \ldots, n, \end{cases} \tag{23}$$

$[u_i, z_i, w_i] \in W^{1,p}(\Omega) \times V^{1,p}(\Omega) \times V^{1,p}(\partial\Omega)$ being such that $z_i(x) \in \gamma(u_i(x))$ a.e. in Ω, $w_i(x) \in \beta(u_i(x))$ a.e. in $\partial\Omega$ and

$$\int_\Omega \mathbf{a}(x, Du_i) \cdot Dv + \int_\Omega \frac{z_i - z_{i-1}}{\epsilon} v + \int_{\partial\Omega} \frac{w_i - w_{i-1}}{\epsilon} v = \int_\Omega f_i v + \int_{\partial\Omega} g_i v \tag{24}$$

for all $v \in W^{1,p}(\Omega)$.

Taking $v = u_i$ as a test function in (24), we get

$$\int_\Omega \mathbf{a}(x, Du_i) \cdot Du_i + \int_\Omega \frac{z_i - z_{i-1}}{\epsilon} u_i + \int_{\partial\Omega} \frac{w_i - w_{i-1}}{\epsilon} u_i = \int_\Omega f_i u_i + \int_{\partial\Omega} g_i u_i. \tag{25}$$

Since $z_i(x) \in \gamma(u_i(x))$ a.e. in Ω and $w_i(x) \in \beta(u_i(x))$ a.e. in $\partial\Omega$, we have

$$\begin{aligned} u_i(x) &\in \gamma^{-1}(z_i(x)) = \partial j_\gamma^*(z_i(x)) && \text{a.e. in } \Omega, \\ u_i(x) &\in \beta^{-1}(w_i(x)) = \partial j_\beta^*(w_i(x)) && \text{a.e. in } \partial\Omega. \end{aligned}$$

Hence,

$$\begin{aligned} j_\gamma^*(z_{i-1}(x)) - j_\gamma^*(z_i(x)) &\geqslant (z_{i-1}(x) - z_i(x))u_i(x) && \text{a.e. in } \Omega, \\ j_\beta^*(w_{i-1}(x)) - j_\beta^*(w_i(x)) &\geqslant (w_{i-1}(x) - w_i(x))u_i(x) && \text{a.e. in } \partial\Omega. \end{aligned}$$

Therefore,

$$\frac{1}{\epsilon}\int_\Omega (j_\gamma^*(z_i) - j_\gamma^*(z_{i-1})) + \frac{1}{\epsilon}\int_{\partial\Omega} (j_\beta^*(w_i) - j_\beta^*(w_{i-1})) \leqslant \int_\Omega \frac{z_i - z_{i-1}}{\epsilon} u_i + \int_{\partial\Omega} \frac{w_i - w_{i-1}}{\epsilon} u_i,$$

and by (25), we get

$$\int_\Omega \mathbf{a}(x, Du_i) \cdot Du_i + \frac{1}{\epsilon}\int_\Omega (j_\gamma^*(z_i) - j_\gamma^*(z_{i-1})) + \frac{1}{\epsilon}\int_{\partial\Omega} (j_\beta^*(w_i) - j_\beta^*(w_{i-1})) \leqslant \int_\Omega f_i u_i + \int_{\partial\Omega} g_i u_i.$$

Then, integrating in time and adding in the last inequality, we obtain that

$$\begin{aligned} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \mathbf{a}(x, Du_i) \cdot Du_i + \int_\Omega (j_\gamma^*(z_n) - j_\gamma^*(z_0)) + \int_{\partial\Omega} (j_\beta^*(w_n) - j_\beta^*(w_0)) \\ \leqslant \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\int_\Omega f_i u_i + \int_{\partial\Omega} g_i u_i \right). \end{aligned}$$

Consequently, if we set $f_\epsilon(t) = f_i$, $g_\epsilon(t) = g_i$ and $u_\epsilon(t) = u_i$ for $t \in]t_{i-1}, t_i]$, $i = 1, \ldots, n$, it follows that

$$\begin{aligned} \int_0^T \int_\Omega \mathbf{a}(x, Du_\epsilon(t)) \cdot Du_\epsilon(t)\, \mathrm{d}t + \int_\Omega (j_\gamma^*(z_n) - j_\gamma^*(z_0)) + \int_{\partial\Omega} (j_\beta^*(w_n) - j_\beta^*(w_0)) \\ \leqslant \int_0^T \int_\Omega f_\epsilon(t) u_\epsilon(t) + \int_0^T \int_{\partial\Omega} g_\epsilon(t) u_\epsilon(t). \end{aligned} \tag{26}$$

Then, having in mind (H_1) and (5), we see that there exists a positive constant C_1 such that

$$\begin{aligned}\lambda\int_0^T\int_\Omega |Du_\epsilon(t)|^p\,\mathrm{d}t &\leqslant \int_0^T\int_\Omega \mathbf{a}(x, Du_\epsilon(t))\cdot Du_\epsilon(t)\,\mathrm{d}t\\ &\leqslant \int_\Omega j_\gamma^*(z_0)+\int_{\partial\Omega} j_\beta^*(w_0)+\int_0^T\int_\Omega f_\epsilon(t)u_\epsilon(t)+\int_0^T\int_{\partial\Omega} g_\epsilon(t)u_\epsilon(t)\\ &\leqslant C_1+\int_0^T \|f_\epsilon(t)\|_{L^{p'}(\Omega)}\|u_\epsilon(t)\|_{L^p(\Omega)}\,\mathrm{d}t+\int_0^T \|g\|_{L^{p'}(\partial\Omega)}\|u_\epsilon(t)\|_{L^p(\partial\Omega)}\,\mathrm{d}t.\end{aligned}$$

Therefore, using Young's inequality, for any $\mu > 0$ there exists $C_2(\mu) > 0$ such that

$$\int_0^T\int_\Omega |Du_\epsilon(t)|^p\,\mathrm{d}t \leqslant C_2(\mu)+\mu\int_0^T (\|u_\epsilon(t)\|^p_{L^p(\Omega)}+\|u_\epsilon(t)\|^p_{L^p(\partial\Omega)})\,\mathrm{d}t.$$

Hence, by the trace theorem, for any $\mu > 0$ there exists $C_3(\mu) > 0$ such that

$$\int_0^T\int_\Omega |Du_\epsilon(t)|^p\,\mathrm{d}t \leqslant C_3(\mu)+\mu\int_0^T (\|u_\epsilon(t)\|^p_{L^p(\Omega)}+\|Du_\epsilon(t)\|^p_{L^p(\Omega)})\,\mathrm{d}t. \tag{27}$$

By (22), if $\mathcal{R}^+_{\gamma,\beta} = +\infty$, there exist $M > 0$ and $n_0 \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]}\int_\Omega z^+_\epsilon(t)+\int_{\partial\Omega} w^+_\epsilon(t) < M \qquad \forall n \geqslant n_0,$$

and if $\mathcal{R}^+_{\gamma,\beta} < +\infty$, there exist $M \in \mathbb{R}$, $h > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\sup_{t\in[0,T]}\int_\Omega z_\epsilon(t)+\int_{\partial\Omega} w_\epsilon(t) < M < \mathcal{R}^+_{\gamma,\beta}$$

and

$$\sup_{t\in[0,T]}\max\left\{\int_{\{x\in\Omega\,:\,z_\epsilon(t)(x)<-h\}}|z_\epsilon(t)|,\int_{\{x\in\partial\Omega\,:\,w_\epsilon(t)(x)<-h\}}|w_\epsilon(t)|\right\} < \frac{\mathcal{R}^+_{\gamma,\beta}-M}{8}.$$

Consequently, from Lemma 4.2, there exists a constant $C_4 > 0$ such that

$$\|u^+_\epsilon(t)\|_{L^p(\Omega)} \leqslant C_4(\|Du^+_\epsilon(t)\|_{L^p(\Omega)}+1) \qquad \text{for all } t\in[0,T]. \tag{28}$$

Similarly, there exists $C_5 > 0$ such that

$$\|u^-_\epsilon(t)\|_{L^p(\Omega)} \leqslant C_5(\|Du^-_\epsilon(t)\|_{L^p(\Omega)}+1) \qquad \text{for all } t\in[0,T]. \tag{29}$$

Consequently, from (27), (28) and (29), choosing μ small enough, we deduce that there exists $C_6 > 0$ such that

$$\int_0^T\int_\Omega |Du_\epsilon(t)|^p\,\mathrm{d}t \leqslant C_6. \tag{30}$$

By (30), (28) and (29), $\{u_\epsilon\}$ is bounded in $L^p(0, T; W^{1,p}(\Omega))$. So, there exists a subsequence, denoted again $\{u_\epsilon\}$, such that

$$u_\epsilon \rightharpoonup u \qquad \text{weakly in } L^p(0, T; W^{1,p}(\Omega)) \text{ as } \epsilon \to 0^+, \tag{31}$$

$$u_\epsilon \rightharpoonup u \qquad \text{weakly in } L^p(S_T) \text{ as } \epsilon \to 0^+. \tag{32}$$

Since $z_\epsilon \in \gamma(u_\epsilon)$ a.e. in Q_T, $w_\epsilon \in \beta(u_\epsilon)$ a.e. on S_T, $z_\epsilon \to z$ in $L^1(Q_T)$ and $w_\epsilon \to w$ in $L^1(S_T)$, having in mind (31), (32) and using a monotonicity argument we conclude that $z \in \gamma(u)$ a.e. in Q_T and $w \in \beta(u)$ a.e. on S_T.

Since $\{Du_\epsilon\}$ is bounded in $L^p(Q_T)$, by (H_2) the sequence $\{|\mathbf{a}(x, Du_\epsilon)|\}$ is bounded in $L^{p'}(Q_T)$, so we can assume that

$$\mathbf{a}(x, Du_\epsilon) \rightharpoonup \Phi \qquad \text{weakly in } L^{p'}(Q_T) \text{ as } \epsilon \to 0^+. \tag{33}$$

From (24), we have

$$\int_\Omega \mathbf{a}(x, Du_\epsilon(t)) \cdot Dv + \int_\Omega \frac{z_\epsilon(t) - z_\epsilon(t-\epsilon)}{\epsilon} v + \int_{\partial\Omega} \frac{w_\epsilon(t) - w_\epsilon(t-\epsilon)}{\epsilon} v = \int_\Omega f_\epsilon(t) v + \int_{\partial\Omega} g_\epsilon(t) v \tag{34}$$

for all $v \in W^{1,p}(\Omega)$. Then, given $\psi \in W^{1,1}(0, T; W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$, $\psi(0) = \psi(T) = 0$, from (34), we get

$$\int_0^T \int_\Omega \mathbf{a}(x, Du_\epsilon(t)) \cdot D\psi + \int_\Omega \int_0^T \frac{z_\epsilon(t) - z_\epsilon(t-\epsilon)}{\epsilon} \psi(t) + \int_{\partial\Omega} \int_0^T \frac{w_\epsilon(t) - w_\epsilon(t-\epsilon)}{\epsilon} \psi(t) = \int_0^T \int_\Omega f_\epsilon(t) \psi + \int_0^T \int_{\partial\Omega} g_\epsilon(t) \psi. \tag{35}$$

Now,

$$\begin{aligned} \lim_{\epsilon \to 0} \int_\Omega \int_0^T \frac{z_\epsilon(t) - z_\epsilon(t-\epsilon)}{\epsilon} \psi(t) &= \lim_{\epsilon \to 0} \left(- \int_\Omega \int_0^{T-\epsilon} z_\epsilon(t) \frac{\psi(t+\epsilon) - \psi(t)}{\epsilon} + \int_\Omega \int_{T-\epsilon}^T \frac{z_\epsilon(t)\psi(t)}{\epsilon} - \int_\Omega \int_0^\epsilon \frac{z_0 \psi(t)}{\epsilon} \right) \\ &= - \int_0^T \int_\Omega z(t) \psi_t. \end{aligned}$$

Similarly,

$$\lim_{\epsilon \to 0} \int_{\partial\Omega} \int_0^T \frac{w_\epsilon(t) - w_\epsilon(t-\epsilon)}{\epsilon} \psi(t) = - \int_0^T \int_{\partial\Omega} w(t) \psi_t.$$

Therefore, taking the limit in (35) as $\epsilon \to 0^+$, we obtain

$$\int_0^T \int_\Omega \Phi \cdot D\psi - \int_0^T \int_\Omega z(t) \psi_t - \int_0^T \int_{\partial\Omega} w(t) \psi_t = \int_0^T \int_\Omega f(t) \psi + \int_0^T \int_{\partial\Omega} g(t) \psi. \tag{36}$$

Thus, to finish the proof of the existence, we only need to show that $\Phi = \mathbf{a}(x, Du)$. To do that we prove the following inequality:

$$\limsup_{\epsilon \to 0} \int_{Q_T} \mathbf{a}(x, Du_\epsilon) \cdot Du_\epsilon \leqslant \int_{Q_T} \Phi \cdot Du, \tag{37}$$

and we apply Minty–Browder's method. Indeed, if we assume that (37) holds, then for any $\rho \in L^p(0, T; W^{1,p}(\Omega))$,

$$\int_{Q_T} \mathbf{a}(x, D\rho) \cdot D(u_\epsilon - \rho) \leqslant \int_{Q_T} \mathbf{a}(x, Du_\epsilon) \cdot D(u_\epsilon - \rho),$$

so that, passing to the limit and using (37), we get

$$\int_{Q_T} \mathbf{a}(x, D\rho) \cdot D(u - \rho) \leqslant \int_{Q_T} \Phi \cdot D(u - \rho).$$

Then taking $\rho = u \pm \lambda\xi$ for $\lambda > 0$ and $\xi \in L^p(0, T; W^{1,p}(\Omega))$, we get

$$\int_{Q_T} \mathbf{a}(x, D(u + \lambda\rho)) \cdot D\xi = \int_{Q_T} \Phi \cdot D\xi,$$

and by letting $\lambda \to 0$, we obtain

$$\int_{Q_T} \mathbf{a}(x, D(u)) \cdot D\xi = \int_{Q_T} \Phi \cdot D\xi \quad \text{for any } \xi \in L^p(0, T; W^{1,p}(\Omega)),$$

which implies that $\mathbf{a}(x, D(u)) = \Phi$ a.e. in Q.

Now, let us prove (37). Thanks to (26) and Fatou's lemma, we have

$$\limsup_{\epsilon \to 0} \int_0^T \int_\Omega \mathbf{a}(x, Du_\epsilon(t)) \cdot Du_\epsilon(t)\, \mathrm{d}t \leqslant -\int_\Omega \bigl(j_\gamma^*(z(T)) - j_\gamma^*(z_0)\bigr) - \int_{\partial\Omega} \bigl(j_\beta^*(w(T)) - j_\beta^*(w_0)\bigr) + \int_0^T \int_\Omega fu + \int_0^T \int_{\partial\Omega} gu. \tag{38}$$

On the other hand, (36) can be rewritten as

$$\int_0^T \int_\Omega z(t)\psi_t + \int_0^T \int_{\partial\Omega} w(t)\psi_t = \int_0^T (F(t), \psi(t))\, \mathrm{d}t,$$

where F is given by

$$(F(t), \psi(t)) = \int_\Omega \Phi(t) \cdot D\psi(t) - \int_\Omega f(t)\psi(t) - \int_{\partial\Omega} g(t)\psi(t).$$

Then, by Lemma 4.1 applied to the above F, $H(x, r) = r$ and $\psi(t, x) = \xi(x)\phi(t)$, $\xi \in \mathcal{D}(\mathbb{R}^N)$, $\xi = 1$ in Ω, $\phi \in \mathcal{D}(]0, T[)$, we get

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega j_\gamma^*(z) - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} j_\beta^*(w) = (F, u) \quad \text{in } \mathcal{D}'(]0, T[). \tag{39}$$

Therefore,

$$\int_\Omega j_\gamma^*(z) + \int_{\partial\Omega} j_\beta^*(w) \in W^{1,1}(]0,T[).$$

So, integrating from 0 to T in (39), we have

$$\int_0^T \int_\Omega \Phi \cdot Du = -\int_\Omega (j_\gamma^*(z(T)) - j_\gamma^*(z_0)) - \int_{\partial\Omega} (j_\beta^*(w(T)) - j_\beta^*(w_0)) + \int_0^T \int_\Omega fu + \int_0^T \int_{\partial\Omega} gu.$$

Hence, using (38) we obtain (37). □

5. Contraction principle

Our main tool to prove the contraction principle is the concept of *integral solution* due to Ph. Bénilan (see [10], [14]).

DEFINITION 5.1 A function $V = (z, w) \in C([0,T]; X)$ is an *integral solution* of (10) in $[0,T]$ if for every $(\hat{f}, \hat{g}) \in \mathcal{B}^{\gamma,\beta}(\hat{z}, \hat{w})$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |z(t)-\hat{z}| + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} |w(t)-\hat{w}| &\leqslant \int_\Omega (f(t)-\hat{f})\,\mathrm{sign}_0(z(t)-\hat{z}) + \int_{\{x\in\Omega\,:\,z(t)=\hat{z}\}} |f(t)-\hat{f}| \\ &\quad + \int_{\partial\Omega} (g(t)-\hat{g})\,\mathrm{sign}_0(w(t)-\hat{w}) + \int_{\{x\in\partial\Omega\,:\,w(t)=\hat{w}\}} |g(t)-\hat{g}|\end{aligned}$$

in $\mathcal{D}'(]0,T[)$, and $V(0) = (z_0, w_0)$.

Since $\mathcal{B}^{\gamma,\beta}$ is accretive in X, it is well known (see [10], [14]) that mild solutions and integral solutions of problem (10) coincide and a contraction principle holds. We shall prove in Theorem 5.3 that a weak solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0,T]$ is an integral solution of (10). Consequently, since, in fact, $\mathcal{B}^{\gamma,\beta}$ is T-accretive in X, the contraction principle (3) given in Theorem 2.4 follows.

In the proof of Theorem 5.3, the main difficulties are due to the nonlinear and nonhomogeneous boundary conditions and to the jumps of γ and β. In [18], to obtain the L^1-contraction principle for a similar problem in the case $\beta = \{0\} \times \mathbb{R}$, and for γ having a set of jumps without density points, the authors give an improvement of the “hole filling” argument of [22] and use the doubling variable technique. This technique can be adapted to our problem. Now, by nonlinear semigroup theory, we are able to simplify the proof without using the doubling variable technique and without imposing any condition on the jumps of γ and β.

LEMMA 5.2 Let (z, w) be a weak solution of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0,T]$. Let $u \in L^p(0,T; W^{1,p}(\Omega))$ be such that $z \in \gamma(u)$ a.e. in Q_T and $w \in \beta(u)$ a.e. on S_T as in Definition 2.2. Let $\hat{z}, \hat{f} \in L^1(\Omega)$ and $\hat{u} \in W^{1,p}(\Omega)$ with $\hat{z} \in \gamma(\hat{u})$ a.e. in Ω be such that

$$\int_\Omega \mathbf{a}(x, D\hat{u}) \cdot D\psi = \int_\Omega \hat{f}\psi \quad \forall \psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{40}$$

Then, for any $\psi \in \mathcal{D}(\Omega)$, $\psi \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|z(t)-\hat{z}|\psi + \int_{\Omega}\mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t))-\mathbf{a}(x,D\hat{u}))\cdot D\psi \leqslant \int_{\Omega}(f(t)-\hat{f})\,\mathrm{sign}_0(z(t)-\hat{z})\psi + \int_{\{x\in\Omega\,:\,z(t)=\hat{z}\}}|f(t)-\hat{f}|\psi$$

in $\mathcal{D}'(]0,T[)$.

Proof. In Lemma 4.1, take the function F given by

$$(F(t),\psi(t)) = \int_{\Omega}\mathbf{a}(x,Du(t))\cdot D\psi(t) - \int_{\Omega}f(t)\psi(t) - \int_{\partial\Omega}g(t)\psi(t)$$

for all $\psi \in W^{1,1}(0,T;W^{1,1}(\Omega)\cap L^{\infty}(\Omega))\cap L^p(0,T;W^{1,p}(\Omega))$ with $\psi(0)=\psi(T)=0$, and

$$H(x,r) = \frac{1}{k}T_k(r-\hat{u}(x)+k\rho(x)),$$

where $\rho \in W^{1,p}(\Omega)$, $-1 \leqslant \rho \leqslant 1$. Then, for any $\psi \in \mathcal{D}(\Omega)$, $\psi \geqslant 0$, having in mind (40), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left(\int_{\hat{z}}^{z(t)}\frac{1}{k}T_k((\gamma^{-1})^0(\tau)-\hat{u}+k\rho)\,\mathrm{d}\tau\right)\psi &+ \int_{\Omega}(\mathbf{a}(x,Du(t))-\mathbf{a}(x,D\hat{u}))\cdot D\left(\frac{1}{k}T_k(u(t)-\hat{u}+k\rho)\psi\right)\\ &= \int_{\Omega}(f(t)-\hat{f})\frac{1}{k}T_k(u(t)-\hat{u}+k\rho)\psi \quad \text{in } \mathcal{D}'(]0,T[).\end{aligned} \tag{41}$$

Now, it is easy to see that

$$\begin{aligned}&\lim_{k\to 0}\int_{\hat{z}}^{z(t)}\frac{1}{k}T_k((\gamma^{-1})^0(\tau)-\hat{u}+k\rho))\,\mathrm{d}\tau = \int_{\hat{z}}^{z(t)}[\mathrm{sign}_0((\gamma^{-1})^0(\tau)-\hat{u})+\rho\chi_{\{\tau\,:\,(\gamma^{-1})^0(\tau)=\hat{u}\}}]\,\mathrm{d}\tau\\ &= \int_{\hat{z}}^{z(t)}[\mathrm{sign}_0(\tau-\hat{z})+(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau\,:\,(\gamma^{-1})^0(\tau)=\hat{u}\}}+\mathrm{sign}_0((\gamma^{-1})^0(\tau)-\hat{u})\chi_{\{\tau=\hat{z}\}}]\,\mathrm{d}\tau\\ &= \int_{\hat{z}}^{z(t)}[\mathrm{sign}_0(\tau-\hat{z})+(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau\,:\,(\gamma^{-1})^0(\tau)=\hat{u}\}}]\,\mathrm{d}\tau\\ &= |z(t)-\hat{z}| + \int_{\hat{z}}^{z(t)}(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau\,:\,(\gamma^{-1})^0(\tau)=\hat{u}\}}.\end{aligned}$$

Hence, taking limits in (41) as k goes to 0, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left(|z(t)-\hat{z}|+\int_{\hat{z}}^{z(t)}(\rho-\mathrm{sign}_0(\tau-\hat{z}))\chi_{\{\tau\,:\,(\gamma^{-1})^0(\tau)=\hat{u}\}}\right)\psi + \int_{\Omega}\mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t))-\mathbf{a}(x,D\hat{u}))\cdot D\psi$$

$$\leqslant \int_{\Omega} (f(t) - \hat{f})(\mathrm{sign}_0(z(t) - \hat{z}) + \mathrm{sign}_0(u(t) - \hat{u})\chi_{\{x\in\Omega\,:\,z(t)=\hat{z}\}})\psi + \int_{\Omega} (f(t) - \hat{f})(\rho - \mathrm{sign}_0(z(t) - \hat{z}))\chi_{\{x\in\Omega\,:\,u(t)=\hat{u}\}}\psi,$$

and integrating between $\hat{t}, t \in]0, T[$, we get

$$\begin{aligned} \int_{\Omega} |z(t) - \hat{z}|\psi - \int_{\Omega} |z(\hat{t}) - \hat{z}|\psi + \int_{\Omega}\int_{z(\hat{t})}^{z(t)} (\rho - \mathrm{sign}_0(\tau - \hat{z}))\chi_{\{\tau\,:\,(\gamma^{-1})^0(\tau)=\hat{u}\}}\psi \\ + \int_{\hat{t}}^{t}\int_{\Omega} \mathrm{sign}_0(u(\tau) - \hat{u})(\mathbf{a}(x, Du(\tau)) - \mathbf{a}(x, D\hat{u})) \cdot D\psi \\ \leqslant \int_{\hat{t}}^{t}\int_{\Omega} (f(\tau) - \hat{f})(\mathrm{sign}_0(z(\tau) - \hat{z}) + \mathrm{sign}_0(u(\tau) - \hat{u})\chi_{\{x\in\Omega\,:\,z(\tau)=\hat{z}\}})\psi \\ + \int_{\hat{t}}^{t}\int_{\Omega} (f(\tau) - \hat{f})(\rho - \mathrm{sign}_0(z(\tau) - \hat{z}))\chi_{\{x\in\Omega\,:\,u(\tau)=\hat{u}\}}\psi. \end{aligned}$$

Since in the last expression there are no space derivatives of ρ, we can take, for each t fixed, $\rho = \mathrm{sign}_0(z(t) - \hat{z})$. Then the second term in the above expression is positive and we have, for any $\hat{t}, t \in]0, T[$,

$$\begin{aligned} \int_{\Omega} |z(t) - \hat{z}|\psi - \int_{\Omega} |z(\hat{t}) - \hat{z}|\psi + \int_{\hat{t}}^{t}\int_{\Omega} \mathrm{sign}_0(u(\tau) - \hat{u})(\mathbf{a}(x, Du(\tau)) - \mathbf{a}(x, D\hat{u})) \cdot D\psi \\ \leqslant \int_{\hat{t}}^{t}\int_{\Omega} (f(\tau) - \hat{f})(\mathrm{sign}_0(z(\tau) - \hat{z}) + \mathrm{sign}_0(u(\tau) - \hat{u}))\chi_{\{x\in\Omega\,:\,z(\tau)=\hat{z}\}}\psi \\ + \int_{\hat{t}}^{t}\int_{\Omega} (f(\tau) - \hat{f})(\mathrm{sign}_0(z(t) - \hat{z}) - \mathrm{sign}_0(z(\tau) - \hat{z}))\chi_{\{x\in\Omega\,:\,u(\tau)=\hat{u}\}}\psi. \end{aligned} \tag{42}$$

Let

$$\begin{aligned} \varphi_1(t) &:= \int_{\Omega} |z(t) - \hat{z}|\psi, \\ \varphi_2(\tau) &:= -\int_{\Omega} \mathrm{sign}_0(u(\tau) - \hat{u})(\mathbf{a}(x, Du(\tau)) - \mathbf{a}(x, D\hat{u}))D\psi \\ &\quad + \int_{\Omega} (f(\tau) - \hat{f})(\mathrm{sign}_0(z(\tau) - \hat{z}) + \mathrm{sign}_0(u(\tau) - \hat{u}))\chi_{\{x\in\Omega\,:\,z(\tau)(x)=\hat{z}(x)\}}\psi \end{aligned}$$

and

$$\varphi_3(t, \tau) := \int_{\Omega} (f(\tau) - \hat{f})(\mathrm{sign}_0(z(t) - \hat{z}) - \mathrm{sign}_0(z(\tau) - \hat{z}))\chi_{\{x\in\Omega\,:\,u(\tau)(x)=\hat{u}(x)\}}\psi.$$

Then, taking in (42) $\hat{t} = t - h$, $h > 0$, dividing by h and letting h go to 0, for any $\eta \in \mathcal{D}(]0, T[)$, $\eta \geqslant 0$, we get

$$
\begin{aligned}
-\int_0^T \varphi_1(t)\eta_t(t)\,\mathrm{d}t &= -\lim_{h\to 0^+}\int_0^T \varphi_1(t)\frac{\eta(t+h)-\eta(t)}{h}\,\mathrm{d}t \\
&= \lim_{h\to 0^+}\int_0^T \frac{\varphi_1(t)-\varphi_1(t-h)}{h}\eta(t)\,\mathrm{d}t \\
&\leqslant \lim_{h\to 0^+}\left(\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_2(\tau)\,\mathrm{d}\tau\right)\eta(t)\,\mathrm{d}t + \int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_3(t,\tau)\,\mathrm{d}\tau\right)\eta(t)\,\mathrm{d}t\right). \qquad (43)
\end{aligned}
$$

Now, by the dominated convergence theorem,

$$
\begin{aligned}
\lim_{h\to 0^+}\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_2(\tau)\,\mathrm{d}\tau\right)\eta(t)\,\mathrm{d}t &= -\lim_{h\to 0^+}\int_0^T\left(\int_0^t \varphi_2(\tau)\,\mathrm{d}\tau\right)\frac{\eta(t+h)-\eta(t)}{h}\,\mathrm{d}t \\
&= -\int_0^T\left(\int_0^t \varphi_2(\tau)\,\mathrm{d}\tau\right)\eta_t(t)\,\mathrm{d}t = \int_0^T \varphi_2(t)\eta(t)\,\mathrm{d}t.
\end{aligned}
$$

On the other hand, for h small enough,

$$
\int_0^T \frac{1}{h}\left(\int_{t-h}^t \varphi_3(t,\tau)\,\mathrm{d}\tau\right)\eta(t)\,\mathrm{d}t = \int_0^T \frac{1}{h}\left(\int_\tau^{\tau+h} \varphi_3(t,\tau)\eta(t)\,\mathrm{d}t\right)\mathrm{d}\tau.
$$

Now,

$$
\begin{aligned}
&\left|\int_0^T \frac{1}{h}\left(\int_\tau^{\tau+h} \varphi_3(t,\tau)\eta(t)\,\mathrm{d}t\right)\mathrm{d}\tau\right| \\
&\quad\leqslant \int_0^T \frac{1}{h}\left(\int_\tau^{\tau+h}\int_\Omega |f(\tau)-\hat{f}|\,|\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})|\eta(t)\psi(x)\,\mathrm{d}x\,\mathrm{d}t\right)\mathrm{d}\tau \\
&\quad\leqslant \|\psi\|_{L^\infty(\Omega)}\|\eta\|_{L^\infty(0,T)}\int_0^T\left[\int_\Omega |f(\tau)-\hat{f}|\,\mathrm{d}x\right. \\
&\qquad\qquad \left.\times\frac{1}{h}\int_\tau^{\tau+h}\|\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})\|_{L^\infty(\Omega)}\,\mathrm{d}t\right]\mathrm{d}\tau.
\end{aligned}
$$

Moreover, for all Lebesgue points of the $L^1(0,T;L^\infty(\Omega))$-function $\mathrm{sign}_0(z(\cdot)-\hat{z})$, and so for a.e. $\tau\in\,]0,T[$, we have

$$
\lim_{h\to 0^+}\frac{1}{h}\int_\tau^{\tau+h}\|\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})\|_{L^\infty(\Omega)}\,\mathrm{d}t = 0.
$$

Consequently, since

$$
\begin{aligned}
&\left(\int_\Omega |f(\tau)-\hat{f}|\,\mathrm{d}x\right)\frac{1}{h}\int_\tau^{\tau+h}\|\mathrm{sign}_0(z(t)-\hat{z})-\mathrm{sign}_0(z(\tau)-\hat{z})\|_{L^\infty(\Omega)}\,\mathrm{d}t \\
&\qquad\qquad \leqslant 2\int_\Omega |f(\tau)-\hat{f}|\,\mathrm{d}x,
\end{aligned}
$$

which is a function in $L^1(0,T)$, by the dominated convergence theorem, we get

$$\lim_{h\to 0^+}\int_0^T \frac{1}{h}\left(\int_{t-h}^{t} \varphi_3(t,\tau)\,\mathrm{d}\tau\right)\eta(t)\,\mathrm{d}t = \lim_{h\to 0^+}\int_0^T \frac{1}{h}\left(\int_{\tau}^{\tau+h} \varphi_3(t,\tau)\eta(t)\,\mathrm{d}t\right)\mathrm{d}\tau = 0.$$

Therefore, from (43) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |z(t)-\hat{z}|\psi + \int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D\psi \leqslant \int_\Omega (f(t)-\hat{f})(\mathrm{sign}_0(z(t)-\hat{z})+\mathrm{sign}_0(u(t)-\hat{u})\chi_{\{x\in\Omega\,:\,z(t)=\hat{z}\}})\psi$$

in $\mathcal{D}'(]0,T[)$. □

THEOREM 5.3 Let (z,w) be a weak solution of $P_{\gamma,\beta}(f,g,z_0,w_0)$ in $[0,T]$. Let $(\hat{f},\hat{g}) \in \mathcal{B}^{\gamma,\beta}(\hat{z},\hat{w})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |z(t)-\hat{z}| + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} |w(t)-\hat{w}| \leqslant \int_\Omega (f(t)-\hat{f})\,\mathrm{sign}_0(z(t)-\hat{z}) + \int_{\{x\in\Omega\,:\,z(t)=\hat{z}\}} |f(t)-\hat{f}| + \int_{\partial\Omega}(g(t)-\hat{g})\,\mathrm{sign}_0(w(t)-\hat{w}) + \int_{\{x\in\partial\Omega\,:\,w(t)=\hat{w}\}} |g(t)-\hat{g}|$$

in $\mathcal{D}'(]0,T[)$, that is, since $(z(0),w(0))=(z_0,w_0)$, (z,w) is an integral solution of (10) in $[0,T]$.

Proof. Let $u\in L^p(0,T;W^{1,p}(\Omega))$ be such that $z\in\gamma(u)$ a.e. in Q_T and $w\in\beta(u)$ a.e. on S_T as in Definition 2.2, and let $\hat{u}\in W^{1,p}(\Omega)$ be such that $\hat{z}\in\gamma(\hat{u})$ a.e. in Ω and $\hat{w}\in\gamma(\hat{u})$ a.e. in $\partial\Omega$ as in the definition of $\mathcal{B}^{\gamma,\beta}$.

Thanks to Lemma 5.2, for any $\psi\in\mathcal{D}(\Omega)$, $0\leqslant\psi\leqslant 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |z(t)-\hat{z}|\psi + \int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D\psi \leqslant \int_\Omega (f(t)-\hat{f})\,\mathrm{sign}_0(z(t)-\hat{z})\psi + \int_{\{x\in\Omega\,:\,z(t)=\hat{z}\}} |f(t)-\hat{f}|\psi \tag{44}$$

in $\mathcal{D}'(]0,T[)$. Now, for the second term on the left hand side we have

$$\int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D\psi = \int_\Omega \mathrm{sign}_0(u(t)-\hat{u})(\mathbf{a}(x,Du(t)-\mathbf{a}(x,D\hat{u}))\cdot D(\psi-1) \geqslant \lim_{k\to 0}\int_\Omega (\mathbf{a}(x,Du(t))-\mathbf{a}(x,D\hat{u}))\cdot D\left(\frac{1}{k}T_k(u(t)-\hat{u}+k\rho)(\psi-1)\right), \tag{45}$$

where $\rho \in W^{1,p}(\Omega)$, $-1 \leqslant \rho \leqslant 1$. Using again Lemma 4.1 we get

$$\begin{aligned}
&\int_\Omega (\mathbf{a}(x, Du(t) - \mathbf{a}(x, D\hat{u})) \cdot D\left(\frac{1}{k} T_k(u(t) - \hat{u} + k\rho)(\psi - 1)\right) \\
&\qquad = -\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\int_{\hat{z}}^{z(t)} \frac{1}{k} T_k((\gamma^{-1})^0(s) - \hat{u} + k\rho)\, \mathrm{d}s \right)(\psi - 1) \\
&\qquad\quad + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} \left(\int_{\hat{w}}^{w(t)} \frac{1}{k} T_k((\beta^{-1})^0(s) - \hat{u} + k\rho)\, \mathrm{d}s \right) \\
&\qquad\quad + \int_\Omega (f(t) - \hat{f}) \frac{1}{k} T_k(u(t) - \hat{u} + k\rho)(\psi - 1) \\
&\qquad\quad - \int_{\partial\Omega} (g(t) - \hat{g}) \frac{1}{k} T_k(u(t) - \hat{u} + k\rho),
\end{aligned} \tag{46}$$

which converges as k goes to 0 to

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(|z(t) - \hat{z}| - \int_{\hat{z}}^{z(t)} (\rho - \mathrm{sign}_0(s - \hat{z}))\chi_{\{s\,:\,(\gamma^{-1})^0(s) = \hat{u}\}} \right)(\psi - 1) \\
&+ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} \left(|w(t) - \hat{w}| + \int_{\hat{w}}^{w(t)} (\rho - \mathrm{sign}_0(s - \hat{w}))\chi_{\{s\,:\,(\beta^{-1})^0(s) = \hat{u}\}} \right) \\
&+ \int_\Omega (f(t) - \hat{f})(\mathrm{sign}_0(z(t) - \hat{z}) + \mathrm{sign}_0(u(t) - \hat{u})\chi_{\{x\in\Omega\,:\,z(t) = \hat{z}\}})(\psi - 1) \\
&+ \int_\Omega (f(t) - \hat{f})(\rho - \mathrm{sign}_0(z(t) - \hat{z}))\chi_{\{x\in\Omega\,:\,u(t) = \hat{u}\}}(\psi - 1) \\
&- \int_{\partial\Omega} (g(t) - \hat{g})(\mathrm{sign}_0(w(t) - \hat{w}) + \mathrm{sign}_0(u(t) - \hat{u})\chi_{\{x\in\partial\Omega\,:\,w(t) = \hat{w}\}}) \\
&- \int_{\partial\Omega} (g(t) - \hat{g})(\rho - \mathrm{sign}_0(w(t) - \hat{w}))\chi_{\{x\in\partial\Omega\,:\,u(t) = \hat{u}\}}.
\end{aligned}$$

Therefore, taking into account (45) and (46) in (44), replacing ψ by ψ_n such that $L^1(\Omega)$-$\lim_n \psi_n = 1$, and taking limits as n goes to $+\infty$ we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |z(t) - \hat{z}| + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} \left(|w(t) - \hat{w}| + \int_{\hat{w}}^{w(t)} (\rho - \mathrm{sign}_0(s - \hat{w}))\chi_{\{s\,:\,(\beta^{-1})^0(s) = \hat{u}\}} \right) \\
&\qquad \leqslant \int_\Omega (f(t) - \hat{f})\, \mathrm{sign}_0(z(t) - \hat{z}) + \int_{\{x\in\Omega\,:\,z(t) = \hat{z}\}} |f(t) - \hat{f}| \\
&\qquad\quad + \int_{\partial\Omega} (g(t) - \hat{g})(\mathrm{sign}_0(w(t) - \hat{w}) + \mathrm{sign}_0(u(t) - \hat{u})\chi_{\{x\in\partial\Omega\,:\,w(t) = \hat{w}\}}) \\
&\qquad\quad + \int_{\partial\Omega} (g(t) - \hat{g})(\rho - \mathrm{sign}_0(w(t) - \hat{w}))\chi_{\{x\in\partial\Omega\,:\,u(t) = \hat{u}\}}.
\end{aligned}$$

Finally, by a similar argument to the one used in Lemma 5.2, we finish the proof. □

REMARK 5.4 It is easy to see that Theorem 2.5 also holds for data $(z_0, w_0) \in V^{1,p}(\Omega) \times V^{1,p}(\partial\Omega)$ and $(f, g) \in L^{p'}(0, T; V^{1,p}(\Omega)) \times L^{p'}(0, T; V^{1,p}(\partial\Omega))$ satisfying conditions (4)–(6), in particular, if $p > N$, for $(z_0, w_0) \in L^1(\Omega) \times L^1(\partial\Omega)$ and $(f, g) \in L^1(0, T; L^1(\Omega)) \times L^1(0, T; L^1(\partial\Omega))$ satisfying (4)–(6).

6. Appendix

This appendix is devoted to the proof of Theorem 3.5. For that we need some lemmas.

LEMMA 6.1 Assume $\gamma, \beta : \mathbb{R} \to \mathbb{R}$ are strictly increasing functions with $\mathrm{Ran}(\gamma) = \mathrm{Ran}(\beta) = \mathbb{R}$. Let $\phi \in L^\infty(\Omega)$ and $\psi \in L^\infty(\partial\Omega)$. Then, if $[u, z, w]$ is a weak solution of problem $(S_{\phi,\psi}^{\gamma,\beta})$, we have

$$\inf\{\gamma^{-1}(\inf\phi), \beta^{-1}(\inf\psi)\} \leqslant u \leqslant \max\{\gamma^{-1}(\sup\phi), \beta^{-1}(\sup\psi)\}.$$

Proof. By Theorem 3.2(ii), if

$$a := \inf\{\gamma^{-1}(\inf\phi), \beta^{-1}(\inf\psi)\}, \qquad b := \max\{\gamma^{-1}(\sup\phi), \beta^{-1}(\sup\psi)\},$$

we have

$$\int_\Omega (\gamma(a) - z)^+ + \int_{\partial\Omega} (\beta(a) - w)^+ \leqslant \int_\Omega (\gamma(a) - \phi)^+ + \int_{\partial\Omega} (\beta(a) - \psi)^+,$$
$$\int_\Omega (z - \gamma(b))^+ + \int_{\partial\Omega} (w - \beta(b))^+ \leqslant \int_\Omega (\phi - \gamma(b))^+ + \int_{\partial\Omega} (\psi - \beta(b))^+,$$

and hence the result follows. □

Now, for $m, n, r, l \in \mathbb{N}$, let

$$\gamma_l^{m,n}(s) = \gamma_l(s) + \frac{1}{l}|s|^{p-2}s + \frac{1}{m}s^+ - \frac{1}{n}s^-, \qquad \beta_r^{m,n}(s) = \beta_r(s) + \frac{1}{m}s^+ - \frac{1}{n}s^-,$$

where γ_l and β_r are the Yosida approximations of γ and β respectively. Then, by the above lemma, if $[u_{r,l}^{m,n}, z_{r,l}^{m,n}, w_{r,l}^{m,n}]$ is a weak solution of $(S_{\phi,\psi}^{\gamma_l^{m,n},\beta_r^{m,n}})$ for $\phi \in L^\infty(\Omega)$ and $\psi \in L^\infty(\partial\Omega)$, then

$$\inf\{(\gamma_{r,l}^{m,n})^{-1}(\inf\phi), (\beta_{r,l}^{m,n})^{-1}(\inf\psi)\} \leqslant u_{r,l}^{m,n} \leqslant \sup\{(\gamma_l^{m,n})^{-1}(\sup\phi), (\beta_r^{m,n})^{-1}(\sup\psi)\}.$$

Since

$$\gamma^{m,n}(s) := (\liminf_{l\to+\infty} \gamma_l^{m,n})(s) = \gamma(s) + \frac{1}{m}s^+ - \frac{1}{n}s^-,$$
$$\beta^{m,n}(s) := (\liminf_{r\to+\infty} \beta_r^{m,n})(s) = \beta(s) + \frac{1}{m}s^+ - \frac{1}{n}s^-,$$

the next lemma follows.

LEMMA 6.2 Assume $\lim_l \lim_r u_{r,l}^{m,n} = u^{m,n}$ a.e. in Ω or $\lim_r u_{r,r}^{m,n} = u^{m,n}$ a.e. in Ω. Let $\phi \in L^\infty(\Omega)$ and $\psi \in L^\infty(\partial\Omega)$. Then

$$\inf\{\inf(\gamma^{m,n})^{-1}(\inf\phi), \inf(\beta^{m,n})^{-1}(\inf\psi)\} \leqslant u^{m,n}$$
$$\leqslant \sup\{\sup(\gamma^{m,n})^{-1}(\sup\phi), \sup(\beta^{m,n})^{-1}(\sup\psi)\}.$$

Let $n(m)$ be a subsequence in $\mathbb{N}$. Since

$$\liminf_{m\to\infty} \gamma^{m,n(m)} = \gamma, \qquad \liminf_{m\to\infty} \beta^{m,n(m)} = \beta,$$

the following result holds.

LEMMA 6.3 Assume $\lim_{m\to\infty} u^{m,n(m)} = u$ a.e. in Ω. If $a_0 \leqslant \phi \leqslant a_1$ and $b_0 \leqslant \psi \leqslant b_1$, where

- $\gamma_- < a_0 < 0$ if $\gamma_- < 0$ and $0 \leqslant a_0$ if $\gamma_- = 0$,
- $0 < a_1 < \gamma_+$ if $\gamma_+ > 0$ and $a_1 \leqslant 0$ if $\gamma_+ = 0$,
- $\beta_- < b_0 < 0$ if $\beta_- < 0$ and $0 \leqslant b_0$ if $\beta_- = 0$,
- $0 < b_1 < \beta_+$ if $\beta_+ > 0$ and $b_1 \leqslant 0$ if $\beta_+ = 0$,

then

$$\inf\{A_0, B_0\} \leqslant u \leqslant \sup\{A_1, B_1\},$$

where $A_0 = \inf \gamma^{-1}(a_0)$ if $\gamma_- < 0$, $A_0 = 0$ if $\gamma_- = 0$, $B_0 = \inf \beta^{-1}(b_0)$ if $\beta_- < 0$, $B_0 = 0$ if $\beta_- = 0$, $A_1 = \sup \gamma^{-1}(a_1)$ if $\gamma_+ > 0$, $A_1 = 0$ if $\gamma_+ = 0$, $B_1 = \sup \beta^{-1}(b_1)$ if $\beta_+ > 0$ and $B_1 = 0$ if $\beta_+ = 0$.

Proof of Theorem 3.5. It is obvious that

$$\overline{D(\mathcal{B}^{\gamma,\beta})}^{L^1(\Omega)\times L^1(\partial\Omega)} \subset \{(z, w) \in L^1(\Omega) \times L^1(\partial\Omega) : \gamma_- \leqslant z \leqslant \gamma_+,\ \beta_- \leqslant w \leqslant \beta_+\}.$$

To obtain the other inclusion, it is enough to take $(z, w) \in L^\infty(\Omega) \times L^\infty(\partial\Omega)$ with $a_0 \leqslant z \leqslant a_1$ and $b_0 \leqslant w \leqslant b_1$, where the constants a_i, b_i, $i = 0, 1$, satisfy

- $\gamma_- < a_0 < 0$ if $\gamma_- < 0$ and $a_0 = 0$ if $\gamma_- = 0$,
- $0 < a_1 < \gamma_+$ if $\gamma_+ > 0$ and $a_1 = 0$ if $\gamma_+ = 0$,
- $\beta_- < b_0 < 0$ if $\beta_- < 0$ and $b_0 = 0$ if $\beta_- = 0$,
- $0 < b_1 < \beta_+$ if $\beta_+ > 0$ and $b_1 = 0$ if $\beta_+ = 0$,

and to prove that $(z, w) \in \overline{D(\mathcal{B}^{\gamma,\beta})}^X$.

Given $(z, w) \in L^\infty(\Omega) \times L^\infty(\partial\Omega)$ with $a_0 \leqslant z \leqslant a_1$ and $b_0 \leqslant w \leqslant b_1$, we set

$$(z_n, w_n) = \left(I + \frac{1}{n}\mathcal{B}^{\gamma,\beta}\right)^{-1}(z, w), \qquad n \in \mathbb{N}.$$

Let us see that there exists a subsequence, denoted (z_n, w_n) again, such that

$$(z_n, w_n) \to (z, w) \quad \text{in } L^1(\Omega) \times L^1(\partial\Omega),$$

which implies that $(z, w) \in \overline{D(\mathcal{B}^{\gamma,\beta})}^X$.

Since $((z_n, w_n), n(z - z_n, w - w_n)) \in \mathcal{B}^{\gamma,\beta}$, there exist $u_n \in W^{1,p}(\Omega)$ such that $[u_n, z_n, w_n]$ is a weak solution of problem $(S^{\gamma,\beta}_{z_n+n(z-z_n),w_n+n(w-w_n)})$. Hence, $z_n(x) \in \gamma(u_n(x))$ a.e. in Ω, $w_n(x) \in \beta(u_n(x))$ a.e. in $\partial\Omega$ and

$$\frac{1}{n}\int_\Omega \mathbf{a}(x, Du_n)\cdot D\phi + \int_\Omega z_n\phi + \int_{\partial\Omega} w_n\phi = \int_\Omega z\phi + \int_{\partial\Omega} w\phi \tag{47}$$

for all $\phi \in W^{1,p}(\Omega)$.

Note that if $\mathbf{a}_n(x,\xi) := \frac{1}{n}\mathbf{a}(x,\xi)$, then $[u_n, z_n, w_n]$ is a weak solution of the problem

$$({}_{\mathbf{a}_n}S_{z,w}^{\gamma,\beta}) \quad \begin{cases} -\operatorname{div}\mathbf{a}_n(x,Du) + \gamma(u) \ni z & \text{in } \Omega, \\ \mathbf{a}_n(x,Du)\cdot\eta + \beta(u) \ni w & \text{on } \partial\Omega, \end{cases}$$

and by uniqueness, we can assume that $[u_n, z_n, w_n]$ is the weak solution of problem $({}_{\mathbf{a}_n}S_{z,w}^{\gamma,\beta})$ given in Theorem 3.3. This construction is done as follows (see [6]). Firstly, we find a weak solution $[(u_n)_r^{m,k}, (z_n)_r^{m,k}, (w_n)_r^{m,k}]$ of $({}_{\mathbf{a}_n}S_{z,w}^{\gamma_r^{m,k},\beta_r^{m,k}})$ in case $\operatorname{Dom}(\beta)=\mathbb{R}$, and $[(u_n)_{r,l}^{m,k}, (z_n)_{r,l}^{m,k}, (w_n)_{r,l}^{m,k}]$ of $({}_{\mathbf{a}_n}S_{z,w}^{\gamma_l^{m,k},\beta_r^{m,k}})$ in case $\mathbf{a}$ is smooth. If $\operatorname{Dom}(\beta)=\mathbb{R}$, taking limits as r goes to $+\infty$, we have

$$\begin{aligned} \lim_r (u_n)_r^{m,k} &= (u_n)^{m,k} && \text{in } L^1(\Omega), \\ \lim_r (z_n)_r^{m,k} &= (z_n)^{m,k} && \text{weakly in } L^1(\Omega), \\ \lim_r (w_n)_r^{m,k} &= (w_n)^{m,k} && \text{weakly in } L^1(\partial\Omega), \end{aligned}$$

$[(u_n)^{m,k}, (z_n)^{m,k}, (w_n)^{m,k}]$ being a weak solution of $(S_{z,w}^{\gamma^{m,k},\beta^{m,k}})$; in case $\mathbf{a}$ is smooth, taking limits as l goes to $+\infty$ we get

$$\begin{aligned} \lim_l (u_n)_{r,l}^{m,k} &= (u_n)_r^{m,k} && \text{in } L^1(\Omega), \\ \lim_l (z_n)_{r,l}^{m,k} &= (z_n)_r^{m,k} && \text{weakly in } L^1(\Omega), \\ \lim_l (w_n)_{r,l}^{m,k} &= (w_n)_r^{m,k} && \text{weakly in } L^1(\partial\Omega), \end{aligned}$$

$[(u_n)_r^{m,k}, (z_n)_r^{m,k}, (w_n)_r^{m,k}]$ being a weak solution of $(S_{z,w}^{\gamma^{m,k},\beta_r^{m,k}})$, and taking limits as r goes to $+\infty$, we obtain

$$\begin{aligned} \lim_r (u_n)_r^{m,k} &= (u_n)^{m,k} && \text{in } L^1(\Omega), \\ \lim_r (z_n)_r^{m,k} &= (z_n)^{m,k} && \text{weakly in } L^1(\Omega), \\ \lim_r (w_n)_r^{m,k} &= (w_n)^{m,k} && \text{weakly in } L^1(\partial\Omega), \end{aligned}$$

$[(u_n)^{m,k}, (z_n)^{m,k}, (w_n)^{m,k}]$ being a weak solution of $(S_{z,w}^{\gamma^{m,k},\beta^{m,k}})$. Moreover, in the case of $\mathbf{a}$ smooth,

$$(w_n)^{m,k} \ll w - \mathbf{a}_n(x, D(\hat{u}_n)^{m,k})\cdot\eta,$$

$[(\hat{u}_n)^{m,k}, (\hat{z}_n)^{m,k}]$ being a weak solution of

$$\begin{cases} -\operatorname{div}\mathbf{a}_n(x, D(\hat{u}_n)^{m,k}) + \gamma((\hat{u}_n)^{m,k}) + \dfrac{1}{m}((\hat{u}_n)^{m,k})^+ - \dfrac{1}{k}((\hat{u}_n)^{m,k})^- \ni z & \text{in } \Omega, \\ (\hat{u}_n)^{m,k} = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, passing to the limit in m for a suitable subsequence $\{k(m)\}$ in $\mathbb{N}$, we have

$$\begin{aligned} \lim_{m\to\infty} (u_n)^{m,k(m)} &= u_n && \text{in } L^1(\Omega), \\ \lim_{m\to\infty} (z_n)^{m,k(m)} &= z_n && \text{in } L^1(\Omega), \\ \lim_{m\to\infty} (w_n)^{m,k(m)} &= w_n && \text{in } L^1(\partial\Omega). \end{aligned} \tag{48}$$

If **a** is smooth,

$$\begin{aligned}
\lim_{m\to\infty} (\hat{u}_n)^{m,k(m)} &= \hat{u}_n && \text{in } L^1(\Omega),\\
\lim_{m\to\infty} (\hat{z}_n)^{m,k(m)} &= \hat{z}_n && \text{in } L^1(\Omega),\\
\lim_{m\to\infty} \mathbf{a}_n(x, D(\hat{u}_n)^{m,k(m)}) \cdot \eta &= \mathbf{a}_n(x, D\hat{u}_n) \cdot \eta && \text{in } L^1(\partial\Omega),
\end{aligned}$$

$[\hat{u}_n, \hat{z}_n]$ being a weak solution of

$$\begin{cases} -\mathrm{div}\, \mathbf{a}_n(x, D\hat{u}_n) + \gamma(\hat{u}_n) \ni z & \text{in } \Omega,\\ \hat{u}_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover (see [6]),

$$\hat{z}_n \ll z, \tag{49}$$

$$w_n \ll w - \mathbf{a}_n(x, D\hat{u}_n) \cdot \eta, \tag{50}$$

and

$$\int_{\partial\Omega} |\mathbf{a}_n(x, D\hat{u}_n) \cdot \eta| \leqslant \int_{\Omega} |z - \hat{z}_n|. \tag{51}$$

Observe that, by Lemmas 6.1–6.3, $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$; similarly, $\{\hat{u}_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Therefore, since $\mathrm{Dom}(\gamma) = \mathbb{R}$, $\{z_n\}$ and $\{\hat{z}_n\}$ are uniformly bounded in $L^\infty(\Omega)$, so there exists a subsequence (not relabelled) such that z_n and $\hat{z}_n$ are weakly convergent in $L^1(\Omega)$. Also, in the case $\mathrm{Dom}(\beta) = \mathbb{R}$, there exists a subsequence (not relabelled) such that w_n is weakly convergent in $L^1(\partial\Omega)$.

We now claim that

$$\lim_{n\to\infty} \int_{\Omega} z_n \phi = \int_{\Omega} z\phi \quad \text{for every } \phi \in \mathcal{D}(\Omega). \tag{52}$$

Taking $\phi = u_n$ in (47), since $z_n(x) \in \gamma(u_n(x))$ a.e. in Ω, $w_n(x) \in \beta(u_n(x))$ a.e. in $\partial\Omega$, and $\{u_n\}$ is bounded in $L^\infty(\Omega)$, we get

$$\int_{\Omega} \mathbf{a}(x, Du_n) \cdot Du_n \leqslant n\left(\int_{\Omega} z u_n + \int_{\partial\Omega} w u_n\right) \leqslant nC.$$

Now, using (H_1) and (H_2), we have

$$\begin{aligned}
\left(\int_{\Omega} |\mathbf{a}(x, Du_n)|^{p'}\right)^{1/p'} &\leqslant \sigma\left(\int_{\Omega} (\varrho(x) + |Du_n|^{p-1})^{p'}\right)^{1/p'}\\
&\leqslant \sigma\left(\left(\int_{\Omega} \varrho(x)^{p'}\right)^{1/p'} + \left(\int_{\Omega} |Du_n|^{p}\right)^{1/p'}\right)\\
&\leqslant \sigma\left(\left(\int_{\Omega} \varrho(x)^{p'}\right)^{1/p'} + \left(\frac{1}{\lambda}\int_{\Omega} \mathbf{a}(x, Du_n) \cdot Du_n\right)^{1/p'}\right)\\
&\leqslant \sigma\|\varrho\|_{L^{p'}(\Omega)} + \sigma\left(\frac{C}{\lambda} n\right)^{1/p'}.
\end{aligned}$$

Consequently,

$$\left(\int_{\Omega} \left| \frac{1}{n} \mathbf{a}(x, Du_n) \right|^{p'} \right)^{1/p'} \leqslant \frac{\sigma \|\varrho\|_{L^{p'}(\Omega)}}{n} + \sigma \left(\frac{C/\lambda}{n^{p'-1}} \right)^{1/p'}. \tag{53}$$

On the other hand, taking $\phi \in \mathcal{D}(\Omega)$ in (47) we have

$$\frac{1}{n} \int_{\Omega} \mathbf{a}(x, Du_n) \cdot D\phi + \int_{\Omega} z_n \phi = \int_{\Omega} z\phi.$$

By (53), we get (52). Consequently,

$$z_n \rightharpoonup z \quad \text{weakly in } L^1(\Omega). \tag{54}$$

Having in mind (54) and (53), it follows from (47) that

$$\int_{\partial\Omega} w_n \phi \to \int_{\partial\Omega} w\phi \quad \text{for any } \phi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega). \tag{55}$$

Therefore, in the case $\mathrm{Dom}(\beta) = \mathbb{R}$, by (55) we find that

$$w_n \rightharpoonup w \quad \text{weakly in } L^1(\partial\Omega).$$

Similarly, $\hat{z}_n \rightharpoonup z$ weakly in $L^1(\Omega)$, hence by (49), $\hat{z}_n \to z$ in $L^1(\Omega)$. Therefore, in the case of $\mathbf{a}$ smooth, from (50), (51) and a similar argument, we infer that

$$w_n \rightharpoonup w \quad \text{weakly in } L^1(\partial\Omega).$$

Observe that for any $b \geqslant 0$ and $c \geqslant 0$, we also have

$$(z_n - b)^+ \rightharpoonup z_b \geqslant (z - b)^+, \quad (w_n - c)^+ \rightharpoonup w_c \geqslant (w - c)^+.$$

Now, if $c \notin \mathrm{Ran}(\beta)$, then $\int_{\partial\Omega} (w_n - c)^+ \leqslant 0$, therefore $\int_{\partial\Omega} (w - c)^+ \leqslant \int_{\partial\Omega} w_c \leqslant 0$, and so

$$w_c = (w - c)^+.$$

On the other hand, if $c \in \mathrm{Ran}(\beta)$, there exists $a \geqslant 0$ such that $c \in \beta(a)$; taking $b \in \gamma(a)$, since $[a, b, c]$ is an entropy solution of problem $({}_{\mathbf{a}_n} S_{b,c}^{\gamma,\beta})$, we have

$$\int_{\Omega} (z_n - b)^+ + \int_{\partial\Omega} (w_n - c)^+ \leqslant \int_{\Omega} (z - b)^+ + \int_{\partial\Omega} (w - c)^+. \tag{56}$$

Taking limits in (56), we get

$$\int_{\Omega} (z - b)^+ + \int_{\partial\Omega} (w - c)^+ \leqslant \int_{\Omega} z_b + \int_{\partial\Omega} w_c \leqslant \int_{\Omega} (z - b)^+ + \int_{\partial\Omega} (w - c)^+,$$

hence again

$$w_c = (w - c)^+.$$

Consequently, for any $c \geqslant 0$,

$$(w_n - c)^+ \rightharpoonup (w - c)^+ \quad \text{weakly in } L^1(\partial\Omega). \tag{57}$$

Working similarly, we also get

$$(w_n + c)^- \rightharpoonup (w + c)^- \quad \text{weakly in } L^1(\partial\Omega). \tag{58}$$

By (57) and (58), proceeding as in the proof of [13, Proposition 2.11], we conclude that

$$w_n \to w \quad \text{in } L^1(\partial\Omega).$$

For $b \geqslant 0$, we have

$$(z_n - b)^+ \rightharpoonup z_b \geqslant (z - b)^+.$$

Now, if $b \notin \mathrm{Ran}(\gamma)$, that $\int_\Omega (z-b)^+ \leqslant \int_\Omega z_b \leqslant 0$, hence $z_b = (z-b)^+$. On the other hand, if $b \in \mathrm{Ran}(\gamma)$, then there exists $a \geqslant 0$ such that $b \in \gamma(a)$. If $a \in \mathrm{Dom}(\beta)$, taking $c \in \beta(a)$, we obtain

$$\int_\Omega (z_n - b)^+ + \int_{\partial\Omega} (w_n - c)^+ \leqslant \int_\Omega (z - b)^+ + \int_{\partial\Omega} (w - c)^+. \tag{59}$$

And if $a \notin \mathrm{Dom}(\beta)$ (so assuming **a** is smooth), we take $b^m = b + \frac{1}{m}a$, which belongs to $\gamma^{m,k(m)}(a)$ and satisfies

$$\lim_{m\to\infty} b^m = b.$$

Now, since $[(u_n)_r^{m,k}, (z_n)_r^{m,k}, (w_n)_r^{m,k}]$ is a weak solution of $(S_{z,w}^{\gamma^{m,k},\beta_r^{m,k}})$, we have

$$\int_\Omega ((z_n)_r^{m,k} - b^m)^+ + \int_{\partial\Omega} ((w_n)_r^{m,k(m)} - \beta_r^{m,k}(a))^+ \leqslant \int_\Omega (z - b^m)^+ + \int_{\partial\Omega} (w - \beta_r^{m,k}(a))^+.$$

Then, letting r go to $+\infty$ and having in mind that $\lim_r \beta_r^{m,k}(a) = +\infty$, we get

$$\int_\Omega ((z_n)^{m,k} - b^m)^+ \leqslant \int_\Omega (z - b^m)^+. \tag{60}$$

Take the subsequence $k(m)$ used in (48). Then, taking limits when m goes to $+\infty$ in (60) with $k = k(m)$, we get

$$\int_\Omega (z_n - b)^+ \leqslant \int_\Omega (z - b)^+. \tag{61}$$

Now, letting n go to $+\infty$ in (59) and (61), we have

$$\int_\Omega z_b \leqslant \int_\Omega (z - b)^+,$$

and therefore $z_b = (z-b)^+$. Hence, for any $b \geqslant 0$,

$$(z_n - b)^+ \rightharpoonup (z - b)^+ \quad \text{weakly in } L^1(\Omega).$$

Similarly, we can get

$$(z_n + b)^- \rightharpoonup (z + b)^- \quad \text{weakly in } L^1(\Omega).$$

From these convergences we deduce that $z_n \to z$ in $L^1(\Omega)$, and the proof is complete. □

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