AN OPTIMAL MATCHING PROBLEM FOR THE EUCLIDEAN DISTANCE

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To the memory of Vicent Caselles, an outstanding mathematician and friend.

Abstract. We deal with an optimal matching problem, that is, we want to transport two measures to a given place (the target set), where they will match, minimizing the total transport cost that in our case is given by the sum of the Euclidean distance that each measure is transported. We show that such a problem has a solution with matching measure concentrated on the boundary of the target set. Furthermore we perform a method to approximate the solution of the problem taking limit as $p \to \infty$ in a system of PDE's of $p-$Laplacian type.

Key words. Optimal matching problem, Monge-Kantorovich's mass transport theory, p-Laplacian systems

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1. Introduction.

We are interested in an optimal matching problem (see [9], [8]) that consists in transporting two commodities (say nuts and screws, we assume that we have the same total number of nuts and screws) to a prescribed location, the target set (say factories where we ensemble the nuts and the screws) in such a way that they match there (each factory receive the same number of nuts and of screws) and the total cost of the operation, measured in terms of the Euclidean distance that the commodities are transported, is minimized.

Optimal matching problems for uniformly convex cost where analyzed in [5], [6], [8], [9] and have implications in economic theory (hedonic markets and equilibria), see [9], [10], [11], [12], [8] and references therein. However, when one considers the Euclidean distance as cost new difficulties appear since we deal with a non-uniformly convex cost.

Clearly, the optimal matching problem under consideration is related to the classical Monge-Kantorovich's mass transport problem. By using tools from this theory, it follows the existence of a solution of the optimal matching problem. The existence of solution is true for any norm in $\mathbb{R}^N$. We show the existence of a matching measure concentrated on the boundary of the target set. Next, our main contribution in this paper is to perform a method to solve the problem taking limit as $p \to \infty$ in a system of PDE's of $p-$Laplacian type, which allows us to give more information about the matching measure and the Kantorovich potentials for the involved transport. This procedure to solve mass transport problems (taking limit as $p \to \infty$ in a $p-$Laplacian equation) was introduced by Evans and Gangbo in [15] and remains quite fruitful, see [2], [20], [17]. We have to remark that the limit as $p \to \infty$ in the system requires some care since the system is nontrivially coupled and therefore the estimates for one component are related to the ones for the other, and we believe that it is interesting in its own right.

1.1. The optimal matching problem.

To write the optimal matching problem in mathematical terms, we fix two non-negative compactly supported functions

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$f^+,\ f^- \in L^\infty$, with supports $X_+,\ X_-$, respectively, satisfying the mass balance condition

$$M_0 := \int_{X_+} f^+ = \int_{X_-} f^-.$$

We also consider a compact set D (the target set). Then we take a large bounded domain Ω such that it contains all the relevant sets, the supports of f_+ and f_-, X_+, X_- and the target set D. For simplicity we will assume that Ω is a convex C^2 bounded open set. We also assume that

$$X_+ \cap\ X_- = \emptyset, \quad \big(X_+ \cup X_-\big) \cap D = \emptyset \quad \text{and} \quad \big(X_+ \cup X_-\big) \cup D \subset\subset \Omega. \tag{1.1}$$

Whenever T is a map from a measure space (X, μ) to an arbitrary space Y, we denote by $T\#\mu$ the pushforward measure of μ by T. Explicitly, $(T\#\mu)[B] = \mu[T^{-1}(B)]$. When we write $T\#f = g$, where f and g are nonnegative functions, this means that the measure having density f is pushed-forward to the measure having density g.

For Borel functions $T_\pm : \Omega \to \Omega$ such that $T_+\#f^+ = T_-\#f^-$, we consider the functional

$$\mathcal{F}(T_+, T_-) := \int_\Omega |x - T_+(x)| f^+(x) dx + \int_\Omega |y - T_-(y)| f^-(y) dy,$$

where $|\cdot|$ denotes the Euclidean norm. The optimal matching problem can be stated as the minimization problem

$$\min_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+, T_-), \tag{1.2}$$

where

$$\mathcal{A}_D(f^+, f^-) :=$$
$$\Big\{(T_+, T_-) :\ T_\pm : \Omega \to \Omega \text{ are Borel functions, } T_\pm(X_\pm) \subset D,\ T_+\#f^+ = T_-\#f^-\Big\}.$$

If $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ is a minimizer of the optimal matching problem (1.2), we shall call the measure $\mu^* := T_+^*\#f^+ = T_-^*\#f^-$ a *matching measure* to the problem. Note that there is no reason why a matching measure should be absolutely continuous with respect to the Lebesgue measure. In fact we shall see examples of matching measures that are singular (see Example 4.1).

We denote by $\mathcal{M}(\Omega)$ the set of all Radon measures on Ω and by $\mathcal{M}^+(\Omega)$ the non-negative elements of $\mathcal{M}(\Omega)$. Given $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition

$$\mu(\Omega) = \nu(\Omega) \tag{1.3}$$

we denote by $\mathcal{A}(\mu, \nu)$ the set of transport maps pushing μ to ν, that is, the set of Borel maps $T : \Omega \to \Omega$ such that $T\#\mu = \nu$. In the case $\mu = f\mathcal{L}^N \llcorner \Omega$ and $\nu = g\mathcal{L}^N \llcorner \Omega$, we shall write $\mathcal{A}(f, g)$.

We denote by

$$\mathcal{M}(D, M_0) := \{\mu \in \mathcal{M}^+(\Omega)\ :\ \text{supp}(\mu) \subset D,\ \mu(\Omega) = M_0\}$$

the set of all possible matching measures. Given $\mu \in \mathcal{M}(D, M_0)$, we have that

$$\begin{aligned} \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+,T_-) &= \inf_{\mu\in\mathcal{M}(D,M_0)} \inf_{(T_+,T_-)\in\mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+,T_-) \\ &= \inf_{\mu\in\mathcal{M}(D,M_0)} \Big\{ W_1(f_+,\mu) + W_1(f_-,\mu) \Big\}. \end{aligned} \tag{1.4}$$

where $\mathcal{A}(f^+, f^-, \mu) := \{(T_+, T_-) \ : \ T_+ \in \mathcal{A}(f^+, \mu),\ T_- \in \mathcal{A}(f^-, \mu)\}$, and where $W_1(\cdot,\cdot)$ denotes the 1-Wasserstein distance (its definition is given in (1.6) below). Indeed, observe that given $(T_+, T_-) \in \mathcal{A}_D(f^+, f^-)$, if we define $\mu := T_+\#f_+$, we have that $\mu \in \mathcal{M}(D, M_0)$ and $(T_+, T_-) \in \mathcal{A}(f^+, f^-, \mu)$. By convenience we will call

$$W^D_{f^\pm} := \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+,T_-).$$

Note that on the right-hand side of (1.4) we are considering all possible measures supported in D with total mass M_0 and then we minimize the total transport cost. This is probably the most natural way of looking at the optimal matching problem and, as shown above, it is equivalent to our previous formulation.

We have the following existence theorem.

THEOREM 1.1. *The optimal matching problem* (1.2) *has a solution, that is, there exist Borel functions* $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ *such that*

$$\mathcal{F}(T_+^*, T_-^*) = \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+,T_-).$$

Moreover, we can obtain a solution $(\tilde{T}_+, \tilde{T}_-)$ *of the optimal matching problem* (1.2) *with a matching measure supported on the boundary of* D.

REMARK 1.2. We note that the fact that there is an optimal matching measure supported on ∂D greatly simplifies the problem, since it allows to reduce the target set to its boundary. Moreover we will show that the matching measure is not unique in general. For less degenerate cost functions than the Euclidean one, the existence of a matching measure supported on the boundary of D is not true in general, even if it is absolutely continuous with respect to Lebesgue measure, and unique, see for instance [9, 6]. See also [21, 22] for related problems.

We provide two different proofs of the existence theorem. The first one is more direct but does not provide a constructive way of getting the optimal matching measure in D, which is one of the unknowns in this problem; consequently, the construction of optimal transport maps (that are proved to exist) remains a difficult task. The main tool in this first proof is the use of ingredients from the classical Mass Transport Theory. The second proof is by approximation of the associated Kantorovich potentials by a system of $p-$Laplacian type problems when p goes to ∞. This approach provides an approximation of the potentials but also allows us to obtain the optimal measure in the limit. In addition we present several examples (that show that, in general, there is no uniqueness of the optimal configuration) and characterize when the optimal matching measure is a Dirac delta.

Let us now introduce some notations, concepts and results from the Monge-Kantorovich Mass Transport Theory (see [2], [14], [23] and [24]) that will be used in the rest of the paper.

1.2. Monge-Kantorovich's Mass Transport Theory.

The Monge problem. *Given $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition* (1.3)*. The Monge problem, associated with the measures μ and ν, is to find a map $T^* \in \mathcal{A}(\mu,\nu)$ which minimizes the cost functional*

$$\tilde{\mathcal{F}}(T) := \int_\Omega |x - T(x)| \, d\mu(x)$$

in the set $\mathcal{A}(\mu,\nu)$. A map $T^* \in \mathcal{A}(\mu,\nu)$ satisfying $\tilde{\mathcal{F}}(T^*) = \min\{\tilde{\mathcal{F}}(T) \,:\, T \in \mathcal{A}(\mu,\nu)\}$, is called an optimal transport map of μ to ν.

In general, the Monge problem is ill-posed. To overcome the difficulties of the Monge problem, in 1942, L. V. Kantorovich ([18]) proposed to study a relaxed version of the Monge problem and, what is more relevant here, introduced a dual variational principle.

Let us define $\pi_t(x,y) := (1-t)x + ty$. Given a Radon measure γ in $\Omega \times \Omega$, its marginals are defined by $proj_x(\gamma) := \pi_0 \# \gamma$, $proj_y(\gamma) := \pi_1 \# \gamma$.

The Monge-Kantorovich problem. *Fix $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition* (1.3)*. The Monge-Kantorovich problem is the minimization problem*

$$\int_{\Omega\times\Omega} |x-y| \, d\gamma^*(x,y) = \min\left\{ \int_{\Omega\times\Omega} |x-y| \, d\gamma(x,y) \,:\, \gamma \in \Pi(\mu,\nu) \right\},$$

where

$$\Pi(\mu,\nu) := \{\textit{Radon measures } \gamma \textit{ in } \Omega\times\Omega : \pi_0\#\gamma = \mu, \pi_1\#\gamma = \nu\}.$$

The elements $\gamma \in \Pi(\mu,\nu)$ are called transport plans between μ and ν, and a minimizer γ^* an optimal transport plan. These minimizers always exist.

The Monge-Kantorovich problem has a dual formulation that can be stated in this case as follows (see for instance [23, Theorem 1.14]).

Kantorovich-Rubinstein Theorem. *Let $\mu, \nu \in \mathcal{M}(\Omega)$ be two measures satisfying the mass balance condition* (1.3)*. Then,*

$$\begin{aligned} &\min\left\{ \int_{\Omega\times\Omega} |x-y| \, d\gamma(x,y) \,:\, \gamma \in \Pi(\mu,\nu) \right\} \\ &\qquad = \sup\left\{ \int_\Omega u \, d(\mu-\nu) \,:\, u \in K_1(X) \right\}, \end{aligned} \tag{1.5}$$

where $K_1(\Omega) := \{u : X \to \mathbb{R} \;:\; |u(x)-u(y)| \le |x-y| \;\; \forall x,y \in \Omega\}$ is the set of 1-Lipschitz functions in Ω. The maximizers u^* of the right hand side of (1.5) are called *Kantorovich potentials*.

For two measures $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition (1.3), the 1-Wasserstein distance (also called Kantorovich-Rubinstein distance) between μ and ν is defined as

$$W_1(\mu,\nu) := \inf\left\{ \int_{\Omega\times\Omega} |x-y| \, d\gamma(x,y) \,:\, \gamma \in \Pi(\mu,\nu) \right\}.$$

In the case μ has no atom, by [2, Theorem 2.1], we have that

$$W_1(\mu,\nu) = \inf\left\{ \int_\Omega |x - T(x)| \, d\mu(x) \,:\, T \in \mathcal{A}(\mu,\nu) \right\}. \tag{1.6}$$

Let us briefly summarize the contents of this paper. Section 2 is devoted to the proof of Theorem 1.1; in Section 3 we study the limit as $p \to \infty$ in a $p-$Laplacian system obtaining more information about the solution of the matching problem; in Section 4 we describe some examples and characterize the geometrical configurations for which the matching measure is a point mass, finally, in Section 5 we collect final remarks.

2. Proof of Theorem 1.1.

Proof. The existence part follows by standard arguments in Mass Transport Theory. Indeed, take in (1.4) a minimizing sequence $\mu_n \in \mathcal{M}(D, M_0)$, then by the weak compactness of $\mathcal{M}(D, M_0)$ there exist a subsequence, denoted equal, that converges weakly in the sense of measures to a $\mu_\infty \in \mathcal{M}(D, M_0)$. Hence, by the weakly lower semi-continuity of the function $\nu \mapsto W_1(\mu, \nu)$, we have

$$W_1(f_+, \mu_\infty) + W_1(f_-, \mu_\infty) \leq \lim_n \left(W_1(f_+, \mu_n) + W_1(f_-, \mu_n)\right) = W^D_{f^\pm}.$$

Therefore,

$$W_1(f_+, \mu_\infty) + W_1(f_-, \mu_\infty) = W^D_{f^\pm}.$$

Now, by [2, Theorem 6.2], which states the existence of an optimal transport map T^*_+ transferring f^+ to μ_∞, and an optimal transport map T^*_- transferring f^- to μ_∞, we obtain that

$$\mathcal{F}(T^*_+, T^*_-) = W^D_{f^\pm}.$$

This finishes the proof of the existence.

Now, let us show that we can restrict ourselves to matching measures supported on ∂D. Let us consider a minimizer (T^*_+, T^*_-) of the matching problem and $h_\infty = T^*_+ \# f_+$ the corresponding matching measure. Let us see that we can obtain a matching measure supported on ∂D. For $x \in \text{supp}(f^+)$, let

$$\alpha(x) := \min\{\alpha \in [0,1] \ : \ (1-\alpha)x + \alpha T^*_+(x) \in D\}.$$

Applying [4, Corollary 1] to the function $f : \text{supp}(f^+) \times [0,1] \to]-\infty, +\infty]$ defined by

$$f(x,a) := \begin{cases} +\infty & \text{if } (1-a)x + aT^*(x) \notin D, \\ a & \text{if } (1-a)x + aT^*(x) \in D, \end{cases}$$

we get easily that α is Borel measurable function. We define

$$\tilde{T}_+(x) := (1-\alpha(x))x + \alpha(x) T^*_+(x),$$

that is, $\tilde{T}_+(x)$ is the first point in D of the segment that goes from x to $T^*_+(x)$ (remember that we are under condition (1.1)). Then,

$$\begin{aligned} &\int_\Omega |x - T^*_+(x)| f^+(x) dx \\ &= \int_\Omega |x - \tilde{T}_+(x)| f^+(x) dx + \int_\Omega |\tilde{T}_+(x) - T^*_+(x)| f^+(x) dx \qquad (2.1) \\ &= \int_{\Omega\times\Omega} |x-y| d((Id \times \tilde{T}_+)\# f^+)(x,y) + \int_{\Omega\times\Omega} |x-y| d((\tilde{T}_+ \times T^*_+)\# f^+)(x,y). \end{aligned}$$

If we define the measure $\tilde{h}_\infty := \tilde{T}_+ \# f^+$, which is supported on ∂D, we have that $(Id \times \tilde{T}_+)\# f^+$ is a transport plan induced by the map $\tilde{T}_+$ between f^+ and the measure $\tilde{h}_\infty$. On the other hand, a simple computation shows that

$$\tilde{\gamma}(x,y) := ((\tilde{T}_+ \times T_+^*)\# f^+)(x,y)$$

is a transport plan between $\tilde{h}_\infty$ and h_∞. Now, by (2.1), $(Id \times \tilde{T}_+)\# f^+$ is an optimal transport plan between f^+ and $\tilde{h}_\infty$, and $\tilde{\gamma}$ is an optimal transport plan between $\tilde{h}_\infty$ and h_∞.

By [2, Theorem 6.2], there exists an optimal transport map $\tilde{T}_-$ transferring f^- to $\tilde{h}_\infty$. Let us see that $(\tilde{T}_+, \tilde{T}_-)$ is a solution, for the matching problem, that is,

$$\mathcal{F}(\tilde{T}_+, \tilde{T}_-) = \mathcal{F}(T_+^*, T_-^*). \tag{2.2}$$

Indeed, by (2.1) and the triangular inequality for the 1-Wasserstein distance, we have

$$\begin{aligned}\mathcal{F}(T_+^*, T_-^*) &= W_1(f_+, h_\infty) + W_1(h_\infty, f_-) = W_1(f_+, \tilde{h}_\infty) + W_1(\tilde{h}_\infty, h_\infty) + W_1(h_\infty, f_-)\\ &\geq W_1(f_+, \tilde{h}_\infty) + W_1(\tilde{h}_\infty, f_-) = \mathcal{F}(\tilde{T}_+, \tilde{T}_-).\end{aligned}$$

Therefore, (2.2) holds and $(\tilde{T}_+, \tilde{T}_-)$ is a solution for the matching problem with matching measure $\tilde{h}_\infty$ supported on ∂D. □

Remark 2.1. Having in mind the results in [7], let us remark that Theorem 1.1 is also true in the case that we change in the cost function the Euclidean norm by any norm in $\mathbb{R}^N$.

We also point out that the existence part of Theorem 1.1 is essentially already known since it is contained in a very general result given in [6] although with a different formulation.

3. The limit as $p \to \infty$ in a $p-$Laplacian system. In this section we show that we can follow the ideas of Evans-Gangbo, [15], to get the matching measure, and Kantorovich potentials for the transports involved at the same time. Let us begin with the following proposition.

Proposition 3.1.

$$W_{f^\pm}^D := \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+,T_-) = \sup_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega)\\ |\nabla v|_\infty,\, |\nabla w|_\infty \,\le\, 1\\ v \,\le\, w \text{ in } D}} \int_\Omega v f^+ - w f^- .$$

Proof. For a fixed $\mu \in \mathcal{M}(D, M_0)$, it is well known (see for instance [2, 23]) that

$$\max_{u\in W^{1,\infty}(\Omega),\, |\nabla u|_\infty \le 1} \int_\Omega u(f^+ - \mu) = \min_{T\in\mathcal{A}(f^+,\mu)} \int_\Omega |x - T(x)| f^+(x)dx,$$

and

$$\max_{u\in W^{1,\infty}(\Omega),\, |\nabla u|_\infty \le 1} \int_\Omega u(f^- - \mu) = \min_{T\in\mathcal{A}(f^-,\mu)} \int_\Omega |x - T(x)| f^-(x)dx.$$

Therefore,

$$\sup_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega)\\ |\nabla v|_\infty,\, |\nabla w|_\infty \,\le\, 1}} \int_\Omega v f^+ - w f^- + (w - v)\mu = \min_{(T_+,T_-)\in\mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+,T_-)\,. \tag{3.1}$$

Since

$$\begin{aligned}
&\int_\Omega vf^+ - wf^- + (w-v)\mu \\
&= \int_\Omega vf^+ - (w - \min_D(w-v))f^- + (w - \min_D(w-v) - v)\mu \\
&\leq \sup_{\substack{\tilde v,\, \tilde w \,\in\, W^{1,\infty}(\Omega) \\ |\nabla \tilde v|_\infty,\, |\nabla \tilde w|_\infty \leq 1 \\ \tilde v \leq \tilde w \text{ in } D}} \int_\Omega \tilde v f^+ - \tilde w f^- + (\tilde w - \tilde v)\mu,
\end{aligned}$$

we have

$$\begin{aligned}
&\sup_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega) \\ |\nabla v|_\infty,\, |\nabla w|_\infty \leq 1 \\ v \leq w \text{ in } D}} \int_\Omega vf^+ - wf^- + (w-v)\mu \\
&= \min_{(T_+,T_-)\in\mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+,T_-)\,.
\end{aligned} \tag{3.2}$$

Hence,

$$\begin{aligned}
&\inf_{\mu\in\mathcal{M}(D,M_0)} \sup_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega) \\ |\nabla v|_\infty,\, |\nabla w|_\infty \leq 1 \\ v \leq w \text{ in } D}} \int_\Omega vf^+ - wf^- + (w-v)\mu \\
&= \inf_{\mu\in\mathcal{M}(D,M_0)} \min_{(T_+,T_-)\in\mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+,T_-) = W^D_{f^\pm}.
\end{aligned}$$

Now, by Fan's Minimax Theorem ([16]), we can interchange inf sup by sup inf in the first part of the above expression and, since

$$\begin{aligned}
&\sup_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega) \\ |\nabla v|_\infty,\, |\nabla w|_\infty \leq 1 \\ v \leq w \text{ in } D}} \min_{\mu\in\mathcal{M}(D,M_0)} \int_\Omega vf^+ - wf^- + (w-v)\mu \\
&= \sup_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega) \\ |\nabla v|_\infty,\, |\nabla w|_\infty \leq 1 \\ v \leq w \text{ in } D}} \int_\Omega vf^+ - wf^-\,,
\end{aligned}$$

we get the desired conclusion. □

This result is the starting point of our variational approach to the problem via a $p-$Laplacian system in this section.

Take $p > N$ in this section and recall that, for simplicity, we assumed that Ω is a convex C^2 bounded open set.

We will use the following result whose proof follows standard Functional Analysis arguments.

LEMMA 3.2 (A Poincaré's type inequality). *There exists a constant $C > 0$ such that*

$$\|(f,g)\|_p \leq C\left(\|(\nabla f, \nabla g)\|_p + \left|\int_\Omega (f+g)\right|\right)$$

for all $(f,g) \in W^{1,p}(\Omega)\times W^{1,p}(\Omega)$, $f(x_0) = g(x_0)$ for some $x_0 \in D$.

REMARK 3.3. The constant that appears in Lemma 3.2 may depend on p. It is not our aim in this paper to make this dependence precise, then we are not allowed to use this result in the passage to the limit as $p \to \infty$. Lemma 3.2 is only used to show existence and uniqueness of a solution of the elliptic system under consideration. To pass to the limit we rely on a local Morrey's inequality, see the proof of Theorem 3.5 below.

Let us consider the following variational problem

$$\min_{\substack{(v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v \le w \text{ in } D}} \frac{1}{p} \int_\Omega |Dv|^p + \frac{1}{p} \int_\Omega |Dw|^p - \int_\Omega v f^+ + \int_\Omega w f^-. \tag{3.3}$$

Our next result in this section deals with existence and uniqueness of solutions for the variational problem (3.3).

THEOREM 3.4. *There exists a minimizer* (v_p, w_p) *of* (3.3). *In addition any two minimizers differ by a constant, that is, if* (v_p, w_p) *and* $(\tilde v_p, \tilde w_p)$ *are minimizers then there exists a constant* c *with* $v_p = \tilde v_p + c$ *and* $w_p = \tilde w_p + c$.

Proof. Set

$$\Psi_p(v,w) := \frac{1}{p} \int_\Omega |Dv|^p + \frac{1}{p} \int_\Omega |Dw|^p - \int_\Omega v f^+ + \int_\Omega w f^- .$$

Let us begin by observing that, since the functions in $W^{1,p}(\Omega)$ are continuous, it is easy to see that

$$\min_{\substack{(v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v \le w \text{ in } D}} \Psi_p(v,w) = \min_{\substack{(v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v \le w \text{ in } D \\ \exists x_0 \in D,\ v(x_0) = w(x_0)}} \Psi_p(v,w). \tag{3.4}$$

Moreover, since

$$\Psi_p(v,w) = \Psi_p(v-c, w-c) \quad \text{for any constant } c,$$

by taking

$$c = \frac{1}{2} \left(\frac{1}{|\Omega|} \int_\Omega v + \frac{1}{|\Omega|} \int_\Omega w \right),$$

we can minimize $\Psi_p(v,w)$ between functions (v,w) with

$$\int_\Omega v + \int_\Omega w = 0.$$

Now, by Lemma 3.2,

$$\Psi_p(v,w) := \frac{1}{p} \int_\Omega |Dv|^p + \frac{1}{p} \int_\Omega |Dw|^p - \int_\Omega v f^+ + \int_\Omega w f^-$$

is a finite lower semicontinuous and coercive convex functional for the closed convex subset of $W^{1,p}(\Omega) \times W^{1,p}(\Omega)$

$$\mathcal{B} := \Big\{ (v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) :\ v \le w \text{ in } D,\ v(x_0) = w(x_0) \\ \text{for some } x_0 \in D,\ \int_\Omega (v+w) = 0 \Big\}.$$

Then, by [3, Corollary 3.23], Ψ_p attains its infimum on $\mathcal{B}$, which is equivalent to say that

$$\inf_{\substack{(v,\,w)\,\in\, W^{1,p}(\Omega)\times W^{1,p}(\Omega)\\ v\,\le\, w \text{ in } D}} \Psi_p(v,w)$$

is attained.

Finally, let us show uniqueness of the minimizer up to an additive constant. Equivalently, we prove uniqueness of the minimizer when we impose the constraint

$$\int_\Omega v + \int_\Omega w = 0.$$

Assume that we have two pairs (v_p, w_p) and $(\tilde{v}_p, \tilde{w}_p)$ of minimizers and that

$$\int_\Omega v_p + \int_\Omega w_p = \int_\Omega \tilde{v}_p + \int_\Omega \tilde{w}_p = 0. \tag{3.5}$$

By the strict convexity of the function $\xi \mapsto \|\xi\|_p$ (we have $1 < p < \infty$) we obtain that $Dv_p = D\tilde{v}_p$ and $Dw_p = D\tilde{w}_p$. Then there are constants c_1 and c_2 such that $v_p = \tilde{v}_p + c_1$ and $w_p = \tilde{w}_p + \tilde{c}_2$. Hence, from (3.5) we get that

$$c_1 + c_2 = 0.$$

Therefore, we obtain

$$\Psi_p(v_p, w_p) = \Psi_p(\tilde{v}_p, \tilde{w}_p) - c_1\left(\int_\Omega f^+ + \int_\Omega f^-\right)$$

and we conclude that

$$c_1 = c_2 = 0$$

from the fact that both pairs are minimizers. □

Now we prove that we can pass to the limit as $p \to \infty$ in the sequence of minimizer functions.

THEOREM 3.5. *Let (v_p, w_p) be minimizer functions of (3.3). Then, up to a subsequence,*

$$\lim_{p\to\infty} (v_p, w_p) = (v_\infty, w_\infty) \qquad \textit{uniformly,}$$

where (v_∞, w_∞) is a solution of the variational problem

$$\max_{\substack{v,\,w\,\in\, W^{1,\infty}(\Omega)\\ |\nabla v|_\infty,\,|\nabla w|_\infty\,\le\,1\\ v\,\le\, w \text{ in } D}} \int_\Omega v f^+ - w f^-. \tag{3.6}$$

REMARK 3.6. As we will see, the limit (v_∞, w_∞) gives a pair of Kantorovich potentials for our optimal matching problem. But in fact this limit procedure gives much more since it allows us to identify the optimal matching measure (see Theorem 3.9 below).

Proof. [Proof of Theorem 3.5] Let us take $(v_p, w_p) \in \mathcal{B}$ a minimizer of (3.3). For $(v,w) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$, with $|\nabla v|_\infty, |\nabla w|_\infty \le 1$ and $v \le w$ in D, we have that

$$\begin{aligned} -\int_\Omega v_p f^+ + \int_\Omega w_p f^- &\le \frac{1}{p}\int_\Omega |Dv_p|^p + \frac{1}{p}\int_\Omega |Dw_p|^p - \int_\Omega v_p f^+ + \int_\Omega w_p f^- \\ &\le \frac{1}{p}\int_\Omega |Dv|^p + \frac{1}{p}\int_\Omega |Dw|^p - \int_\Omega v f^+ + \int_\Omega w f^- \\ &\le 2\frac{|\Omega|}{p} - \int_\Omega v f^+ + \int_\Omega w f^- . \end{aligned} \tag{3.7}$$

Now, by (3.4), we can assume that there exists $x_p \in D$ such that $v_p(x_p) = w_p(x_p)$. We can also assume that $v_p(z_\infty) = 0$ for all p, for any fixed $z_\infty \in \Omega$. Hence, as $p > N$, we have:

$$\|v_p\|_\infty \le C_1 \|Dv_p\|_p, \tag{3.8}$$

and

$$\|w_p\|_\infty \le C_1 \left(\|Dw_p\|_p + \|Dv_p\|_p\right), \tag{3.9}$$

with C_1 not depending on p. Indeed, since Ω is C^2, for a fixed $x \in \Omega$, there exists $x = x_0, x_1, ..., x_m = z_\infty$ and m balls $Q_i \subset \Omega$ $(i = 1, 2, ..., m)$ of certain fixed diameter $r > 0$, such that $x_i, x_{i+1} \in Q_{i+1}$ and m is bounded independently of x, z_∞. Then, local Morrey's inequality (see, e.g., the Remark in page 268 of [13] or [3]), implies

$$\begin{aligned} |v_p(x)| = |v_p(x) - v_p(z_\infty)| &\le \sum_{i=1}^m |v_p(x_i) - v_p(x_{i+1})| \\ &\le C_0 r^{1-N/p} m \|\nabla v_p\|_p \le C_1 \|\nabla v_p\|_p, \end{aligned}$$

being C_i independent of p. With the same argument, but changing the extreme points and the function, we obtain

$$|w_p(x)| = |w_p(x) - w_p(x_p)| + |v_p(x_p)| \le C_1 \|\nabla w_p\|_p + |v_p(x_p)|.$$

From (3.7), using Hölder's inequality and having in mind (3.8) and (3.9), we get

$$\begin{aligned} \frac{1}{p}\int_\Omega |Dv_p|^p + \frac{1}{p}\int_\Omega |Dw_p|^p &\le C_2(\|v_p\|_{L^p(\Omega)} + \|w_p\|_{L^p(\Omega)} + 1) \\ &\le C_3(\|Dv_p\|_{L^p(\Omega)} + \|Dw_p\|_{L^p(\Omega)} + 1), \end{aligned}$$

with C_i independent of p. Hence,

$$\|\nabla v_p\|_{L^p(\Omega)}^{p-1}, \|\nabla w_p\|_{L^p(\Omega)}^{p-1} \le p\, C_4 \qquad \forall p > N, \tag{3.10}$$

with C_4 independent of p.

Therefore, $\|v_p\|_{W^{1,p}(\Omega)}$ and $\|w_p\|_{W^{1,p}(\Omega)}$ are bounded uniformly in p, and, by Morrey's inequality (e.g. [3] or [13])

$$\begin{cases} |v_p(x) - v_p(y)| \le C_5 |x-y|^{1-\frac{N}{p}}, \\ |w_p(x) - w_p(y)| \le C_5 |x-y|^{1-\frac{N}{p}}, \end{cases}$$

for some constant C_5 not depending on p. Then, by Arzela-Ascoli's compactness criterion we can extract a sequence $p_i \to \infty$ such that

$$v_{p_i} \to v_\infty \quad \text{uniformly in } \overline{\Omega},$$

$$w_{p_i} \to w_\infty \quad \text{uniformly in } \overline{\Omega},$$

and, so, $v_\infty \le w_\infty$ in D. Moreover, by (3.10), we have

$$\|\nabla v_\infty\|_\infty,\ \|\nabla w_\infty\|_\infty \le 1.$$

Finally, passing to the limit in (3.7), we get

$$\int_\Omega v_\infty f^+ - w_\infty f^- \quad = \sup_{\substack{v,\, w\, \in\, W^{1,\infty}(\Omega) \\ |\nabla v|_\infty,\, |\nabla w|_\infty \le 1 \\ v \le w \text{ in D}}} \int_\Omega v f^+ - w f^- .$$

This ends the proof. □

REMARK 3.7. Remark that the convergence as $p \to \infty$ is only along a subsequence. The main content of our result is that there is enough compactness to pass to the limit along subsequences and moreover that all possible limits are solutions to the maximization limit problem.

We now prove some properties of the minimizers and their limits that show that we have found (in the limit) Kantorovich potentials and an optimal matching measure for our matching problem.

We divide the proof of these properties into a series of lemmas.

LEMMA 3.8. *Let (v_p, w_p) be minimizer functions of problem* (3.3). *Then, there exists a non-negative Radon measure h_p of mass M_0 such that*

1.

$$\begin{cases} -\Delta_p v_p = f^+ - h_p & \text{in } \ \Omega, \\ |\nabla v_p|^{p-2}\nabla v_p \cdot \eta = 0 & \text{on } \ \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta_p w_p = h_p - f^- & \text{in } \ \Omega, \\ |\nabla w_p|^{p-2}\nabla w_p \cdot \eta = 0 & \text{on } \ \partial\Omega. \end{cases}$$

2. The non-negative Radon measure h_p is supported on $\{x \in D : v_p(x) = w_p(x)\}$.

Proof. Recall that since $p > N$, we have $W^{1,p}(\Omega) \subset C(\overline{\Omega})$. For any $\varphi, \psi \in W^{1,p}(\Omega)$ such that $\varphi = \psi$ in D, since (v_p, w_p) is a minimizer of Ψ_p in the set

$$\{(v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \ : \ v \le w \text{ in } D\},$$

the function

$$I_1(t) := \Psi_p(v_p + t\varphi, w_p + t\psi)$$

has a minimum at $t = 0$. Therefore, $I_1'(0) = 0$, from where it follows that

$$\int_\Omega |\nabla v_p|^{p-2}\nabla v_p \nabla\varphi + \int_\Omega |\nabla w_p|^{p-2}\nabla w_p \nabla\psi = \int_\Omega f^+\varphi - \int_\Omega f^-\psi. \tag{3.11}$$

Observe that, taking $\psi = \varphi$ in (3.11), we get that

$$\begin{cases} -\Delta_p v_p - \Delta_p w_p = f^+ - f^- & \text{in } \Omega, \\ |\nabla v_p|^{p-2}\nabla v_p \cdot \eta + |\nabla w_p|^{p-2}\nabla w_p \cdot \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Similarly, for any $\varphi \in W^{1,p}(\Omega)$, $\varphi \geq 0$, and any $t > 0$, we have

$$I_2(t) := \Psi_p(v_p - t\varphi, w_p) - \Psi_p(v_p, w_p) \geq 0$$

and

$$I_3(t) := \Psi_p(v_p, w_p + t\varphi) - \Psi_p(v_p, w_p) \geq 0.$$

Then, by taking limits in $\frac{I_i(t)}{t}$, $i = 2, 3$, as $t \to 0$, we get

$$\begin{cases} -\Delta_p v_p \leq f^+ & \text{in } \mathcal{D}'(\Omega), \\ -\Delta_p w_p \geq -f^- & \text{in } \mathcal{D}'(\Omega). \end{cases}$$

Hence, $h_p := \Delta_p v_p + f^+$ is a non-negative distribution and therefore defines a non-negative Radon measure which, thanks to (3.12), is equal to $f^- - \Delta_p w_p$. The fact that h_p is supported on $\{x \in D : v_p(x) = w_p(x)\}$ follows from the fact that, for $\varphi \in \mathcal{D}(\Omega)$ supported on $\Omega \setminus \{x \in D : v_p(x) = w_p(x)\}$ and $t \neq 0$ small enough,

$$I_4(t) := \Psi_p(v_p + t\varphi, w_p) - \Psi_p(v_p, w_p) \geq 0.$$

Again, by taking limits in $\frac{I_4(t)}{t}$ as $t \to 0$, we conclude. This gives the proof of (2).

Given $\varphi \in \mathcal{D}(\mathbb{R}^N)$, if we take $\psi \in \mathcal{D}(\Omega)$ such that $\varphi = \psi$ en D, (3.11) says that

$$\int_\Omega |\nabla v_p|^{p-2}\nabla v_p \nabla\varphi + \int_\Omega |\nabla w_p|^{p-2}\nabla w_p \nabla\psi = \int_\Omega f^+\varphi - \int_\Omega f^-\psi.$$

But, since $\psi \in \mathcal{D}(\Omega)$ and $\operatorname{supp}(h_p) \subset D$, we have

$$\int_\Omega |\nabla w_p|^{p-2}\nabla w_p \nabla\psi = \int_\Omega \psi dh_p - \int_\Omega f^-\psi = \int_\Omega \varphi dh_p - \int_\Omega f^-\psi.$$

Then, from the two above expressions, by density we obtain that

$$\int_\Omega |\nabla v_p|^{p-2}\nabla v_p \nabla\varphi = \int_\Omega f^+\varphi - \int_\Omega \varphi dh_p, \quad \forall \varphi \in W^{1,p}(\Omega), \tag{3.13}$$

which shows the first statement in (1) for the first problem. Similarly, we obtain the second one. From here, now, it is an easy consequence that (just take $\varphi = 1$ in (3.13))

$$\int_\Omega dh_p = M_0,$$

and the proof concludes. □

THEOREM 3.9. *Let h_p be the measure obtained in Lemma 3.8 and (v_∞, w_∞) the pair obtained in Theorem 3.5. Then:*

1. Up to a subsequence,

$$h_p \rightharpoonup h_\infty \quad \textit{as } p \to \infty, \textit{ weakly}^* \textit{ as measures,}$$

with h_∞ a non-negative Radon measure of mass M_0 supported on $\{x \in D : v_\infty(x) = w_\infty(x)\}$.

2. (v_∞, w_∞) satisfies:

$$v_\infty \text{ is a Kantorovich potential for the transport of } f^+ \text{ to } h_\infty,$$

$$w_\infty \text{ is a Kantorovich potential for the transport of } h_\infty \text{ to } f^-,$$

with respect to the Euclidean distance.

3. The measure h_∞ is a matching measure to the optimal matching problem (1.2).

Proof. From the last equality in the proof of the previous lemma,

$$\int_\Omega dh_p = M_0,$$

we can assume that there exists a non-negative Radon measure h_∞ of mass M_0 such that, up to a subsequence,

$$h_p \rightharpoonup h_\infty.$$

Let $\varphi \in \mathcal{D}(\Omega)$ be supported on $\Omega \setminus \{x \in D : v_\infty(x) = w_\infty(x)\}$. Then, since

$$\lim_p (v_p, w_p) = (v_\infty, w_\infty) \quad \text{uniformly},$$

there exists $p_0 > N$ such that φ is supported on $\Omega \setminus \{x \in D : v_p(x) = w_p(x)\}$ for all $p \geq p_0$. Therefore,

$$\int_\Omega \varphi dh_\infty = \lim_{p\to\infty} \int_\Omega \varphi dh_p = 0.$$

Consequently, h_∞ is supported on $\{x \in D : v_\infty = w_\infty\}$.

Since $|\xi|^p - |\eta|^p \leq p|\xi|^{p-2}\xi \cdot (\xi - \eta)$ for any ξ, $\eta \in \mathbb{R}^N$, we have

$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla v_p|^p - \int_\Omega (f^+ - dh_p)v_p + \int_\Omega |\nabla v_p|^{p-2}\nabla v_p \cdot (\nabla\varphi - \nabla v_p) - \int_\Omega (f^+ - dh_p)(\varphi - v_p) \\ \leq \frac{1}{p}\int_\Omega |\nabla \varphi|^p - \int_\Omega (f^+ - dh_p)\varphi\end{aligned}$$

for every $\varphi \in W^{1,p}(\Omega)$. Then, having in mind (3.13), we have

$$\int_\Omega |\nabla v_p|^{p-2}\nabla v_p \cdot (\nabla\varphi - \nabla v_p) - \int_\Omega (f^+ - dh_p)(\varphi - v_p) = 0,$$

and we arrive to

$$\frac{1}{p}\int_\Omega |\nabla v_p|^p - \int_\Omega (f^+ - dh_p)v_p \leq \frac{1}{p}\int_\Omega |\nabla \varphi|^p - \int_\Omega (f^+ - dh_p)\varphi \quad \forall\, \varphi \in W^{1,p}(\Omega). \tag{3.14}$$

Therefore, for any $v \in W^{1,\infty}(\Omega)$, $|\nabla v|_\infty \leq 1$,

$$\begin{aligned}-\int_\Omega (f^+ - dh_p)v_p \quad &\leq \frac{1}{p}\int_\Omega |\nabla v_p|^p - \int_\Omega (f^+ - dh_p)v_p \\ &\leq \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega (f^+ - dh_p)v \quad \leq \frac{1}{p}|\Omega| - \int_\Omega (f^+ - dh_p)v.\end{aligned}$$

Taking limit as $p \to \infty$ in the last inequality, we get

$$\int_\Omega (f^+ - dh_\infty)v \le \int_\Omega (f^+ - dh_\infty)v_\infty,$$

from where it follows that

$$\int_\Omega (f^+ - dh_\infty)v_\infty = \sup_{\substack{v \in W^{1,\infty}(\Omega) \\ |\nabla v|_\infty \le 1}} \int_\Omega v(f^+ - dh_\infty),$$

and consequently, v_∞ is a Kantorovich potential for the transport of f^+ to h_∞, with respect to the Euclidean distance. The proof for w_∞ is similar.

Let us now prove *3*. We have

$$W_1(f_+, h_\infty) = \int_\Omega v_\infty(f^+ - h_\infty) \quad \text{and} \quad W_1(f_-, h_\infty) = \int_\Omega w_\infty(h_\infty - f^-).$$

Therefore, by Theorem 3.5, Proposition 3.1 and the fact that h_∞ is supported on $\{x \in D : v_\infty(x) = w_\infty(x)\}$, we get

$$W^D_{f^\pm} = W_1(f_+, h_\infty) + W_1(f_-, h_\infty),$$

which finishes the proof. □

Observe that the above result gives an alternative proof for the first statement in Theorem 1.1. We will see in Theorem 3.13 that in some cases this approach also selects a matching measure supported on the boundary of the target set, which is the second statement of Theorem 1.1.

Remark 3.10. For any matching measure μ_∞ and any optimal pair solution (u_∞, w_∞) of (3.6) we have that

$$\int_\Omega (w_\infty - v_\infty)\mu_\infty = 0, \tag{3.15}$$

which implies that μ_∞ is, in fact, supported where $v_\infty = w_\infty$ in D.

Indeed, using (3.2) for $\mu = \mu_\infty$,

$$W^D_{f^\pm} = \inf_{\mu \in \mathcal{M}(D,M_0)} \ \min_{(T_+,T_-) \in \mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+, T_-)$$

$$= \min_{(T_+,T_-) \in \mathcal{A}(f^+,f^-,\mu_\infty)} \mathcal{F}(T_+, T_-) = \sup_{\substack{v,\, w \in W^{1,\infty}(\Omega) \\ |\nabla v|_\infty, |\nabla w|_\infty \le 1 \\ v \le w \text{ in } D}} \int_\Omega vf^+ - wf^- + (w - v)\mu_\infty$$

$$\ge \int_\Omega v_\infty f^+ - w_\infty f^- + (w_\infty - v_\infty)\mu_\infty \ge \int_\Omega v_\infty f^+ - w_\infty f^- = W^D_{f^\pm},$$

which implies (3.15).

Remark 3.11. Using the terminology and definitions given by Carlier and Ekeland in [6], let us point out that

$$((-v_\infty, w_\infty), ((Id \times T^*_+)\#f^+, (Id \times T^*_-)\#f^-), h_\infty)$$

can be seen as a pure matching equilibria for the marriage problem, that is, for the *matching for two teams* problem, when the cost function is the Euclidean distance.

See also Section 5 for the matching problem for more than two commodities which can be related with the general matching for teams problem.

For the proof of Theorem 3.13, we will use the following result.

LEMMA 3.12. *If (v_p, w_p) is a pair solving the equations in Lemma 3.8 for a non-negative Radon measure h_p, and*

$$v_p \le w_p \quad \text{in} \quad D,$$

$$supp(h_p) \subset \{x \in D : v_p(x) = w_p(x)\},$$

then, (v_p, w_p) is a minimizer in the minimization problem (3.3).

Proof. We have that (see (3.14))

$$\frac{1}{p}\int_\Omega |\nabla v_p|^p - \int_\Omega (f^+ - dh_p)v_p \le \frac{1}{p}\int_\Omega |\nabla \varphi|^p - \int_\Omega (f^+ - dh_p)\varphi \quad \forall\, \varphi \in W^{1,p}(\Omega),$$

and, similarly,

$$\frac{1}{p}\int_\Omega |\nabla w_p|^p - \int_\Omega (dh_p - f^-)w_p \le \frac{1}{p}\int_\Omega |\nabla \psi|^p - \int_\Omega (dh_p - f^-)\psi \quad \forall\, \psi \in W^{1,p}(\Omega).$$

Adding up both expressions, since h_p is supported in D where $v_p = w_p$, and $v_p \le w_p$ in D,

$$\frac{1}{p}\int_\Omega |\nabla v_p|^p + \frac{1}{p}\int_\Omega |\nabla w_p|^p - \int_\Omega f^+ v_p + \int_\Omega f^- w_p$$

$$\le \frac{1}{p}\int_\Omega |\nabla \varphi|^p + \frac{1}{p}\int_\Omega |\nabla \psi|^p - \int_\Omega f^+ \varphi + \int_\Omega f^- \psi$$

for all φ, $\psi \in W^{1,p}(\Omega)$, $\varphi \le \psi$ in D. □

THEOREM 3.13. *Assume that D is the closure of a smooth domain Θ, then h_p is supported on ∂D and hence h_∞ is concentrated on the boundary of D.*

Proof. Let $\tilde v_p, \tilde w_p$ be minimizers of

$$\min_{\substack{(v,\, w)\, \in\, W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v \le w \text{ in } \partial D}} \Psi_p(v, w),$$

and let $\tilde h_p$ be a non-negative Radon measure, $\mathrm{supp}(\tilde h_p) \subset \{x \in \partial D : \tilde v_p(x) = \tilde w_p(x)\}$, such that

$$\begin{cases} -\Delta_p \tilde v_p = f^+ - \tilde h_p & \text{in } \Omega, \\ |\nabla \tilde v_p|^{p-2}\nabla \tilde v_p \cdot \eta = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p \tilde w_p = \tilde h_p - f^- & \text{in } \Omega, \\ |\nabla \tilde w_p|^{p-2}\nabla \tilde w_p \cdot \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

Set now $\overline{v}_p = \tilde v_p$ in $\Omega \setminus D$, and define $\overline{v}_p$ in D as the solution of

$$\begin{cases} -\Delta_p v = 0 & \text{in } \Theta, \\ v = \tilde v_p & \text{on } \partial D. \end{cases}$$

Similarly we define $\overline{w}_p$. Observe that, by the Maximum Principle,

$$\overline{v}_p \le \overline{w}_p \quad \text{in } D$$

and also

$$\int_\Theta |D\overline{v}_p|^p \le \int_\Theta |D\tilde{v}_p|^p, \qquad \int_\Theta |D\overline{w}_p|^p \le \int_\Theta |D\tilde{w}_p|^p.$$

Then, we get

$$\Psi_p(\overline{v}_p, \overline{w}_p) \le \Psi_p(\tilde{v}_p, \tilde{w}_p).$$

But, in fact, since $\overline{v}_p \le \overline{w}_p$ in D,

$$\Psi_p(\overline{v}_p, \overline{w}_p) = \Psi_p(\tilde{v}_p, \tilde{w}_p).$$

Hence, by Theorem 3.4, there exists a constant c such that $(\overline{v}_p, \overline{w}_p) = (\tilde{v}_p + c, \tilde{w}_p + c)$, and consequently,

$$\begin{cases} -\Delta_p \overline{v}_p = f^+ - \tilde{h}_p & \text{in } \Omega, \\ |\nabla \overline{v}_p|^{p-2}\nabla \overline{v}_p \cdot \eta = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p \overline{w}_p = \tilde{h}_p - f^- & \text{in } \Omega, \\ |\nabla \overline{w}_p|^{p-2}\nabla \overline{w}_p \cdot \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, since $\overline{v}_p \le \overline{w}_p$ in D, by Lemma 3.12, we have $(\overline{v}_p, \overline{w}_p)$ is a minimizer of Problem (3.3). Therefore, by Theorem 3.4, there exists a constant $\overline{c}$ such that $(\overline{v}_p, \overline{w}_p) = (v_p + \overline{c}, w_p + \overline{c})$, and consequently, $h_p = \tilde{h}_p$, which implies that h_∞ is supported on ∂D. $\square$

Remark 3.14. It is easy to see that the following duality also holds:

$$W^D_{f^\pm} = \min_{\mu \in \mathcal{M}(D, M_0)} \ \min_{(\gamma_+, \gamma_-) \in \Pi(f^+, f^-, \mu)} \int_{\Omega\times\Omega} |x-y| d\gamma_+ + \int_{\Omega\times\Omega} |x-y| d\gamma_-$$

$$= \min_{(\gamma_+, \gamma_-) \in \Pi_D(f^+, f^-)} \int_{\Omega\times\Omega} |x-y| d\gamma_+ + \int_{\Omega\times\Omega} |x-y| d\gamma_-,$$

where

$$\Pi(f^+, f^-, \mu) := \left\{ (\gamma_+, \gamma_-) \in \mathcal{M}^+(\Omega\times\Omega)^2 : \gamma_+ \in \Pi(f^+, \mu), \gamma_- \in \Pi(f^-, \mu) \right\}$$

and

$$\Pi_D(f^+, f^-) :=$$

$$\left\{ (\gamma_+, \gamma_-) \in \mathcal{M}^+(\Omega\times\Omega)^2 : \pi_0\#\gamma_\pm = f^\pm, \ \pi_1\#\gamma_+ = \pi_1\#\gamma_-, \operatorname{supp}(\pi_1\#\gamma_\pm) \subset D \right\}.$$

4. Examples. Let us first compute some examples that illustrate our results and next characterize when the optimal matching measure is a delta.

Example 4.1. Consider the optimal matching problem for the data: $\Omega =]-4, 4[$, $f^+ = b\chi_{]-3,-2[} + (1-b)\chi_{]2,3[}$, $f^- = \chi_{]-2,-1[}$ and $D = [0, 1]$, where $0 \le b \le 1$ is fixed. Then, any matching measure in D is of the form $b\delta_0 + \mu$, for any non-negative Radon measure μ, of mass $1 - b$, supported on D. Indeed, it is easy to see that, for

$$T^*_+(x) = \begin{cases} 0 & \text{if } -3 < x < -2 \\ t^*_+(x) & \text{in other case,} \end{cases}$$

where t_+^* is any optimal transport map transporting $(1-b)\chi_{]2,3[}$ to μ, and

$$T_-^*(x) = \begin{cases} 0 & \text{if } -2 < x < -2+b \\ t_-^*(x) & \text{in other case,} \end{cases}$$

where t_-^* is any optimal transport map transporting $\chi_{]-2+b,-1[}$ to μ,

$$\mathcal{F}(T_+^*, T_-^*) = 4.$$

Also, for

$$v^*(x) := \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x \geq 0, \end{cases}$$

and

$$w^*(x) = x,$$

$$\int_\Omega v^*(x) f^+(x)\, dx - w^*(x) f^-(x)\, dx = 4.$$

Then, our assertion follows from

$$\begin{aligned} &\int_\Omega v^*(x) f^+(x)\, dx - w^*(x) f^-(x)\, dx \\ &\quad \leq \sup_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega) \\ |\nabla v|_\infty,\, |\nabla w|_\infty \,\leq\, 1 \\ v \,\leq\, w \text{ in } D}} \int_\Omega v f^+ - w f^- \\ &\quad = \inf_{(T_+,T_-)\in \mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+, T_-) \leq \mathcal{F}(T_+^*, T_-^*). \end{aligned}$$

Observe also that, in this case, the cost for the usual transport of f^+ to f^- is $(b-2)^2$.

We distinguish three cases:

1. If $b = 1$, δ_0 is the unique matching measure.

2. If $0 < b < 1$, there are infinitely many matching measures but all of them with singular part.

3. If $b = 0$, we have that any non-negative Radon measure of mass 1 supported on D is a matching measure. Moreover, only in this case, the cost of the matching problem is the same as the cost of the classical transport problem of f^+ to f^-.

So we can not expect uniqueness of h_∞ in general, but it may hold for some special configurations of the masses and the target set. Uniqueness of h_∞ holds in one-dimension if and only if the target set D is located to the left or to the right from the supports of f^+ and f^-, while if there is some mass of f^+ to the left of D and some mass of f^- to the right (or viceversa) then there are infinitely many optimal measures h_∞.

Moreover, in one dimension there is necessarily a singular part in the optimal measure h_∞ if the masses f_+ and f_- has some part of both of them to the left or to the right of D, while if f_+ is completely on the right and f_- completely on the left of D then there are optimal h_∞ without singular part.

Now, let us come back to the symmetric situation given in the case $b = 0$. In this case we can also compute optimal pairs (v_p, w_p). Let

$$z_p(x) = \frac{p-1}{p}|x|^{\frac{1}{p-1}}x.$$

This antisymmetric function z_p is a solution to $-(|z'|^{p-2}z')'(x) = -1$ for $x > 0$ with $z'(0) = 0$. Note that $(z_p)'(1) = 1$ and $z_p(1) = \frac{p-1}{p}$. Also note that

$$z_p(x) \to x \qquad \text{as } p \to \infty.$$

With the aid of this z_p let us define $v_{p,c}$ and $w_{p,c}$ as follows. For any $c \in [0,1]$ we consider the functions

$$v_{p,c}(x) = \begin{cases} 0, & -4 \le x \le 0, \\ cx, & 0 \le x \le 1, \\ x + c - 1, & 1 \le x \le 2, \\ z_p(x-3) + \frac{2p-1}{p} + c, & 2 \le x \le 3, \\ \frac{2p-1}{p} + c, & 3 \le x \le 4, \end{cases}$$

and

$$w_{p,c}(x) = \begin{cases} -\frac{2p-1}{p}, & -4 \le x \le -2, \\ z_p(x+2) - \frac{2p-1}{p}, & -2 \le x \le -1, \\ x, & -1 \le x \le 0, \\ cx, & 0 \le x \le 1, \\ c, & 1 \le x \le 4. \end{cases}$$

A simple computation gives

$$-(|(v_{p,c})'|^{p-2}(v_{p,c})')' = f^+ - (c^{p-1}\delta_0 + (1 - c^{p-1})\delta_1)$$

and

$$-(|(w_{p,c})'|^{p-2}(w_{p,c})')' = (1 - c^{p-1})\delta_0 + c^{p-1}\delta_1 - f^-.$$

Hence, taking

$$c = \left(\frac{1}{2}\right)^{\frac{1}{p-1}},$$

if we define $v_p := v_{p,c}$ and $w_p := w_{p,c}$, we have

$$-\Delta_p v_p = f^+ - h_p \quad \text{and} \quad -\Delta_p w_p = h_p - f^-,$$

being $h_p := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Moreover $v_p \le w_p$ in D and h_p is supported on $\{x \in D : v_p(x) = w_p(x)\}$. Therefore we have obtained a sequence of minimizers (v_p, w_p) that gives in the limit the matching measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. In addition it can be checked that the optimal Kantorovich potentials that appear in this limit procedure are just given by

$$v_\infty(x) = \begin{cases} 0, & -4 \le x \le 0, \\ x, & 0 \le x \le 3, \\ 3, & 3 \le x \le 4, \end{cases}$$

and

$$w_\infty(x) = \begin{cases} -2, & -4 \le x \le -2, \\ x, & -2 \le x \le 1, \\ 1, & 1 \le x \le 4. \end{cases}$$

Note that (v_p, w_p) is unique, up to a constant, that is, any other minimizer is of the form $(v_p + c, w_p + c)$, c constant. Therefore, this example shows that not every possible optimal matching measure can be obtained using this procedure.

Let us characterize now, in any space dimension, the set of configurations for which the matching measure is a delta concentrated at a point $z_0 \in D$.

THEOREM 4.2. *Assume that there is a point $z_0 \in D$ such that for any pair of points $x \in X_+$ and $y \in X_-$ we have*

$$\min_{z \in D}\{|x - z| + |y - z|\} = |x - z_0| + |y - z_0|, \tag{4.1}$$

then the measure $M_0\delta_{z_0}$ is an optimal matching measure.

Conversely, if $M_0\delta_{z_0}$ is an optimal matching measure, then for any pair of points $x \in X_+$ and $y \in X_-$ we have (4.1).

Proof. Let $\hat{a}(x) := |x - z_0|$ for $x \in X_+$ and $\hat{b}(x) = -|x - z_0|$ for $x \in X_-$. Both are 1–Lipschitz functions.

Let now $a(x) := \sup_{y \in X_+}\{\hat{a}(x) - |x - y|\}$ for $x \in \Omega$, and $b(x) := \inf_{y \in X_-}\{\hat{b}(x) + |x - y|\}$ for $x \in \Omega$, the lower 1–Lipschitz extension of a to Ω and the upper 1–Lipschitz extension of b to Ω, respectively (in fact these are the McShane and Whitney extensions, see [19, 25]).

Let us see that $a \le b$ on D. By (4.1) we have that, for $z \in D$,

$$|x - z_0| - |x - z| \le -|y - z_0| + |y - z| \quad \forall x \in X_+ \text{ and } \forall y \in X_-;$$

therefore, taking the supremum in x and the infimum in y we get that $a(z) \le b(z)$.

Let us see now that (a, b) is a maximizer of (3.6). Let (v, w) a pair of test functions, then

$$v(x) \le v(z_0) + |x - z_0| \qquad \forall x \in X_+$$

and

$$w(y) \ge w(z_0) - |y - z_0| \qquad \forall y \in X_-.$$

Therefore, using that $v \le w$ in D, we get

$$\int_\Omega v f^+ - \int_\Omega w f^- \le (v(z_0) - w(z_0)) \int_\Omega f^+ + \int_\Omega |x - z_0| f^+(x)dx + \int_\Omega |y - z_0| f^-(y)dy$$

$$\leq \int_\Omega |x-z_0|f^+(x)dx + \int_\Omega |y-z_0|f^-(y)dy = \int_\Omega af^+ - \int_\Omega bf^-.$$

Observe now that, setting $T_+^*(x)=z_0$ for $x\in X_+$ and $T_-^*(x)=z_0$ for $x\in X_-$,

$$\int_\Omega af^+ - \int_\Omega bf^- = \int_\Omega |x-T_+^*(x)|f^+(x)dx + \int_\Omega |y-T_-^*(y)|f^-(y)dy\,.$$

Therefore, $M_0\delta_{z_0}$ is an optimal matching measure.

To see the converse we argue by contradiction. Hence, assume that $M_0\delta_{z_0}$ is an optimal matching measure and that there are two points $x_0\in X_+$ and $y_0\in X_-$ such that (4.1) does not hold, that is, there exists $z_1\in D$ such that

$$|x_0-z_1|+|y_0-z_1| < |x_0-z_0|+|y_0-z_0|.$$

By continuity we can find a positive number η and two small radii r_1 and r_2 such that

$$|x-z_1|+|y-z_1| < |x-z_0|+|y-z_0|-\eta, \tag{4.2}$$

for every $x\in B_{r_1}(x_0)$ and every $y\in B_{r_2}(y_0)$ and such that

$$\int_{B_{r_1}(x_0)} f_+(x)\,dx = \int_{B_{r_2}(y_0)} f_-(y)\,dy = k>0. \tag{4.3}$$

Note that, thanks to this mass balance condition (4.3), we have an optimal transport map $x=S(y)$ that sends $f_-\chi_{B_{r_2}(y_0)}$ to $f_+\chi_{B_{r_1}(x_0)}$. In particular S satisfies

$$\int_{B_{r_1}(x_0)} A(x)f_+(x)\,dx = \int_{B_{r_2}(y_0)} A(S(y))f_-(y)\,dy$$

for every continuous function A. Hence,

$$\int_{B_{r_1}(x_0)} |x-z_i|f^+(x)dx = \int_{B_{r_2}(y_0)} |S(y)-z_i|f^-(y)dy, \quad i=0,1;$$

and using (4.2), we obtain that

$$\begin{aligned}
&\int_{B_{r_1}(x_0)} |x-z_1|f^+(x)dx + \int_{B_{r_2}(y_0)} |y-z_1|f^-(y)dy \\
&= \int_{B_{r_2}(y_0)} (|S(y)-z_1|+|y-z_1|)f^-(y)dy \\
&\leq \int_{B_{r_2}(y_0)} (|S(y)-z_0|+|y-z_0|)f^-(y)dy - k\eta \\
&= \int_{B_{r_1}(x_0)} |x-z_0|f^+(x)dx + \int_{B_{r_2}(y_0)} |y-z_0|f^-(y)dy - k\eta.
\end{aligned}$$

Now let us define

$$\tilde{T}_+(x) = \begin{cases} z_0, & x\in X_+\setminus B_{r_1}(x_0), \\ z_1, & x\in B_{r_1}(x_0), \end{cases}$$

and

$$\tilde{T}_-(y) = \begin{cases} z_0, & y\in X_-\setminus B_{r_2}(y_0), \\ z_1, & y\in B_{r_2}(y_0). \end{cases}$$

This pair corresponds to the transport of f_+ and f_- to the measure $(M_0-k)\delta_{z_0}+k\delta_{z_1}$ that is supported in D. We have

$$
\begin{aligned}
&\int_\Omega |x-\tilde{T}_+(x)|f^+(x)dx+\int_\Omega |y-\tilde{T}_-(y)|f^-(y)dy \\
&=\int_{X_+\setminus B_{r_1}(x_0)} |x-z_0|f^+(x)dx+\int_{X_-\setminus B_{r_2}(y_0)} |y-z_0|f^-(y)dy \\
&\quad +\int_{B_{r_1}(x_0)} |x-z_1|f^+(x)dx+\int_{B_{r_2}(y_0)} |y-z_1|f^-(y)dy \\
&\le \int_{X_+} |x-z_0|f^+(x)dx+\int_{X_-} |y-z_0|f^-(y)dy-k\eta,
\end{aligned}
$$

a contradiction with the fact that $M_0\delta_{z_0}$ is an optimal matching measure. □

It is easy to see that, for D convex, condition (4.1) is equivalent to:

$$\left\langle \frac{x-z_0}{|x-z_0|}+\frac{y-x_0}{|y-z_0|}, z-z_0 \right\rangle \le 0 \quad \text{for all } x\in X_+,\ y\in X_- \text{ and } z\in D$$

(note that z_0 may belong to ∂D).

Remark 4.3. Since we know that the target set in this problem can be reduced to the boundary, it is worth searching for a $z_0\in\partial D$ such that, for any pair of points $x\in X_+$ and $y\in X_-$,

$$\min_{z\in\partial D}\{|x-z|+|y-z|\}=|x-z_0|+|y-z_0|;$$

which also ensures the existence of a matching measure $M_0\delta_{z_0}$, now concentrated on the boundary of D.

5. Extensions. With the same ideas we can also consider the situation in which the cost is different for the transport of f^+ to the set D and for f^- to the set D. In fact we can consider the following cost functional

$$\int_\Omega \frac{1}{A}|x-T_+(x)|f^+(x)dx+\int_\Omega \frac{1}{B}|x-T_-(x)|f^-(x)dx.$$

With the constants A and B we are taking into account that the cost of transporting nuts and screws can be different (for example due to a difference in the weight).

For this kind of problems we only have to modify the $p-$Laplacian approximation replacing the L^p-norm of the gradient with

$$\frac{1}{p}\int A^p|Dv|^p.$$

In fact, doing this we are lead to consider variational problems of the form

$$\min_{\substack{(v,w)\in W^{1,p}(\Omega)\times W^{1,p}(\Omega)\\ v\le w \text{ in } D}} \frac{1}{p}\int_\Omega A^p|Dv|^p+\frac{1}{p}\int_\Omega B^p|Dw|^p-\int_\Omega vf^+ +\int_\Omega wf^-,$$

and when we pass to the limit as $p\to\infty$ we arrive to

$$\max_{\substack{v,w\in W^{1,\infty}(\Omega)\\ A|\nabla v|_\infty,\, B|\nabla w|_\infty\le 1\\ v\le w \text{ in D}}} \int_\Omega vf^+ - wf^-.$$

Note that the constraint $A|\nabla v|_\infty, B|\nabla w|_\infty \le 1$ is equivalent to

$$|v(x)-v(y)| \le \frac{1}{A}|x-y|, \qquad |w(x)-w(y)| \le \frac{1}{B}|x-y|.$$

Hence we find Kantorovich potentials for the optimal matching problem of minimizing

$$\int_\Omega \frac{1}{A}|x-T_+(x)|f^+(x)dx + \int_\Omega \frac{1}{B}|x-T_-(x)|f^-(x)dx.$$

Another possible extension is the following. We can consider a matching problem with more than two commodities. Let f^1, f^2, ... ,f^n, be nonnegative functions with the same total mass, that is,

$$\int f^i = M_0$$

for every i. Given a target set D we can look at the minimization problem

$$\min_{(T_i)\in\mathcal{A}_D} \sum_{i=1}^n \int_\Omega |x-T_i(x)|f^i(x)dx.$$

where

$$\begin{aligned}\mathcal{A}_D \quad := \Big\{ & (T_i) : T_i : \Omega \to \Omega \text{ are Borel functions, } T_i(\mathrm{supp}(f^i)) \subset D, \\ & \int_{T_i^{-1}(E)} f^i = \int_{T_j^{-1}(E)} f^j \text{ for all Borel subset } E \text{ of } \Omega \Big\}.\end{aligned}$$

To handle this situation, say for three commodities, the minimization problem to take into account is given by

$$\min_{\substack{(v,\,w,\,z)\,\in\,(W^{1,p}(\Omega))^3\\ v\,+\,w\,+\,z\,\le\,0\text{ in }D}} \frac{1}{p}\int_\Omega |Dv|^p + \frac{1}{p}\int_\Omega |Dw|^p + \frac{1}{p}\int_\Omega |Dz|^p - \int_\Omega vf^1 - \int_\Omega wf^2 - \int_\Omega zf^3.$$

Note that this is similar to what we did before since (3.3) can be rewritten as

$$\min_{\substack{(v,\,w)\,\in\,W^{1,p}(\Omega)\times W^{1,p}(\Omega)\\ v\,+\,w\,\le\,0\text{ in }D}} \frac{1}{p}\int_\Omega |Dv|^p + \frac{1}{p}\int_\Omega |Dw|^p - \int_\Omega vf^+ - \int_\Omega wf^-.$$

We presented the details for only two masses since this simpler case shows how to handle the main mathematical difficulties.

Remark 5.1. One can try to solve the optimal matching problem for the Euclidean distance by taking the optimal matching measures for the cost $|x-y|^r$ with $r>1$ (these are uniformly convex costs) and then take the limit as $r\to 1$. This passage to the limit seems delicate and hence we preferred to perform instead the $p-$Laplacian approximation since it gives us not only the optimal matching measure but also gives the Kantorovich potentials.

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