Research Article

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Mass transport problems obtained as limits of p-Laplacian type problems with spatial dependence

Abstract: We consider the following problem: given a bounded convex domain $\Omega \subset \mathbb{R}^N$ we consider the limit as $p \to \infty$ of solutions to

$$\begin{cases} -\operatorname{div}(b_p^{-p}|Du|^{p-2}Du) = f_+ - f_- & \text{in } \Omega, \\ b_p^{-p}|Du|^{p-2}\dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Under appropriate assumptions on the coefficients b_p that in particular verify that $\lim_{p\to\infty} b_p = b$ uniformly in $\overline{\Omega}$, we prove that there is a uniform limit of u_{p_j} (along a sequence $p_j \to \infty$) and that this limit is a Kantorovich potential for the optimal mass transport problem of f_+ to f_- with cost $c(x, y)$ given by the formula $c(x, y) = \inf_{\sigma(0)=x,\, \sigma(1)=y} \int_\sigma b\, ds$.

Keywords: Mass transport, Monge–Kantorovich problems, p-Laplacian equation

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1 Introduction

Taking limits as $p \to \infty$ in p-Laplacian type problems to find solutions to optimal mass transport problems (an idea from Evans and Gangbo [5]) has recently been used in [6, 8–10]. Our main goal in the present paper is to see what are the optimal transport problems that can be approximated when one considers a spatially dependent coefficient in the p-Laplacian approximations. Namely, we consider the following p-Laplacian type problem:

$$\begin{cases} -\operatorname{div}(b_p^{-p}|Du|^{p-2}Du) = f & \text{in } \Omega, \\ b_p^{-p}|Du|^{p-2}\dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded and convex C^2 domain in $\mathbb{R}^N$, $p > N$, $f = f_+ - f_- \in L^\infty(\Omega)$ has zero mean in Ω, $\int_\Omega f = 0$ (otherwise this problem does not have solutions), and the diffusion coefficient b_p is a continuous positive function $\overline{\Omega}$ such that

$$0 < \frac{b(x)}{(c_2)^{\frac{1}{p}}} \le b_p(x) \le \frac{B}{(c_1)^{\frac{1}{p}}} \quad \text{for all } x \in \Omega \text{ and all } p > N, \tag{1.2}$$

and

$$\lim_{p\to\infty} b_p = b \quad \text{uniformly in } \overline{\Omega} \tag{1.3}$$

for some continuous positive function b in $\overline{\Omega}$ and constants $B, c_1, c_2 > 0$.

The simplest example of positive b_p that verifies (1.3) is to consider b_p independent of p, $b_p(x) = b(x)$, for a fixed positive continuous function on $\overline{\Omega}$.

Existence and uniqueness (up to an additive constant) for this problem of a continuous weak solution in the Sobolev space $W^{1,p}(\Omega)$, $p > N$, can be easily obtained from variational arguments. It turns out that this weak solution is also a viscosity solution, see [7].

Limits as $p \to \infty$ of similar type problems are related to optimal mass transport problems for the Euclidean distance. In fact, this relation was the key to the first complete proof of the existence of an optimal transport map for the classical Monge problem (here the transport cost of one unit of mass between x and y is the Euclidean distance $|x - y|$) given by Evans and Gangbo in [5]. Note that the usual Euclidean distance is not a strictly convex cost. This makes this optimal mass transport different from the strictly convex cost case in which there is existence of a convex function (solution to a Monge–Ampere type problem) whose gradient provides an optimal transport map, see [11]. For notation and general results on Mass Transport Theory we refer to [1, 2, 4, 5, 11, 12].

In our case, we can pass to the limit in (1.1) and obtain that, for a sequence $p_j \to \infty$, $u_{p_j} \to u_\infty$ uniformly in $\overline{\Omega}$. It turns out that this limit u_∞ is a Kantorovich potential for the optimal transport problem that we describe below.

An optimal mass transport problem with a non-standard cost. Assume that we have some production in a domain Ω encoded in f_+ and some consumption encoded in f_-. To transport one unit of material from x to y we pay as transport cost $c(x, y)$ (we may take into account that the cost is not translation invariant in this transport operation) that in our case is given in terms of b by the formula

$$c(x, y) = \inf_{\sigma \in \Gamma(x,y)} \int_\sigma b \, ds, \tag{1.4}$$

where $\Gamma(x, y) := \{\sigma \in C^1([0, 1], \Omega) : \sigma(0) = x, \ \sigma(1) = y\}$.

The Monge transport problem is to find a Borel map T such that the push-forward of f_+ by T is f_- and minimizes

$$\int_\Omega c(x, T(x)) f_+(x) \, dx.$$

In its relaxed version (Monge–Kantorovich problem), this optimal transport problem reads as follows: Let $\Pi(f_+, f_-)$ be the set of transport plans between f_+ and f_-, that is, the set of non-negative Radon measures μ in $\Omega \times \Omega$ such that $\mathrm{proj}_x(\mu) = f_+ \, dx$ and $\mathrm{proj}_y(\mu) = f_- \, dx$; the aim is to find a measure $\mu^* \in \Pi(f_+, f_-)$ which minimizes the cost functional

$$\mathcal{K}_c(\mu) := \int_{\Omega \times \Omega} c(x, y) \, d\mu(x, y)$$

in the set $\Pi(f_+, f_-)$.

We prove the following result:

Theorem 1.1. *Let u_p be the unique solution to* (1.1) *which satisfies $\int_\Omega u_p = 0$ and assume* (1.2) *and* (1.3). *Then, there is a sequence $p_j \to \infty$ such that the uniform limit u_∞ of the solutions u_{p_j} is a Kantorovich potential for the optimal transport problem of f_+ to f_- with the cost given by $c(x, y)$ in* (1.4), *that is,*

$$\min\{\mathcal{K}_c(\mu) : \mu \in \Pi(f_+, f_-)\} = \sup\left\{\int_\Omega v f : v \in K_c(\Omega)\right\} = \int_\Omega u_\infty f,$$

where

$$K_c(\Omega) := \{v : \Omega \to \mathbb{R} : |v(x) - v(y)| \le c(x, y) \text{ for all } x, y \in \Omega\}.$$

Note that we can approximate the total transport cost since we have that

$$\lim_{p_j \to \infty} \int_\Omega u_{p_j} f = \int_\Omega u_\infty f.$$

Let us end the introduction with a brief description of the main techniques used in the proofs. Concerning approximations using p-Laplacian type operators, we quote [3], from where the main idea to show the key bounds for the L^p-norm of the gradient is taken. Once we have a uniform in p bound for the L^p-norm of the gradients we can extract a subsequence that converge uniformly and show that this limit is a maximizer of $\int_\Omega v f \, dx$ in $K_c(\Omega)$. From this the proof follows using the general duality argument that can be found, for example, in [11].

When b_p is of the form $b_p = Be^{-\eta}$, $\eta > 0$, for the study of the limit equation (in the viscosity sense) when $p \to \infty$ we refer to [7]. Here we focus our attention on the mass transport problem obtained in this limit procedure rather than in the equation that is verified by the limit.

The paper is organized as follows: In Section 2 we prove that there is a sequence of solutions to (1.1) that converges uniformly; in Section 3 we prove that the uniform limit is a solution (Kantorovich potential) to the optimal mass transport problem.

2 A p-Laplacian limit

Recall that we are considering problem (1.1). First, we show existence and uniqueness to it. The proof is standard, but we include the details for the sake of completeness.

Lemma 2.1. *Let $p > N$ be fixed. Then there exists a unique continuous solution to the variational problem*

$$\min_{S_p}\left\{\int_\Omega \frac{1}{b_p^p}\frac{|Du|^p}{p} - \int_\Omega uf\right\},$$

where

$$S_p = \left\{u \in W^{1,p}(\Omega) : \int_\Omega u = 0\right\}.$$

This minimum is a weak solution of problem (1.1)*, that is, it verifies*

$$\int_\Omega \frac{1}{b_p^p}|Du|^{p-2}DuD\phi = \int_\Omega f\phi \quad \text{for all } \phi \in C^\infty(\Omega).$$

Proof. By our assumptions we have that b_p^{-p} is bounded from below and above, $0 < c_1B^{-p} \le b_p^{-p} \le c_{2,p} < \infty$ (note that even c_1 can depend on p here since p is fixed along this proof). Hence, we obtain that for every $u \in W^{1,p}(\Omega)$ there holds

$$c_1B^{-p}\int_\Omega \frac{|Du|^p}{p} \le \int_\Omega \frac{1}{b_p^p}\frac{|Du|^p}{p} \le c_{2,p}\int_\Omega \frac{|Du|^p}{p}$$

and then the functional

$$\Theta(u) = \int_\Omega \frac{1}{b_p^p}\frac{|Du|^p}{p} - \int_\Omega uf$$

is well defined in the set S_p, which is convex, weakly closed and non-empty.

On the other hand, since $\int u = 0$ on S_p, by the inequalities of Poincaré, Hölder and Young, there exist positive constant c, C, independent of u, such that

$$c\|u\|_{W^{1,p}(\Omega)} \le \Theta(u) + C,$$

hence Θ is coercive and bounded from below, moreover it is weakly lower semicontinuous in S_p. Therefore, there is a minimizing sequence $u_n \in S_p \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u \in S_p$ and

$$\inf_S \Theta = \liminf_{n\to+\infty} \Theta(u_n) \ge \Theta(u).$$

Hence the minimum of Θ in S_p is attained. From the strict convexity of Θ in S_p we obtain that u_p is the unique minimum of Θ in S_p. Finally, u_p, the unique minimizer, is a weak solution of (1.1). The fact that u_p is continuous follows from the fact that $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ since $p > N$. □

Remark 2.2. Note that we have imposed that $\int_\Omega u = 0$ just to obtain uniqueness of the solution. As usually happens for homogeneous Neumann problems there are infinitely many solutions to (1.1), but any two of them differ by an additive constant.

Remark 2.3. Following the ideas in [7] it can be proved that a continuous weak solution to (1.1) is a viscosity solution to the same equation.

Our next step is to prove that we can extract a sequence of solutions to (1.1), u_{p_j} with $p_j \to \infty$, that converges uniformly as $j \to \infty$.

Lemma 2.4. *Let u_p be a solution to* (1.1), $p > N$. *There exists a sequence $p_j \to \infty$ such that $u_{p_j} \to u_\infty$ uniformly in $\overline{\Omega}$. Moreover, the limit u_∞ is Lipschitz continuous.*

Proof. Along this proof we will denote by C a constant independent of p that may change from one line to another.

Our first aim is to prove that the L^p-norm of the gradient of u_p is bounded independently of p. We already proved in the previous Lemma 2.1 that u_p is a minimizer of Θ in S_p. Then,

$$\Theta(u_p) = \int_\Omega \frac{1}{b_p^p} \frac{|Du_p|^p}{p} - \int_\Omega f u_p \le \Theta(0) = 0.$$

That is,

$$\int_\Omega \frac{1}{b_p^p} \frac{|Du_p|^p}{p} \le \int_\Omega f u_p.$$

Now,

$$\int_\Omega f u_p \le C \|Du_p\|_{L^p(\Omega)}.$$

Indeed, since $\int_\Omega u_p = 0$, there exists a point $x_p \in \Omega$ such that $u_p(x_p) = 0$. Then, since Ω is a bounded convex C^2 domain, for a fixed $x \in \Omega$, there exist $x = x_0, x_1, \dots, x_m = x_p$ and m balls $Q_i \subset \Omega$ $(i = 1, 2, \dots, m)$ of certain fixed diameter $r > 0$ such that $x_i, x_{i+1} \in Q_{i+1}$ and m is bounded independently of x, x_p. Then, the local Morrey's inequality (see, e.g., [4, Remark, p. 268]) implies

$$|u_p(x)| = |u_p(x) - u_p(x_p)| \le \sum_{i=1}^{m} |u_p(x_i) - u_p(x_{i+1})| \le C_0 r^{1-\frac{N}{p}} m \|\nabla u_p\|_p \le C_1 \|\nabla u_p\|_p,$$

being C_i independent of p.

Then we get

$$\int_\Omega \frac{1}{b_p^p} \frac{|Du_p|^p}{p} \le C \|Du_p\|_{L^p(\Omega)}. \tag{2.1}$$

Now we use that $b_p^{-p} \ge c_1 B^{-p}$ to obtain

$$\int_\Omega \left| \frac{Du_p}{B} \right|^p \le pC + pC \left\| \frac{Du_p}{B} \right\|_{L^p(\Omega)}.$$

From this inequality and using that $(pC)^{\frac{1}{p-1}} \to 1$ (since C is independent of p), we obtain that

$$\left(\int_\Omega |Du_p|^p \right)^{\frac{1}{p}} \le C_1, \tag{2.2}$$

with C_1 a constant independent of p.

Now, using this uniform bound, we prove uniform convergence of a sequence u_{p_j}. In fact, we take m such that $N < m \le p$ and obtain the bound

$$\|Du_p\|_{L^m(\Omega)} = \left(\int_\Omega |Du_p|^m \cdot 1 \right)^{\frac{1}{m}} \le \left[\left(\int_\Omega |Du_p|^p \right)^{\frac{m}{p}} \left(\int_\Omega 1 \right)^{\frac{p-m}{p}} \right]^{\frac{1}{m}} \le C_1 |\Omega|^{\frac{p-m}{pm}} \le C,$$

the constant C being independent of p. We have proved that the sequence $\{u_p\}_{p>N}$ is bounded in $W^{1,m}(\Omega)$, and we know that $\int_\Omega u_p = 0$, so we can obtain a weakly convergent sequence $u_{p_j} \rightharpoonup u_\infty \in W^{1,m}(\Omega)$ with $p_j \to +\infty$.

Since $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ and $u_{p_j} \rightharpoonup u_\infty \in W^{1,p}(\Omega)$, we obtain $u_{p_j} \to u_\infty$ in $C^{0,\alpha}(\Omega)$, and in particular $u_{p_j} \to u_\infty$ uniformly in $\overline{\Omega}$. As $u_{p_j} \in C(\overline{\Omega})$, it follows that $u_\infty \in C(\overline{\Omega})$. Using a diagonal procedure, we conclude the existence of a sequence u_{p_j} that is weakly convergent in $W^{1,m}(\Omega)$ for every m.

Finally, let us show that the limit function u_∞ is Lipschitz. In fact, we proved that

$$\left(\int_\Omega |Du_\infty|^m\right)^{\frac{1}{m}} \le \liminf_{p_j \to +\infty} \left(\int_\Omega |Du_{p_j}|^m\right)^{\frac{1}{m}} \le C_1 |\Omega|^{\frac{1}{m}} \le C.$$

Now, we take $m \to \infty$ to obtain $\|Du_\infty\|_{L^\infty(\Omega)} \le C$. So, we have proved $u_\infty \in W^{1,\infty}(\Omega)$, that is, u_∞ is a Lipschitz function. □

3 Mass transport interpretation of the limit

The goal of this section is to show that u_∞ is a Kantorovich potential for the mass transport problem of f_+ to f_- with the cost $c(x, y)$ given by

$$c(x, y) = \inf_{\sigma \in \Gamma(x,y)} \int_\sigma b \, ds,$$

that is,

$$c(x, y) = \inf_{\sigma \in \Gamma(x,y)} \int_0^1 L(\sigma(t), \sigma'(t)) \, dt,$$

with L the Lagrangian given by $L(z, \xi) = b(z)|\xi|$.

The key idea to identify the cost is as follows: if we have a Lipschitz continuous function u such that

$$|Du(x)| \le b(x) \quad \text{a.e. in } \Omega,$$

then choosing a path σ with $\sigma(0) = x$, $\sigma(1) = y$ and

$$c(x, y) \ge \int_\sigma b \, ds - \varepsilon,$$

we have

$$|u(x) - u(y)| = \left| \int_0^l \langle Du(\sigma(t)), \sigma'(t) \rangle \, dt \right| \le \int_0^l b(\sigma(t)) |\sigma'(t)| \, dt \le c(x, y) + \varepsilon.$$

Hence, we conclude that

$$|u(x) - u(y)| \le c(x, y).$$

Conversely, if we have

$$|u(x) - u(y)| \le c(x, y),$$

then

$$|Du(x)| \le b(x) \quad \text{a.e. in } \Omega. \tag{3.1}$$

In fact, for $\xi \in \mathbb{R}^N$ and $h \in \mathbb{R}$ with $|h|$ small enough, if we just consider the path $\sigma : [0, 1] \to \Omega$ given by

$$\sigma(t) = x + t(hb^{-1}(x)\xi),$$

we have

$$|\langle b^{-1}(x) Du(x), \xi \rangle| = |\langle Du(x), b^{-1}(x)\xi \rangle| = \lim_{h \to 0} \frac{|u(x) - u(x + hb^{-1}(x)\xi)|}{|h|} \le \liminf_{h \to 0} \frac{c(x, x + hb^{-1}(x)\xi)}{|h|}$$

$$\le \liminf_{h \to 0} \frac{1}{|h|} \int_\sigma b \, ds = \liminf_{h \to 0} \int_0^1 b(x + thb^{-1}(x)\xi) b^{-1}(x) |\xi| \, dt = |\xi|,$$

from where we get (3.1).

Therefore, if c and b are related by

$$c(x, y) = \inf_{\sigma \in \Gamma(x,y)} \int_\sigma b \, ds,$$

then the set of functions

$$K_c(\Omega) = \left\{ u : \Omega \to \mathbb{R} : |u(x) - u(y)| \le c(x, y) \text{ and } \int_\Omega u = 0 \right\}$$

coincides with the set

$$\tilde{K}_b(\Omega) := \left\{ u : \Omega \to \mathbb{R} : |Du(x)| \le b(x) \text{ and } \int_\Omega u = 0 \right\}.$$

Hence, we have that

$$\sup\left\{ \int_\Omega vf : v \in K_c(\Omega) \right\} = \sup\left\{ \int_\Omega vf : v \in \tilde{K}_b(\Omega) \right\}. \tag{3.2}$$

Lemma 3.1. *Any uniform limit u_∞ of a sequence u_{p_j} is a Kantorovich potential for the optimal transport problem of f_+ to f_- with the cost given by*

$$c(x, y) = \inf_{\sigma \in \Gamma(x,y)} \int_\sigma b \, ds,$$

that is, it holds that

$$\min\{\mathcal{K}_c(\mu) : \mu \in \Pi(f_+, f_-)\} = \sup\left\{ \int_\Omega vf : v \in K_c(\Omega) \right\} = \int_\Omega u_\infty f.$$

Proof. The equality

$$\min\{\mathcal{K}_c(\mu) : \mu \in \Pi(f_+, f_-)\} = \sup\left\{ \int_\Omega vf : v \in K_c(\Omega) \right\}$$

follows by well-known duality arguments, using that c is a distance, see [11]. Therefore, due to (3.2), we just need to show that

$$\sup\left\{ \int_\Omega vf : v \in \tilde{K}_b(\Omega) \right\} = \int_\Omega u_\infty f.$$

Now, we have that, for every Lipschitz function v with $|Dv| \le b$ a.e. in Ω and $\int_\Omega v = 0$,

$$\Theta(u_p) \le \Theta(v),$$

and then

$$-\int_\Omega f u_p \le \int_\Omega \frac{1}{b_p^p} \frac{|Du_p|^p}{p} - \int_\Omega f u_p \le \int_\Omega \frac{1}{b_p^p} \frac{|Dv|^p}{p} - \int_\Omega fv \le \int_\Omega \frac{b^p}{p b_p^p} - \int_\Omega fv \le \frac{c_2}{p}|\Omega| - \int_\Omega fv,$$

where we have used (1.2). Taking limits as $p \to \infty$ we obtain

$$\int_\Omega f u_\infty \ge \sup\left\{ \int_\Omega vf : v \in \tilde{K}_b(\Omega) \right\}.$$

Then we just need to show that $u_\infty \in \tilde{K}_b(\Omega)$. From the uniform convergence of u_{p_j} to u_∞, we immediately conclude that

$$\int_\Omega u_\infty = 0.$$

Now, using again the computations of the proof of Lemma 2.4, (2.1) and (2.2), we have

$$\int_\Omega \frac{1}{b_p^p} \frac{|Du_p|^p}{p} \le C,$$

with C independent of p. Hence,

$$\left(\int_\Omega \left|\frac{Du_p}{b_p}\right|^p\right)^{\frac{1}{p}} \le (pC)^{\frac{1}{p}}.$$

To finish, let us argue as in the final part of the proof of Lemma 2.4. Let $N < m < p$. We get

$$\frac{Du_p}{b_p} \rightharpoonup \frac{Du_\infty}{b} \quad \text{in } L^m(\Omega)$$

and

$$\left(\int_\Omega \left|\frac{Du_\infty}{b}\right|^m\right)^{\frac{1}{m}} \le \liminf_{p_j\to\infty}\left(\int_\Omega \left|\frac{Du_{p_j}}{b_{p_j}}\right|^m\right)^{\frac{1}{m}}.$$

Now,

$$\left(\int_\Omega \left|\frac{Du_{p_j}}{b_{p_j}}\right|^m\right)^{\frac{1}{m}} \le |\Omega|^{\frac{p_j-m}{mp_j}}\left(\int_\Omega \left|\frac{Du_{p_j}}{b_{p_j}}\right|^{p_j}\right)^{\frac{1}{p_j}} \le |\Omega|^{\frac{p_j-m}{mp_j}}(p_jC)^{\frac{1}{p_j}}.$$

Hence, since C is independent of p_j, we have $(p_jC)^{\frac{1}{p_j}} \to 1$ and

$$\left(\int_\Omega \left|\frac{Du_\infty}{b}\right|^m\right)^{\frac{1}{m}} \le |\Omega|^{\frac{1}{m}}.$$

Taking now $m \to \infty$, we get

$$\|b^{-1}Du_\infty\|_{L^\infty(\Omega)} \le 1,$$

that is,

$$|Du_\infty(x)| \le b(x)$$

a.e. in Ω and we conclude that $u_\infty \in \tilde{K}_b(\Omega)$. □

Remark 3.2. In one space dimension, that is, $\Omega = (a, b) \subset \mathbb{R}$, it is easy to see that

$$\min\{\mathcal{K}_c(\mu) : \mu \in \Pi(f_+, f_-)\} = \sup\left\{\int_\Omega vf : v \in K_c(\Omega)\right\} = \int_\Omega u_\infty f = \sup\left\{\int_\Omega vf : v \in K_{\tilde{c}}(\Omega)\right\},$$

where

$$\tilde{c}(x, y) := \int_0^1 b((1-t)x + ty)\,dt\,|x-y| = \int_0^{|x-y|} b\left(x + t\frac{y-x}{|x-y|}\right)dt,$$

being the last term 0 when $x = y$. Nevertheless, in general, for dimension greater than one, this total cost is strictly less than

$$\min\{\mathcal{K}_{\tilde{c}}(\mu) : \mu \in \Pi(f_+, f_-)\}.$$

In one dimension, both total costs coincide; indeed, if we set

$$d(r) = \int_0^r b(s)\,ds,$$

then

$$c(x, y) = \tilde{c}(x, y) = |d(x) - d(y)|.$$

In higher dimensions this is not true in general since with the cost $\tilde{c}$ we are using straight lines to go from x to y and we can have functions b for which a straight line is not the optimal one when computing the cost $c(x, y)$ given by (1.4).

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