SIAM J. OPTIM.
Vol. 25, No. 3, pp. 1609–1632

OPTIMAL MASS TRANSPORT ON METRIC GRAPHS*

J. M. MAZÓN†, J. D. ROSSI‡, AND J. TOLEDO†

Abstract. We study an optimal mass transport problem between two equal masses on a metric graph where the cost is given by the distance in the graph. To solve this problem we find a Kantorovich potential as the limit of p-Laplacian–type problems in the graph where at the vertices we impose zero total flux boundary conditions. In addition, the approximation procedure allows us to find a transport density that encodes how much mass has to be transported through a given point in the graph, and also provides a simple formula of convex optimization for the total cost.

Key words. p-Laplacian, metric graphs, optimal transport, convex optimization

AMS subject classifications. 35J62, 49J99, 35J92

DOI. 10.1137/140995611

1. Introduction.

In this paper we are interested in the Monge–Kantorovich mass transport problem on metric graphs. That is, we want to transport a certain amount of material in the graph to a prescribed final distribution minimizing a cost given by the distance inside the graph. Our approach to this problem is based on an idea by Evans and Gangbo [7] that approximates a Kantorovich potential for a transport problem in the Euclidean space with cost given by the Euclidean distance using the limit as p goes to infinity of a family of p-Laplacian–type problems. This limit procedure turns out to be quite flexible and allows us to deal with different transport problems (like optimal matching problems, problems with taxes, etc.) in which the cost is given by the Euclidean distance or variants of it. See [8, 11, 12, 13, 14]. Here we apply these ideas to the optimal transport problem on a metric graph, showing again that this approximation procedure is quite powerful since it provides all the relevant information for the transport problem.

To put our optimal mass transport in modern mathematical terms we have to introduce some notation. Let Γ be a metric graph (see section 2 for a precise definition) and consider two nonnegative measures on the graph μ and ν with the same total mass, that is,

$$\int_\Gamma \mu = \int_\Gamma \nu.$$

The associated optimal transport problem (in its relaxed version, also known as the Monge–Kantorovich mass transport problem) reads as follows: find an optimal transport plan, that is, a measure $\gamma(x,y)$ that solves the minimization problem

$$\min_{\gamma\in\Pi(\mu,\nu)} \int_\Gamma\int_\Gamma d_\Gamma(x,y)\, d\gamma(x,y),$$

*Received by the editors November 13, 2014; accepted for publication (in revised form) June 24, 2015; published electronically August 11, 2015.
http://www.siam.org/journals/siopt/25-3/99561.html
†Departament d'Anàlisi Matemàtica, Universitat de València, 46100 Burjassot, Valencia, Spain (mazon@uv.es, toledojj@uv.es). The research of these authors was partially supported by MTM2012-31103 (Spain).
‡CONICET and Depto. de Matemática, FCEyN, Universidad de Buenos Aires, Pab. I, Ciudad. Universitaria (1428), Buenos Aires, Argentina (jrossi@dm.uba.ar). This author's research was partially supported by MTM2011-27998 (Spain).

where $d_\Gamma(\cdot,\cdot)$ is the distance in the metric graph and $\Pi(\mu,\nu)$ is the set of measures that have marginals μ and ν in the first and the second variable, respectively. A simple argument using a minimizing sequence shows that there exists an optimal transport plan γ^* (see [16] or [1]). We will denote by $W_{\mu,\nu}$ the value of the minimum above and refer to it as the total transport cost.

This minimization problem has a dual formulation: to find a Kantorovich potential u, that is, a function that solves the maximization problem

$$\max_{u\in K_{d_\Gamma}(\Gamma)} \int_\Gamma u\,d\eta,$$

where

$$\eta = \mu - \nu$$

and $K_{d_\Gamma}(\Gamma)$ is the set of 1-Lipschitz functions on Γ, that is, functions $u:\Gamma\mapsto\mathbb{R}$ such that $|u(x)-u(y)|\le d_\Gamma(x,y)$ for every $x,y\in\Gamma$.

We can find a Kantorovich potential by an approximation procedure using a sequence of solutions u_p to p-Laplacian–type problems and taking the limit as $p\to\infty$. To be more precise, let us consider solutions to the variational problem

$$\min_{u\in S_p} \frac{1}{p}\int_\Gamma |u'|^p - \int_\Gamma u\,d\eta,$$

where

$$S_p := \left\{ u\in W^{1,p}(\Gamma) \ :\ \int_\Gamma u = 0 \right\}.$$

Here and in the remainder of the paper we will integrate with respect to the Lebesgue measure (unless other measures arise, in which case we will make this explicit by writing, for example, $\int_\Gamma u\,d\mu$). Such minimizers u_p are, in fact, weak solutions to the following p-Laplacian problem on the graph (see the notation afterwards):

$$\text{(1.1)}\qquad \begin{cases} -(|u'|^{p-2}u')' = \eta & \text{on each edge } \mathrm{e}\in \mathrm{E}_\mathrm{v}(\Gamma),\\ \displaystyle\sum_{\mathrm{e}\in \mathrm{E}_\mathrm{v}(\Gamma)} \left|\frac{\partial u}{\partial x_\mathrm{e}}\right|^{p-2} \frac{\partial u}{\partial x_\mathrm{e}}(\mathrm{v}) = 0 & \text{on each vertex } \mathrm{v}\in \mathrm{V}(\Gamma). \end{cases}$$

Our first result establishes the existence of a subsequence $p_j\to\infty$ such that u_{p_j} converges uniformly in Γ to a Kantorovich potential u_∞ for the optimal mass transport problem between μ and ν.

Observe that in (1.1) we associate a differential law with each edge that models the interaction between the vertices defining such edges (we denote by v the vertices and by e the edges in what follows). Therefore we are dealing with what is known in the literature as quantum graphs. The use of quantum graphs (in contrast to more elementary graph models, such as simple unweighted or weighted graphs) opens up the possibility of modeling the interactions between agents identified by the graph's vertices in a far more detailed manner than with standard graphs. Quantum graphs are now widely used in physics, chemistry, and engineering (nanotechnology) problems, but can also be used, in principle, in the analysis of complex phenomena taking place on large complex networks, including social and biological networks. Such graphs are characterized by highly skewed degree distributions, small diameter, and high clustering coefficients, and they have topological and spectral properties that are quite different from those of the highly regular graphs or lattices arising in physics and chemistry applications. Quantum graphs are also used to model thin tubular structures, so-called graph-like spaces; they are their natural limits when the radius of a graph-like space tends to zero. On both the graph-like spaces and the metric graph, we can naturally define Laplace-like differential operators [2, 3, 10, 15].

Next, as a second result, we show that our approximation procedure gives much more; it allows us to construct a transport density and to provide simple formulas for the total cost. Decompose η as a measure supported on the edges plus a sum of deltas supported on the vertices, that is,

$$\eta = \underline{\mu} + \sum_{\mathrm{v}\in \mathrm{V}(\Gamma)} a_{\mathrm{v}}\delta_{\mathrm{v}}, \tag{1.2}$$

where $a_{\mathrm{v}} \in \mathbb{R}$ and $\underline{\mu}$ is a Radon measure on Γ of the form

$$\langle \underline{\mu}, \varphi\rangle = \sum_{\mathrm{e}\in \mathrm{E}(\Gamma)} \int_0^{\ell_{\mathrm{e}}} [\varphi]_{\mathrm{e}} d\underline{\mu}_{\mathrm{e}} \qquad \text{for } \varphi \in C(\Gamma),$$

with $\underline{\mu}_{\mathrm{e}}$ a Radon measure in $(0, \ell_{\mathrm{e}})$. Here and in what follows we denote by $[\varphi]_{\mathrm{e}}$ the restriction of the function φ to the edge e. Observe that the mass balance condition satisfied by η reads as

$$\int_\Gamma d\underline{\mu} + \sum_{\mathrm{v}\in \mathrm{V}(\Gamma)} a_{\mathrm{v}} = 0.$$

Then, for such a $\underline{\mu}$, we prove that the optimal total transport cost, $W_{\mu,\nu}$, can be obtained as the sum of optimal transport costs along the edges (expressed as Wasserstein distances in $\mathbb{R}$) constrained to an underdetermined linear system that balances the masses that enter/leave the edges via the vertices during the optimal transport process.

Remark 1.1. There exists a large amount of literature dealing with optimal transport problems in networks; see, for instance, the two excellent works [4] and [5]. We want to point out that the problems studied in these monographs are different from the one we face here. In fact, in those references the authors try to find an optimal network that minimizes some energy functional associated with the network. In contrast, here the network is given, and our aim is to describe the Kantorovich potential and the transport density of a Monge–Kantorovich mass transport problem on the graph represented by the network.

The paper is organized as follows: in section 2 we collect some preliminaries; in section 3 we study the limit as $p \to \infty$ in our p-Laplacian approximation and prove the main results. At the end of section 3 we collect examples that illustrate these results.

2. Preliminaries.

2.1. Quantum graph. We recall here some basic knowledge about quantum graphs; see, for instance, [3] and references therein.

A graph Γ consists of a finite or countable infinite set of vertices $\mathrm{V}(\Gamma) = \{\mathrm{v}_i\}$ and a set of edges $\mathrm{E}(\Gamma) = \{\mathrm{e}_j\}$ connecting the vertices. A graph Γ is said to be a finite graph if the number of edges and the number of vertices are finite. An edge and a vertex on that edge are called incident. We will denote $\mathrm{v} \in \mathrm{e}$ when the edge e and the vertex v are incident. We define $\mathrm{E_v}(\Gamma)$ as the set of all edges incident to v.

We will assume the absence of loops, since if these are present, one can break them into pieces by introducing new intermediate vertices. We also assume the absence of multiple edges.

A *walk* is a sequence of edges $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \dots\}$ in which, for each i (except the last), the end of e_i is the beginning of e_{i+1}. A *trail* is a walk in which no edge is repeated. A *path* is a trail in which no vertex is repeated.

From now on we will deal with a connected, compact, and metric graph Γ as follows.

- A graph Γ is a metric graph if
    1. Each edge e is assigned a positive length $\ell_{\mathrm{e}} \in (0, +\infty]$.
    2. For each edge e, a coordinate is assigned to each point of the edge including the vertices. For that purpose each edge e is identified with an ordered pair $(\mathrm{i_e}, \mathrm{f_e})$ of vertices, with $\mathrm{i_e}$ and $\mathrm{f_e}$ being the initial and terminal vertex of e, respectively, but allows one to define coordinates by means of an increasing function

$$\begin{array}{cccc} c_{\mathrm{e}}: & \mathrm{e} & \to & [0, \ell_{\mathrm{e}}] \\ & x & \rightsquigarrow & x_{\mathrm{e}} \end{array}$$

    such that, letting $c_{\mathrm{e}}(\mathrm{i_e}) := 0$ and $c_{\mathrm{e}}(\mathrm{f_e}) := \ell_{\mathrm{e}}$, is exhaustive; x_{e} is called the coordinate of the point $x \in \mathrm{e}$.
- A graph is said to be connected if a path exists between every pair of vertices, that is, a graph which is connected in the usual topological sense.
- A compact metric graph is a finite metric graph whose edges all have finite length.

If a sequence of edges $\{\mathrm{e}_j\}_{j=1}^n$ forms a path, its length is defined as $\sum_{j=1}^n \ell_{\mathrm{e}_j}$. The length of a metric graph, denoted $\ell(\Gamma)$, is the sum of the lengths of all its edges.

For two vertices v and $\hat{\mathrm{v}}$, the distance between v and $\hat{\mathrm{v}}$, $d_\Gamma(\mathrm{v}, \hat{\mathrm{v}})$, is defined as the minimal length of the paths connecting them. Let us be more precise and consider x, y, two points in the graph Γ.

– If $x, y \in \mathrm{e}$ (they belong to the same edge, note that they can be vertices), we define the *distance in the path* e between x and y as

$$\mathrm{dist}_{\mathrm{e}}(x, y) := |y_{\mathrm{e}} - x_{\mathrm{e}}|.$$

– If $x \in \mathrm{e}_a$, $y \in \mathrm{e}_b$, with e_a and e_b different edges, let $P = \{\mathrm{e}_a, \mathrm{e}_1, \dots, \mathrm{e}_n, \mathrm{e}_b\}$ be a path ($n \geq 0$) connecting them. Let us say that $\mathrm{e}_0 = \mathrm{e}_a$ and $\mathrm{e}_{n+1} = \mathrm{e}_b$. Following the definition given above for a path, set v_0 as the vertex that is the end of e_0 and the beginning of e_1 (note that these vertices need not be the terminal and the initial vertices of the edges that are taken into account), and v_n as the vertex that is the end of e_n and the beginning of e_{n+1}. We will say that the *distance in the path* P between x and y is equal to

$$\mathrm{dist}_{\mathrm{e}_0}(x, \mathrm{v}_0) + \sum_{1 \leq j \leq n} \ell_{\mathrm{e}_j} + \mathrm{dist}_{\mathrm{e}_{n+1}}(\mathrm{v}_n, y).$$

We define the distance between x and y, which we denote by $d_\Gamma(x,y)$, as the infimum of all the *distances in paths* between x and y, that is,

$$d_\Gamma(x,y) = \inf\Big\{\mathrm{dist}_{\mathrm{e}_0}(x,\mathrm{v}_0) + \sum_{1\le j\le n} \ell_{\mathrm{e}_j} + \mathrm{dist}_{\mathrm{e}_{n+1}}(\mathrm{v}_n,y) : \\ \{\mathrm{e}_0,\mathrm{e}_1,\dots,\mathrm{e}_n,\mathrm{e}_{n+1}\} \text{ path connecting } x \text{ and } y\Big\}.$$

Note that the distance between two points x and y belonging to the same edge e can be strictly smaller than $|y_{\mathrm{e}} - x_{\mathrm{e}}|$. This happens when there is a path connecting them (using more edges than e) with length smaller than $|y_{\mathrm{e}} - x_{\mathrm{e}}|$.

A function u on a metric graph Γ is a collection of functions $[u]_{\mathrm{e}}$ defined on $(0,\ell_{\mathrm{e}})$ for all $\mathrm{e}\in \mathrm{E}(\Gamma)$, not just at the vertices as in discrete models.

Throughout this work, $\int_\Gamma u(x)dx$ or $\int_\Gamma u$ denotes $\sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \int_0^{\ell_{\mathrm{e}}} [u]_{\mathrm{e}}(x_{\mathrm{e}})\,dx_{\mathrm{e}}$.

Let $1\le p\le +\infty$. We say that u belongs to $L^p(\Gamma)$ if $[u]_{\mathrm{e}}$ belongs to $L^p(0,\ell_{\mathrm{e}})$ for all $\mathrm{e}\in\mathrm{E}(\Gamma)$ and

$$\|u\|^p_{L^p(\Gamma)} \coloneqq \sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \|[u]_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} < +\infty.$$

The Sobolev space $W^{1,p}(\Gamma)$ is defined as the space of continuous functions u on Γ such that $[u]_{\mathrm{e}}\in W^{1,p}(0,\ell_{\mathrm{e}})$ for all $\mathrm{e}\in\mathrm{E}(\Gamma)$ and

$$\|u\|^p_{W^{1,p}(\Gamma)} \coloneqq \sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \|[u]_{\mathrm{e}}\|^p_{L^p(0,\ell_{\mathrm{e}})} + \|[u]_{\mathrm{e}}{}'\|^p_{L^p(0,\ell_{\mathrm{e}})} < +\infty.$$

The space $W^{1,p}(\Gamma)$ is a Banach space for $1\le p\le\infty$. It is reflexive for $1<p<\infty$ and separable for $1\le p<\infty$. Observe that the continuity condition in the definition of $W^{1,p}(\Gamma)$ means that for each $\mathrm{v}\in\mathrm{V}(\Gamma)$, the function on all edges $\mathrm{e}\in\mathrm{E}_{\mathrm{v}}(\Gamma)$ assumes the same value at v.

Let Γ be a compact graph, and let $1<p<+\infty$. Since for an interval I in the real line we have that $W^{1,p}(I)\subset C(I)$ compactly, we have that the injection $W^{1,p}(\Gamma)\subset C(\Gamma)$ is compact (just note that if $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Gamma)$, then it is bounded in $W^{1,p}(\mathrm{e})$ for any edge, and therefore we can extract a subsequence that converges uniformly in e for any edge in Γ).

A quantum graph is a metric graph Γ equipped with a differential operator acting on the edges accompanied by vertex conditions. In this work, we will consider the p-Laplacian differential operators given by

$$-\Delta_p u(x) := -(|u'(x)|^{p-2}u'(x))', \qquad \text{with } p>1,$$

on each edge.

THE MONGE–KANTOROVICH PROBLEM. *Fix $\mu,\nu\in\mathcal{M}^+(\Gamma)$ satisfying the mass balance condition*

$$\mu(\Gamma)=\nu(\Gamma). \tag{2.1}$$

The Monge–Kantorovich problem is the minimization problem

$$\min\left\{\int_{\Gamma\times\Gamma} d_\Gamma(x,y)\,d\gamma(x,y) \,:\, \gamma\in\Pi(\mu,\nu)\right\},$$

where $\Pi(\mu,\nu) := \{$*Radon measures* γ *in* $\Gamma\times\Gamma : \pi_1\#\gamma = \mu, \pi_2\#\gamma = \nu\}$*. The elements* $\gamma \in \Pi(\mu,\nu)$ *are called transport plans between* μ *and* ν*, and a minimizer* γ^* *is an optimal transport plan. Since* d *is lower semicontinuous, there are optimal plans, that is,*

$$\exists\, \gamma^* \in \arg\min_{\gamma\in\Pi(\mu,\nu)} \int_{\Gamma\times\Gamma} d_\Gamma(x,y)\, d\gamma(x,y).$$

The Monge–Kantorovich problem has a dual formulation that can be stated in this case as follows (see for instance [16, Theorem 1.14]).

Kantorovich–Rubinstein Theorem. *Let* $\mu,\nu \in \mathcal{M}^+(\Gamma)$ *be two measures satisfying the mass balance condition* (2.1)*. Then,*

$$\min\left\{\int_{\Gamma\times\Gamma} d_\Gamma(x,y)\, d\gamma(x,y) \,:\, \gamma\in\Pi(\mu,\nu)\right\} = \sup\left\{\int_\Gamma u\, d(\mu-\nu) \,:\, u \in K_{d_\Gamma}(\Gamma)\right\},$$

where

$$K_{d_\Gamma}(\Gamma) := \{u : \Gamma \mapsto \mathbb{R} \,:\, |u(y)-u(x)| \le d_\Gamma(y,x)\}\,. \tag{2.2}$$

Moreover, there exists $u \in K_{d_\Gamma}(\Gamma)$ *such that*

$$\int_\Gamma u\, d(\mu-\nu) = \sup\left\{\int_\Gamma v\, d(\mu-\nu) : v\in K_{d_\Gamma}(\Gamma)\right\}.$$

Such maximizers are called Kantorovich potentials.

Example. As an example of a metric graph in connection with this mass transport problem we can consider the road network in a country, which can be considered as a metric graph Γ in which $V(\Gamma)$ corresponds to the set of important cities and $E(\Gamma)$ to the roads connecting these cities. Then, for $x \in \mathrm{e} \in E(\Gamma)$, $x_{\mathrm{e}} \in (0,\ell_{\mathrm{e}})$, which represents the cost of transporting a unit mass from the city $\mathrm{i_e}$ to x on the road e, could be given by the distance on that road, but we can also take into account the possible tolls by changing the parametrization of the edges, expanding them according to the size of the toll on that road.

3. The *p*-Laplacian approximation.

Theorem 3.1. *Let* $\mu,\nu \in \mathcal{M}^+(\Gamma)$ *be two measures satisfying the mass balance condition* (2.1)*. Take* $\eta = \mu-\nu$*. Consider the functional* $\mathcal{F}_p : W^{1,p}(\Gamma) \to \mathbb{R}$ *defined by*

$$\mathcal{F}_p(u) := \frac{1}{p}\int_\Gamma |u'|^p - \int_\Gamma u\, d\eta.$$

Then, there exists a minimizer u_p *of the functional* $\mathcal{F}_p$ *in the set*

$$S_p := \left\{u \in W^{1,p}(\Gamma) \;:\; \int_\Gamma u = 0\right\}.$$

Moreover, a minimizer u_p *is a weak solution of the problem*

$$\begin{cases} -(|u'|^{p-2}u')' = \eta & \text{on edges,} \\ \displaystyle\sum_{\mathrm{e}\in \mathrm{E_v}(\Gamma)} \left|\frac{\partial u}{\partial x_{\mathrm{e}}}\right|^{p-2} \frac{\partial u}{\partial x_{\mathrm{e}}}(\mathrm{v}) = 0 & \text{on vertices,} \end{cases} \tag{3.1}$$

in the sense that $u_p \in W^{1,p}(\Gamma)$ *and*

$$\int_\Gamma |u_p'|^{p-2} u_p' \varphi' = \int_\Gamma \varphi \, d\eta$$

for every $\varphi \in W^{1,p}(\Gamma)$.

Observe that in the boundary condition in (3.1) the derivatives are taken in the direction away from the vertex.

Proof. Let $\{u_n\}$ be a minimizing sequence in S_p. Since $\int_\Gamma u_n = 0$ and u_n is continuous, there exists $x_n \in \Gamma$ such that $u_n(x_n) = 0$. Suppose $x_n \in \mathrm{e} \in E(\Gamma)$. Then, for $x \in \mathrm{e}$,

$$\begin{aligned} |u_n(x)| = |[u_n]_\mathrm{e}(x_\mathrm{e}) - [u_n]((x_n)_\mathrm{e})| &= \left| \int_{(x_n)_\mathrm{e}}^{x_\mathrm{e}} [u_n]_\mathrm{e}' \right| \\ &\le |x_\mathrm{e} - (x_n)_\mathrm{e}|^{1/p'} \|[u_n]_\mathrm{e}'\|_{L^p(0,\ell_\mathrm{e})} \le C_1 \|u_n'\|_{L^p(\Gamma)}, \end{aligned}$$

with C_1 independent of p. Now, since u_n is continuous, we can apply the above argument in the edges that share a vertex with e. Since Γ is connected and compact, doing this at all edges, we get

$$\sup_{x\in\Gamma} |u_n(x)| \le C_2 \|u_n'\|_{L^p(\Gamma)}, \tag{3.2}$$

with C_2 independent of p.

Then, as we have

$$\left| \int_\Gamma u_n \, d\eta \right| \le C_3 \|u_n\|_{L^\infty(\Gamma)} \le C_3 C_2 \|u_n\|_{W^{1,p}(\Gamma)},$$

with C_3 also independent of p, we get that $\{u_n\}$ is bounded in $W^{1,p}(\Gamma)$, and hence (using that $\int_\Gamma u_n = 0$) we can extract a subsequence $u_{n_j} \to u_p$ weakly in $W^{1,p}(\Gamma)$ and uniformly in Γ. Then, we have $\int_\Gamma u_p = 0$ and

$$\mathcal{F}_p(u_p) \le \liminf_j \mathcal{F}_p(u_{n_j}),$$

and we conclude that u_p is the desired minimizer.

To prove that a minimizer is a weak solution to (3.1) we just observe that when we differentiate with respect to t at $t = 0$ the function $\mathcal{F}_p(u_p + t\varphi)$, we obtain $u_p \in W^{1,p}(\Gamma)$ and

$$\int_\Gamma |u_p'|^{p-2} u_p' \varphi' = \int_\Gamma \varphi \, d\eta \qquad \forall \varphi \in W^{1,p}(\Gamma).$$

This ends the proof. ∎

Now, we show that there is a limit as $p \to \infty$ of the minimizers u_p.

LEMMA 3.2. *Let u_p be a minimizer of the functional $\mathcal{F}_p$ on S_p, $p > 1$. There exists a subsequence $p_j \to \infty$ such that*

$$u_{p_j} \to u_\infty$$

uniformly in Γ. *Moreover, the limit u_∞ is Lipschitz continuous and $\|u_\infty'\|_{L^\infty(\Gamma)} \le 1$.*

Proof. Our first aim is to prove that the L^p-norm of the gradient of u_p is bounded independently of p. Since u_p is a minimizer of $\mathcal{F}_p$ on S_p,

$$\mathcal{F}_p(u_p) = \int_\Gamma \frac{\left|u_p'\right|^p}{p} - \int_\Gamma u_p \, d\eta \le \mathcal{F}_p(0) = 0.$$

That is,

$$\int_\Gamma \frac{\left|u_p'\right|^p}{p} \le \int_\Gamma u_p \, d\eta.$$

Now, from the same arguments leading to (3.2), we obtain

$$\sup_{x\in\Gamma} |u_p(x)| \le C_1 \|u_p'\|_{L^p(\Gamma)} \tag{3.3}$$

and

$$\int_\Gamma u_p \, d\eta \le C_1 \|u_p'\|_{L^p(\Gamma)},$$

with C_1 independent of p. Then we get

$$\int_\Gamma \frac{\left|u_p'\right|^p}{p} \le C_1 \|u_p'\|_{L^p(\Gamma)}.$$

From this inequality, we obtain that

$$\|u_p'\|_{L^p(\Gamma)} \le (pC_1)^{\frac{1}{p-1}}. \tag{3.4}$$

Now, by (3.3) and (3.4), we obtain that

$$\sup_{x\in\Gamma} |u_p(x)| \le C_1 (pC_1)^{\frac{1}{p-1}}. \tag{3.5}$$

Using this uniform bound, we prove uniform convergence of a sequence u_{p_j}. In fact, we take m such that $1 < m \le p$ and obtain the bound

$$\begin{aligned} \|u_p'\|_{L^m(\Gamma)} &= \left(\int_\Gamma \left|u_p'\right|^m \cdot 1 \right)^{\frac{1}{m}} \\ &\le \left[\left(\int_\Gamma \left|u_p'\right|^p \right)^{\frac{m}{p}} \left(\int_\Gamma 1 \right)^{\frac{p-m}{p}} \right]^{\frac{1}{m}} \\ &\le (pC_1)^{\frac{1}{p-1}} \ell(\Gamma)^{\frac{p-m}{pm}}, \end{aligned} \tag{3.6}$$

with this last upper bound bounded on p. We have proved that $\{u_p\}_{p>1}$ is bounded in $W^{1,m}(\Gamma)$, so we can obtain a weakly convergent sequence $u_{p_j} \rightharpoonup u_\infty \in W^{1,m}(\Gamma)$ with $p_j \to +\infty$. Since $W^{1,p}(\Gamma)$ is compactly embedded in $C(\Gamma)$ and $u_{p_j} \rightharpoonup u_\infty \in W^{1,p}(\Gamma)$, we obtain $u_{p_j} \to u_\infty$ uniformly in Γ. Using a diagonal procedure we conclude the existence of sequence u_{p_j} which is weakly convergent in $W^{1,m}(\Gamma)$ for every m.

Finally, let us show that the limit function u_∞ is Lipschitz. In fact, using (3.6), we have that

$$\left(\int_\Gamma \left|u_\infty'\right|^m \right)^{\frac{1}{m}} \le \liminf_{p_j \to +\infty} \left(\int_\Gamma \left|u_{p_j}'\right|^m \right)^{\frac{1}{m}} \le \ell(\Gamma)^{\frac{1}{m}}.$$

Now, we take $m \to \infty$ to obtain $\|u'_\infty\|_{L^\infty(\Gamma)} \le 1$. So, we have proved that $u_\infty \in W^{1,\infty}(\Gamma)$, that is, u_∞ is a Lipschitz function and $\|u'_\infty\|_{L^\infty(\Gamma)} \le 1$. ◻

THEOREM 3.3. *Any uniform limit u_∞ of a sequence u_{p_j} is a Kantorovich potential for the optimal transport problem of μ to ν with the cost given as d_Γ; that is, it holds that*

$$\min\left\{\int_{\Gamma\times\Gamma} d_\Gamma(x,y)\,d\gamma(x,y) \,:\, \gamma\in\Pi(\mu,\nu)\right\}$$
$$= \sup\left\{\int_\Gamma u\,d(\nu-\mu) \,:\, u\in K_{d_\Gamma}(\Gamma)\right\} = \int_\Gamma u_\infty\,d(\mu-\nu).$$

Proof. By the Kantorovich–Rubinstein theorem, we need only show the last equality. To do that, first let us see that $K_{d_\Gamma}(\Gamma)$ given in (2.2) verifies

$$K_{d_\Gamma}(\Gamma) = \left\{u\in W^{1,\infty}(\Gamma) \;:\; \|u'\|_{L^\infty(\Gamma)} \le 1\right\}. \tag{3.7}$$

It is easy to see that

$$K_{d_\Gamma}(\Gamma) \subset \mathrm{Lip}_{d_\Gamma}(\Gamma) = \left\{u\in W^{1,\infty}(\Gamma) \;:\; \|u'\|_{L^\infty(\Gamma)} \le 1\right\}.$$

Let us see the reverse inclusion. Let

$$u \in \left\{u\in W^{1,\infty}(\Gamma) \;:\; \|u'\|_{L^\infty(\Gamma)} \le 1\right\}$$

and $x, y\in\Gamma$ with $x\in \mathrm{e}_a$ and $y\in\mathrm{e}_b$. Suppose that $d_\Gamma(x,y)$ is attained at the path $\{\mathrm{e}_0,\mathrm{e}_1,\ldots,\mathrm{e}_n,\mathrm{e}_{n+1}\}$, $n\ge -1$, where $\mathrm{e}_a=\mathrm{e}_0$ and $\mathrm{e}_b=\mathrm{e}_{n+1}$. Suppose $\mathrm{f}_{\mathrm{e}_0}$ is the vertex that is the end of e_0 and the beginning of e_1, and $\mathrm{i}_{\mathrm{e}_{n+1}}$ is the vertex that is the end of e_n and the beginning of e_{n+1}; for other cases the argument is similar. Then, we have

$$d_\Gamma(x,y) = |y_{\mathrm{e}_0} - x_{\mathrm{e}_0}| \qquad \text{if } n=-1,$$
$$d_\Gamma(x,y) = l_{\mathrm{e}_0} - x_{\mathrm{e}_0} + \sum_{1\le i\le n} \ell_{\mathrm{e}_i} + y_{\mathrm{e}_{n+1}} \qquad \text{if } n\ge -1.$$

Now, if $n=-1$,

$$|u(x)-u(y)| = |[u]_{\mathrm{e}_0}(x_{\mathrm{e}_0}) - [u]_{\mathrm{e}_0}(y_{\mathrm{e}_0})| \le |y_{\mathrm{e}_0} - x_{\mathrm{e}_0}| = d_\Gamma(x,y),$$

and if $n\ge 0$,

$$|u(x)-u(\mathrm{f}_{\mathrm{e}_0})| = |[u]_{\mathrm{e}_0}(x_{\mathrm{e}_0}) - [u]_{\mathrm{e}_0}(\ell_{\mathrm{e}_0})| \le \ell_{\mathrm{e}_0} - x_{\mathrm{e}_0},$$

$$|u(\mathrm{i}_{\mathrm{e}_i}) - u(\mathrm{f}_{\mathrm{e}_i})| = |[u]_{\mathrm{e}_i}(0) - [u]_{\mathrm{e}_i}(\ell_{\mathrm{e}_i})| \le \ell_{\mathrm{e}_{i+1}}, \quad 1\le i\le n,$$

and

$$|u(\mathrm{i}_{\mathrm{e}_{n+1}}) - u(y)| = |[u]_{\mathrm{e}_{n+1}}(0) - [u]_{\mathrm{e}_{n+1}}(y_{\mathrm{e}_{n+1}})| \le y_{\mathrm{e}_{n+1}}.$$

Hence,

$$|u(x)-u(y)| \le |u(x)-u(\mathrm{f}_{\mathrm{e}_0})| + \sum_{1\le i\le n} |u(\mathrm{i}_{\mathrm{e}_i}) - u(\mathrm{f}_{\mathrm{e}_i})| + |u(\mathrm{i}_{\mathrm{e}_{n+1}}) - u(y)|$$
$$\le \ell_{\mathrm{e}_0} - x_{\mathrm{e}_0} + \sum_{1\le i\le n} \ell_{\mathrm{e}_i} + y_{\mathrm{e}_{n+1}} = d_\Gamma(x,y).$$

Consequently, (3.7) holds.

Due to (3.7), we just need to show that

$$\sup\left\{\int_\Gamma v\,d(\mu-\nu)\,:\,v\in \mathrm{Lip}_{d_\Gamma}\right\}=\int_\Gamma u_\infty\,d(\mu-\nu). \tag{3.8}$$

Given $v\in \mathrm{Lip}_{d_\Gamma}(\Gamma)$, we define

$$\tilde v := v-\frac{1}{\ell(\Gamma)}\int_\Gamma v.$$

We have $\tilde v\in S_p$, then

$$\begin{aligned}\mathcal{F}_p(u_p)&=\int_\Gamma\frac{\left|u_p'\right|^p}{p}-\int_\Gamma u_p\,d(\mu-\nu)\le \mathcal{F}_p(\tilde v)\\&=\int_\Gamma\frac{|v'|^p}{p}-\int_\Gamma v\,d(\mu-\nu)\le\frac1p\ell(\Gamma)-\int_\Gamma v\,d(\mu-\nu).\end{aligned}$$

Therefore,

$$-\int_\Gamma u_p\,d(\mu-\nu)\le\int_\Gamma\frac{\left|u_p'\right|^p}{p}-\int_\Gamma u_p\,d(\mu-\nu)$$

$$\le\int_\Gamma\frac{|v'|^p}{p}-\int_\Gamma v\,d(\mu-\nu)\le\frac1p\ell(\Gamma)-\int_\Gamma v\,d(\mu-\nu).$$

Taking limits as $p\to\infty$, we obtain

$$\int_\Gamma u_\infty\,d(\mu-\nu)\ge\int_\Gamma v\,d(\mu-\nu),$$

and consequently, we get

$$\int_\Gamma u_\infty\,d(\mu-\nu)\ge\sup\left\{\int_\Gamma v\,d(\mu-\nu)\,:\,v\in\mathrm{Lip}_{d_\Gamma}(\Gamma)\right\},$$

from which follows (3.8), since $u_\infty\in\mathrm{Lip}_d(\Gamma)$. ∎

In order to find the transport density we need the following result.

LEMMA 3.4. *Let $\underline{\mu}=\underline{\mu}^+-\underline{\mu}^-$, where $\underline{\mu}^+$ and $\underline{\mu}^-$ are positive Radon measures in (a,b), and let $\alpha,\beta\in\mathbb{R}$ satisfying*

$$\int_a^b d\underline{\mu}+\alpha+\beta=0.$$

(i) *For any $p>1$, let v_p be a weak solution of the problem*

$$\begin{cases}-\left(|v'|^{p-2}v'\right)'=\underline{\mu}\quad in\ (a,b),\\ \left(|v'|^{p-2}v'\right)(a)=-\alpha,\\ \left(|v'|^{p-2}v'\right)(b)=\beta.\end{cases} \tag{3.9}$$

If

$$v_p\to v_\infty$$

uniformly in $[a,b]$ with $\|v_\infty'\|_{L^\infty(a,b)}\le 1$, then v_∞ is a Kantorovich potential for the optimal transport problem of $\eta^+:=\underline{\mu}^++(\alpha^+\delta_a+\beta^+\delta_b)$ to $\eta^-:=\underline{\mu}^-+(\alpha^-\delta_a+\beta^-\delta_b)$ with the cost given by the Euclidean distance.

(ii) *If there exist a nonnegative function* $\mathbf{a} \in L^\infty(a,b)$ *and a Lipschitz continuous function* u*, with* $\|u'\|_{L^\infty(a,b)} \leq 1$*, such that* u *is a weak solution of*

$$\begin{cases} -(\mathbf{a}u')' = \underline{\mu} \quad \text{in } (a,b), \\ \mathbf{a}u'(a) = -\alpha, \\ \mathbf{a}u'(b) = \beta, \end{cases} \tag{3.10}$$

in the sense that

$$\int_a^b \mathbf{a}u'\varphi' - \int_a^b \varphi \, d\underline{\mu} = \alpha\varphi(a) + \beta\varphi(a)$$

for all $\varphi \in W^{1,\infty}(0,\ell_{\mathrm{e}})$*, and verifies*

$$|u'| = 1 \quad \text{on } \{\mathbf{a} > 0\},$$

then u *is a Kantorovich potential for the optimal transport problem of* η^+ *to* η^- *with the cost given by the Euclidean distance.*

Proof. (i) Since v_p is a weak solution of the problem (3.9), we have

$$\int_a^b \left(|v_p'|^{p-2}v_p'\right) w' = \int_a^b w \, d\underline{\mu} + \alpha w(a) + \beta w(b), \qquad \forall w \in W^{1,p}(\,]a,b[\,). \tag{3.11}$$

Taking $w = v_p$ in (3.11), we obtain

$$\int_a^b |v_p'|^p = \int_a^b v_p \, d\underline{\mu} + \alpha v_p(a) + \beta v_p(b) \leq C.$$

Given $v \in W^{1,\infty}(\,]a,b[\,)$ with $\|v'\|_\infty \leq 1$, taking $w = v_p - v$ in (3.11), we get

$$\int_a^b |v_p'|^p - \int_a^b \left(|v_p'|^{p-2}v_p'\right) v' \\ = \int_a^b (v_p - v) d\underline{\mu} + \alpha(v_p(a) - v(a)) + \beta(v_p(b) - v(b)).$$

Hence

$$\int_a^b v_p \, d\eta - \int_a^b v \, d\eta = \int_a^b |v_p'|^p - \int_a^b \left(|v_p'|^{p-2}v_p'\right) v'.$$

Now, by Young's inequality, we have

$$\int_a^b \left(|v_p'|^{p-2}v_p'\right) v' \leq \frac{p-1}{p} \int_a^b |v_p'|^p + \frac{1}{p} \int_a^b |v'|^p \\ \leq \frac{p-1}{p} \int_a^b |v_p'|^p + \frac{1}{p}(b-a).$$

Therefore, we obtain

$$\int_a^b v_p \, d\eta - \int_a^b v \, d\eta \geq \frac{1}{p} \int_a^b |v_p'|^p - \frac{1}{p}(b-a) \geq -\frac{1}{p}(b-a).$$

Taking limits as $p \to \infty$ we get

$$\int_a^b v_\infty \, d\eta \geq \int_a^b v \, d\eta,$$

and, consequently,

$$\int_a^b v_\infty \, d\eta = \sup \left\{ \int_a^b v d\eta \ : \ v \in W^{1,\infty}(\,]a,b[\,) \ \text{ with } \|v'\|_\infty \leq 1 \right\}.$$

This ends the proof of (i).

(ii) Taking u as test function in (3.10), we get that

$$\int_a^b u d\underline{\mu} + \alpha u(0) + \beta u(b) = \int_a^b \mathbf{a}.$$

Now take a Lipschitz continuous function $\hat{u}$, with $\|\hat{u}'\|_{L^\infty(a,b)} \leq 1$, as test function in (3.10). Then

$$\begin{aligned} \int_a^b \hat{u} d\underline{\mu} + \alpha \hat{u}(a) + \beta \hat{u}(b) &= \int_a^b \mathbf{a} u' \hat{u}' \\ &\leq \int_a^b \mathbf{a} = \int_a^b u d\underline{\mu} + \alpha u(0) + \beta u(b), \end{aligned}$$

which gives the assertion. $\square$

Suppose now that η is as in (1.2) and take u_p as in Theorem 3.1. Then, $u_p \in W^{1,p}(\Gamma)$ and

$$\int_\Gamma |u_p'|^{p-2} u_p' \varphi' = \sum_{\mathrm{e} \in \mathrm{E_v}(\Gamma)} \int_0^{\ell_\mathrm{e}} [\varphi]_\mathrm{e} d\underline{\mu}_\mathrm{e} + \sum_{\mathrm{v} \in \mathrm{V}(\Gamma)} a_\mathrm{v} \varphi(\mathrm{v})$$

for every $\varphi \in W^{1,p}(\Gamma)$. For a fixed edge $\mathrm{e} \in E(\Gamma)$, we define the distribution $\eta_{p,\mathrm{e}}$ in $\mathbb{R}$ as

$$\langle \eta_{p,\mathrm{e}}, \varphi \rangle := \int_0^{\ell_\mathrm{e}} |[u_p]_\mathrm{e}'|^{p-2} [u_p]_\mathrm{e}' \varphi' - \int_0^{\ell_\mathrm{e}} \varphi d\underline{\mu}_\mathrm{e} \qquad \forall \varphi \in C_c^\infty(\mathbb{R}).$$

THEOREM 3.5. *For u_p as above, measures $\eta_{p,\mathrm{e}}$, and u_∞ as in Lemma 3.2, we have the following:*

1. *For each edge $\mathrm{e} \in E(\Gamma)$ the following facts hold:*
   (a) *$\eta_{p,\mathrm{e}}$ is a Radon measure on $\mathbb{R}$ supported on $\{0, \ell_\mathrm{e}\}$, and consequently*

$$\eta_{p,\mathrm{e}} = a_{p,\mathrm{e},\mathrm{i_e}} \delta_0 + a_{p,\mathrm{e},\mathrm{f_e}} \delta_{\ell_\mathrm{e}}, \quad a_{p,\mathrm{e},\mathrm{i_e}}, a_{p,\mathrm{e},\mathrm{f_e}} \in \mathbb{R}.$$

   (b) *$[u_p]_\mathrm{e}$ is a weak solution of*

$$(3.12) \qquad \begin{cases} -\left(|u'|^{p-2} u'\right)' = \underline{\mu}_\mathrm{e} & \text{in } (0, \ell_\mathrm{e}), \\ \left|\dfrac{\partial u}{\partial x_\mathrm{e}}\right|^{p-2} \dfrac{\partial u}{\partial x_\mathrm{e}}(0) = -a_{p,\mathrm{e},\mathrm{i_e}}, \\ \left|\dfrac{\partial u}{\partial x_\mathrm{e}}\right|^{p-2} \dfrac{\partial u}{\partial x_\mathrm{e}}(\ell_\mathrm{e}) = a_{p,\mathrm{e},\mathrm{f_e}}. \end{cases}$$

(c) *For a subsequence* $p_i \to +\infty$,

$$(a_{p_i,\mathrm{e},\mathrm{i_e}}, a_{p_i,\mathrm{e},\mathrm{f_e}}) \to (a_{\infty,\mathrm{e},\mathrm{i_e}}, a_{\infty,\mathrm{e},\mathrm{f_e}}).$$

(d) $[u_\infty]_\mathrm{e}$ *is a Kantorovich potential for the optimal transport problem of* $\underline{\mu}_\mathrm{e}^+ + (a_{\infty,\mathrm{e},\mathrm{i_e}})^+\delta_0 + (a_{\infty,\mathrm{e},\mathrm{f_e}})^+\delta_{\ell_\mathrm{e}}$ *to* $\underline{\mu}_\mathrm{e}^- + (a_{\infty,\mathrm{e},\mathrm{i_e}})^-\delta_0 + (a_{\infty,\mathrm{e},\mathrm{f_e}})^-\delta_{\ell_\mathrm{e}}$ *with the cost given by the Euclidean distance.*

2. $\sum_{\mathrm{e}\in \mathrm{E_v}(\Gamma)} a_{\infty,\mathrm{e},\mathrm{v}} = a_\mathrm{v}$ *for any* $\mathrm{v} \in \mathrm{V}(\Gamma)$.

Proof. 1(a) Given $\varphi \in C_c^\infty(\mathbb{R})$ supported on $\mathbb{R} \setminus \{0, \ell_\mathrm{e}\}$, we have

$$\int_0^{\ell_\mathrm{e}} |[u_p]_\mathrm{e}'|^{p-2}[u_p]_\mathrm{e}'\varphi' - \int_0^{\ell_\mathrm{e}} \varphi\, d\underline{\mu}_\mathrm{e} = 0.$$

Therefore, $\eta_{p,\mathrm{e}}$ defines a Radon measure on $\mathbb{R}$ supported on $\{0, \ell_\mathrm{e}\}$, and consequently there exist $a_{p,\mathrm{i_e}}, a_{p,\mathrm{f_e}} \in \mathbb{R}$ such that

$$\eta_{p,\mathrm{e}} = a_{p,\mathrm{e},\mathrm{i_e}}\delta_0 + a_{p,\mathrm{e},\mathrm{f_e}}\delta_{\ell_\mathrm{e}}.$$

1(b) Given $\varphi \in W^{1,p}(0, \ell_\mathrm{e})$, let $\varphi_n \in C_c^\infty(\mathbb{R})$ such that

$$\varphi_{n|(0,\ell_\mathrm{e})} \to \varphi$$

in $W^{1,p}(0, \ell_\mathrm{e})$. Then

$$\int_0^{\ell_\mathrm{e}} |[u_p]_\mathrm{e}'|^{p-2}[u_p]_\mathrm{e}'\varphi_n' - \int_0^{\ell_\mathrm{e}} \varphi_n d\underline{\mu}_\mathrm{e} = a_{p,\mathrm{e},\mathrm{f_e}}\varphi_n(\ell_\mathrm{e}) + a_{p,\mathrm{e},\mathrm{i_e}}\varphi_n(0).$$

Hence, taking limits as $n \to \infty$, we obtain that

$$\int_0^{\ell_\mathrm{e}} |[u_p]_\mathrm{e}'|^{p-2}[u_p]_\mathrm{e}'\varphi' - \int_0^{\ell_\mathrm{e}} \varphi d\underline{\mu}_\mathrm{e} = a_{p,\mathrm{e},\mathrm{f_e}}\varphi(\ell_\mathrm{e}) + a_{p,\mathrm{e},\mathrm{i_e}}\varphi(0). \tag{3.13}$$

Therefore, $[u_p]_\mathrm{e}$ is a weak solution of (3.12).

1(c) Let us see that $\{a_{p,\mathrm{e},\mathrm{i_e}}\}_{p>1}$ and $\{a_{p,\mathrm{e},\mathrm{f_e}}\}_{p>1}$ are bounded. Taking in (3.13) $\varphi = [u_p]_\mathrm{e}$, since by (3.5), $[u_p]_\mathrm{e}$ is uniformly bounded independently of p, we have

$$\int_0^{\ell_\mathrm{e}} |[u_p]_\mathrm{e}'|^p \le \int_0^{\ell_\mathrm{e}} [u_p]_\mathrm{e} d\underline{\mu}_\mathrm{e} + a_{p,\mathrm{e},\mathrm{f_e}}[u_p]_\mathrm{e}(\ell_\mathrm{e}) + a_{p,\mathrm{e},\mathrm{i_e}}[u_p]_\mathrm{e}(0) \le C \tag{3.14}$$

for every $p > 1$.

On the other hand, taking $\varphi(x) = x$ in (3.13), we get

$$a_{p,\mathrm{e},\mathrm{f_e}}\ell_\mathrm{e} = \int_0^{\ell_\mathrm{e}} |[u_p]_\mathrm{e}'|^{p-2}[u_p]_\mathrm{e}' - \int_0^{\ell_\mathrm{e}} x\, d\underline{\mu}_\mathrm{e}(x).$$

Then, by (3.14) and using Hölder's inequality, we get that $\{a_{p,\mathrm{e},\mathrm{f_e}}\}_{p>1}$ is bounded. Finally, taking $\varphi = 1$ in (3.13) we get that $\{a_{p,\mathrm{e},\mathrm{i_e}}\}$ is bounded.

1(d) Recall that since we have a solution to (3.12), the compatibility condition associated to this problem must hold (just take $\varphi = 1$ in (3.13)):

$$\int_0^{\ell_\mathrm{e}} d\underline{\mu}_\mathrm{e} + a_{p,\mathrm{e},\mathrm{i_e}} + a_{p,\mathrm{e},\mathrm{f_e}} = 0.$$

Taking limits as $p \to \infty$, we obtain

$$\int_0^{\ell_e} d\underline{\mu}_e + a_{\infty,e,i_e} + a_{\infty,e,f_e} = 0.$$

Then, by Lemma 3.4, we have that $[u_\infty]_e$ is a Kantorovich potential for the optimal transport problem of $[\underline{\mu}^+]_e + (a_{\infty,e,i_e})^+\delta_0 + (a_{\infty,e,f_e})^+\delta_{\ell_e}$ to $[\underline{\mu}^-]_e + (a_{\infty,e,i_e})^-\delta_0 + (a_{\infty,e,f_e})^-\delta_{\ell_e}$ with the cost given by the Euclidean distance.

(2) Given $\varphi \in W^{1,p}(\Gamma)$, since u_p is a weak solution of (3.1), adding (3.13) for all $e \in E(\Gamma)$, we have

$$\sum_{e\in E(\Gamma)} (a_{p,e,f_e}[\varphi]_e(\ell_e) + a_{p,e,i_e}[\varphi]_e(0)) = \sum_{v\in V(\Gamma)} a_v \varphi(v).$$

Letting $p \to +\infty$, we get

$$\sum_{e\in E(\Gamma)} (a_{\infty,e,f_e}[\varphi]_e(\ell_e) + a_{\infty,e,i_e}[\varphi]_e(0)) = \sum_{v\in V(\Gamma)} a_v \varphi(v).$$

Then, rearranging the terms,

$$\sum_{v\in V(\Gamma)} \sum_{e\in E_v(\Gamma)} a_{\infty,e,v}\varphi(v) = \sum_{v\in V(\Gamma)} a_v \varphi(v),$$

from which we get the desired conclusion. ◻

Observe that $a_{\infty,e,v}$ are solutions of the following system of linear equations for unknowns $a_{e,v}$:

$$(L_{\underline{\mu},a_v}) \begin{cases} \displaystyle\sum_{v\in e} a_{e,v} = -\int_0^{\ell_e} d\underline{\mu}_e \quad \forall e \in E(\Gamma), \\ \displaystyle\sum_{e\in E_v(\Gamma)} a_{e,v} = a_v \quad \forall v \in V(\Gamma). \end{cases}$$

For any solution $\{a_{e,v}\}$ of $(L_{\underline{\mu},a_v})$, we define

$$\mathcal{C}_{\underline{\mu},a_v}(\{a_{e,v}\}) := \sum_{e\in E(\Gamma)} W_1\left(\left(\underline{\mu}_e + \sum_{v\in e}[a_{e,v}\delta_v]_e\right)^+, \left(\underline{\mu}_e + \sum_{v\in e}[a_{e,v}\delta_v]_e\right)^- \right),$$

where

$$\sum_{v\in e}[a_{e,v}\delta_v]_e = a_{e,i_e}\delta_0 + a_{e,f_e}\delta_{\ell_e},$$

and W_1 is the Wasserstein distance in $\mathbb{R}$ w.r.t. the cost $c(x,y) = |x-y|$. In the next result we shall see how to solve the transport problem in graphs through $\{a_{\infty,e,v}\}$ and the functional $\mathcal{C}_{\underline{\mu},a_v}$.

THEOREM 3.6. *Let η be a measure given by* (1.2). *Under the assumptions of Theorem* 3.5 *and for* $\{a_{\infty,e,v}\}_{\substack{e \in E(\Gamma) \\ v \in e}}$ *given there, we have that*

$$\{a_{\infty,e,v}\}_{\substack{e \in E(\Gamma) \\ v \in e}} \in \operatorname*{arg\,min}_{\{a_{e,v}\} \text{ solve } (L_{\underline{\mu},a_v})} \mathcal{C}_{\underline{\mu},a_v}(\{a_{e,v}\}). \tag{3.15}$$

Moreover, the total cost $W_{\mu,\nu}$ *satisfies*

$$
\begin{aligned}
W_{\mu,\nu} &= \min\left\{ \int_{\Gamma\times\Gamma} d_\Gamma(x,y)d\sigma(x,y) \ : \ \sigma \in \Pi\left(\eta^+,\eta^-\right) \right\} \\
&= \int_\Gamma u_\infty \left(\underline{\mu} + \sum_{\mathrm{v}\in\mathrm{V}(\Gamma)} a_{\mathrm{v}}\delta_{\mathrm{v}} \right) \\
&= \min_{\{a_{\mathrm{e,v}}\} \ \mathit{solve}\ (L_{\underline{\mu},a_{\mathrm{v}}})} \mathcal{C}_{\underline{\mu},a_{\mathrm{v}}}(\{a_{\mathrm{e,v}}\}) \\
&= \mathcal{C}_{\underline{\mu},a_{\mathrm{v}}}(\{a_{\infty,\mathrm{e,v}}\}).
\end{aligned} \tag{3.16}
$$

Proof. Since $a_{\infty,\mathrm{e,v}}$ are solutions of $(L_{\underline{\mu},a_{\mathrm{v}}})$, we have

$$
\int_\Gamma u_\infty \left(\sum_{\mathrm{v}\in\mathrm{V}(\Gamma)} a_{\mathrm{v}}\delta_{\mathrm{v}} \right) = \sum_{\mathrm{v}\in\mathrm{V}(\Gamma)} a_{\mathrm{v}} u_\infty(\mathrm{v}) = \sum_{\mathrm{v}\in\mathrm{V}(\Gamma)} \left(\sum_{\mathrm{e}\in\mathrm{E}_{\mathrm{v}}(\Gamma)} a_{\infty,\mathrm{e,v}} \right) u_\infty(\mathrm{v})
$$

and, rearranging,

$$
\int_\Gamma u_\infty \left(\sum_{\mathrm{v}\in\mathrm{V}(\Gamma)} a_{\mathrm{v}}\delta_{\mathrm{v}} \right) = \sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \sum_{\mathrm{v}\in\mathrm{e}} a_{\infty,\mathrm{e,v}} u_\infty(\mathrm{v}).
$$

Therefore,

$$
\begin{aligned}
&\int_\Gamma u_\infty \left(\underline{\mu} + \sum_{\mathrm{v}\in\mathrm{V}(\Gamma)} a_{\mathrm{v}}\delta_{\mathrm{v}} \right) \\
&\quad = \sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} \left(\int_0^{\ell_{\mathrm{e}}} [u_\infty]_{\mathrm{e}} d\underline{\mu}_{\mathrm{e}} + a_{\infty,\mathrm{e,i_e}}[u_\infty]_{\mathrm{e}}(0) + a_{\infty,\mathrm{e,f_e}}[u_\infty]_{\mathrm{e}}(\ell_{\mathrm{e}}) \right).
\end{aligned}
$$

Now, since $[u_\infty]_{\mathrm{e}}$ is a Kantorovich potential for the optimal transport problem of $[\underline{\mu}^+]_{\mathrm{e}} + (a_{\infty,\mathrm{e,i_e}})^+\delta_0 + (a_{\infty,\mathrm{e,f_e}})^+\delta_{\ell_{\mathrm{e}}}$ to $[\underline{\mu}^-]_{\mathrm{e}} + (a_{\infty,\mathrm{e,i_e}})^-\delta_0 + (a_{\infty,\mathrm{e,f_e}})^-\delta_{\ell_{\mathrm{e}}}$ with the cost given by the Euclidean distance, we have

$$
\begin{aligned}
&\int_0^{\ell_{\mathrm{e}}} [u_\infty]_{\mathrm{e}} d\underline{\mu}_{\mathrm{e}} + a_{\infty,\mathrm{e,i_e}}[u_\infty]_{\mathrm{e}}(0) + a_{\infty,\mathrm{e,f_e}}[u_\infty]_{\mathrm{e}}(\ell_{\mathrm{e}}) \\
&\quad = W_1\left(\left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}} [a_{\infty,\mathrm{e,v}}\delta_{\mathrm{v}}]_{\mathrm{e}} \right)^+, \left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}} [a_{\infty,\mathrm{e,v}}\delta_{\mathrm{v}}]_{\mathrm{e}} \right)^- \right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\int_\Gamma u_\infty \left(\underline{\mu} + \sum_{\mathrm{v}\in\mathrm{V}(\Gamma)} a_{\mathrm{v}}\delta_{\mathrm{v}} \right) \\
&\quad = \sum_{\mathrm{e}\in\mathrm{E}(\Gamma)} W_1\left(\left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}} [a_{\infty,\mathrm{e,v}}\delta_{\mathrm{v}}]_{\mathrm{e}} \right)^+, \left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}} [a_{\infty,\mathrm{e,v}}\delta_{\mathrm{v}}]_{\mathrm{e}} \right)^- \right) \\
&\quad = \mathcal{C}_{\underline{\mu},a_{\mathrm{v}}}(\{a_{\infty,\mathrm{e,v}}\}).
\end{aligned}
$$

Hence, for

$$\{\tilde a_{\rm e,v}\}_{\substack{\rm e\,\in\,E(\Gamma)\\ \rm v\,\in\,e}} \in \mathop{\arg\min}_{\{a_{\rm e,v}\}\ \text{solve}\ (L_{\underline{\mu},a_{\rm v}})} \mathcal{C}_{\underline{\mu},a_{\rm v}}(\{a_{\rm e,v}\}),$$

we have that

$$\begin{aligned}&\int_\Gamma u_\infty\left(\underline{\mu}+\sum_{\rm v\in V(\Gamma)} a_{\rm v}\delta_{\rm v}\right)\\ &\quad\geq \sum_{\rm e\in E(\Gamma)} W_1\left(\left(\underline{\mu}_{\rm e}+\sum_{\rm v\in e}[\tilde a_{\rm e,v}\delta_{\rm v}]_{\rm e}\right)^+,\left(\underline{\mu}_{\rm e}+\sum_{\rm v\in e}[\tilde a_{\rm e,v}\delta_{\rm v}]_{\rm e}\right)^-\right)\\ &\quad= \mathcal{C}_{\underline{\mu},a_{\rm v}}(\{\tilde a_{\rm e,v}\}).\end{aligned}$$

Given $\rm e\in E(\Gamma)$, let $\gamma_{\rm e}$ be an optimal transport plan between $\left(\underline{\mu}\llcorner {\rm e}+\sum_{\rm v\in e}\tilde a_{\rm e,v}\delta_{\rm v}\right)^+$ and $\left(\underline{\mu}\llcorner {\rm e}+\sum_{\rm v\in e}\tilde a_{\rm e,v}\delta_{\rm v}\right)^-$ w.r.t. the cost given by the metric d_Γ (here we refer to $\underline{\mu}\llcorner A$ as the restriction of $\underline{\mu}$ to the set A). Then we have

$$W_1\left(\left(\underline{\mu}_{\rm e}+\sum_{\rm v\in e}[\tilde a_{\rm e,v}\delta_{\rm v}]_{\rm e}\right)^+,\left(\underline{\mu}_{\rm e}+\sum_{\rm v\in e}[\tilde a_{\rm e,v}\delta_{\rm v}]_{\rm e}\right)^-\right)\geq \int_{\Gamma\times\Gamma} d_\Gamma(x,y)d\gamma_{\rm e}(x,y).$$

Hence, if $\gamma := \sum_{\rm e\in E(\Gamma)}\gamma_{\rm e}$, we get

$$\begin{aligned}&\mathcal{C}_{\underline{\mu},a_{\rm v}}(\{\tilde a_{\rm e,v}\}) = \sum_{\rm e\in E(\Gamma)} W_1\left(\left(\underline{\mu}_{\rm e}+\sum_{\rm v\in e}[\tilde a_{\rm e,v}\delta_{\rm v}]_{\rm e}\right)^+,\left(\underline{\mu}_{\rm e}+\sum_{\rm v\in e}[\tilde a_{\rm e,v}\delta_{\rm v}]_{\rm e}\right)^-\right)\\ &\quad\geq \int_{\Gamma\times\Gamma} d_\Gamma(x,y)d\gamma(x,y).\end{aligned}$$

Now, given $\varphi\in C(\Gamma)$, we have

$$\begin{aligned}&\int_\Gamma \varphi(x)d\pi_1\#\gamma(x) = \int_{\Gamma\times\Gamma}\varphi(x)d\gamma(x,y) = \sum_{\rm e\in E(\Gamma)}\int_{\Gamma\times\Gamma}\varphi(x)d\gamma_{\rm e}(x,y)\\ &\quad= \sum_{\rm e\in E(\Gamma)}\int_\Gamma \varphi(x)d\left(\underline{\mu}\llcorner {\rm e}+\sum_{\rm v\in e}\tilde a_{\rm e,v}\delta_{\rm v}\right)^+\\ &\quad= \sum_{\rm e\in E(\Gamma)}\left(\int_0^{\ell_{\rm e}}[\varphi]_{\rm e}(x)d\underline{\mu}_{\rm e}+\sum_{\rm v\in e}\tilde a^+_{\rm e,v}\varphi({\rm v})\right).\end{aligned}$$

Since

$$\sum_{\rm e\in E(\Gamma)}\sum_{\rm v\in e}\tilde a^+_{\rm e,v}\varphi({\rm v}) = \sum_{{\rm v}\in V(\Gamma)} a^+_{\rm v}\varphi({\rm v}) = \int_\Gamma\varphi(x)d\left(\sum_{\rm v\in V(\Gamma)} a_{\rm v}\delta_{\rm v}\right)^+,$$

we get

$$\pi_1\#\gamma = \left(\underline{\mu}+\sum_{\rm v\in V(\Gamma)} a_{\rm v}\delta_{\rm v}\right)^+.$$

Similarly, we obtain

$$\pi_2 \# \gamma = \left(\underline{\mu} + \sum_{\mathrm{v} \in \mathrm{V}(\Gamma)} a_{\mathrm{v}} \delta_{\mathrm{v}} \right)^{-}.$$

Consequently,

$$\gamma \in \prod \left(\left(\underline{\mu} + \sum_{\mathrm{v} \in \mathrm{V}(\Gamma)} a_{\mathrm{v}} \delta_{\mathrm{v}} \right)^{+}, \left(\underline{\mu} + \sum_{\mathrm{v} \in \mathrm{V}(\Gamma)} a_{\mathrm{v}} \delta_{\mathrm{v}} \right)^{-} \right).$$

Therefore,

$$\begin{aligned} &\int_\Gamma u_\infty \left(\underline{\mu} + \sum_{\mathrm{v} \in \mathrm{V}(\Gamma)} a_{\mathrm{v}} \delta_{\mathrm{v}} \right) \\ &\quad = \min \left\{ \int_{\Gamma \times \Gamma} d_\Gamma(x,y) d\sigma(x,y) \ : \ \sigma \in \Pi\left(\eta^+, \eta^-\right) \right\} \\ &\quad \le \int_{\Gamma \times \Gamma} d_\Gamma(x,y) d\gamma(x,y) \le \mathcal{C}_{\underline{\mu}, a_{\mathrm{v}}}(\{\tilde{a}_{\mathrm{e,v}}\}) \\ &\quad \le \mathcal{C}_{\underline{\mu}, a_{\mathrm{v}}}(\{a_{\infty,\mathrm{e,v}}\}) \\ &\quad = \int_\Gamma u_\infty \left(\underline{\mu} + \sum_{\mathrm{v} \in \mathrm{V}(\Gamma)} a_{\mathrm{v}} \delta_{\mathrm{v}} \right), \end{aligned}$$

from which we get (3.15) and (3.16). □

Remark 3.7. Observe that (3.16) implies that each $a_{\infty,\mathrm{e,v}}$ is the mass that enters/leaves e via the vertex $\mathrm{v} \in \mathrm{e}$, depending on its positive/negative sign, during the optimal transport process.

Remark 3.8. In the particular case of $\underline{\mu} \equiv 0$ we get the even simpler formula

$$\{a_{\infty,\mathrm{e,v}}\}_{\substack{\mathrm{e} \in \mathrm{E}(\Gamma) \\ \mathrm{v} \in \mathrm{e}}} \in \arg\min \left\{ \frac{1}{2} \sum_{\mathrm{e} \in \mathrm{E}(\Gamma)} \left(\sum_{\mathrm{v} \in \mathrm{e}} |a_{\mathrm{e,v}}| \right) \ell_{\mathrm{e}} : \{a_{\mathrm{e,v}}\} \text{ solve } (L_{0,a_{\mathrm{v}}}) \right\},$$

and

$$W_{0,a_{\mathrm{v}}} = \min \left\{ \frac{1}{2} \sum_{\mathrm{e} \in \mathrm{E}(\Gamma)} \left(\sum_{\mathrm{v} \in \mathrm{e}} |a_{\mathrm{e,v}}| \right) \ell_{\mathrm{e}} : \{a_{\mathrm{e,v}}\} \text{ solve } (L_{0,a_{\mathrm{v}}}) \right\}$$

for the optimal total cost of the transport problem. Observe that we can rewrite this as follows:

$$W_{0,a_{\mathrm{v}}} = \min \left\{ \sum_{\mathrm{e} \in \mathrm{E}(\Gamma)} |a_{\mathrm{e,i_e}}| \ell_{\mathrm{e}} : \{a_{\mathrm{e,v}}\} \text{ solve } (L_{0,a_{\mathrm{v}}}) \right\}.$$

Moreover, since

$$a_{p,\mathrm{e,i_e}} + a_{p,\mathrm{e,f_e}} = 0,$$

we have that $[u_p]_{\rm e}$ is a weak solution of

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = 0 & \text{in } (0,\ell_{\rm e}),\\ \left|\dfrac{\partial u}{\partial x_{\rm e}}\right|^{p-2}\dfrac{\partial u}{\partial x_{\rm e}}(0) = -a_{p,{\rm e},{\rm i}_{\rm e}}, \\ \left|\dfrac{\partial u}{\partial x_{\rm e}}\right|^{p-2}\dfrac{\partial u}{\partial x_{\rm e}}(\ell_{\rm e}) = -a_{p,{\rm e},{\rm i}_{\rm e}}. \end{cases}$$

Therefore, up to a constant,

$$[u_p]_{\rm e}(x) = -\text{sign}(a_{p,{\rm e},{\rm i}_{\rm e}})|a_{p,{\rm e},{\rm i}_{\rm e}}|^{\frac{1}{p-1}}x,$$

and consequently, up to a constant,

$$[u_\infty]_{\rm e}(x) = -\text{sign}(a_{\infty,{\rm e},{\rm i}_{\rm e}})x.$$

A remarkable fact is that our results show that these functions can be glued continuously on the graph Γ.

In the next result we shall see that also in the case $\underline{\mu} \neq 0$ it is possible to get a simple formula for the total cost. We also find a transport density for the transport problem.

THEOREM 3.9. *Let η be a measure given by* (1.2). *Under the assumptions of Theorem* 3.5 *and for* $\{a_{\infty,{\rm e},{\rm v}}\}_{\substack{{\rm e}\,\in\,{\rm E}(\Gamma)\\ {\rm v}\,\in\,{\rm e}}}$ *given there, there exists a nonnegative function* $\mathbf{a} \in L^\infty(\Gamma)$ *such that* $[u_\infty]_{\rm e}$ *is a weak solution of*

$$\begin{cases} -([\mathbf{a}]_{\rm e}[u_\infty]_{\rm e}')' = \underline{\mu}_{\rm e} \quad \textit{in } (0,\ell_{\rm e}),\\ [\mathbf{a}]_{\rm e}[u_\infty]_{\rm e}'(0) = -a_{\infty,{\rm e},{\rm i}_{\rm e}},\\ [\mathbf{a}]_{\rm e}[u_\infty]_{\rm e}'(\ell_{\rm e}) = a_{\infty,{\rm e},{\rm f}_{\rm e}}, \end{cases} \tag{3.17}$$

in the sense

$$\int_0^{\ell_{\rm e}} [\mathbf{a}]_{\rm e}[u_\infty]_{\rm e}'\varphi' - \int_0^{\ell_{\rm e}} \varphi\, d\underline{\mu}_{\rm e} = a_{\infty,{\rm e},{\rm f}_{\rm e}}\varphi(\ell_{\rm e}) + a_{\infty,{\rm e},{\rm i}_{\rm e}}\varphi(0)$$

for all $\varphi \in W^{1,\infty}(0,\ell_{\rm e})$. *Furthermore,*

$$|[u_\infty]_{\rm e}'| = 1 \qquad \textit{on} \quad \{[\mathbf{a}]_{\rm e} > 0\}. \tag{3.18}$$

More precisely, we have

$$[\mathbf{a}]_{\rm e}(x) = \left| a_{\infty,{\rm e},{\rm i}_{\rm e}} + \int_0^x d\underline{\mu}_{\rm e}(y)\right| \qquad \textit{for } x \in (0,\ell_{\rm e}) \tag{3.19}$$

and

$$[u_\infty]_{\rm e}'(x) = \text{sign}\left(-a_{\infty,{\rm e},{\rm i}_{\rm e}} - \int_0^x d\underline{\mu}_{\rm e}(y)\right) \qquad \textit{for } x \in (0,\ell_{\rm e}). \tag{3.20}$$

Moreover, it holds that the optimal total cost is given by

$$\begin{aligned} W_{\mu,\nu} &= \min\left\{ \sum_{{\rm e}\in{\rm E}(\Gamma)} \int_0^{\ell{\rm e}} \left| a_{{\rm e},{\rm i}_{\rm e}} + \int_0^x d\underline{\mu}_{\rm e}(y)\right| dx : \{a_{{\rm e},{\rm v}}\} \textit{ solve } (L_{\underline{\mu},a_{\rm v}}) \right\} \\ &= \sum_{{\rm e}\in{\rm E}(\Gamma)} \int_0^{\ell{\rm e}} \left| a_{\infty,{\rm e},{\rm i}_{\rm e}} + \int_0^x d\underline{\mu}_{\rm e}(y)\right| dx. \end{aligned} \tag{3.21}$$

Proof. Fix $\mathrm{e} \in E(\Gamma)$, and take $[u_p]_{\mathrm{e}}$, the weak solution obtained above. Then, from (3.12) we have

$$-\left(|[u_p]'_{\mathrm{e}}|^{p-2}[u_p]'_{\mathrm{e}}\right)' = \underline{\mu}_{\mathrm{e}} \qquad \text{in } \mathcal{D}'(0,\ell_{\mathrm{e}}).$$

Now, we let

$$w_{\mu,\mathrm{e}}(x) := \underline{\mu}_{\mathrm{e}}((0,x)) = \int_0^x d\underline{\mu}_{\mathrm{e}}(y)$$

for $x \in (0,\ell_{\mathrm{e}})$, and we observe that $(w_{\mu,\mathrm{e}})' = \underline{\mu}_{\mathrm{e}}$ in $\mathcal{D}'(0,\ell_{\mathrm{e}})$. Hence, there exists a constant $\alpha_{p,\mathrm{e}} \in \mathbb{R}$ such that $|[u_p]'_{\mathrm{e}}|^{p-2}[u_p]'_{\mathrm{e}} = -w_{\mu,\mathrm{e}} + \alpha_{p,\mathrm{e}}$. Then, since

$$-a_{p,\mathrm{e},\mathrm{i_e}} = |[u_p]'_{\mathrm{e}}|^{p-2}[u_p]'_{\mathrm{e}}(0) = -w_{\mu,\mathrm{e}}(0) + \alpha_{p,\mathrm{e}} = \alpha_{p,\mathrm{e}},$$

we get

$$|[u_p]'_{\mathrm{e}}|^{p-2}[u_p]'_{\mathrm{e}}(x) = -w_{\mu,\mathrm{e}}(x) - a_{p,\mathrm{e},\mathrm{i_e}}.$$

Now we observe that $-w_{\mu,\mathrm{e}}(x) - a_{p,\mathrm{e},\mathrm{i_e}}$ is bounded in L^∞ uniformly in p, and that, for a subsequence,

$$|[u_p]'_{\mathrm{e}}(x)|^{p-2} = |w_{\mu,\mathrm{e}}(x) + a_{p,\mathrm{e},\mathrm{i_e}}|^{\frac{p-2}{p-1}} \to |w_{\mu,\mathrm{e}}(x) + a_{\infty,\mathrm{e},\mathrm{i_e}}| =: [\mathbf{a}]_{\mathrm{e}}(x) \quad \forall x \in (0,\ell_{\mathrm{e}}).$$

Moreover, by (3.14), we can assume that

$$[u_p]'_{\mathrm{e}} \rightharpoonup [u_\infty]'_{\mathrm{e}} \quad \text{weakly in } L^2((0,\ell_{\mathrm{e}})).$$

Therefore, we have that

$$|[u_p]'_{\mathrm{e}}|^{p-2}[u_p]'_{\mathrm{e}} \to |w_{\mu,\mathrm{e}} + a_{\infty,\mathrm{e},\mathrm{i_e}}|\,[u_\infty]'_{\mathrm{e}} \quad \text{in } L^1((0,\ell_{\mathrm{e}})),$$

but also

$$|[u_p]'_{\mathrm{e}}|^{p-2}[u_p]'_{\mathrm{e}} \to -w_{\mu,\mathrm{e}} - a_{\infty,\mathrm{e},\mathrm{i_e}}.$$

Consequently, (3.18), (3.19), and (3.20) are proved.

On the other hand, by (3.13), for any $\varphi \in W^{1,\infty}(0,\ell_{\mathrm{e}})$, we have

$$\int_0^{\ell_{\mathrm{e}}} |[u_p]'_{\mathrm{e}}|^{p-2}[u_p]'_{\mathrm{e}}\varphi' - \int_0^{\ell_{\mathrm{e}}} \varphi d\underline{\mu}_{\mathrm{e}} = a_{p,\mathrm{e},\mathrm{f_e}}\varphi(\ell_{\mathrm{e}}) + a_{p,\mathrm{e},\mathrm{i_e}}\varphi(0). \tag{3.22}$$

Then, taking limits in (3.22) when $p \to \infty$, we get

$$\int_0^{\ell_{\mathrm{e}}} [\mathbf{a}]_{\mathrm{e}}[u_\infty]'_{\mathrm{e}}\varphi' - \int_0^{\ell_{\mathrm{e}}} \varphi d\underline{\mu}_{\mathrm{e}} = a_{\infty,\mathrm{e},\mathrm{f_e}}\varphi(\ell_{\mathrm{e}}) + a_{\infty,\mathrm{e},\mathrm{i_e}}\varphi(0)$$

for all $\varphi \in W^{1,\infty}(0,\ell_{\mathrm{e}})$, and thus (3.17) holds.

Let $\{a_{\mathrm{e},\mathrm{v}}\}$ be a solution of $(L_{\mu,a_{\mathrm{v}}})$. Then, if we define

$$\tilde{\mathbf{a}}(x) = \left| a_{e,\mathrm{i_e}} + \int_0^x d\underline{\mu}_{\mathrm{e}}(y) \right| \quad \text{for } x \in (0,\ell_{\mathrm{e}}),$$

and $\tilde{u}$ by

$$\tilde{u}'(x) = \operatorname{sign}\left(-a_{e,\mathrm{i_e}} - \int_0^x d\underline{\mu}_{\mathrm{e}}(y)\right) \quad \text{for } x \in (0, \ell_{\mathrm{e}}),$$

we have that $0 \le \tilde{\mathbf{a}} \in L^\infty(0,\ell_{\mathrm{e}})$ and $\tilde{u}$ is a Lipschitz function with $\|\tilde{u}'\|_{L^\infty(0,\ell_{\mathrm{e}})} \le 1$ such that $\tilde{u}$ is a weak solution of

$$\begin{cases} -(\tilde{\mathbf{a}}\tilde{u}')' = \underline{\mu}_{\mathrm{e}} & \text{in } (0,\ell_{\mathrm{e}}), \\ \tilde{\mathbf{a}}\tilde{u}'(0) = -a_{\mathrm{e},\mathrm{i_e}}, \\ \tilde{\mathbf{a}}\tilde{u}'(\ell_{\mathrm{e}}) = a_{\mathrm{e},\mathrm{f_e}}, \end{cases} \tag{3.23}$$

and

$$|\tilde{u}'| = 1 \quad \text{ on } \{\tilde{\mathbf{a}} > 0\}.$$

Then by (ii) in Lemma 3.4, $\tilde{u}$ is a Kantorovich potential for the transport of the measure $\left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}}[a_{\mathrm{e},\mathrm{v}}\delta_{\mathrm{v}}]_{\mathrm{e}}\right)^+$ to $\left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}}[a_{\mathrm{e},\mathrm{v}}\delta_{\mathrm{v}}]_{\mathrm{e}}\right)^-$. Therefore, taking $\tilde{u}$ as test function in (3.23), we get

$$\int_0^{\ell_{\mathrm{e}}} \tilde{\mathbf{a}}|\tilde{u}'|^2 - \int_0^{\ell_{\mathrm{e}}} \tilde{u} d\underline{\mu}_{\mathrm{e}} = a_{\mathrm{e},\mathrm{f_e}}\tilde{u}(\ell_{\mathrm{e}}) + a_{\mathrm{e},\mathrm{i_e}}\tilde{u}(0),$$

and consequently

$$\begin{aligned} \int_0^{\ell_{\mathrm{e}}} \left| a_{e,\mathrm{i_e}} + \int_0^x d\underline{\mu}_{\mathrm{e}}(y)\right| dx &= -\int_0^{\ell_{\mathrm{e}}} \tilde{u} d\underline{\mu}_{\mathrm{e}} + a_{\mathrm{e},\mathrm{f_e}}\tilde{u}(\ell_{\mathrm{e}}) + a_{\mathrm{e},\mathrm{i_e}}\tilde{u}(0) \\ &= W_1\left(\left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}}[a_{\mathrm{e},\mathrm{v}}\delta_{\mathrm{v}}]_{\mathrm{e}}\right)^+, \left(\underline{\mu}_{\mathrm{e}} + \sum_{\mathrm{v}\in\mathrm{e}}[a_{\mathrm{e},\mathrm{v}}\delta_{\mathrm{v}}]_{\mathrm{e}}\right)^-\right). \end{aligned}$$

Adding, we obtain

$$\sum_{\mathrm{e}\in \mathrm{E}(\Gamma)} \int_0^{\ell \mathrm{e}} \left| a_{\mathrm{e},\mathrm{i_e}} + \int_0^x d\underline{\mu}_{\mathrm{e}}(y)\right| dx = \mathcal{C}_{\mu,a_{\mathrm{v}}}(\{a_{\mathrm{e},\mathrm{v}}\}),$$

from which (3.21) follows, keeping in mind (3.16). ◻

Remark 3.10. The first part of the above result is similar to Lemma 7.2 in [1]; the main difference is that here we have to consider fluxes at the ends of the edges.

Finally, let us present some simple examples to illustrate our results.

Example 3.11. Let Γ be the metric graph with edges $E(\Gamma) = \{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4\}$ and vertices $V(\Gamma) = \{\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4\}$ such that $(\mathrm{i}_{\mathrm{e}_1},\mathrm{f}_{\mathrm{e}_1}) = (\mathrm{v}_1,\mathrm{v}_2)$, $(\mathrm{i}_{\mathrm{e}_2},\mathrm{f}_{\mathrm{e}_2}) = (\mathrm{v}_2,\mathrm{v}_3)$, $(\mathrm{i}_{\mathrm{e}_3},\mathrm{f}_{\mathrm{e}_3}) = (\mathrm{v}_2,\mathrm{v}_4)$, $(\mathrm{i}_{\mathrm{e}_4},\mathrm{f}_{\mathrm{e}_4}) = (\mathrm{v}_3,\mathrm{v}_4)$ (see Figure 1).

Consider the measure

$$\eta = \underline{\mu} + \sum_{\mathrm{v}\in V(\Gamma)} a_{\mathrm{v}}\delta_{\mathrm{v}},$$

with $\mu^+_{\mathrm{e}_1} = \chi_{(0,\frac{\ell_{\mathrm{e}_1}}{2})}$, all the other $\mu^{\pm}_{\mathrm{e}_i} = 0$, and $a_{\mathrm{v}_1} = a_{\mathrm{v}_2} = 0$, $a_{\mathrm{v}_3} = a_{\mathrm{v}_4} = -\frac{\ell_{\mathrm{e}_1}}{4}$. By Theorem 3.9, we have that

$$\begin{aligned} W_{\mu,\nu} = \min_{\{a_{ij}\} \text{ solve } (L_{\mu,a_{\mathrm{v}}})} \Bigg(& \int_0^{\frac{l_1}{2}} |a_{11} + x|\, dx + \int_{\frac{l_1}{2}}^{l_1} \left| a_{11} + \frac{l_1}{2}\right| dx \\ & + |a_{22}| l_2 + |a_{32}| l_3 + |a_{43}| l_4 \Bigg), \end{aligned}$$

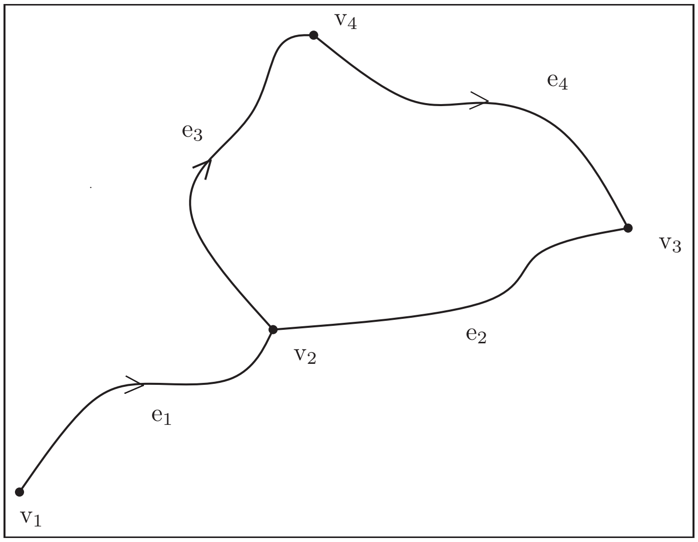

FIG. 1. *Optimal transport path for $l_2 > l_3 + l_4$ (here arrows indicate mass transport directions).*

where $a_{ij} = a_{\mathrm{e}_i,\mathrm{v}_j}$ and $l_i = \ell_{\mathrm{e}_i}$. Now, a simple calculation shows that

$$a_{11} = 0, \quad a_{12} = -\frac{l_1}{2}, \quad a_{23} = -a_{22}, \quad a_{32} = \frac{l_1}{2} - a_{22},$$

and

$$a_{34} = a_{22} - \frac{l_1}{2}, \quad a_{43} = a_{22} - \frac{l_1}{4}, \quad a_{44} = \frac{l_1}{4} - a_{22}.$$

Hence, the total cost is given by

$$W_{\mu,\nu} = \min_{a_{22}\in\mathbb{R}} \left(\frac{3}{8} l_1^2 + |a_{22}| l_2 + \left| a_{22} - \frac{l_1}{2} \right| l_3 + \left| a_{22} - \frac{l_1}{4} \right| l_4 \right).$$

We have different values for the minimum depending on the values of l_2, l_3, l_4.

(1) If $l_2 > l_3 + l_4$, then the minimum is attained at $a_{22} = 0$. Consequently, as expected, the route of the optimal transport is given in Figure 1. Note that we are not sending mass through e_2, but instead we use e_3 until we reach v_4, where we deposit half of the mass and then continue through e_4 until reaching v_3 with the other half of the mass.

In this case, the optimal transport cost is given by

$$W_{\mu,\nu} = \frac{3}{8} {l_1}^2 + \frac{1}{2} l_1 l_2 + \frac{1}{4} l_1 l_4 = \frac{1}{2} l_1 \frac{3}{4} l_1 + \frac{1}{4} l_1 l_3 + \frac{1}{4} l_1 (l_3 + l_4).$$

(2) If $l_3 > l_2 + l_4$, then the minimum is attained at $a_{22} = \frac{l_1}{2}$. Hence $a_{32} = 0$, and, consequently, in this case, the best strategy is not to use e_3 to transport mass.

(3) If $l_2 \leq l_3 + l_4$ and $l_3 \leq l_2 + l_4$, the minimum is attained at $a_{22} = \frac{l_1}{4}$. Hence $a_{43} = a_{44} = 0$; therefore, here we split the mass into two equal parts when we arrive at v_2 and send them to v_3 and v_4 using e_2 and e_3. Now we are not using e_4.

Example 3.12. Let Γ be the metric graph with edges $E(\Gamma) = \{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ and vertices $V(\Gamma) = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ such that $(\mathrm{i}_{\mathrm{e}_1}, \mathrm{f}_{\mathrm{e}_1}) = (\mathrm{v}_1, \mathrm{v}_2)$, $(\mathrm{i}_{\mathrm{e}_2}, \mathrm{f}_{\mathrm{e}_2}) = (\mathrm{v}_2, \mathrm{v}_3)$, $(\mathrm{i}_{\mathrm{e}_3}, \mathrm{f}_{\mathrm{e}_3}) = (\mathrm{v}_3, \mathrm{v}_1)$ (see Figure 2). Take $\ell_{\mathrm{e}_1} = 1$.

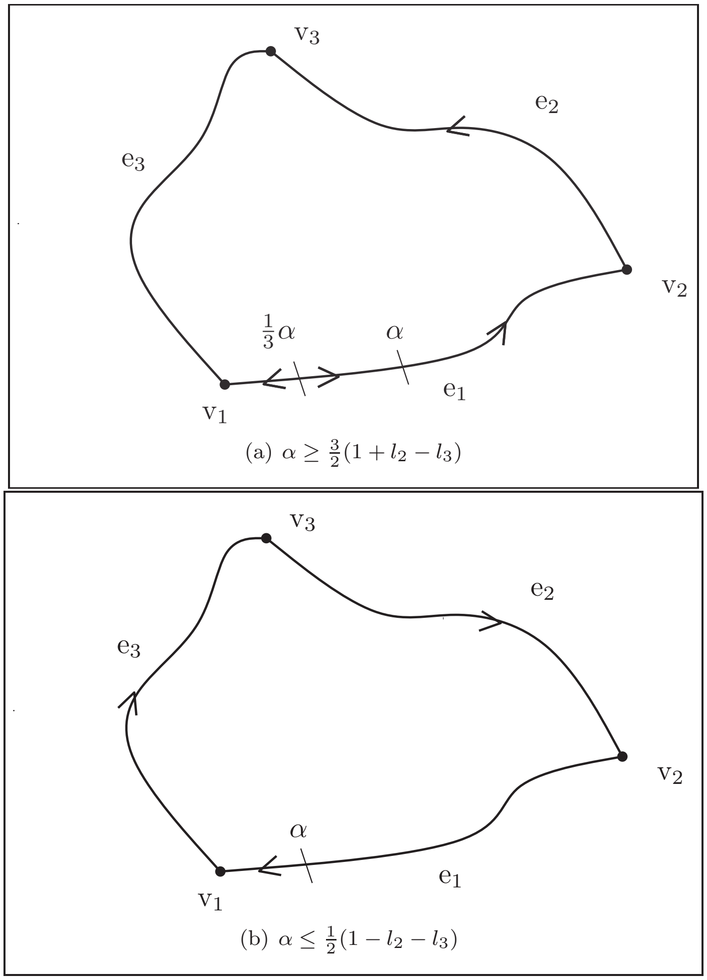

Fig. 2. *Optimal transport paths for different α.*

Consider the measure

$$\eta = \underline{\mu} + \sum_{\mathrm{v} \in V(\Gamma)} a_{\mathrm{v}} \delta_{\mathrm{v}},$$

with $\underline{\mu}_{\mathrm{e}_1}^{+} = \chi_{(0,\alpha)}$, $0 < \alpha \leq 1$, all the other $\underline{\mu}_{\mathrm{e}_i}^{\pm} = 0$, and $a_{\mathrm{v}_1} = a_{\mathrm{v}_2} = a_{\mathrm{v}_3} = -\frac{\alpha}{3}$. By Theorem 3.9, we have that

$$W_{\mu,\nu} = \min_{\{a_{ij}\} \text{ solve } (L_{\mu, a_{\mathrm{v}}})} \left(\int_0^\alpha |a_{11} + x| \, dx \right.$$
$$\left. + |a_{11} + \alpha| \, (1 - \alpha) + |a_{22}| l_2 + |a_{31}| l_3 \right),$$

where $a_{ij} = a_{\mathrm{e}_i, \mathrm{v}_j}$ and $l_i = \ell_{\mathrm{e}_i}$. Now, a simple calculation shows that

$$a_{12} = -a_{11} - \alpha, \quad a_{22} = a_{11} + \frac{2}{3}\alpha, \quad a_{23} = -a_{22}, \quad a_{31} = -a_{11} - \frac{1}{3}\alpha, \quad a_{33} = -a_{31}.$$

Therefore,

$$W_{\mu,\nu} = \min_{a_{11}\in\mathbb{R}} \left(\int_0^\alpha |a_{11}+x|\,dx + \Big|a_{11}+\alpha\Big|(1-\alpha) + \Big|a_{11}+\frac{2}{3}\alpha\Big| l_2 + \Big|a_{11}+\frac{1}{3}\alpha\Big| l_3 \right).$$

We have different values for the minimum depending on the values of l_2, l_3, and α. In fact, preforming a tedious computation, we get the following:

(1) If $\alpha \geq \frac{3}{2}(1+l_2-l_3)$, then the minimum is attained at $a_{11} = -\frac{1}{3}\alpha$. This implies that

$$a_{12} = -\frac{2}{3}\alpha, \quad a_{22} = \frac{1}{3}\alpha, \quad a_{23} = -\frac{1}{3}\alpha, \quad a_{31} = a_{33} = 0.$$

The path of the optimal transport is given in Figure 2(a).

Note that we are not sending mass through e_3, but instead we use e_1 until we reach v_2, where we deposit $\frac{1}{3}\alpha$ of the mass and then continue through e_2 until reaching v_3 with the other $\frac{1}{3}\alpha$ of the mass.

(2) If $\frac{3}{4}(1+l_2-l_3) < \alpha < \frac{3}{2}(1+l_2-l_3)$, then the minimum is attained at $a_{11} = \frac{1}{2}(l_3-l_2-1) \in \left(-\frac{2}{3}\alpha, -\frac{1}{3}\alpha\right)$. Therefore, all the $a_{ij} \neq 0$, and, consequently, in this case it is necessary to use all the edges.

(3) If $\frac{3}{4}(1-l_2-l_3) \leq \alpha \leq \frac{3}{4}(1+l_2-l_3)$, then the minimum is attained at $a_{11} = -\frac{2}{3}\alpha$. Hence,

$$a_{12} = -\frac{1}{3}\alpha, \quad a_{22} = a_{23} = 0, \quad a_{31} = -\frac{1}{3}\alpha \quad a_{33} = \frac{1}{3}\alpha.$$

Consequently, in this case, the best strategy is not to use e_2 to transport mass.

(4) If $\frac{1}{2}(1-l_2-l_3) < \alpha < \frac{3}{4}(1-l_2-l_3)$, then the minimum is attained at $a_{11} = \frac{1}{2}(l_3+l_2-1) \in \left(-\alpha, -\frac{2}{3}\alpha\right)$. In this case, the best strategy is not to use e_3 to transport mass.

(5) If $\alpha \leq \frac{1}{2}(1-l_2-l_3)$, then the minimum is attained at $a_{11} = -\alpha$. Then, we get

$$a_{12} = 0, \quad a_{22} = -\frac{1}{3}\alpha, \quad a_{23} = \frac{1}{3}\alpha, \quad a_{31} = \frac{2}{3}\alpha, \quad a_{33} = -\frac{2}{3}.$$

In this case we are sending all the mass through v_1, and then we use e_3 and e_2 to deliver it to its final destination at v_3 and v_2 (see Figure 2(b)).

REFERENCES

[1] L. Ambrosio, *Lecture notes on optimal transport problems*, in Mathematical Aspects of Evolving Interfaces (Funchal, 2000), Lecture Notes in Math. 1812, Springer, Berlin, 2003, pp. 1–52.

[2] V. Banica and L. I. Ignat, *Dispersion for the Schrödinger equation on networks.* J. Math. Phys., 52 (2011), 083703.

[3] G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, Math. Surveys Monogr. 186, American Mathematical Society, Providence, RI, 2013.

[4] M. Bernot, V. Caselles, and J.-M. Morel, *Optimal Transportation Networks. Models and Theory*, Lecture Notes in Math. 1955, Springer-Verlag, Berlin, 2009.

[5] G. Buttazzo, A. Pratelli, S. Solimini, and E. Stepanov, *Optimal Urban Networks via Mass Transportation*, Lecture Notes in Math. 1961, Springer-Verlag, Berlin, 2009.

[6] L. C. Evans, *Partial Differential Equations*, Grad. Studi. Math. 19, AMS, Providence, RI, 1998.

[7] L. C. Evans and W. Gangbo, *Differential equations methods for the Monge-Kantorovich mass transfer problem*, Mem. Amer. Math. Soc., 137 (1999), no. 653.

[8] N. Igbida, J. M. Mazón, J. D. Rossi, and J. Toledo, *A Monge-Kantorovich mass transport problem for a discrete distance*, J. Funct. Anal., 260 (2011), pp. 3494–3534.

[9] V. Kostrykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A, 32 (1999), pp. 595–630.

[10] P. Kuchment, *Quantum graphs:* I. *Some basic structures*, Waves Random Media, 14 (2004), pp. S107–S128.

[11] J. M. Mazón, J. D. Rossi, and J. Toledo, *An optimal transportation problem with a cost given by the Euclidean distance plus import/export taxes on the boundary*, Rev. Mat. Iberoam., 30 (2014), pp. 277–308.

[12] J. M. Mazón, J. D. Rossi, and J. Toledo, *An optimal matching problem for the Euclidean distance*, SIAM J. Math. Anal., 46 (2014), pp. 233–255.

[13] J. M. Mazón, J. D. Rossi, and J. Toledo, *Mass transport problems for the Euclidean distance obtained as limits of p-Laplacian type problems with obstacles*, J. Differential Equations, 256 (2014), pp. 3208–3244.

[14] J. M. Mazón, J. D. Rossi, and J. Toledo, *Mass transport problems obtained as limits of p-Laplacian type problems with spatial dependence*, Adv. Nonlinear Anal., 3 (2014), pp. 133–140.

[15] O. Post, *Spectral Analysis on Graph-Like Spaces*, Lecture Notes in Math. 2012, Springer, Heidelberg,

[16] C. Villani, *Topics in Optimal Transportation*, Grad. Studi. Math. 58, AMS, Providence, RI, 2003.

[17] C. Villani, *Optimal transport. Old and new*, Grundlehren Math. Wiss. 338, Springer-Verlag, Berlin, 2009.

This paper has been sligthly modified to avoid a misprint: in the original paper there was used μ for two different measures, we have used an underline to minimize the confussion.